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On a system of Riemann–Liouville fractional differential equations with coupled nonlocal boundary conditions

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Abstract

We investigate the existence of solutions for a system of Riemann–Liouville fractional differential equations with nonlinearities dependent on fractional integrals, subject to coupled nonlocal boundary conditions which contain various fractional derivatives and Riemann–Stieltjes integrals. In the proof of our main results, we use some theorems from the fixed point theory.

MSC: 34A08; 45G15

Keywords: Systems of Riemann–Liouville fractional differential equations; Fractional integrals; Coupled nonlocal boundary conditions; Existence of solutions

1 Introduction

We consider the nonlinear system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}x(t)+f(t,x(t),y(t),I_{0+}^{\theta_1}x(t),I_{0+}^{\sigma_1}y(t))=0, & t\in(0,1),\\ D_{0+}^{\beta}y(t)+g(t,x(t),y(t),I_{0+}^{\theta_2}x(t),I_{0+}^{\sigma_2}y(t))=0, & t\in(0,1), \end{cases} \tag{S}$$

with the coupled nonlocal boundary conditions

$$\begin{cases} x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0, & D_{0+}^{\gamma_0}x(1)=\sum_{i=1}^{p}\int_0^1 D_{0+}^{\gamma_i}y(t)\,dH_i(t),\\ y(0)=y'(0)=\cdots=y^{(m-2)}(0)=0, & D_{0+}^{\delta_0}y(1)=\sum_{i=1}^{q}\int_0^1 D_{0+}^{\delta_i}x(t)\,dK_i(t), \end{cases} \tag{BC}$$

where $\alpha,\beta\in\mathbb{R}$, $\alpha\in(n-1,n]$, $\beta\in(m-1,m]$, $n,m\in\mathbb{N}$, $n\geq 2$, $m\geq 2$, $\theta_1,\theta_2,\sigma_1,\sigma_2>0$, $p,q\in\mathbb{N}$, $\gamma_i\in\mathbb{R}$ for all $i=0,\dots,p$, $0\leq\gamma_1<\gamma_2<\cdots<\gamma_p<\beta-1$, $\gamma_0\in[0,\alpha-1)$, $\delta_i\in\mathbb{R}$ for all $i=0,\dots,q$, $0\leq\delta_1<\delta_2<\cdots<\delta_q<\alpha-1$, $\delta_0\in[0,\beta-1)$, D_{0+}^{k} denotes the Riemann–Liouville derivative of order k (for $k=\alpha,\beta,\gamma_0,\gamma_i,i=1,\dots,p,\delta_0,\delta_i,i=1,\dots,q$), I_{0+}^{ζ} is the Riemann–Liouville integral of order ζ (for $\zeta=\theta_1,\sigma_1,\theta_2,\sigma_2$), f and g are nonlinear functions, and the integrals from the boundary conditions (*BC*) are Riemann–Stieltjes integrals with H_i for $i=1,\dots,p$ and K_i for $i=1,\dots,q$ functions of bounded variation.

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In this paper, we show the existence and uniqueness of solutions for problem (S)–(BC), by applying standard fixed point theorems. We prove the existence of a unique solution by using the Banach contraction mapping principle, and five existence results by applying the Leray–Schauder alternative theorem, the Krasnosel'skii theorem for the sum of two operators (for two results), the Schauder fixed point theorem, and the nonlinear alternative of Leray–Schauder-type, respectively. The methods used for proofs are standard, but their applications in this framework of systems of coupled Riemann–Liouville fractional boundary value problems are new.

In the last decades, many authors investigated the existence of positive solutions for Riemann–Liouville fractional differential equations and systems of Riemann–Liouville fractional differential equations, subject to nonlocal boundary conditions. For example, the existence and multiplicity of positive solutions for the equation

$$D_{0+}^{\alpha}u(t) + f\bigl(t, u(t)\bigr) = 0, \quad t \in (0,1), \tag{E}$$

with the nonlocal boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\beta_0}u(1) = \sum_{i=1}^{m}\int_0^1 D_{0+}^{\beta_i}u(t)\,dH_i(t), \tag{BC_1}$$

where $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n, m \in \mathbb{N}$, $n \geq 3$, $\beta_i \in \mathbb{R}$ for all $i = 0, \ldots, m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \beta_0 < \alpha - 1$, and where the function f may change sign and be singular in the points $t = 0, 1$ and/or in the space variable u, was studied in the paper [1]. In the proof of the main results of [1], the authors used various height functions of the nonlinearity of the equation defined on special bounded sets, some properties of the corresponding Green functions, and two theorems from the fixed point index theory. Equation (E) with a positive parameter λ, supplemented with the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{p}u(1) = \sum_{i=1}^{m} a_i D_{0+}^{q}u(\xi_i), \tag{BC_2}$$

where $\xi_i \in \mathbb{R}$, $i = 1, \ldots, m$, $0 < \xi_1 < \cdots < \xi_m < 1$, $p, q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0, p]$, was investigated in [16]. In this paper, the nonlinearity f changes sign and it is singular only for $t = 0, 1$, while the authors used the Guo–Krasnosel'skii fixed point theorem to prove the existence of positive solutions when the parameter belongs to various intervals. For some recent results on the existence, nonexistence, and multiplicity of solutions for fractional differential equations and systems of fractional differential equations subject to various boundary conditions, we refer the reader to the monographs [15, 38] and the papers [1–6, 14, 17–19, 23–25, 30, 35–37]. We also mention the books [8–10, 20, 21, 29, 31, 32], and the papers [7, 11–13, 26–28, 33], for applications of the fractional differential equations in various disciplines.

The main features of the present work are the following. Firstly, the system and the coupled boundary conditions contain Riemann–Liouville fractional derivatives, and secondly, the nonlinearities in the system depend not only on the unknown functions x and y, but also on the Riemann–Liouville fractional integrals of x and y. Thirdly, the obtained solution (x, y) is a general one which can change sign. Section 2 contains an auxiliary lemma which is important to establish our main theorems, some notations, and the operator associated to our problem. The main existence results are presented in Sect. 3, and in Sect. 4 we give some illustrative examples.

2 Auxiliary results

We consider here the system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}x(t) + h(t) = 0, & t \in (0,1), \\ D_{0+}^{\beta}y(t) + k(t) = 0, & t \in (0,1), \end{cases} \tag{1}$$

with the boundary conditions (*BC*), where $h, k \in C(0,1) \cap L^1(0,1)$. We denote by

$$\Delta_1 = \sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s), \qquad \Delta_2 = \sum_{i=1}^{q} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)} \int_0^1 s^{\alpha-\delta_i-1}\, dK_i(s),$$
$$\Delta = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} - \Delta_1\Delta_2.$$

By using similar arguments as those used in the proof of Lemma 2.1 from [34], we obtain the following result.

Lemma 2.1 *If $\Delta \neq 0$, then the unique solution $(x,y) \in C[0,1] \times C[0,1]$ of problem (1)–(BC) is given by*

$$\begin{aligned}
x(t) = {} & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds \\
& + \frac{t^{\alpha-1}}{\Delta} \Bigg[\frac{\Gamma(\beta)}{\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} h(s)\, ds \\
& - \frac{\Gamma(\beta)}{\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1} k(\tau)\, d\tau \right) dH_i(s) \\
& + \frac{\Delta_1}{\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} k(s)\, ds \\
& - \Delta_1 \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} h(\tau)\, d\tau \right) dK_i(s) \right) \Bigg], \\
& t \in [0,1], \\
y(t) = {} & -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} k(s)\, ds \\
& + \frac{t^{\beta-1}}{\Delta} \Bigg[\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} k(s)\, ds \\
& - \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_0)} \sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} h(\tau)\, d\tau \right) dK_i(s) \\
& + \frac{\Delta_2}{\Gamma(\alpha-\gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} h(s)\, ds
\end{aligned} \tag{2}$$

$$- \Delta_2 \left(\sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1} k(\tau)\, d\tau \right) dH_i(s) \right) \Bigg], \quad t \in [0,1].$$

Remark 2.1 If $u \in C[0,1]$ then for $\chi > 0$ we have

$$\left| I_{0+}^{\chi} u(t) \right| \le \frac{\|u\|}{\Gamma(\chi+1)}, \quad \forall t \in [0,1],$$

where $\|u\| = \sup_{t\in[0,1]} |u(t)|$.

We introduce now the assumption $(J1)$ for problem (S)–(BC) that will be used in our main results.

$(J1)$ $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n,m \in \mathbb{N}$, $n \ge 2$, $m \ge 2$, $\theta_1,\theta_2,\sigma_1,\sigma_2 > 0$, $p,q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0,\dots,p$, $0 \le \gamma_1 < \gamma_2 < \cdots < \gamma_p < \beta - 1$, $\gamma_0 \in [0, \beta-1)$, $\delta_i \in \mathbb{R}$ for all $i = 0,\dots,q$, $0 \le \delta_1 < \delta_2 < \cdots < \delta_q < \alpha - 1$, $\delta_0 \in [0,\alpha-1)$, $H_i : [0,1] \to \mathbb{R}$, $i = 1,\dots,p$ and $K_j : [0,1] \to \mathbb{R}$, $j = 1,\dots,q$ are functions of bounded variation, and $\Delta \ne 0$.

