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Emden–Fowler-type neutral differential equations: oscillatory properties of solutions

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Abstract

In this paper, we study the oscillation of a class of fourth-order Emden–Fowler delay differential equations with neutral term. Using the Riccati transformation and comparison method, we establish several new oscillation conditions. These new conditions complement a number of results in the literature. We give examples to illustrate our main results.

MSC: 34C10; 34K11

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1 Introduction

In this paper, we study the oscillatory properties of solutions of the following fourth-order neutral differential equation:

$$X_t' + q(t)x^{(p_2-1)}\bigl(\sigma(t)\bigr) = 0, \quad t \geq t_0, \tag{1}$$

where $X_t = a(t)(y'''(t))^{(p_1-1)}$ and $y(t) := x(t) + r(t)x(\delta(t))$. We make the following assumptions:

N1: $r \in C[t_0,\infty)$, $0 \leq r(t) < r_0 < \infty$,

N2: $\delta, \sigma, q \in C[t_0,\infty)$, $q(t) > 0$, $\delta(t) \leq t$, $\lim_{t\to\infty}\delta(t) = \lim_{t\to\infty}\sigma(t) = \infty$,

N3: $a \in C[t_0,\infty)$, $a(t) > 0$, $a'(t) \geq 0$, and

$$\int_{t_0}^{\infty} \frac{1}{a^{1/(p_1-1)}(s)}\,\mathrm{d}s = \infty, \tag{2}$$

N4: $p_i > 1$, $i = 1,2$, are constants, and

$$p_1 := \begin{cases} 2 & \text{if } p_2 \leq 2, \\ 1 + 2^{\beta-1} & \text{if } p_2 > 2. \end{cases}$$

By a solution of (1) we mean a function $x \in C^3[t,\infty)$, $t \geq t_0$, that has the property $a(t)(y'''(t))^{\alpha} \in C^1[t_0,\infty)$ and satisfies (1) on $[t_0,\infty)$.

The study of differential equations has been the object of many researchers over the last decades. Different approaches and various techniques are adopted to investigate the qualitative properties of their solutions. Recently, driven by their widespread applications, the investigation of fourth-order differential equations has drawn significant attention. The existence uniqueness, stability, and oscillation of solutions were the main features that attracted consideration [1–3].

In spite of the increasing interest in the study of second-order differential equations, the oscillation and nonoscillation of solutions for differential equations are still considered as an open area to investigate [4–9]. Equations with neutral terms are of particular significance as they arise in many applications including systems of control, electrodynamics, mixing liquids, neutron transportation, networks, and population models. In the qualitative analysis of such systems, indeed, the oscillatory behavior of solutions of equations, where the rate of the growth depends not only on the current and the past states but also on the rate of change in the past, play an important role [10–14]. In the light of this motivation and justification, different results have been reported regarding the asymptotic behavior of higher-order differential equations with neutral terms [15–18]. For relevant results on the application of oscillation theory, the reader can consult [19–21]. In the past 20 years, there have been a lot of research results on the oscillation of differential equations. As a matter of fact, Eq. (1) is a natural of the half-linear/Emden–Fowler differential equation (including the related differential equation), which arises in a variety of real-world problems such as the study of p-Laplace equations, non-Newtonian fluid theory, the turbulent flow of a polytrophic gas in a porous medium, and so on; see, for example, the papers [22, 23] for more detail.

The authors in [24, 25] considered the equation

$$y^{(r)}(t) + q(t)x\bigl(\sigma(t)\bigr) = 0$$

and proved that it is oscillatory if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} K(s)\,\mathrm{d}s > 2^{(r-1)(r-2)} \frac{(r-1)}{\mathrm{e}} \tag{3}$$

and

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} K(s)\,\mathrm{d}s > \frac{(r-1)!}{\mathrm{e}}, \tag{4}$$

where $K(t) := \sigma^{r-1}(t)(1 - b(\sigma(t)))q(t)$, and r is an even.

