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Study of a nonlinear multi-terms boundary value problem of fractional pantograph differential equations

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Abstract

In this research work, a class of multi-term fractional pantograph differential equations (FODEs) subject to antiperiodic boundary conditions (APBCs) is considered. The ensuing problem involves proportional type delay terms and constitutes a subclass of delay differential equations known as pantograph. On using fixed point theorems due to Banach and Schaefer, some sufficient conditions are developed for the existence and uniqueness of the solution to the problem under investigation. Furthermore, due to the significance of stability analysis from a numerical and optimization point of view Ulam type stability and its various forms are studied. Here we mention different forms of stability: Hyers–Ulam (HU), generalized Hyers–Ulam (GHU), Hyers–Ulam Rassias (HUR) and generalized Hyers–Ulam–Rassias (GHUR). After the demonstration of our results, some pertinent examples are given.

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1 Introduction

FODEs have many applications in modeling various real world processes and phenomena. Therefore in previous several decades it has been given much attention. In fact, FODEs are definite integrals which include classical differential and integral equations as a special case. In recent time this has become a most intensely studied area for research in mathematics and in other applied sciences like physics, dynamics, electrodynamics and fluid mechanics. The mentioned area has a large number of applications also in mathematical modeling of biological models (for details see [1–4]).

In the past the areas of nonlinear integrals and DEs were given great importance as they have numerous uses in modeling different problems in several fields of technologies and engineering. This is so because using the usual derivative for modeling various real world processes, hereditary and memory descriptions cannot be expressed properly in many situations. So researchers have proved that on using fractional differential operators to describe memory and hereditary processes in many situations has produced very good

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results as compared to integer order derivatives. This fact motivated researchers to study FODEs from different directions. The said area has been investigated from different aspects including qualitative and stability theory, optimization and approximation results. Hence plenty of work can be traced in the literature about existence theory of solutions, we refer to [5–10].

On the other hand, very well performed research has been conducted on the numerical side of FODEs. In this regard plenty of research articles addressing numerical and qualitative analysis have been presented in the past few years (for instance see [11–16] and the references therein). Here we remark that stability analysis is also an important aspect of qualitative analysis. This is so because stability results are the important requirements for numerical and optimization purposes during the investigation of solutions to applied problems. Different kinds of stability results, like exponential, Mittag-Leffler and Lyapunov type, have been studied for classical differential and integral equations. In the previous few years the mentioned theory was greatly updated for FODEs (see for details [17–19]). Establishing these stabilities for nonlinear systems have merits and de-merits in constructions. Some of them need a "pre-defined Lyapunov function" and such a function usually is very hard to construct. Also we have the "exponential and Mittag-Leffler stability [20] involving exponential functions" which often create difficulties when one is to find numerical solutions to certain problems. In this regard another kind of stability has been given much attention by researchers, known as HU stability. This kind of stability was first pointed by Ulam in 1940 during a talk. After that in 1940 Hyers very nicely gave explanations for functional equations (for details we refer to [21–23]). Interesting results have been developed in the last few years (refer to ([24–27] and the references therein).

Study of real world problems with respect to delay problems constitutes a huge class of applied analysis. In this regard, proportional type delay problems constitute a subclass known as pantograph differential equations (PDEs). The aforesaid area is increasingly used to model numerous process. The mentioned type of problems arise in large numbers of applications in electro-dynamics [28]. Therefore, keeping in mind the applications of the said area, many researchers have studied the PDEs from many aspects like existence theory and numerical analysis (see for details [29, 30]). The authors of [31] in 2004 investigated the given delay problem with $t \in [0, \top] = \mathbf{J}$ as

$$\begin{aligned} &\mathrm{w}'(t) = \mathcal{H}\big(t, \mathrm{w}(t), \mathrm{w}(\delta_1 t), \mathrm{w}(\delta_2 t), \dots, \mathrm{w}(\delta_m t)\big), \\ &\mathrm{w}(0) = \mathrm{w}_0, \quad \mathrm{w}_0 \in \mathbf{R}, \end{aligned}$$