We introduce the following constants:

$$\begin{aligned}
&M_1 = 1 + \frac{1}{\Gamma(\theta_1+1)}, \qquad M_2 = 1 + \frac{1}{\Gamma(\sigma_1+1)}, \qquad M_3 = 1 + \frac{1}{\Gamma(\theta_2+1)}, \\
&M_4 = 1 + \frac{1}{\Gamma(\sigma_2+1)}, \qquad M_5 = \max\{M_1, M_2\}, \qquad M_6 = \max\{M_3, M_4\}, \\
&M_7 = \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0+1)\Gamma(\beta-\delta_0)} \\
&\qquad + \frac{1}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \\
&\qquad \times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)} \left| \int_0^1 s^{\alpha-\delta_i}\, dK_i(s) \right| \right), \\
&M_8 = \frac{1}{\Gamma(\beta+1)} + \frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0+1)} \\
&\qquad + \frac{1}{|\Delta|} \left(\sum_{i=1}^{q} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 s^{\alpha-\delta_i-1}\, dK_i(s) \right| \right) \\
&\qquad \times \left(\sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)} \left| \int_0^1 s^{\beta-\gamma_i}\, dH_i(s) \right| \right), \\
&M_9 = \frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)} \sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)} \left| \int_0^1 s^{\alpha-\delta_i}\, dK_i(s) \right| \\
&\qquad + \frac{1}{|\Delta|\Gamma(\alpha-\gamma_0+1)} \left(\sum_{i=1}^{q} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 s^{\alpha-\delta_i-1}\, dK_i(s) \right| \right), \\
&M_{10} = \frac{\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)} \left| \int_0^1 s^{\beta-\gamma_i}\, dH_i(s) \right| \\
&\qquad + \frac{1}{|\Delta|\Gamma(\beta-\delta_0+1)} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right),
\end{aligned} \tag{3}$$

$$M_{11} = M_7 - \frac{1}{\Gamma(\alpha+1)}, \qquad M_{12} = M_8 - \frac{1}{\Gamma(\beta+1)}.$$

We consider the Banach space $X = C[0,1]$ with the supremum norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$, and the Banach space $Y = X \times X$ with the norm $\|(x,y)\|_Y = \|x\| + \|y\|$. We introduce the operator $Q : Y \to Y$ defined by $Q(u,v) = (Q_1(u,v), Q_2(u,v))$ for $(u,v) \in Y$, where the operators $Q_1, Q_2 : Y \to X$ are given by

$$\begin{aligned}
Q_1(x,y)(t) = {}& -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{f}_{xy}(s)\,ds \\
&+ \frac{t^{\alpha-1}\Gamma(\beta)}{\Delta\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} \hat{f}_{xy}(s)\,ds \\
&- \frac{t^{\alpha-1}\Gamma(\beta)}{\Delta\Gamma(\beta-\delta_0)} \sum_{i=1}^p \frac{1}{\Gamma(\beta-\gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1} \hat{g}_{xy}(\tau)\,d\tau \right) dH_i(s) \\
&+ \frac{t^{\alpha-1}\Delta_1}{\Delta\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} \hat{g}_{xy}(s)\,ds \\
&- \frac{t^{\alpha-1}\Delta_1}{\Delta} \left(\sum_{i=1}^q \frac{1}{\Gamma(\alpha-\delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} \hat{f}_{xy}(\tau)\,d\tau \right) dK_i(s) \right), \\
Q_2(x,y)(t) = {}& -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \hat{g}_{xy}(s)\,ds \\
&+ \frac{t^{\beta-1}\Gamma(\alpha)}{\Delta\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} \hat{g}_{xy}(s)\,ds \\
&- \frac{t^{\beta-1}\Gamma(\alpha)}{\Delta\Gamma(\alpha-\gamma_0)} \sum_{i=1}^q \frac{1}{\Gamma(\alpha-\delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} \hat{f}_{xy}(\tau)\,d\tau \right) dK_i(s) \\
&+ \frac{t^{\beta-1}\Delta_2}{\Delta\Gamma(\alpha-\gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} \hat{f}_{xy}(s)\,ds \\
&- \frac{t^{\beta-1}\Delta_2}{\Delta} \left(\sum_{i=1}^p \frac{1}{\Gamma(\beta-\gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1} \hat{g}_{xy}(\tau)\,d\tau \right) dH_i(s) \right),
\end{aligned} \tag{4}$$

for $t \in [0,1]$ and $(x,y) \in Y$, where $\hat{f}_{xy}(s) = f(s, x(s), y(s), I_{0+}^{\theta_1} x(s), I_{0+}^{\sigma_1} y(s))$, $\hat{g}_{xy}(s) = g(s, x(s), y(s), I_{0+}^{\theta_2} x(s), I_{0+}^{\sigma_2} y(s))$ for $s \in [0,1]$.

By using Lemma 2.1, we see that (x,y) is a solution of problem (S)–(BC) if and only if (x,y) is a fixed point of operator Q.

3 Existence of solutions

In this section we will give some existence results for the solutions of our problem (S)–(BC).

Theorem 3.1 *Assume that (J1) and*

(J2) *The functions $f, g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous and there exist $L_1, L_2 > 0$ such that*

$$\left| f(t, u_1, u_2, u_3, u_4) - f(t, v_1, v_2, v_3, v_4) \right| \le L_1 \sum_{i=1}^4 |u_i - v_i|,$$

$$\left|g(t,u_1,u_2,u_3,u_4)-g(t,v_1,v_2,v_3,v_4)\right| \le L_2 \sum_{i=1}^{4} |u_i - v_i|,$$

for all $t \in [0,1]$, $u_i, v_i \in \mathbb{R}, i = 1,\dots,4,$
hold. If $\Xi := L_1M_5(M_7 + M_9) + L_2M_6(M_8 + M_{10}) < 1$, *then problem* (*S*)–(*BC*) *has a unique solution* $(x(t),y(t))$, $t \in [0,1]$, *where* $M_5,\dots,M_{10}$ *are given by* (3).

Proof We consider the positive number r given by

$$r = \left[M_0(M_7+M_9)+\widetilde{M}_0(M_8+M_{10})\right]\left[1-L_1M_5(M_7+M_9)-L_2M_6(M_8+M_{10})\right]^{-1},$$

where $M_0 = \sup_{t\in[0,1]} |f(t,0,0,0,0)|$, $\widetilde{M}_0 = \sup_{t\in[0,1]} |g(t,0,0,0,0)|$. We define the set $\overline{B}_r = \{(x,y) \in Y, \|(x,y)\|_Y \le r\}$ and show firstly that $Q(\overline{B}_r) \subset \overline{B}_r$. Let $(x,y) \in \overline{B}_r$. By using (*J*2) and Remark 2.1, for $\hat{f}_{xy}(t)$ we deduce the following inequalities:

$$\begin{aligned}
\left|\hat{f}_{xy}(t)\right| &\le \left|f\left(t,x(t),y(t),I_{0+}^{\theta_1}x(t),I_{0+}^{\sigma_1}y(t)\right)-f(t,0,0,0,0)\right| + \left|f(t,0,0,0,0)\right| \\
&\le L_1\left(\left|x(t)\right| + \left|y(t)\right| + \left|I_{0+}^{\theta_1}x(t)\right| + \left|I_{0+}^{\sigma_1}y(t)\right|\right) + M_0 \\
&\le L_1\left(\|x\| + \|y\| + \frac{\|x\|}{\Gamma(\theta_1+1)} + \frac{\|y\|}{\Gamma(\sigma_1+1)}\right) + M_0 \\
&= L_1\left(M_1\|x\| + M_2\|y\|\right) + M_0 \\
&\le L_1M_5\left\|(x,y)\right\|_Y + M_0 \le L_1M_5r + M_0, \quad \forall t \in [0,1].
\end{aligned}$$

Arguing as before, we find

$$\begin{aligned}
\left|\hat{g}_{xy}(t)\right| &\le \left|g\left(t,x(t),y(t),I_{0+}^{\theta_2}x(t),I_{0+}^{\sigma_2}y(t)\right)-g(t,0,0,0,0)\right| + \left|g(t,0,0,0,0)\right| \\
&\le L_2\left(\left|x(t)\right| + \left|y(t)\right| + \left|I_{0+}^{\theta_2}x(t)\right| + \left|I_{0+}^{\sigma_2}y(t)\right|\right) + \widetilde{M}_0 \\
&\le L_2\left(\|x\| + \|y\| + \frac{\|x\|}{\Gamma(\theta_2+1)} + \frac{\|y\|}{\Gamma(\sigma_2+1)}\right) + \widetilde{M}_0 \\
&= L_2\left(M_3\|x\| + M_4\|y\|\right) + \widetilde{M}_0 \\
&\le L_2M_6\left\|(x,y)\right\|_Y + \widetilde{M}_0 \le L_2M_6r + \widetilde{M}_0, \quad \forall t \in [0,1].
\end{aligned}$$

Then by the definition of operators Q_1 and Q_2, we conclude

$$\begin{aligned}
&\left|Q_1(x,y)(t)\right| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(L_1M_5r+M_0)\,ds \\
&\qquad+ \frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)}\int_0^1 (1-s)^{\alpha-\gamma_0-1}(L_1M_5r+M_0)\,ds \\
&\qquad+ \frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)}\sum_{i=1}^{p}\frac{1}{\Gamma(\beta-\gamma_i)}\left|\int_0^1\left(\int_0^s (s-\tau)^{\beta-\gamma_i-1}(L_2M_6r+\widetilde{M}_0)\,d\tau\right)dH_i(s)\right| \\
&\qquad+ \frac{t^{\alpha-1}}{|\Delta|}\left(\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} (L_2M_6r + \widetilde{M}_0)\, ds \right) \\
&+ \frac{t^{\alpha-1}}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \\
&\times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} (L_1M_5r + M_0)\, d\tau \right) dK_i(s) \right| \right) \\
= {} & (L_1M_5r + M_0) \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, ds + \frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1}\, ds \right. \\
&+ \frac{t^{\alpha-1}}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \\
&\left. \times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1}\, d\tau \right) dK_i(s) \right| \right) \right] \\
&+ (L_2M_6r + \widetilde{M}_0) \left[\frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1}\, d\tau \right) dH_i(s) \right| \right. \\
&\left. + \frac{t^{\alpha-1}}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \left(\frac{1}{\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1}\, ds \right) \right] \\
= {} & (L_1M_5r + M_0) \left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0+1)\Gamma(\beta-\delta_0)} \right. \\
&\left. + \frac{t^{\alpha-1}}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)} \left| \int_0^1 s^{\alpha-\delta_i}\, dK_i(s) \right| \right) \right] \\
&+ (L_2M_6r + \widetilde{M}_0) \left[\frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)} \left| \int_0^1 s^{\beta-\gamma_i}\, dH_i(s) \right| \right. \\
&\left. + \frac{t^{\alpha-1}}{|\Delta|\Gamma(\beta-\delta_0+1)} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \right], \quad \forall t \in [0,1].
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
&\| Q_1(x,y) \| \\
&\leq (L_1M_5r + M_0) \left[\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0+1)\Gamma(\beta-\delta_0)} \right. \\
&\quad + \frac{1}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \\
&\quad \left. \times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)} \left| \int_0^1 s^{\alpha-\delta_i}\, dK_i(s) \right| \right) \right] \qquad (5) \\
&\quad + (L_2M_6r + \widetilde{M}_0) \left[\frac{\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)} \left| \int_0^1 s^{\beta-\gamma_i}\, dH_i(s) \right| \right.
\end{aligned}
$$

$$+ \frac{1}{|\Delta|\Gamma(\beta-\delta_0+1)}\left(\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right)\Bigg]$$
$$= (L_1M_5r+M_0)M_7 + (L_2M_6r+\widetilde{M}_0)M_{10}.$$