In [26, 27] the authors proved that the equation

$$\bigl(a(t)\bigl(y^{(n-1)}(t)\bigr)^{\alpha}\bigr)' + q(t)x^{\alpha}\bigl(\sigma(t)\bigr) = 0$$

is oscillatory if

$$\liminf_{t\to\infty} \int_{\delta^{-1}(\sigma(t))}^{t} \frac{q(s)}{a(s)} \bigl(s^{n-1}\bigr)^{\alpha}\,\mathrm{d}s > \left(\frac{1}{\sigma_0} + \frac{p_0^{\alpha}}{\sigma_0\delta_0}\right) \frac{((n-1)!)^{\alpha}}{\mathrm{e}} \tag{5}$$

and

$$\liminf_{t\to\infty}\int_{\delta^{-1}(\eta(t))}^{t}\left(\frac{(\tau^{-1}(\eta(s)))^{n-1}}{a^{1/\alpha}(\delta^{-1}(\eta(s)))}\right)^{\alpha} q(s)P_n^{\alpha}\big(\sigma(s)\big)\,\mathrm{d}s > \frac{((n-1)!)^{\alpha}}{\mathrm{e}}, \tag{6}$$

where $\eta \in C^1([t_0,\infty),\mathbb{R})$ and $\widehat{q}(t) := \min\{q(\sigma^{-1}(t)), q(\sigma^{-1}(\delta(t)))\}$.

Li et al. [28] studied the oscillatory and asymptotic behavior of higher-order Emden–Fowler neutral differential equations using the Riccati substitution together with integral averaging technique. Bazighifan [29] established sufficient conditions for the oscillation of all solutions of (1) and used the comparison method with second-order equations. Agarwal et al. [30] gave some results providing information on the asymptotic behavior of solutions of fourth-order Emden–Fowler neutral differential equations. This time the authors used the comparison method with first- and second-order equations. In [31] the authors considered the equation

$$\big(a(t)X\big(y^{(n-1)}(t)\big)\big)' + q(t)F\big(y\big(\sigma(t)\big)\big) = 0,$$

where $F = |s|^{p-2}s$, and obtained some new oscillation conditions.

In particular, there has been interest from many researchers to study the oscillatory behavior of this type of equation; see [32–39].

In this paper, we establish oscillatory properties of solutions of (1) and give some examples for applying the criteria.

2 Preliminaries

We first provide some notations which help us to easily display the results. Moreover, we present some auxiliary lemmas.

Lemma 2.1 ([40]) *If $u^{(i)}(t) > 0$, $i = 0,1,\ldots,j$, and $u^{(j+1)}(t) < 0$ eventually, then, for every $\varepsilon_1 \in (0,1)$, $u(t)/u'(t) \geq \varepsilon_1 t/j$ eventually.*

Lemma 2.2 ([41]) *Let $u \in C^j([t_0,\infty),(0,\infty))$. Assume that $u^{(j)}(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$ and that there exists $t_1 \geq t_0$ such that $u^{(j-1)}(t)u^{(j)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} u(t) \neq 0$, then for every $\mu \in (0,1)$, there exists $t_\mu \geq t_1$ such that*

$$u(t) \geq \frac{\mu}{(j-1)!}t^{j-1}\big|u^{(j-1)}(t)\big| \quad \textit{for } t \geq t_\mu.$$

Lemma 2.3 ([42, Lemmas 1 and 2]) *Let $m_1, m_2 \geq 0$. Then*

$$(m_1+m_2)^{\beta} \leq \begin{cases} 2^{\beta-1}(m_1^{\beta}+m_2^{\beta}) & \textit{for } \beta \geq 1,\\ m_1^{\beta}+m_2^{\beta} & \textit{for } \beta \leq 1.\end{cases}$$

For convenience, we impose the following hypothesis:

(H1) x is an eventually positive solution of (1).