where $0 < \varepsilon \leq 1$, $0 < \delta_j < 1$, $j = 1, 2, \dots, n$ and $\mathcal{H} : \mathbf{J} \times \mathbf{R}^{m+1} \to \mathbf{R}$. By fixed point theory, they investigated the results. The generalized form of the aforementioned problem was investigated in 2013 in [32]. The concerned fractional order PDEs have been studied recently for existence theory in [33, 34]. Also continuous type delay problems of FODEs have been studied recently for theoretical analysis, we refer the reader to [35–37]. Furthermore, in [38], the authors have investigated the following problem under APBCs with $0 < \delta < 1$:

$$\begin{cases} {}^C\mathrm{D}^{\sigma}_{+0}\mathrm{w}(t) = \mathcal{H}(t, \mathrm{w}(t), \mathrm{w}(\delta t), {}^C\mathrm{D}^{\sigma}_{+0}\mathrm{w}(t)), \quad t \in \mathbf{J}, 2 < \sigma \leq 3, \\ \mathrm{w}(0) = -\mathrm{w}(\top), \qquad {}^C\mathrm{D}^{r}_{+0}\mathrm{w}(0) = -{}^C\mathrm{D}^{r}_{+0}\mathrm{w}(\top), \qquad {}^C\mathrm{D}^{s}_{+0}\mathrm{w}(0) = -{}^C\mathrm{D}^{s}_{+0}\mathrm{w}(\top). \end{cases} \tag{1}$$

They have established qualitative results of stability and existence theory as regards the given problem in (1).

Inspired by the aforementioned work, in this research article, we study the given multi-term problem of FODEs involving delay terms

$$\begin{cases} {}^{C}\mathrm{D}^{\sigma}_{+0}\mathrm{w}(t) = \mathcal{H}(t,\mathrm{w}(t),\mathrm{w}(\delta t),\mathrm{w}(\delta_2 t),\dots,\mathrm{w}(\delta_m t)), & t\in\mathbf{J}, 2<\sigma\le 3,\\ \mathrm{w}(0) = -\mathrm{w}(\top), \qquad {}^{C}\mathrm{D}^{r}_{+0}\mathrm{w}(0) = -{}^{C}\mathrm{D}^{r}_{+0}\mathrm{w}(\top), & {}^{C}\mathrm{D}^{s}_{+0}\mathrm{w}(0) = -{}^{C}\mathrm{D}^{s}_{+0}\mathrm{w}(\top), \end{cases} \tag{2}$$

where $0<\delta_j<1$, $0<r<1$, $1<s<2$, $j=1,2,\dots,m$ and the nonlinear function $\mathcal{H}:\mathbf{J}\times\mathbf{R}^{m+1}\to\mathbf{R}$ is continuous, ${}^{C}\mathrm{D}_{+0}$ is the Caputo fractional derivative. In the present paper, we develop the aforementioned results for (2). In developing the existence criteria of solution we apply the Schauder and Banach theorems. In the end some pertinent problems are given to illustrate the results.

2 Preliminaries

In this section, we give some related definitions and results from the given literature. The notation $\mathscr{X} = C(\mathbf{J})$ is used for a Banach space under the norm

$$\|\mathrm{w}\|_{\mathscr{X}} = \max_{t\in\mathbf{J}}\bigl\{\bigl|\mathrm{w}(t)\bigr| : t\in\mathbf{J}\bigr\}. \tag{3}$$

Definition 1 The integral of fractional order for the function $\mathrm{h}\in L^1(\mathbf{J},\mathbf{R}^+)$ of order $\sigma\in\mathbf{R}^+$ is recalled as

$$\mathrm{I}^{\sigma}_{+0}\mathrm{h}(t) = \int_0^t \frac{(t-\ell)^{\sigma-1}}{\Gamma(\sigma)}\mathrm{h}(\ell)\,d\ell, \tag{4}$$

with the integral on the right being point wise on $\mathbf{R}^+$.

Definition 2 The Caputo derivative of the function h over $\mathbf{J}$ is recalled to be

$${}^{C}\mathrm{D}^{\sigma}_{+0}\mathrm{h}(t) = \frac{1}{\Gamma(n-\sigma)}\int_0^t (t-\ell)^{n-\sigma-1}\mathrm{h}^{(n)}(\ell)\,d\ell, \tag{5}$$

with $n=[\sigma]+1$.

Lemma 1 *If* $\sigma>0$, *then*

$$\mathrm{I}^{\sigma}_{+0}\bigl({}^{C}\mathrm{D}^{\sigma}_{+0}\mathrm{h}(t)\bigr) = \mathrm{h}(t) - \sum_{j=0}^{n-1}\mathbf{k}_j t^j, \quad \textit{where } n=[\sigma]+1$$

holds.