In a similar manner, we deduce

$$\begin{aligned}
&\|Q_2(x,y)\| \\
&\le (L_1M_5r+M_0)\Bigg[\frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)}\sum_{i=1}^{q}\frac{1}{\Gamma(\alpha-\delta_i+1)}\left|\int_0^1 s^{\alpha-\delta_i}\,dK_i(s)\right| \\
&\quad + \frac{1}{|\Delta|\Gamma(\alpha-\gamma_0+1)}\left(\sum_{i=1}^{q}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)}\left|\int_0^1 s^{\alpha-\delta_i-1}\,dK_i(s)\right|\right)\Bigg] \\
&\quad + (L_2M_6r+\widetilde{M}_0)\Bigg[\frac{1}{\Gamma(\beta+1)} + \frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0+1)} \\
&\quad + \frac{1}{|\Delta|}\left(\sum_{i=1}^{q}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)}\left|\int_0^1 s^{\alpha-\delta_i-1}\,dK_i(s)\right|\right) \\
&\quad \times\left(\sum_{i=1}^{p}\frac{1}{\Gamma(\beta-\gamma_i+1)}\left|\int_0^1 s^{\beta-\gamma_i}\,dH_i(s)\right|\right)\Bigg] \\
&= (L_1M_5r+M_0)M_9 + (L_2M_6r+\widetilde{M}_0)M_8.
\end{aligned} \tag{6}$$

By relations (5) and (6), we conclude

$$\begin{aligned}
\|Q(x,y)\|_Y &= \|Q_1(x,y)\| + \|Q_2(x,y)\| \\
&\le (L_1M_5r+M_0)(M_7+M_9) + (L_2M_6r+\widetilde{M}_0)(M_8+M_{10}) = r,
\end{aligned}$$

for all $(x,y)\in\overline{B}_r$, which implies that $Q(\overline{B}_r)\subset\overline{B}_r$.

Next we prove that operator Q is a contraction. For $(x_i,y_i)\in\overline{B}_r$, $i=1,2$, and for each $t\in[0,1]$, we have

$$\begin{aligned}
&\left|Q_1(x_1,y_1)(t)-Q_1(x_2,y_2)(t)\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\hat{f}_{x_1y_1}(s)-\hat{f}_{x_2y_2}(s)\right|ds \\
&\quad + \frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)}\int_0^1 (1-s)^{\alpha-\gamma_0-1}\left|\hat{f}_{x_1y_1}(s)-\hat{f}_{x_2y_2}(s)\right|ds \\
&\quad + \frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)}\sum_{i=1}^{p}\frac{1}{\Gamma(\beta-\gamma_i)} \\
&\quad \times\left|\int_0^1\left(\int_0^s (s-\tau)^{\beta-\gamma_i-1}\left|\hat{g}_{x_1y_1}(\tau)-\hat{g}_{x_2y_2}(\tau)\right|d\tau\right)dH_i(s)\right| \\
&\quad + \frac{t^{\alpha-1}}{|\Delta|}\left(\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right) \\
&\quad \times\left(\frac{1}{\Gamma(\beta-\delta_0)}\int_0^1 (1-s)^{\beta-\delta_0-1}\left|\hat{g}_{x_1y_1}(s)-\hat{g}_{x_2y_2}(s)\right|ds\right)
\end{aligned} \tag{7}$$

$$+ \frac{t^{\alpha-1}}{|\Delta|}\left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right)$$
$$\times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)}\left|\int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1}\left|\hat{f}_{x_1y_1}(\tau)-\hat{f}_{x_2y_2}(\tau)\right|d\tau\right)dK_i(s)\right|\right).$$

Because

$$\begin{aligned}
&\left|\hat{f}_{x_1y_1}(s)-\hat{f}_{x_2y_2}(s)\right| \\
&\quad\le L_1\Big(\left|x_1(s)-x_2(s)\right|+\left|y_1(s)-y_2(s)\right| \\
&\qquad+\left|I_{0+}^{\theta_1}x_1(s)-I_{0+}^{\theta_1}x_2(s)\right|+\left|I_{0+}^{\sigma_1}y_1(s)-I_{0+}^{\sigma_1}y_2(s)\right|\Big) \\
&\quad\le L_1\left(\|x_1-x_2\|+\|y_1-y_2\|+\frac{1}{\Gamma(\theta_1+1)}\|x_1-x_2\|+\frac{1}{\Gamma(\sigma_1+1)}\|y_1-y_2\|\right) \\
&\quad= L_1\big(M_1\|x_1-x_2\|+M_2\|y_1-y_2\|\big)\le L_1M_5\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y,\quad \forall s\in[0,1], \\
&\left|\hat{g}_{x_1y_1}(s)-\hat{g}_{x_2y_2}(s)\right| \\
&\quad\le L_2\Big(\left|x_1(s)-x_2(s)\right|+\left|y_1(s)-y_2(s)\right| \\
&\qquad+\left|I_{0+}^{\theta_2}x_1(s)-I_{0+}^{\theta_2}x_2(s)\right|+\left|I_{0+}^{\sigma_2}y_1(s)-I_{0+}^{\sigma_2}y_2(s)\right|\Big) \\
&\quad\le L_2\left(\|x_1-x_2\|+\|y_1-y_2\|+\frac{1}{\Gamma(\theta_2+1)}\|x_1-x_2\|+\frac{1}{\Gamma(\sigma_2+1)}\|y_1-y_2\|\right) \\
&\quad= L_2\big(M_3\|x_1-x_2\|+M_4\|y_1-y_2\|\big)\le L_2M_6\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y,\quad \forall s\in[0,1],
\end{aligned}$$

the inequality (7) gives us

$$\begin{aligned}
&\left|Q_1(x_1,y_1)(t)-Q_1(x_2,y_2)(t)\right| \\
&\quad\le L_1M_5\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y \\
&\qquad\times\Bigg[\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0+1)\Gamma(\beta-\delta_0)} \\
&\qquad+\frac{t^{\alpha-1}}{|\Delta|}\left(\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right)\left(\sum_{i=1}^{q}\frac{1}{\Gamma(\alpha-\delta_i+1)}\left|\int_0^1 s^{\alpha-\delta_i}\,dK_i(s)\right|\right)\Bigg] \\
&\qquad+L_2M_6\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y\Bigg[\frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)}\sum_{i=1}^{p}\frac{1}{\Gamma(\beta-\gamma_i+1)}\left|\int_0^1 s^{\beta-\gamma_i}\,dH_i(s)\right| \\
&\qquad+\frac{t^{\alpha-1}}{|\Delta|\Gamma(\beta-\delta_0+1)}\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\Bigg],\quad \forall t\in[0,1].
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\big\|Q_1(x_1,y_1)-Q_1(x_2,y_2)\big\| \\
&\quad\le\Bigg\{L_1M_5\Bigg[\frac{1}{\Gamma(\alpha+1)}+\frac{\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0+1)\Gamma(\beta-\delta_0)}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{|\Delta|}\left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right) \\
&\times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)}\left|\int_0^1 s^{\alpha-\delta_i}\,dK_i(s)\right|\right)\Bigg] \\
&+ L_2M_6\Bigg[\frac{\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)}\sum_{i=1}^{p}\frac{1}{\Gamma(\beta-\gamma_i+1)}\left|\int_0^1 s^{\beta-\gamma_i}\,dH_i(s)\right| \\
&+ \frac{1}{|\Delta|\Gamma(\beta-\delta_0+1)}\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\Bigg]\Bigg\}\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y \\
&= (L_1M_5M_7 + L_2M_6M_{10})\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y.
\end{aligned} \tag{8}
$$

In a similar manner, we deduce

$$
\begin{aligned}
&\big\|Q_2(x_1,y_1)-Q_2(x_2,y_2)\big\| \\
&\le \Bigg\{L_2M_6\Bigg[\frac{1}{\Gamma(\beta+1)} + \frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0+1)} \\
&+ \frac{1}{|\Delta|}\left(\sum_{i=1}^{q} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)}\left|\int_0^1 s^{\alpha-\delta_i-1}\,dK_i(s)\right|\right) \\
&\times \left(\sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)}\left|\int_0^1 s^{\beta-\gamma_i}\,dH_i(s)\right|\right)\Bigg] \\
&+ L_1M_5\Bigg[\frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)}\sum_{i=1}^{q}\frac{1}{\Gamma(\alpha-\delta_i+1)}\left|\int_0^1 s^{\alpha-\delta_i}\,dK_i(s)\right| \\
&+ \frac{1}{|\Delta|\Gamma(\alpha-\gamma_0+1)}\sum_{i=1}^{q}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)}\left|\int_0^1 s^{\alpha-\delta_i-1}\,dK_i(s)\right|\Bigg]\Bigg\}\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y \\
&= (L_1M_5M_9 + L_2M_6M_8)\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y.
\end{aligned} \tag{9}
$$

Then by using relations (8) and (9), we obtain

$$
\begin{aligned}
&\big\|Q(x_1,y_1)-Q(x_2,y_2)\big\|_Y \\
&= \big\|Q_1(x_1,y_1)-Q_1(x_2,y_2)\big\| + \big\|Q_2(x_1,y_1)-Q_2(x_2,y_2)\big\| \\
&\le \big[L_1M_5(M_7+M_9) + L_2M_6(M_8+M_{10})\big]\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y \\
&= \Xi\big\|(x_1,y_1)-(x_2,y_2)\big\|_Y.
\end{aligned}
$$

By using the condition $\Xi < 1$, we deduce that operator Q is a contraction. By the Banach contraction mapping principle, we conclude that operator Q has a unique fixed point $(x,y) \in \overline{B}_r$, which is the unique solution for problem (S)–(BC) on $[0,1]$. □

Theorem 3.2 *Suppose that* ($J1$) *and*

(*J*3) *The functions* $f, g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *are continuous and there exist real constants* $a_i, b_i \geq 0$, $i = 0, \ldots, 4$, *and at least one of* a_0 *and* b_0 *is positive, such that*

$$\left|f(t,u_1,u_2,u_3,u_4)\right| \leq a_0 + \sum_{i=1}^{4} a_i|u_i|, \qquad \left|g(t,u_1,u_2,u_3,u_4)\right| \leq b_0 + \sum_{i=1}^{4} b_i|u_i|,$$

for all $t \in [0,1]$, $u_i \in \mathbb{R}$, $i = 1, \ldots, 4$,

hold. If $\Xi_1 := \max\{M_{13}, M_{14}\} < 1$, *where* $M_{13} = (a_1 + \frac{a_3}{\Gamma(\theta_1+1)})(M_7 + M_9) + (b_1 + \frac{b_3}{\Gamma(\theta_2+1)})(M_8 + M_{10})$ *and* $M_{14} = (a_2 + \frac{a_4}{\Gamma(\sigma_1+1)})(M_7 + M_9) + (b_2 + \frac{b_4}{\Gamma(\sigma_2+1)})(M_8 + M_{10})$, *then the boundary value problem* (*S*)–(*BC*) *has at least one solution* $(x(t), y(t))$, $t \in [0,1]$.