3 Main results

Theorem 3.1 *Assume that*

$$\big(\sigma^{-1}(t)\big)' \geq \sigma_0 > 0 \quad \textit{and} \quad \delta'(t) \geq \delta_0 > 0. \tag{7}$$

If

$$\eta'(t) + \frac{1}{(p_1-1)}\left(\frac{\mu t^3}{6a^{1/(p_1-1)}(t)}\right)^{(p_2-1)}\left(\frac{\sigma_0\delta_0}{\delta_0 + r_0^{(p_2-1)}}\right)^{(p_2-1)/(p_1-1)} \times \widehat{q}(t)\eta^{(p_2-1)/(p_1-1)}\bigl(\delta^{-1}\bigl(\sigma(t)\bigr)\bigr) \le 0 \tag{8}$$

is oscillatory, where

$$\widehat{q}(t) := \min\bigl\{q\bigl(\sigma^{-1}(t)\bigr), q\bigl(\sigma^{-1}\bigl(\delta(t)\bigr)\bigr)\bigr\},$$

then (1) *is oscillatory.*

Proof Let t be a nonoscillatory solution of (1) on $[t_0,\infty)$. Then $t > 0$, and there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\delta(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$. Since $a'(t) > 0$, we have

$$\begin{aligned} &y(t) > 0, \qquad y'(t) > 0, \qquad y'''(t) > 0, \qquad y^{(4)}(t) < 0, \quad \text{and} \\ &\bigl(a(t)\bigl(y'''(t)\bigr)^{(p_1-1)}\bigr)' \le 0 \end{aligned} \tag{9}$$

for $t \ge t_1$. From (1) we get

$$\frac{1}{(\sigma^{-1}(t))'}\bigl(a\bigl(\sigma^{-1}(t)\bigr)\bigl(y'''\bigl(\sigma^{-1}(t)\bigr)\bigr)^{(p_1-1)}\bigr)' + q\bigl(\sigma^{-1}(t)\bigr)x^{(p_2-1)}(t) = 0. \tag{10}$$

By Lemma 2.3 and the definition of y we obtain

$$\begin{aligned} y^{(p_2-1)}(t) &= \bigl(x(t) + r(t)x\bigl(\delta(t)\bigr)\bigr)^{(p_2-1)} \\ &\le (p_1-1)\bigl(x^{(p_2-1)}(t) + r_0^{(p_2-1)}x^{(p_2-1)}\bigl(\delta(t)\bigr)\bigr). \end{aligned} \tag{11}$$

From (10) and (11) we obtain

$$\begin{aligned} 0 &= \frac{1}{(\sigma^{-1}(t))'}\bigl(a\bigl(\sigma^{-1}(t)\bigr)\bigl(y'''\bigl(\sigma_j^{-1}(t)\bigr)\bigr)^{(p_1-1)}\bigr)' + q\bigl(\sigma^{-1}(t)\bigr)x^{(p_2-1)}(t) \\ &\quad + r_0^{(p_2-1)}\biggl(\frac{1}{(\sigma^{-1}(\delta(t)))'}\bigl(a\bigl(\sigma^{-1}\bigl(\delta(t)\bigr)\bigr)\bigl(y'''\bigl(\sigma^{-1}\bigl(\delta(t)\bigr)\bigr)\bigr)^{(p_1-1)}\bigr)' \\ &\quad + q\bigl(\sigma^{-1}\bigl(\delta(t)\bigr)\bigr)x^{(p_2-1)}\bigl(\delta(t)\bigr)\biggr) \\ &= \frac{(a(\sigma^{-1}(t))(y'''(\sigma^{-1}(t)))^{(p_1-1)})'}{(\sigma^{-1}(t))'} + r_0^{(p_2-1)}\frac{(a(\sigma^{-1}(\delta(t)))(y'''(\sigma^{-1}(\delta(t))))^{(p_1-1)})'}{(\sigma^{-1}(\delta(t)))'} \\ &\quad + q\bigl(\sigma^{-1}(t)\bigr)x^{(p_2-1)}(t) + r_0^{(p_2-1)}q\bigl(\sigma^{-1}\bigl(\delta(t)\bigr)\bigr)x^{(p_2-1)}\bigl(\delta(t)\bigr) \\ &\ge \frac{(a(\sigma^{-1}(t))(y'''(\sigma^{-1}(t)))^{(p_1-1)})'}{(\sigma^{-1}(t))'} + r_0^{(p_2-1)}\frac{(a(\sigma^{-1}(\delta(t)))(y'''(\sigma^{-1}(\delta(t))))^{(p_1-1)})'}{(\sigma^{-1}(\delta(t)))'} \\ &\quad + \frac{1}{(p_1-1)}\widehat{q}(t)y^{(p_2-1)}(t), \end{aligned}$$