Definition 3 The delay FODEs (2) is HU stable if there exists $\mathrm{C}_{\mathcal{H}}>0$ such that, for all $\bar{\epsilon}>0$ and for any solution $\bar{\mathrm{w}}\in\mathscr{X}$ of the inequality

$$\bigl|{}^{C}\mathrm{D}^{\sigma}_{+0}\bar{\mathrm{w}}(t) - \mathcal{H}\bigl(t,\bar{\mathrm{w}}(t),\bar{\mathrm{w}}(\delta_1 t),\bar{\mathrm{w}}(\delta_2 t),\dots,\bar{\mathrm{w}}(\delta_m t)\bigr)\bigr| \le \bar{\epsilon}, \quad \text{for all } t\in\mathbf{J}, \tag{6}$$

there exists at most one solution $\mathrm{w} \in \mathscr{X}$ to problem (2) with

$$\left|\bar{\mathrm{w}}(t) - \mathrm{w}(t)\right| \leq \mathrm{C}_{\mathcal{H}}\bar{\epsilon}, \quad \forall t \in \mathbf{J}.$$

Definition 4 The problem of delay FODEs (2) is GHU stable if there exists $\beta \in C(\mathbf{R}^+, \mathbf{R}^+)$, $\beta(0) = 0$, and also regarding any solution $\bar{\mathrm{w}} \in \mathscr{X}$ of the inequality (6), there is at most one solution $\mathrm{w} \in \mathscr{X}$ of (2) with

$$\left|\bar{\mathrm{w}}(t) - \mathrm{w}(t)\right| \leq \beta(\bar{\epsilon}), \quad \text{for all } t \in \mathbf{J}.$$

Definition 5 The delay FODEs (2) is HUR stable w.r.t. $\xi \in C(\mathbf{J}, \mathbf{R}^+)$, if there exists a real number $\mathrm{C}_{\mathcal{H}} > 0$, and $\bar{\epsilon} > 0$, and also, for any solution $\bar{\mathrm{w}} \in \mathscr{X}$ of the inequality

$$\left|{}^{C}\mathrm{D}^{\sigma}_{+0}\bar{\mathrm{w}}(t) - \mathcal{H}\big(t, \bar{\mathrm{w}}(t), \bar{\mathrm{w}}(\delta_1 t), \bar{\mathrm{w}}(\delta_2 t), \ldots, \bar{\mathrm{w}}(\delta_n t)\big)\right| \leq \xi(t)\bar{\epsilon}, \quad \forall t \in \mathbf{J}, \tag{7}$$

there exists at most one solution $\mathrm{w} \in \mathscr{X}$ of problem (2), such that

$$\left|\bar{\mathrm{w}}(t) - \mathrm{w}(t)\right| \leq \mathrm{C}_{\mathcal{H}}\bar{\epsilon}\xi(t), \quad \text{for all } t \in \mathbf{J}. \tag{8}$$

Definition 6 The delay AODE (2) will be GHUR stable w.r.t. $\xi \in \mathscr{X}$, if for $\mathrm{C}_{\mathcal{H}} > 0$ and any solution $\bar{\mathrm{w}} \in \mathscr{X}$ of the inequality (7) there exists at most one solution $\mathrm{w} \in \mathscr{X}$ of problem (2),

$$\left|\bar{\mathrm{w}}(t) - \mathrm{w}(t)\right| \leq \mathrm{C}_{\mathcal{H}}\xi(t), \quad \text{for all } t \in \mathbf{J}. \tag{9}$$

Remark 1 Let $\bar{\mathrm{w}} \in \mathscr{X}$ be the result of (6); there exists $\psi(t) \in \mathscr{X}$ with

(i) $|\psi(t)| \leq \bar{\epsilon}$, for all $t \in \mathbf{J}$.
(ii) ${}^{C}\mathrm{D}^{\sigma}_{+0}\bar{\mathrm{w}}(t) = \mathcal{H}(t, \bar{\mathrm{w}}(t), \bar{\mathrm{w}}(\delta_1 t), \bar{\mathrm{w}}(\delta_2 t), \ldots, \bar{\mathrm{w}}(\delta_m t)) + \psi(t)$, for all $t \in \mathbf{J}$.