Proof We show that operator Q is completely continuous. Because the functions f and g are continuous, we deduce that the operators Q_1 and Q_2 are continuous, and then Q is a continuous operator. We will prove next that Q is a compact operator, that is, it maps bounded sets into relatively compact sets. Let $\Omega \subset Y$ be a bounded set. Then there exist positive constants L_3 and L_4 such that $|\hat{f}_{xy}(t)| \leq L_3$ and $|\hat{g}_{xy}(t)| \leq L_4$ for all $t \in [0,1]$ and $(x,y) \in \Omega$. Hence we obtain as in the proof of Theorem 3.1 that

$$\left|Q_1(x,y)(t)\right| \leq L_3 M_7 + L_4 M_{10}, \qquad \left|Q_2(x,y)(t)\right| \leq L_3 M_9 + L_4 M_8,$$

for all $t \in [0,1]$ and $(x,y) \in \Omega$. So, we find

$$\left\|Q(x,y)\right\|_Y \leq L_3(M_7 + M_9) + L_4(M_8 + M_{10}), \quad \forall (x,y) \in \Omega,$$

and then $Q(\Omega)$ is uniformly bounded.

We show now that $Q(\Omega)$ are equicontinuous. Let $(x,y) \in \Omega$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\left|Q_1(x,y)(t_2) - Q_1(x,y)(t_1)\right| \\
&\quad \leq \left| -\frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} \hat{f}_{xy}(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \hat{f}_{xy}(s)\, ds \right| \\
&\qquad + \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} \left|\hat{f}_{xy}(s)\right| ds \\
&\qquad + \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1} \left|\hat{g}_{xy}(\tau)\right| d\tau \right) dH_i(s) \right| \\
&\qquad + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \\
&\qquad \times \left(\frac{1}{\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} \left|\hat{g}_{xy}(s)\right| ds \right) \\
&\qquad + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \\
&\qquad \times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} \left|\hat{f}_{xy}(\tau)\right| d\tau \right) dK_i(s) \right| \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{L_3}{\Gamma(\alpha)} \int_0^{t_1} \big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\,ds + \frac{L_3}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,ds \\
&\quad + \frac{L_3(t_2^{\alpha-1}-t_1^{\alpha-1})\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1}\,ds \\
&\quad + \frac{L_4(t_2^{\alpha-1}-t_1^{\alpha-1})\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1}\,d\tau \right) dH_i(s) \right| \\
&\quad + \frac{L_4(t_2^{\alpha-1}-t_1^{\alpha-1})}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s) \right| \right) \\
&\quad \times \left(\frac{1}{\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1}\,ds \right) \\
&\quad + \frac{L_3(t_2^{\alpha-1}-t_1^{\alpha-1})}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s) \right| \right) \\
&\quad \times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1}\,d\tau \right) dK_i(s) \right| \right) \\
&= \frac{L_3}{\Gamma(\alpha+1)} \big(t_2^{\alpha}-t_1^{\alpha}\big) + L_3\big(t_2^{\alpha-1}-t_1^{\alpha-1}\big) \left[\frac{\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0+1)\Gamma(\beta-\delta_0)} \right. \\
&\quad \left. + \frac{1}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s) \right| \right) \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)} \left| \int_0^1 s^{\alpha-\delta_i}\,dK_i(s) \right| \right) \right] \\
&\quad + L_4\big(t_2^{\alpha-1}-t_1^{\alpha-1}\big) \left[\frac{\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)} \left| \int_0^1 s^{\beta-\gamma_i}\,dH_i(s) \right| \right. \\
&\quad \left. + \frac{\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0+1)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s) \right| \right] \\
&= \frac{L_3}{\Gamma(\alpha+1)} \big(t_2^{\alpha}-t_1^{\alpha}\big) + (L_3M_{11}+L_4M_{10})\big(t_2^{\alpha-1}-t_1^{\alpha-1}\big).
\end{aligned}
$$

Hence we infer

$$
\big|Q_1(x,y)(t_2)-Q_1(x,y)(t_1)\big| \to 0, \quad \text{as } t_2 \to t_1, \text{ uniformly with respect to } (x,y) \in \Omega.
$$

In a similar manner, for $(x,y) \in \Omega$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we obtain

$$
\begin{aligned}
&\big|Q_2(x,y)(t_2)-Q_2(x,y)(t_1)\big| \\
&\leq \frac{L_4}{\Gamma(\beta+1)} \big(t_2^{\beta}-t_1^{\beta}\big) + L_4\big(t_2^{\beta-1}-t_1^{\beta-1}\big) \left[\frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0+1)} \right. \\
&\quad \left. + \frac{1}{|\Delta|} \left(\sum_{i=1}^{q} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 s^{\alpha-\delta_i-1}\,dK_i(s) \right| \right) \left(\sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)} \left| \int_0^1 s^{\beta-\gamma_i}\,dH_i(s) \right| \right) \right] \\
&\quad + L_3\big(t_2^{\beta-1}-t_1^{\beta-1}\big) \left[\frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)} \sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)} \left| \int_0^1 s^{\alpha-\delta_i}\,dK_i(s) \right| \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0+1)} \sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 s^{\alpha-\delta_i-1}\,dK_i(s) \right| \Bigg] \\
&= \frac{L_4}{\Gamma(\beta+1)}\bigl(t_2^\beta - t_1^\beta\bigr) + (L_4M_{12}+L_3M_9)\bigl(t_2^{\beta-1}-t_1^{\beta-1}\bigr).
\end{aligned}
$$

So we deduce

$$
\bigl|Q_2(x,y)(t_2)-Q_2(x,y)(t_1)\bigr| \to 0, \quad \text{as } t_2 \to t_1, \text{ uniformly with respect to } (x,y)\in\Omega.
$$

Then $Q_1(\Omega)$ and $Q_2(\Omega)$ are equicontinuous, and so $Q(\Omega)$ is also equicontinuous. Therefore, by Arzela–Ascoli theorem, we conclude that $Q(\Omega)$ is relatively compact, and then Q is compact. We infer that operator Q is completely continuous.

Next we will show that the set $U = \{(x,y)\in Y, (x,y) = \nu Q(x,y), 0<\nu<1\}$ is bounded. Let $(x,y)\in U$, that is, $(x,y)=\nu Q(x,y)$. Then for any $t\in[0,1]$, we get $x(t)=\nu Q_1(x,y)(t)$, $y(t)=\nu Q_2(x,y)(t)$. We denote the following functions:

$$
\begin{aligned}
&F_{xy}(s) = a_0 + a_1\bigl|x(s)\bigr| + a_2\bigl|y(s)\bigr| + a_3\bigl|I_{0+}^{\theta_1}x(s)\bigr| + a_4\bigl|I_{0+}^{\sigma_1}y(s)\bigr|, \quad s\in[0,1], \\
&G_{xy}(s) = b_0 + b_1\bigl|x(s)\bigr| + b_2\bigl|y(s)\bigr| + b_3\bigl|I_{0+}^{\theta_2}x(s)\bigr| + b_4\bigl|I_{0+}^{\sigma_2}y(s)\bigr|, \quad s\in[0,1].
\end{aligned}
$$

By $(J3)$, we find

$$
\begin{aligned}
\bigl|x(t)\bigr| &\le \bigl|Q_1(x,y)(t)\bigr| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F_{xy}(s)\,ds \\
&\quad + \frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)}\int_0^1 (1-s)^{\alpha-\gamma_0-1}F_{xy}(s)\,ds \\
&\quad + \frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)}\sum_{i=1}^{p}\frac{1}{\Gamma(\beta-\gamma_i)}\left|\int_0^1\left(\int_0^s (s-\tau)^{\beta-\gamma_i-1}G_{xy}(\tau)\,d\tau\right)dH_i(s)\right| \\
&\quad + \frac{t^{\alpha-1}}{|\Delta|}\left(\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right) \\
&\quad \times \left(\frac{1}{\Gamma(\beta-\delta_0)}\int_0^1 (1-s)^{\beta-\delta_0-1}G_{xy}(s)\,ds\right) \\
&\quad + \frac{t^{\alpha-1}}{|\Delta|}\left(\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right) \\
&\quad \times \left(\sum_{i=1}^{q}\frac{1}{\Gamma(\alpha-\delta_i)}\left|\int_0^1\left(\int_0^s (s-\tau)^{\alpha-\delta_i-1}F_{xy}(\tau)\,d\tau\right)dK_i(s)\right|\right) \\
&\le \left(a_0 + a_1\|x\| + a_2\|y\| + \frac{a_3}{\Gamma(\theta_1+1)}\|x\| + \frac{a_4}{\Gamma(\sigma_1+1)}\|y\|\right) \\
&\quad \times \Bigg[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0+1)\Gamma(\beta-\delta_0)} \\
&\quad + \frac{t^{\alpha-1}}{|\Delta|}\left(\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)} \left| \int_0^1 s^{\alpha-\delta_i}\, dK_i(s) \right| \right) \Bigg] \\
&+ \left(b_0 + b_1\|x\| + b_2\|y\| + \frac{b_3}{\Gamma(\theta_2+1)}\|x\| + \frac{b_4}{\Gamma(\sigma_2+1)}\|y\| \right) \\
&\times \left[\frac{t^{\alpha-1}\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)} \left| \int_0^1 s^{\beta-\gamma_i}\, dH_i(s) \right| \right. \\
&\left. + \frac{t^{\alpha-1}}{|\Delta|\Gamma(\beta-\delta_0+1)} \sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right], \quad \forall t \in [0,1].
\end{aligned}
$$

Therefore we deduce

$$
\begin{aligned}
\|x\| \le{} & \left(a_0 + a_1\|x\| + a_2\|y\| + \frac{a_3}{\Gamma(\theta_1+1)}\|x\| + \frac{a_4}{\Gamma(\sigma_1+1)}\|y\| \right) M_7 \\
& + \left(b_0 + b_1\|x\| + b_2\|y\| + \frac{b_3}{\Gamma(\theta_2+1)}\|x\| + \frac{b_4}{\Gamma(\sigma_2+1)}\|y\| \right) M_{10}.
\end{aligned} \tag{10}
$$