which, together with (7), gives

$$\begin{aligned}&\frac{1}{\sigma_0}\left(a\left(\sigma^{-1}(t)\right)\left(y'''\left(\sigma_j^{-1}(t)\right)\right)^{(p_1-1)}\right)'\\&\quad+\frac{r_0^{(p_2-1)}}{\sigma_0\delta_0}\left(a\left(\sigma^{-1}\left(\delta(t)\right)\right)\left(y'''\left(\sigma^{-1}\left(\delta(t)\right)\right)\right)^{(p_1-1)}\right)'+\frac{1}{(p_1-1)}\widehat{q}(t)y^{(p_2-1)}(t)\leq 0.\end{aligned}\tag{12}$$

Since $y'(t)>0$, we find $\lim_{t\to\infty}y(t)\neq 0$, and by Lemma 2.2 we obtain

$$y(t)\geq\frac{\mu}{6}t^3y'''(t).\tag{13}$$

Combining (12) and (13), we see that

$$\begin{aligned}&\frac{1}{\sigma_0}\left(a\left(\sigma^{-1}(t)\right)\left(y'''\left(\sigma_j^{-1}(t)\right)\right)^{(p_1-1)}\right)'+\frac{r_0^{(p_2-1)}}{\sigma_0\delta_0}\left(a\left(\sigma^{-1}\left(\delta(t)\right)\right)\left(y'''\left(\sigma^{-1}\left(\delta(t)\right)\right)\right)^{(p_1-1)}\right)'\\&\quad+\frac{1}{(p_1-1)}\widehat{q}(t)\left(\frac{\mu}{6}t^3\right)^{(p_2-1)}\left(y'''(t)\right)^{(p_2-1)}\leq 0.\end{aligned}\tag{14}$$

Setting

$$\eta(t):=\frac{1}{\sigma_0}a\left(\sigma^{-1}(t)\right)\left(y'''\left(\sigma_j^{-1}(t)\right)\right)^{(p_1-1)}+\frac{r_0^{(p_2-1)}}{\sigma_0\delta_0}a\left(\sigma^{-1}\left(\delta(t)\right)\right)\left(y'''\left(\sigma^{-1}\left(\delta(t)\right)\right)\right)^{(p_1-1)},$$

we easily see that

$$\eta\left(\delta^{-1}\left(\sigma(t)\right)\right)\leq\left(\frac{\delta_0+r_0^{(p_2-1)}}{\sigma_0\delta_0}\right)a(t)\left(y'''(t)\right)^{(p_1-1)}.$$

From (14) we find

$$\begin{aligned}&\eta'(t)+\frac{1}{(p_1-1)}\left(\frac{\mu t^3}{6a^{1/(p_1-1)}(t)}\right)^{(p_2-1)}\left(\frac{\sigma_0\delta_0}{\delta_0+r_0^{(p_2-1)}}\right)^{(p_2-1)/(p_1-1)}\\&\quad\times\widehat{q}(t)\eta^{(p_2-1)/(p_1-1)}\left(\delta^{-1}\left(\sigma(t)\right)\right)\leq 0,\end{aligned}$$

which is a contradiction. □

Theorem 3.2 *Assume that* (7) *holds. If*

$$\vartheta'(t)+\frac{1}{(p_1-1)}\left(\frac{\mu t^3}{6a^{1/(p_1-1)}(t)}\right)^{(p_2-1)}\left(\frac{\sigma_0\delta_0}{\delta_0+r_0^{(p_2-1)}}\right)\widehat{q}(t)\vartheta^{(p_2-1)/(p_1-1)}\left(\sigma(t)\right)\leq 0\tag{15}$$

is oscillatory, then (1) *is oscillatory.*

Proof It is known that (14) holds in the proof of Theorem 3.1. If we set

$$\vartheta(t):=a\left(\sigma^{-1}(t)\right)\left(y'''\left(\sigma^{-1}(t)\right)\right)^{(p_1-1)},$$

then ϑ is a positive solution of (15), which is a contradiction. □

Corollary 3.1 *Let $p_1 = p_2$, and let (7) hold. If $a(t) \leq t$ and*

$$\liminf_{t\to\infty} \int_{a(t)}^{t} \frac{s^{3(p_1-1)}}{a(s)} \widehat{q}(s)\,\mathrm{d}s > \left(\frac{\delta_0 + r_0^{(p_1-1)}}{\sigma_0\delta_0}\right) \frac{(p_1-1)6^{(p_1-1)}}{\mathrm{e}}, \tag{16}$$

where $a(t) = \delta^{-1}(\sigma(t))$ or $\sigma(t)$, then (1) is oscillatory.

Theorem 3.3 *Let $r_0 < 1$ and $\sigma(t) \leq t$. If for some $\mu \in (0,1)$,*

$$\psi'(t) + (1-r_0)^{(p_2-1)} \left(\frac{\mu\sigma^3(t)}{6a^{1/(p_1-1)}(\sigma(t))}\right)^{(p_2-1)} q(t)\psi^{(p_2-1)/(p_1-1)}\bigl(\sigma(t)\bigr) = 0 \tag{17}$$

is oscillatory, then (1) is oscillatory.

Proof It is known that (9) holds in the proof of Theorem 3.1. By the definition of y we find

$$\begin{aligned} x(t) &\geq y(t) - r_0 x\bigl(\delta(t)\bigr) \geq y(t) - r_0 y\bigl(\delta(t)\bigr) \\ &\geq (1-r_0)y(t), \end{aligned}$$

which, together with (1), gives

$$\bigl(a(t)\bigl(y'''(t)\bigr)^{(p_1-1)}\bigr)' + q(t)(1-r_0)^{(p_2-1)} y^{(p_2-1)}\bigl(\sigma(t)\bigr) \leq 0. \tag{18}$$

From Lemma 2.2 we obtain

$$y(t) \geq \frac{\mu}{6} t^3 y'''(t). \tag{19}$$

Combining (18) and (19), we get

$$\bigl(a(t)\bigl(y'''(t)\bigr)^{(p_1-1)}\bigr)' + q(t)(1-r_0)^{(p_2-1)} \left(\frac{\mu}{6}\sigma^3(t)\right)^{(p_2-1)} \bigl(y'''\bigl(\sigma(t)\bigr)\bigr)^{(p_2-1)} \leq 0.$$

If we set $\psi := a(y''')^{(p_1-1)}$, then

$$\psi'(t) + (1-r_0)^{(p_2-1)} \left(\frac{\mu\sigma^3(t)}{6a^{1/(p_1-1)}(\sigma(t))}\right)^{(p_2-1)} q(t)\psi^{(p_2-1)/(p_1-1)}\bigl(\sigma(t)\bigr) \leq 0.$$

In view of [37, Corollary 1], equation (17) also has a positive solution, which is a contradiction. □

Corollary 3.2 *Let $p_1 = p_2$, $r_0 < 1$, and $\sigma(t) \leq t$. If*

$$\liminf_{t\to\infty} \int_{\sigma_j(t)}^{t} \frac{\sigma^{3(p_1-1)}(s)}{a(\sigma(s))} q(s)\,\mathrm{d}s > \frac{6^{(p_1-1)}}{(1-r_0)^{(p_1-1)}\mathrm{e}}, \tag{20}$$

then (1) is oscillatory.

Lemma 3.1 *If* $(H1)$ *holds, then*

$$\phi_1'(t) \le \frac{\varpi_1'(t)}{\varpi_1(t)}\phi_1(t) - \varpi_1(t)q(t)(1-r_0)^{(p_2-1)}y^{p_2-p_2}(t)\varepsilon_1\left(\frac{\sigma_j(t)}{t}\right)^{3(p_2-1)} \\ - (p_1-1)\mu_1\frac{t^2}{2a^{1/(p_1-1)}(t)\varpi_1^{1/(p_1-1)}(t)}\phi_1^{\frac{p_1}{(p_1-1)}}(t) \tag{21}$$

for some $\mu_1, \varepsilon_1 \in (0,1)$ *and every* $M_1 > 0$, *where*

$$\Psi(t) := M_1^{p_2-p_1}\varpi_1(t)q(t)(1-r_0)^{(p_2-1)}\left(\frac{\sigma(t)}{t}\right)^{3(p_2-1)}.$$