In a similar manner, we obtain

$$
\begin{aligned}
\|y\| \le{} & \left(a_0 + a_1\|x\| + a_2\|y\| + \frac{a_3}{\Gamma(\theta_1+1)}\|x\| + \frac{a_4}{\Gamma(\sigma_1+1)}\|y\| \right) M_9 \\
& + \left(b_0 + b_1\|x\| + b_2\|y\| + \frac{b_3}{\Gamma(\theta_2+1)}\|x\| + \frac{b_4}{\Gamma(\sigma_2+1)}\|y\| \right) M_8.
\end{aligned} \tag{11}
$$

By (10) and (11), we infer

$$
\begin{aligned}
&\|(x,y)\|_Y \\
&\quad = \|x\| + \|y\| \le a_0(M_7+M_9) + b_0(M_8+M_{10}) \\
&\qquad + \left[a_1(M_7+M_9) + \frac{a_3}{\Gamma(\theta_1+1)}(M_7+M_9) + b_1(M_8+M_{10}) \right. \\
&\qquad \left. + \frac{b_3}{\Gamma(\theta_2+1)}(M_8+M_{10}) \right] \|x\| \\
&\qquad + \left[a_2(M_7+M_9) + \frac{a_4}{\Gamma(\sigma_1+1)}(M_7+M_9) + b_2(M_8+M_{10}) \right. \\
&\qquad \left. + \frac{b_4}{\Gamma(\theta_2+1)}(M_8+M_{10}) \right] \|y\| \\
&\quad = a_0(M_7+M_9) + b_0(M_8+M_{10}) + M_{13}\|x\| + M_{14}\|y\| \\
&\quad \le a_0(M_7+M_9) + b_0(M_8+M_{10}) + \Xi_1 \|(x,y)\|_Y.
\end{aligned}
$$

Because $\Xi_1 < 1$, we find

$$
\|(x,y)\|_Y \le \left[a_0(M_7+M_9) + b_0(M_8+M_{10}) \right] (1-\Xi_1)^{-1}, \quad \forall (x,y) \in U.
$$

So we deduce that the set U is bounded.

By using the Leray–Schauder alternative theorem, we conclude that operator Q has at least one fixed point, which is a solution for our problem (S)–(BC). □

Theorem 3.3 *Assume that (J1), (J2), and*

(J4) *There exist the functions* $\psi_1, \psi_2 \in C([0,1],[0,\infty))$ *such that*

$$\left|f(t,u_1,u_2,u_3,u_4)\right| \le \psi_1(t), \qquad \left|g(t,u_1,u_2,u_3,u_4)\right| \le \psi_2(t),$$

for all $t \in [0,1]$, $u_i \in \mathbb{R}$, $i = 1,\dots,4$,

hold. If $\Xi_2 := L_1 M_5 \frac{1}{\Gamma(\alpha+1)} + L_2 M_6 \frac{1}{\Gamma(\beta+1)} < 1$, *then problem (S)–(BC) has at least one solution on* $[0,1]$.

Proof We fix $r_1 > 0$ such that $r_1 \ge (M_7 + M_9)\|\psi_1\| + (M_8 + M_{10})\|\psi_2\|$. We consider the set $\overline{B}_{r_1} = \{(x,y) \in Y, \|(x,y)\|_Y \le r_1\}$, and introduce the operators $D = (D_1, D_2) : \overline{B}_{r_1} \to Y$ and $E = (E_1, E_2) : \overline{B}_{r_1} \to Y$, where $D_1, D_2, E_1, E_2 : \overline{B}_{r_1} \to X$ are defined by

$$\begin{aligned}
D_1(x,y)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{f}_{xy}(s)\,ds,\\
E_1(x,y)(t) &= \frac{t^{\alpha-1}\Gamma(\beta)}{\Delta\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} \hat{f}_{xy}(s)\,ds\\
&\quad - \frac{t^{\alpha-1}\Gamma(\beta)}{\Delta\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1} \hat{g}_{xy}(\tau)\,d\tau \right) dH_i(s)\\
&\quad + \frac{t^{\alpha-1}\Delta_1}{\Delta\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} \hat{g}_{xy}(s)\,ds\\
&\quad - \frac{t^{\alpha-1}\Delta_1}{\Delta} \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} \hat{f}_{xy}(\tau)\,d\tau \right) dK_i(s) \right),\\
D_2(x,y)(t) &= -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \hat{g}_{xy}(s)\,ds,\\
E_2(x,y)(t) &= \frac{t^{\beta-1}\Gamma(\alpha)}{\Delta\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} \hat{g}_{xy}(s)\,ds\\
&\quad - \frac{t^{\beta-1}\Gamma(\alpha)}{\Delta\Gamma(\alpha-\gamma_0)} \sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} \hat{f}_{xy}(\tau)\,d\tau \right) dK_i(s)\\
&\quad + \frac{t^{\beta-1}\Delta_2}{\Delta\Gamma(\alpha-\gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} \hat{f}_{xy}(s)\,ds\\
&\quad - \frac{t^{\beta-1}\Delta_2}{\Delta} \left(\sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i)} \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1} \hat{g}_{xy}(\tau)\,d\tau \right) dH_i(s) \right),
\end{aligned} \tag{12}$$

for all $t \in [0,1]$ and $(x,y) \in \overline{B}_{r_1}$. So $Q_1 = D_1 + E_1$, $Q_2 = D_2 + E_2$, and $Q = D + E$.

By using (J4), we find for all $(x_1,y_1), (x_2,y_2) \in \overline{B}_{r_1}$ as in the proof of Theorem 3.1 that

$$\begin{aligned}
&\left\|D(x_1,y_1) + E(x_2,y_2)\right\|_Y\\
&\quad \le \left\|D(x_1,y_1)\right\|_Y + \left\|E(x_2,y_2)\right\|_Y\\
&\quad = \left\|D_1(x_1,y_1)\right\| + \left\|D_2(x_1,y_1)\right\| + \left\|E_1(x_2,y_2)\right\| + \left\|E_2(x_2,y_2)\right\|\\
&\quad \le \frac{1}{\Gamma(\alpha+1)}\|\psi_1\| + \frac{1}{\Gamma(\beta+1)}\|\psi_2\| + \left(M_{11}\|\psi_1\| + M_{10}\|\psi_2\|\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \left(M_9\|\psi_1\| + M_{12}\|\psi_2\|\right) \\
&= (M_7 + M_9)\|\psi_1\| + (M_8 + M_{10})\|\psi_2\| \le r_1.
\end{aligned}
$$

So $D(x_1,y_1) + E(x_2,y_2) \in \overline{B}_{r_1}$ for all $(x_1,y_1),(x_2,y_2) \in \overline{B}_{r_1}$.

The operator D is a contraction because

$$
\begin{aligned}
&\left\|D(x_1,y_1) - D(x_2,y_2)\right\|_Y \\
&\quad= \left\|D_1(x_1,y_1) - D_1(x_2,y_2)\right\| + \left\|D_2(x_1,y_1) - D_2(x_2,y_2)\right\| \\
&\quad\le \left(L_1M_5\frac{1}{\Gamma(\alpha+1)} + L_2M_6\frac{1}{\Gamma(\beta+1)}\right)\left(\|x_1-x_2\| + \|y_1-y_2\|\right) \\
&\quad= \Xi_2\left\|(x_1,y_1) - (x_2,y_2)\right\|_Y,
\end{aligned}
$$

for all $(x_1,y_1),(x_2,y_2) \in \overline{B}_{r_1}$, and $\Xi_2 < 1$.

Because the functions f and g are continuous, we obtain that operator E is continuous on $\overline{B}_{r_1}$. We show next that E is compact. The functions from E are uniformly bounded on $\overline{B}_{r_1}$ because

$$
\begin{aligned}
&\left\|E(x,y)\right\|_Y = \left\|E_1(x,y)\right\| + \left\|E_2(x,y)\right\| \le (M_{11}+M_9)\|\psi_1\| + (M_{10}+M_{12})\|\psi_2\|, \\
&\forall (x,y) \in \overline{B}_{r_1}.
\end{aligned}
$$

We prove next that the functions from $E(\overline{B}_{r_1})$ are equicontinuous. We denote by

$$
\begin{aligned}
\Psi_{r_1} &= \sup\left\{\left|f(t,x,y,u,v)\right|, t\in[0,1], |x|\le r_1, |y|\le r_1, |u|\le \frac{r_1}{\Gamma(\theta_1+1)},\right. \\
&\quad\left. |v| \le \frac{r_1}{\Gamma(\sigma_1+1)}\right\}, \\
\Theta_{r_1} &= \sup\left\{\left|g(t,x,y,u,v)\right|, t\in[0,1], |x|\le r_1, |y|\le r_1, |u|\le \frac{r_1}{\Gamma(\theta_2+1)},\right. \\
&\quad\left. |v| \le \frac{r_1}{\Gamma(\sigma_2+1)}\right\}.
\end{aligned}
\tag{13}
$$

Then for $(x,y) \in \overline{B}_{r_1}$ and $t_1,t_2 \in [0,1]$ with $t_1 < t_2$, we deduce

$$
\begin{aligned}
&\left|E_1(x,y)(t_2) - E_1(x,y)(t_1)\right| \\
&\quad\le \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)}\int_0^1 (1-s)^{\alpha-\gamma_0-1}\Psi_{r_1}\,ds \\
&\quad\quad+ \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)}\sum_{i=1}^{p}\frac{1}{\Gamma(\beta-\gamma_i)}\left|\int_0^1\left(\int_0^s (s-\tau)^{\beta-\gamma_i-1}\Theta_{r_1}\,d\tau\right)dH_i(s)\right| \\
&\quad\quad+ \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta|}\left(\sum_{i=1}^{p}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)}\left|\int_0^1 s^{\beta-\gamma_i-1}\,dH_i(s)\right|\right) \\
&\quad\quad\times\left(\frac{1}{\Gamma(\beta-\delta_0)}\int_0^1 (1-s)^{\beta-\delta_0-1}\Theta_{r_1}\,ds\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \\
&\times \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} \Psi_{r_1}\, d\tau \right) dK_i(s) \right| \right) \\
&= \Psi_{r_1} \left(t_2^{\alpha-1} - t_1^{\alpha-1} \right) \left[\frac{\Gamma(\beta)}{|\Delta|\Gamma(\alpha-\gamma_0+1)\Gamma(\beta-\delta_0)} \right. \\
&+ \left. \frac{1}{|\Delta|} \left(\sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right) \left(\sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)} \left| \int_0^1 s^{\alpha-\delta_i}\, dK_i(s) \right| \right) \right] \\
&+ \Theta_{r_1} \left(t_2^{\alpha-1} - t_1^{\alpha-1} \right) \left[\frac{\Gamma(\beta)}{|\Delta|\Gamma(\beta-\delta_0)} \sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)} \left| \int_0^1 s^{\beta-\gamma_i}\, dH_i(s) \right| \right. \\
&+ \left. \frac{1}{|\Delta|\Gamma(\beta-\delta_0+1)} \sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s) \right| \right] \\
&= M_{11} \Psi_{r_1} \left(t_2^{\alpha-1} - t_1^{\alpha-1} \right) + M_{10} \Theta_{r_1} \left(t_2^{\alpha-1} - t_1^{\alpha-1} \right),
\end{aligned}
$$

$$
\begin{aligned}
&\left| E_2(x,y)(t_2) - E_2(x,y)(t_1) \right| \\
&\leq \frac{(t_2^{\beta-1} - t_1^{\beta-1})\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0)} \int_0^1 (1-s)^{\beta-\delta_0-1} \Theta_{r_1}\, ds \\
&+ \frac{(t_2^{\beta-1} - t_1^{\beta-1})\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)} \sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\alpha-\delta_i-1} \Psi_{r_1}\, d\tau \right) dK_i(s) \right| \\
&+ \frac{t_2^{\beta-1} - t_1^{\beta-1}}{|\Delta|} \left(\sum_{i=1}^{q} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 s^{\alpha-\delta_i-1}\, dK_i(s) \right| \right) \\
&\times \left(\frac{1}{\Gamma(\alpha-\gamma_0)} \int_0^1 (1-s)^{\alpha-\gamma_0-1} \Psi_{r_1}\, ds \right) \\
&+ \frac{t_2^{\beta-1} - t_1^{\beta-1}}{|\Delta|} \left(\sum_{i=1}^{q} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 s^{\alpha-\delta_i-1}\, dK_i(s) \right| \right) \\
&\times \left(\sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i)} \left| \int_0^1 \left(\int_0^s (s-\tau)^{\beta-\gamma_i-1} \Theta_{r_1}\, d\tau \right) dH_i(s) \right| \right) \\
&= \Psi_{r_1} \left(t_2^{\beta-1} - t_1^{\beta-1} \right) \left[\frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)} \sum_{i=1}^{q} \frac{1}{\Gamma(\alpha-\delta_i+1)} \left| \int_0^1 s^{\alpha-\delta_i}\, dK_i(s) \right| \right. \\
&+ \left. \frac{1}{|\Delta|\Gamma(\alpha-\gamma_0+1)} \sum_{i=1}^{q} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 s^{\alpha-\delta_i-1}\, dK_i(s) \right| \right] \\
&+ \Theta_{r_1} \left(t_2^{\beta-1} - t_1^{\beta-1} \right) \left[\frac{\Gamma(\alpha)}{|\Delta|\Gamma(\alpha-\gamma_0)\Gamma(\beta-\delta_0+1)} \right. \\
&+ \left. \frac{1}{|\Delta|} \left(\sum_{i=1}^{q} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\delta_i)} \left| \int_0^1 s^{\alpha-\delta_i-1}\, dK_i(s) \right| \right) \left(\sum_{i=1}^{p} \frac{1}{\Gamma(\beta-\gamma_i+1)} \left| \int_0^1 s^{\beta-\gamma_i}\, dH_i(s) \right| \right) \right] \\
&= M_9 \Psi_{r_1} \left(t_2^{\beta-1} - t_1^{\beta-1} \right) + M_{12} \Theta_{r_1} \left(t_2^{\beta-1} - t_1^{\beta-1} \right).
\end{aligned}
$$

Therefore we infer

$$\left|E_1(x,y)(t_2) - E_1(x,y)(t_1)\right| \to 0, \qquad \left|E_2(x,y)(t_2) - E_2(x,y)(t_1)\right| \to 0,$$

as $t_2 \to t_1$, uniformly with respect to $(x,y) \in \overline{B}_{r_1}$. Then $E_1(\overline{B}_{r_1})$ and $E_2(\overline{B}_{r_1})$ are equicontinuous, and so $E(\overline{B}_{r_1})$ is also equicontinuous. By applying Arzela–Ascoli theorem, we conclude that the set $E(\overline{B}_{r_1})$ is relatively compact. Hence E is a compact operator on $\overline{B}_{r_1}$. By using the Krasnosel'skii theorem for the sum of two operators (see [22]), we deduce that there exists a fixed point of operator $D+E(=Q)$, which is a solution of problem (S)–(BC). □

Theorem 3.4 *Suppose that* ($J1$), ($J2$), *and* ($J4$) *hold. If* $\Xi_3 := L_1M_5(M_9+M_{11})+L_2M_6(M_{10}+M_{12}) < 1$, *then problem* ($S$)–($BC$) *has at least one solution* (x,y) *on* $[0,1]$.

Proof We consider again a positive number $r_1 \geq (M_7+M_9)\|\psi_1\| + (M_8+M_{10})\|\psi_2\|$ and the operators D and E defined on $\overline{B}_{r_1}$ given by (12). As in the proof of Theorem 3.3, we have $D(x_1,y_1)+E(x_2,y_2) \in \overline{B}_{r_1}$ for all $(x_1,y_1),(x_2,y_2) \in \overline{B}_{r_1}$.

The operator E is a contraction because

$$\begin{aligned}
&\left\|E(x_1,y_1) - E(x_2,y_2)\right\|_Y \\
&\quad = \left\|E_1(x_1,y_1) - E_1(x_1,y_1)\right\| + \left\|E_2(x_1,y_1) - E_2(x_2,y_2)\right\| \\
&\quad \leq (L_1M_5M_{11} + L_2M_6M_{10})\left\|(x_1,y_1) - (x_2,y_2)\right\|_Y \\
&\qquad + (L_1M_5M_9 + L_2M_6M_{12})\left\|(x_1,y_1) - (x_2,y_2)\right\|_Y \\
&\quad = \bigl(L_1M_5(M_9+M_{11}) + L_2M_6(M_{10}+M_{12})\bigr)\left\|(x_1,y_1) - (x_2,y_2)\right\|_Y \\
&\quad = \Xi_3\left\|(x_1,y_1) - (x_2,y_2)\right\|_Y,
\end{aligned}$$

for all $(x_1,y_1),(x_2,y_2) \in \overline{B}_{r_1}$, with $\Xi_3 < 1$.

In what follows, the continuity of functions f and g implies that operator D is continuous on $\overline{B}_{r_1}$. We prove now that D is a compact operator. The functions from $D(\overline{B}_{r_1})$ are uniformly bounded because

$$\left\|D(x,y)\right\|_Y = \left\|D_1(x,y)\right\| + \left\|D_2(x,y)\right\| \leq \frac{1}{\Gamma(\alpha+1)}\|\psi_1\| + \frac{1}{\Gamma(\beta+1)}\|\psi_2\|, \quad \forall (x,y) \in \overline{B}_{r_1}.$$

Now we show that the functions from $D(\overline{B}_{r_1})$ are equicontinuous. By using Ψ_{r_1} and Θ_{r_1} defined by (13), we deduce that for $(x,y) \in \overline{B}_{r_1}$ and $t_1,t_2 \in [0,1]$ with $t_1 < t_2$ that

$$\begin{aligned}
\left|D_1(x,y)(t_2) - D_1(x,y)(t_1)\right| &\leq \frac{\Psi_{r_1}}{\Gamma(\alpha+1)}\bigl(t_2^\alpha - t_1^\alpha\bigr), \\
\left|D_2(x,y)(t_2) - D_2(x,y)(t_1)\right| &\leq \frac{\Theta_{r_1}}{\Gamma(\beta+1)}\bigl(t_2^\beta - t_1^\beta\bigr).
\end{aligned}$$

Therefore we conclude

$$\left|D_1(x,y)(t_2) - D_1(x,y)(t_1)\right| \to 0, \qquad \left|D_2(x,y)(t_2) - D_2(x,y)(t_1)\right| \to 0,$$

as $t_2 \to t_1$, uniformly with respect to $(x,y) \in \overline{B}_{r_1}$. We infer that $D_1(\overline{B}_{r_1})$ and $D_2(\overline{B}_{r_1})$ are equicontinuous, and so $D(\overline{B}_{r_1})$ is equicontinuous. By using Arzela–Ascoli theorem, we deduce that the set $D(\overline{B}_{r_1})$ is relatively compact. Then D is a compact operator on $\overline{B}_{r_1}$. By using the Krasnosel'skii theorem, we conclude that there exists a fixed point of operator $D + E(= Q)$, which is a solution of problem (*S*)–(*BC*). □

Theorem 3.5 *Assume that* $(J1)$ *and*

$(J5)$ *The functions* $f, g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *are continuous and there exist the constants* $c_i \geq 0$, $i = 0, \dots, 4$ *with at least one nonzero constant, the constants* $d_i \geq 0$, $i = 0, \dots, 4$ *with at least one nonzero constant, and* $l_i, m_i \in (0,1)$, $i = 1, \dots, 4$ *such that*

$$\bigl|f(t,u_1,u_2,u_3,u_4)\bigr| \leq c_0 + \sum_{i=1}^{4} c_i |u_i|^{l_i},$$

$$\bigl|g(t,u_1,u_2,u_3,u_4)\bigr| \leq d_0 + \sum_{i=1}^{4} d_i |u_i|^{m_i},$$

for all $t \in [0,1]$, $u_i \in \mathbb{R}$, $i = 1, \dots, 4$,

hold. Then problem (*S*)–(*BC*) *has at least one solution.*

Proof Let $\overline{B}_R = \{(x,y) \in Y, \|(x,y)\|_Y \leq R\}$, where

$$\begin{aligned} R \geq \max\Biggl\{ & 20c_0M_7, (20c_1M_7)^{\frac{1}{1-l_1}}, (20c_2M_7)^{\frac{1}{1-l_2}}, \\ & \biggl(\frac{20c_3M_7}{(\Gamma(\theta_1+1))^{l_3}}\biggr)^{\frac{1}{1-l_3}}, \biggl(\frac{20c_4M_7}{(\Gamma(\sigma_1+1))^{l_4}}\biggr)^{\frac{1}{1-l_4}}, \\ & 20d_0M_{10}, (20d_1M_{10})^{\frac{1}{1-m_1}}, (20d_2M_{10})^{\frac{1}{1-m_2}}, \\ & \biggl(\frac{20d_3M_{10}}{(\Gamma(\theta_2+1))^{m_3}}\biggr)^{\frac{1}{1-m_3}}, \biggl(\frac{20d_4M_{10}}{(\Gamma(\sigma_2+1))^{m_4}}\biggr)^{\frac{1}{1-m_4}}, \\ & 20c_0M_9, (20c_1M_9)^{\frac{1}{1-l_1}}, (20c_2M_9)^{\frac{1}{1-l_2}}, \\ & \biggl(\frac{20c_3M_9}{(\Gamma(\theta_1+1))^{l_3}}\biggr)^{\frac{1}{1-l_3}}, \biggl(\frac{20c_4M_9}{(\Gamma(\sigma_1+1))^{l_4}}\biggr)^{\frac{1}{1-l_4}}, \\ & 20d_0M_8, (20d_1M_8)^{\frac{1}{1-m_1}}, (20d_2M_8)^{\frac{1}{1-m_2}}, \\ & \biggl(\frac{20d_3M_8}{(\Gamma(\theta_2+1))^{m_3}}\biggr)^{\frac{1}{1-m_3}}, \biggl(\frac{20d_4M_8}{(\Gamma(\sigma_2+1))^{m_4}}\biggr)^{\frac{1}{1-m_4}} \Biggr\}. \end{aligned}$$