Proof Let $(H1)$ hold. In the case where $y''(t) > 0$, let

$$\phi_1(t) := \varpi_1(t)\frac{a(t)(y'''(t))^{(p_1-1)}}{y^{(p_1-1)}(t)} > 0.$$

From (18) we find

$$\phi_1'(t) \le \frac{\varpi_1'(t)}{\varpi_1(t)}\phi_1(t) - \varpi_1(t)q(t)(1-r_0)^{(p_2-1)}\frac{y^{(p_2-1)}(\sigma(t))}{y^{(p_1-1)}(t)} \\ - (p_1-1)\varpi_1(t)\frac{a(t)(y'''(t))^{(p_1-1)}}{y^{p_1}(t)}y'(t). \tag{22}$$

Using Lemma 2.1, we obtain $y(t) \ge \frac{t}{3}y'(t)$, and hence

$$\frac{y(\sigma_j(t))}{y(t)} \ge \varepsilon_1\frac{\sigma^3(t)}{t^3}. \tag{23}$$

Using Lemma 2.2, we get

$$y'(t) \ge \frac{\mu_1}{2}t^2y'''(t) \tag{24}$$

for all $\mu_1 \in (0,1)$. Thus by (22), (23), and (24) we obtain

$$\phi_1'(t) \le \frac{\varpi_1'(t)}{\varpi_1(t)}\phi_1(t) - \varpi_1(t)q(t)(1-r_0)^{(p_2-1)}y^{p_2-p_2}(t)\varepsilon_1\left(\frac{\sigma_j(t)}{t}\right)^{3(p_2-1)} \\ - (p_1-1)\mu_1\frac{t^2}{2a^{1/(p_1-1)}(t)\varpi_1^{1/(p_1-1)}(t)}\phi_1^{\frac{p_1}{(p_1-1)}}(t).$$

This completes the proof. □

Lemma 3.2 *If* $(H1)$ *holds, then*

$$\phi_2'(t) \le -\Psi_1(t) + \frac{\varpi'(t)}{\varpi(t)}\vartheta(t) - \frac{1}{\varpi(t)}\phi_2^2(t) \tag{25}$$

for some $\varepsilon_1 \in (0,1)$ *and every* $M_2 > 0$, *where*

$$\Psi_1(t) := \big((1-r_0)\varepsilon_1\big)^{(p_2-1)/(p_1-1)} \varpi(t) M_2^{(p_2-p_1)/(p_1-1)} \times \int_t^\infty \left(\frac{1}{a(t)} \int_t^\infty q(s) \frac{\sigma^{(p_2-1)}(s)}{s^{(p_2-1)}} \,\mathrm{d}s\right)^{1/(p_1-1)} \mathrm{d}t,$$

Proof Let $(H1)$ hold. In the case where $y''(t) < 0$, integrating (18) from t to t, we find

$$a(t)\big(y'''(t)\big)^{(p_1-1)} - a(t)\big(y'''(t)\big)^{(p_1-1)} \le -\int_t^t q(s)(1-r_0)^{(p_2-1)} y^{(p_2-1)}\big(\sigma(s)\big)\,\mathrm{d}s. \tag{26}$$

By Lemma 2.1 we get $y(t) \ge xy'(t)$, and hence

$$y\big(\sigma(t)\big) \ge \varepsilon_1 \frac{\sigma(t)}{t} y(t). \tag{27}$$

For (26), letting $t \to \infty$ and using (27), we get

$$a(t)\big(y'''(t)\big)^{(p_1-1)} \ge \big((1-r_0)\varepsilon_1\big)^{(p_2-1)} y^{(p_2-1)}(t) \int_t^\infty q(s) \frac{\sigma_j^{(p_2-1)}(s)}{s^{(p_2-1)}} \,\mathrm{d}s. \tag{28}$$