We prove that $Q : \overline{B}_R \to \overline{B}_R$. For $(x,y) \in \overline{B}_R$, we have

$$\begin{aligned} \bigl|Q_1(x,y)(t)\bigr| \leq{} & \biggl(c_0 + c_1R^{l_1} + c_2R^{l_2} + c_3\frac{R^{l_3}}{(\Gamma(\theta_1+1))^{l_3}} + c_4\frac{R^{l_4}}{(\Gamma(\sigma_1+1))^{l_4}}\biggr)M_7 \\ & + \biggl(d_0 + d_1R^{m_1} + d_2R^{m_2} + d_3\frac{R^{m_3}}{(\Gamma(\theta_2+1))^{m_3}} + d_4\frac{R^{m_4}}{(\Gamma(\sigma_2+1))^{m_4}}\biggr)M_{10} \\ \leq{} & \frac{R}{2}, \end{aligned}$$

$$
\begin{aligned}
\left|Q_2(x,y)(t)\right| &\le \left(c_0 + c_1R^{l_1} + c_2R^{l_2} + c_3\frac{R^{l_3}}{(\Gamma(\theta_1+1))^{l_3}} + c_4\frac{R^{l_4}}{(\Gamma(\sigma_1+1))^{l_4}}\right)M_9 \\
&\quad + \left(d_0 + d_1R^{m_1} + d_2R^{m_2} + d_3\frac{R^{m_3}}{(\Gamma(\theta_2+1))^{m_3}} + d_4\frac{R^{m_4}}{(\Gamma(\sigma_2+1))^{m_4}}\right)M_8 \\
&\le \frac{R}{2},
\end{aligned}
$$

for all $t \in [0,1]$. Then we obtain

$$
\|Q(x,y))\|_Y = \|Q_1(x,y)\| + \|Q_2(x,y)\| \le R, \quad \forall (x,y) \in \overline{B}_R,
$$

which implies that $Q(\overline{B}_R) \subset \overline{B}_R$.

By using the fact that the functions f and g are continuous, we deduce that that operator Q is continuous on $\overline{B}_R$. Besides, the functions from $Q(\overline{B}_R)$ are uniformly bounded and equicontinuous. Indeed, by using the notations (13) with r_1 replaced by R, we find for any $(x,y) \in \overline{B}_R$ and $t_1, t_2 \in [0,1]$, $t_1 < t_2$ that

$$
\begin{aligned}
\left|Q_1(x,y)(t_2) - Q_1(x,y)(t_1)\right| &\le \frac{\Psi_R}{\Gamma(\alpha+1)}\left(t_2^\alpha - t_1^\alpha\right) + (\Psi_R M_{11} + \Theta_R M_{10})\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right), \\
\left|Q_2(x,y)(t_2) - Q_2(x,y)(t_1)\right| &\le \frac{\Theta_R}{\Gamma(\beta+1)}\left(t_2^\beta - t_1^\beta\right) + (\Psi_R M_9 + \Theta_R M_{12})\left(t_2^{\beta-1} - t_1^{\beta-1}\right).
\end{aligned}
$$

Therefore we obtain

$$
\left|Q_1(x,y)(t_2) - Q_1(x,y)(t_1)\right| \to 0, \qquad \left|Q_2(x,y)(t_2) - Q_2(x,y)(t_1)\right| \to 0, \quad \text{as } t_2 \to t_1,
$$

uniformly with respect to $(x,y) \in \overline{B}_R$. By Arzela–Ascoli theorem, we conclude that $Q(\overline{B}_R)$ is relatively compact, and then Q is a compact operator. By using the Schauder fixed point theorem, we infer that operator Q has at least one fixed point (x,y) in $\overline{B}_R$, which is a solution of our problem (S)–(BC). □

Theorem 3.6 *Suppose that (J1) and*

(J6) *The functions $f, g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous and there exist the constants $p_i \ge 0$, $i = 0, \dots, 4$ with at least one nonzero constant, the constants $q_i \ge 0$, $i = 0, \dots, 4$ with at least one nonzero constant, and nondecreasing functions $\xi_i, \eta_i \in C([0,\infty),[0,\infty))$ $i = 1, \dots, 4$ such that*

$$
\begin{aligned}
\left|f(t,u_1,u_2,u_3,u_4)\right| &\le p_0 + \sum_{i=1}^{4} p_i\xi_i\left(|u_i|\right), \\
\left|g(t,u_1,u_2,u_3,u_4)\right| &\le q_0 + \sum_{i=1}^{4} q_i\eta_i\left(|u_i|\right),
\end{aligned}
$$

for all $t \in [0,1]$, $u_i \in \mathbb{R}$, $i = 1, \dots, 4$, *hold. If there exists* $\Xi_0 > 0$ *such that*

$$\begin{aligned}&\left(p_0 + p_1\xi_1(\Xi_0) + p_2\xi_2(\Xi_0) + p_3\xi_3\left(\frac{\Xi_0}{\Gamma(\theta_1+1)}\right) + p_4\xi_4\left(\frac{\Xi_0}{\Gamma(\sigma_1+1)}\right)\right)(M_7+M_9)\\&\quad+\left(q_0 + q_1\eta_1(\Xi_0) + q_2\eta_2(\Xi_0) + q_3\eta_3\left(\frac{\Xi_0}{\Gamma(\theta_2+1)}\right)\right.\\&\quad\left.+q_4\eta_4\left(\frac{\Xi_0}{\Gamma(\sigma_2+1)}\right)\right)(M_8+M_{10}) < \Xi_0,\end{aligned} \tag{14}$$

then problem (S)–(BC) *has at least one solution on* $[0,1]$.

Proof We consider the set $\overline{B}_{\Xi_0} = \{(x,y) \in Y, \|(x,y)\|_Y \le \Xi_0\}$, where Ξ_0 is given in the theorem. We will show that $Q : \overline{B}_{\Xi_0} \to \overline{B}_{\Xi_0}$. For $(x,y) \in \overline{B}_{\Xi_0}$ and $t \in [0,1]$, we obtain

$$\begin{aligned}&\left|Q_1(x,y)(t)\right|\\&\quad\le\left(p_0 + p_1\xi_1(\Xi_0) + p_2\xi_2(\Xi_0) + p_3\xi_3\left(\frac{\Xi_0}{\Gamma(\theta_1+1)}\right) + p_4\xi_4\left(\frac{\Xi_0}{\Gamma(\sigma_1+1)}\right)\right)M_7\\&\qquad+\left(q_0 + q_1\eta_1(\Xi_0) + q_2\eta_2(\Xi_0) + q_3\eta_3\left(\frac{\Xi_0}{\Gamma(\theta_2+1)}\right) + q_4\eta_4\left(\frac{\Xi_0}{\Gamma(\sigma_2+1)}\right)\right)M_{10},\\&\left|Q_2(x,y)(t)\right|\\&\quad\le\left(p_0 + p_1\xi_1(\Xi_0) + p_2\xi_2(\Xi_0) + p_3\xi_3\left(\frac{\Xi_0}{\Gamma(\theta_1+1)}\right) + p_4\xi_4\left(\frac{\Xi_0}{\Gamma(\sigma_1+1)}\right)\right)M_9\\&\qquad+\left(q_0 + q_1\eta_1(\Xi_0) + q_2\eta_2(\Xi_0) + q_3\eta_3\left(\frac{\Xi_0}{\Gamma(\theta_2+1)}\right) + q_4\eta_4\left(\frac{\Xi_0}{\Gamma(\sigma_2+1)}\right)\right)M_8,\end{aligned}$$

and then, for all $(x,y) \in \overline{B}_{\Xi_0}$, we find

$$\begin{aligned}&\left\|Q(x,y)\right\|_Y\\&\quad\le\left(p_0 + p_1\xi_1(\Xi_0) + p_2\xi_2(\Xi_0) + p_3\xi_3\left(\frac{\Xi_0}{\Gamma(\theta_1+1)}\right) + p_4\xi_4\left(\frac{\Xi_0}{\Gamma(\sigma_1+1)}\right)\right)(M_7+M_9)\\&\qquad+\left(q_0 + q_1\eta_1(\Xi_0) + q_2\eta_2(\Xi_0) + q_3\eta_3\left(\frac{\Xi_0}{\Gamma(\theta_2+1)}\right)\right.\\&\qquad\left.+q_4\eta_4\left(\frac{\Xi_0}{\Gamma(\sigma_2+1)}\right)\right)(M_8+M_{10}) < \Xi_0.\end{aligned}$$

Hence $Q(\overline{B}_{\Xi_0}) \subset \overline{B}_{\Xi_0}$. Using a similar approach as in the proof of Theorem 3.5, we can show that operator Q is completely continuous.