Integrating (28) from t to ∞, we get

$$y''(t) \le -\big((1-r_0)\varepsilon_1\big)^{(p_2-1)/(p_1-1)} y^{(p_2-1)/(p_1-1)}(t) \times \int_t^\infty \left(\frac{1}{a(t)} \int_t^\infty q(s) \frac{\sigma^{(p_2-1)}(s)}{s^{(p_2-1)}} \,\mathrm{d}s\right)^{1/(p_1-1)} \mathrm{d}t \tag{29}$$

for all $\varepsilon_1 \in (0,1)$. Now we define

$$\phi_2(t) = \varpi(t) \frac{y'(t)}{y(t)}.$$

Then $\phi_2(t) > 0$ for $t \ge t_1$. Using (32) and (29), we obtain

$$\begin{aligned}
\phi_2'(t) &= \frac{\varpi'(t)}{\varpi(t)} \phi_2(t) + \varpi(t) \frac{y''(t)}{y(t)} - \varpi(t) \left(\frac{y'(t)}{y(t)}\right)^2 \\
&\le \frac{\varpi'(t)}{\varpi(t)} \phi_2(t) - \frac{1}{\varpi(t)} \phi_2^2(t) \\
&\quad - \big((1-r_0)\varepsilon_1\big)^{(p_2-1)/(p_1-1)} \varpi(t) y^{(p_2-1)/(p_1-2)}(t) \\
&\quad \times \int_t^\infty \left(\frac{1}{a(t)} \int_t^\infty q(s) \frac{\sigma^{(p_2-1)}(s)}{s^{(p_2-1)}} \,\mathrm{d}s\right)^{1/(p_1-1)} \mathrm{d}t.
\end{aligned}$$

Thus we find

$$\phi_2'(t) \le -\Psi_1(t) + \frac{\varpi'(t)}{\varpi(t)} \vartheta(t) - \frac{1}{\varpi(t)} \phi_2^2(t).$$

This completes the proof. □

Theorem 3.4 *Assume that $r_0 < 1$ and $\sigma(t) \leq t$. If there exist two positive functions $\varpi_1, \varpi \in C^1([t_0,\infty))$ such that*

$$\int_{t_0}^{\infty} \left(\Psi(s) - \frac{2^{(p_1-1)}}{p_1^{p_1}} \frac{a(s)(\varpi_1'(s))^{p_1}}{\mu_1^{(p_1-1)} s^{2(p_1-1)} \varpi_1^{(p_1-1)}(s)} \right) \mathrm{d}s = \infty \tag{30}$$

and

$$\int_{t_0}^{\infty} \left(\Psi_1(s) - \frac{(\varpi'(s))^2}{4\varpi(s)} \right) \mathrm{d}s = \infty, \tag{31}$$

then (1) is oscillatory.

Proof It is known that (9) and (18) hold in the proof of Theorem 3.3. From (9) we have that y'' is of one sign. From Lemma 3.1 we get that (21) holds.

Since $y'(t) > 0$, there exist $t_2 \geq t_1$ and a constant $M > 0$ such that

$$y(t) > M \tag{32}$$

for all $t \geq t_2$. From the inequality

$$Ew - Fw^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} E^{\alpha+1} F^{-\alpha}, \quad F > 0,$$

with $E = \varpi_1'(t)/\varpi_1(t)$, $F = (p_1 - 1)\mu t^2 / 2a^{1/(p_1-1)}(t)\varpi_1^{1/(p_1-1)}(t)$, and $x = \phi_1$, we get

$$\phi_1'(t) \leq -\Psi(t) + \frac{2^{(p_1-1)}}{p_1^{p_1}} \frac{a(t)(\varpi_1'(t))^{p_1}}{\mu_1^{(p_1-1)} t^{2(p_1-1)} \varpi_1^{(p_1-1)}(t)}.$$

This implies that

$$\int_{t_1}^{t} \left(\Psi(s) - \frac{2^{(p_1-1)}}{p_1^{p_1}} \frac{a(s)(\varpi_1'(s))^{p_1}}{\mu_1^{(p_1-1)} s^{2(p_1-1)} \varpi_1^{(p_1-1)}(s)} \right) \mathrm{d}s \leq \phi_1(t_1),$$

which contradicts (30).

From Lemma 3.2 we get that (25) holds. This implies that

$$\phi_2'(t) \leq -\Psi_1(t) + \frac{(\varpi'(t))^2}{4\varpi(t)}.$$

Then we obtain

$$\int_{t_1}^{t} \left(\Psi_1(s) - \frac{(\varpi'(t))^2}{4\varpi(t)} \right) \mathrm{d}s \leq \phi_2(t_1),$$

which contradicts (31). This completes the proof. □

Example 3.1 Consider the equation

$$\big(x(t) + (7/8)x(t/e)\big)^{(4)} + q_0 u^{-4} x\big(t/e^2\big) = 0, \quad u \geq 1, \tag{33}$$

Table 1 Conditions comparison

Condition	(3)	(4)	(5)	(6)
Criterion	$q_0 > 113981.3$	$q_0 > 3561.9$	$q_0 > 3008.5$	$q_0 > 587.93$

where $q_0 > 0$ is a constant, and

$$p_1 = 2, \qquad a(t) = 1, \qquad r(t) = 7/8, \qquad \delta(t) = u/e, \qquad q(t) = q_0 u^{-4}, \qquad \sigma(t) = t/e^2.$$

Applying conditions (3), (4), (5), and (6) to Eq. (33), we obtain Table 1.

Therefore we see that [27] enriched the results in [24–26]. Furthermore, we easily find that the conditions for oscillation in [24–26] cannot be applied to (35) and (36). Therefore our results are new.

Example 3.2 Consider the differential equation

$$\left(\left(\left(x + r_0 x(\varpi t)\right)'''\right)^{(p_1-1)}\right)' + \frac{q_0}{t^{3p_1-2}} x(\lambda t) = 0, \quad t \geq 1, \tag{34}$$

where $\varpi, \lambda \in (0,1]$ and $r_0, q_0 > 0$. Let $a(t) = 1$, $r(t) = r_0$, $\delta(t) = \varpi t$, $\sigma(t) = \lambda t$, and $q(t) = q_0/t^{3p_1-2}$. We easily see that

$$\widehat{q}(t) = q_0 \lambda^{3p_1-2} \frac{1}{t^{3p_1-2}}.$$

By Corollary 3.1 equation (34) is oscillatory if

$$q_0 \ln \frac{1}{\lambda} > (p_1 - 1)\left(\frac{\varpi + r_0^{(p_1-1)}}{\varpi}\right) \frac{6^{(p_1-1)}}{\lambda^{3(p_1-1)} \mathrm{e}}. \tag{35}$$

By Corollary 3.2, if

$$q_0 \ln \frac{1}{\lambda} > \frac{1}{(1-r_0)^{(p_1-1)}} \frac{6^{(p_1-1)}}{\lambda^{3(p_1-1)} \mathrm{e}}, \tag{36}$$

then (34) is oscillatory.

Finally, setting $\varpi_1(s) := t^{3(p_1-1)}$ and $\varpi(t) := t^2$, we have

$$\Psi(t) = q_0 (1 - r_0)^{(p_1-1)} \lambda^{3(p_1-1)} \frac{1}{s}$$

and

$$\Psi_1(t) := \frac{1}{2}\left(\frac{q_0}{3(p_1-1)}\right)^{1/(p_1-1)} (1 - r_0)\lambda,$$

By Theorem 3.4 equation (34) is oscillatory if

$$q_0 (1 - r_0)^{(p_1-1)} \lambda^{3(p_1-1)} > 2^{(p_1-1)} 3^{p_1} \left(\frac{(p_1-1)}{p_1}\right)^{p_1} \tag{37}$$

and

$$q_0 > \left(\frac{2}{(1-r_0)\lambda}\right)^{(p_1-1)} 3(p_1-1). \tag{38}$$

4 Conclusion

In this paper, we consider the oscillation and asymptotic behavior of a class of fourth-order Emden–Fowler neutral differential equations. Using the Riccati transformation and comparison method, we establish new oscillation conditions for the solutions of fourth-order neutral differential equations. Our results unify and extend some known results for differential equations. In the future work, we will discuss the oscillatory behavior of these equations by using comparing technique with second-order equations.

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