We suppose now that there exists $(x,y) \in \partial B_{\Xi_0}$ such that $(x,y) = \nu Q(x,y)$ for some $\nu \in (0,1)$. Arguing as above, we deduce $\|(x,y)\|_Y \le \|Q(x,y)\|_Y < \Xi_0$, which is a contradiction, because $(x,y) \in \partial B_{\Xi_0}$. Then by using the nonlinear alternative of Leray–Schauder type, we conclude that operator Q has a fixed point $(x,y) \in \overline{B}_{\Xi_0}$, and so problem ($S$)–($BC$) has at least one solution. □

4 Examples

Let $\alpha = \frac{3}{2}$ $(n = 2)$, $\beta = \frac{7}{3}$ $(m = 3)$, $\theta_1 = \frac{1}{4}$, $\sigma_1 = \frac{6}{5}$, $\theta_2 = \frac{17}{4}$, $\sigma_2 = \frac{1}{3}$, $p = 1$, $q = 2$, $\gamma_0 = \frac{1}{6}$, $\gamma_1 = \frac{3}{4}$, $\delta_0 = \frac{8}{7}$, $\delta_1 = \frac{1}{5}$, $\delta_2 = \frac{1}{3}$, $H_1(t) = \{0, t \in [0, \frac{1}{2}); 3, t \in [\frac{1}{2}, 1]\}$, $K_1(t) = -t^2, t \in [0, 1]$, $K_2(t) = \{0, t \in [0, \frac{1}{3}); 4, t \in [\frac{1}{3}, 1]\}$.

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{3/2} x(t) + f(t, x(t), y(t), I_{0+}^{1/4} x(t), I_{0+}^{6/5} y(t)) = 0, \quad t \in (0, 1), \\ D_{0+}^{7/3} y(t) + g(t, x(t), y(t), I_{0+}^{17/4} x(t), I_{0+}^{1/3} y(t)) = 0, \quad t \in (0, 1), \end{cases} \tag{S_0}$$

with the boundary conditions

$$\begin{cases} x(0) = 0, \qquad D_{0+}^{1/6} x(1) = 3 D_{0+}^{3/4} y(\frac{1}{2}), \\ y(0) = y'(0) = 0, \qquad D_{0+}^{8/7} y(1) = -2 \int_0^1 t D_{0+}^{1/5} x(t)\, dt + 4 D_{0+}^{1/3} x(\frac{1}{3}). \end{cases} \tag{BC_0}$$

We obtain $\Delta \approx -4.92715202 \neq 0$. So assumption $(J1)$ is satisfied. In addition, we have $M_1 \approx 2.10326265$, $M_2 \approx 1.90760368$, $M_3 \approx 1.02839972$, $M_4 \approx 2.11984652$, $M_5 = M_1$, $M_6 = M_4$, $M_7 \approx 1.81109405$, $M_{10} \approx 0.68108088$, $M_9 \approx 0.9999811$, $M_8 \approx 1.12515265$, $M_{11} \approx 1.05884127$, and $M_{12} \approx 0.76520198$.

Example 1 We consider the functions

$$\begin{aligned} & f(t, u_1, u_2, u_3, u_4) \\ & \quad = \frac{1}{\sqrt{9 + t^3}} - \frac{t}{10} \arctan u_1 + \frac{|u_2|}{(t+2)^4 (1 + |u_2|)} + \frac{1}{3(t+8)} \sin^2 u_3 - \frac{t^2}{t + 12} \cos u_4, \\ & g(t, u_1, u_2, u_3, u_4) \\ & \quad = \frac{3t}{t^2 + 4} - \frac{|u_1|}{6(2 + |u_1|)} + \frac{1}{15} \sin u_2 + \frac{t}{t + 24} \cos^2 u_3 - \frac{1}{12} \arctan u_4, \end{aligned}$$

for all $t \in [0, 1]$, $u_i \in \mathbb{R}$, $i = 1, \ldots, 4$. We find the inequalities

$$\begin{aligned} & \bigl| f(t, u_1, u_2, u_3, u_4) - f(t, v_1, v_2, v_3, v_4) \bigr| \\ & \quad \leq \frac{1}{10} |u_1 - v_1| + \frac{1}{16} |u_2 - v_2| + \frac{1}{12} |u_3 - v_3| \\ & \qquad + \frac{1}{13} |u_4 - v_4| \leq \frac{1}{10} \sum_{i=1}^{4} |u_i - v_i|, \\ & \bigl| g(t, u_1, u_2, u_3, u_4) - g(t, v_1, v_2, v_3, v_4) \bigr| \\ & \quad \leq \frac{1}{12} |u_1 - v_1| + \frac{1}{15} |u_2 - v_2| + \frac{2}{25} |u_3 - v_3| \\ & \qquad + \frac{1}{12} |u_4 - v_4| \leq \frac{1}{12} \sum_{i=1}^{4} |u_i - v_i|, \end{aligned}$$

for all $t \in [0, 1]$, $u_i, v_i \in \mathbb{R}, i = 1, \ldots, 4$. So we have $L_1 = \frac{1}{10}$, $L_2 = \frac{1}{12}$ and $\Xi = L_1 M_5 (M_7 + M_9) + L_2 M_6 (M_8 + M_{10}) \approx 0.91032 < 1$. Therefore assumption $(J2)$ is satisfied, and, by Theorem 3.1, we deduce that problem (S_0)–(BC_0) has at least one solution $(x(t), y(t))$, $t \in [0, 1]$.

Example 2 We consider the functions

$$f(t,u_1,u_2,u_3,u_4) = \frac{t+2}{t^2+5}\left(2\sin t + \frac{1}{5}\cos u_1\right) - \frac{1}{(t+5)^2}u_2 - \frac{t}{6}\arctan u_3 + \frac{1}{7}\sin u_4,$$
$$g(t,u_1,u_2,u_3,u_4) = \frac{e^{-t}}{2+t^3} + \frac{1}{4}\cos^2 u_2 - \frac{1}{5}\sin u_3 + \frac{1}{9}\arctan u_4,$$

for all $t \in [0,1]$, $u_i \in \mathbb{R}, i = 1,\dots,4$. Because we have

$$\left|f(t,u_1,u_2,u_3,u_4)\right| \le \frac{11}{10} + \frac{1}{25}|u_2| + \frac{1}{6}|u_3| + \frac{1}{7}|u_4|,$$
$$\left|g(t,u_1,u_2,u_3,u_4)\right| \le \frac{3}{4} + \frac{1}{5}|u_3| + \frac{1}{9}|u_4|,$$

for all $t \in [0,1]$, $u_i \in \mathbb{R}, i = 1,\dots,4$, the assumption $(J3)$ is satisfied with $a_0 = \frac{11}{10}$, $a_1 = 0$, $a_2 = \frac{1}{25}$, $a_3 = \frac{1}{6}$, $a_4 = \frac{1}{7}$, $b_0 = \frac{3}{4}$, $b_1 = b_2 = 0$, $b_3 = \frac{1}{5}$, and $b_4 = \frac{1}{9}$. In addition, we obtain $M_{13} \approx 0.52715168$, $M_{14} \approx 0.70166538$, and $\Xi_1 = \max\{M_{13}, M_{14}\} = M_{14} < 1$. Then, by Theorem 3.2, we conclude that problem (S_0)–(BC_0) has at least one solution $(x(t), y(t)), t \in [0,1]$.

Example 3 We consider the functions

$$f(t,u_1,u_2,u_3,u_4) = -\frac{1}{5}|u_2|^{3/4} + \frac{1}{3(1+t^2)}\arctan|u_3|^{1/2},$$
$$g(t,u_1,u_2,u_3,u_4) = \frac{e^{-t}}{1+t^3} - \frac{1}{3}u_1^{4/5} + \sin u_4^{2/3},$$

for all $t \in [0,1]$, $u_i \in \mathbb{R}, i = 1,\dots,4$. Because we obtain

$$\left|f(t,u_1,u_2,u_3,u_4)\right| \le \frac{1}{5}|u_2|^{3/4} + \frac{1}{3}|u_3|^{1/2},$$
$$\left|g(t,u_1,u_2,u_3,u_4)\right| \le 1 + \frac{1}{3}|u_1|^{4/5} + |u_4|^{2/3},$$

for all $t \in [0,1], u_i \in \mathbb{R}, i = 1,\dots,4$, the assumption $(J5)$ is satisfied with $c_0 = c_1 = 0$, $c_2 = \frac{1}{5}$, $c_3 = \frac{1}{3}$, $c_4 = 0$, $d_0 = 1$, $d_1 = \frac{1}{3}$, $d_2 = d_3 = 0$, $d_4 = 1$, $l_2 = \frac{3}{4}$, $l_3 = \frac{1}{2}$, $m_1 = \frac{4}{5}$, and $m_4 = \frac{2}{3}$. Therefore, by Theorem 3.5, we deduce that problem (S_0)–(BC_0) has at least one solution $(x(t), y(t)), t \in [0,1]$.

Example 4 We consider the functions

$$f(t,u_1,u_2,u_3,u_4) = \frac{t^3}{25} + \frac{e^{-t}u_1^4}{20(1+u_2^2)} - \frac{t^2 u_4^{1/3}}{10},$$
$$g(t,u_1,u_2,u_3,u_4) = \frac{(1-t)^4}{20} - \frac{1-t^2}{15}u_2^2 - \frac{1}{25}u_3^{2/5},$$

for all $t \in [0,1], u_i \in \mathbb{R}, i = 1,\dots,4$. Because we have

$$\left|f(t,u_1,u_2,u_3,u_4)\right| \le \frac{1}{25} + \frac{1}{20}|u_1|^4 + \frac{1}{10}|u_4|^{1/3},$$

$$\left|g(t,u_1,u_2,u_3,u_4)\right| \le \frac{1}{20} + \frac{1}{15}|u_2|^2 + \frac{1}{25}|u_3|^{2/5},$$

for all $t \in [0,1], u_i \in \mathbb{R}, i = 1,\dots,4$, the assumption $(J6)$ is satisfied with $p_0 = \frac{1}{25}$, $p_1 = \frac{1}{20}$, $p_2 = p_3 = 0$, $p_4 = \frac{1}{10}$, $q_0 = \frac{1}{20}$, $q_1 = 0$, $q_2 = \frac{1}{15}$, $q_3 = \frac{1}{25}$, $q_4 = 0$, $\xi_1(x) = x^4$, $\xi_4(x) = x^{1/3}$, $\eta_2(x) = x^2$, and $\eta_3(x) = x^{2/5}$ for $x \ge 0$. For $\Xi_0 = 1$, the condition (14) is satisfied because $(\frac{1}{25} + \frac{1}{20} + \frac{1}{10}(\frac{1}{\Gamma(11/5)})^{1/3})(M_7 + M_9) + (\frac{1}{20} + \frac{1}{15} + \frac{1}{25}(\frac{1}{\Gamma(21/4)})^{2/5})(M_8 + M_{10}) \approx 0.75328 < 1$. Then, by Theorem 3.6, we conclude that problem (S_0)–(BC_0) has at least one solution $(x(t),y(t)), t \in [0,1]$.

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