Remark 2 Let $\bar{\mathrm{w}} \in \mathscr{X}$ be the result of (7); there exists $\psi(t) \in C(\mathbf{J}, \mathbf{R})$ with

(i) $|\psi(t)| \leq \bar{\epsilon}\xi(t)$ for all $t \in \mathbf{J}$;
(ii) ${}^{C}\mathrm{D}^{\sigma}_{+0}\bar{\mathrm{w}}(t) = \mathcal{H}(t, \bar{\mathrm{w}}(t), \bar{\mathrm{w}}(\delta_1 t), \bar{\mathrm{w}}(\delta_2 t), \ldots, \bar{\mathrm{w}}(\delta_m t)) + \psi(t)$, for all $t \in \mathbf{J}$.

3 Criteria for existence of solution

Theorem 1 *Let* $\mathrm{g} \in C(\mathbf{J})$, *then the solution*

$$\begin{cases} {}^{C}\mathrm{D}^{\sigma}_{+0}\mathrm{w}(t) = \mathrm{g}(t), \quad \textit{for } t \in \mathbf{J}, 2 < \sigma \leq 3, \\ \mathrm{w}(0) = -\mathrm{w}(\top), \qquad {}^{C}\mathrm{D}^{r}_{+0}\mathrm{w}(0) = -{}^{C}\mathrm{D}^{r}_{+0}\mathrm{w}(\top), \qquad {}^{C}\mathrm{D}^{s}_{+0}\mathrm{w}(0) = -{}^{C}\mathrm{D}^{s}_{+0}\mathrm{w}(\top), \end{cases} \tag{10}$$

is given by

$$\mathrm{w}(t) = \int_0^{\top} \mathscr{K}(t, \ell) y(\ell)\, d\ell, \tag{11}$$

while Green's function $\mathscr{K}(t,\ell)$ may be provided as

$$\mathscr{K}(t,\ell) = \begin{cases} \frac{(t-\ell)^{\sigma-1}-\frac{1}{2}(\top-\ell)^{\sigma-1}}{\Gamma(\sigma)} + \frac{\Gamma(2-r)(\top-2t)(\top-\ell)^{\sigma-r-1}}{2\Gamma(\sigma-r)\top^{1-r}} & \\ \quad - \frac{[r\top^2-4\top t+2(2-r)t^2\Gamma(3-s)(\top-\ell)^{\sigma-s-1}]}{4(2-r)\Gamma(\sigma-r)\top^{2-s}}, & 0 \le \ell \le t \le \top, \\ \frac{(\top-\ell)^{\sigma-1}}{2\Gamma(\sigma)} + \frac{\Gamma(2-r)(\top-2t)(\top-\ell)^{\sigma-r-1}}{2\Gamma(\sigma-r)\top^{1-r}} & \\ \quad - \frac{[r\top^2-4\top t+2(2-r)t^2\Gamma(3-s)(\top-\ell)^{\sigma-s-1}]}{4(2-r)\Gamma(\sigma-r)\top^{2-s}}, & 0 \le t \le \ell \le \top. \end{cases} \tag{12}$$

Proof 1 The proof of this theorem may be similarly obtained to [38, Theorem 1].

Corollary 1 *By Theorem 1, the proposed problem (2) is equivalent to the following integral equation:*

$$\mathrm{w}(t) = \int_0^\top \mathscr{K}(t,\ell)\mathcal{H}\big(\ell,\mathrm{w}(\ell),\mathrm{w}(\delta_1\ell),\mathrm{w}(\delta_2\ell),\ldots,\mathrm{w}(\delta_n\ell)\big)\,d\ell.$$

Lemma 2 *The function $\mathscr{K}(t,\ell)$, given in (12) has the given characteristics:*
($\mathscr{P}_1$) *$\mathscr{K}(t,\ell)$ is continuous over $\mathbf{J}^2$ and $\mathscr{K}(t,\ell) \ge 0$, for all $t,\ell \in \mathbf{J}$;*
($\mathscr{P}_2$) *the following inequality holds:*

$$\begin{aligned} \max_{t\in\mathbf{J}} \int_0^\top \mathscr{K}(t,\ell)\,d\ell &\le \left[\frac{1}{\Gamma(\sigma+1)} + \frac{\Gamma(2-r)}{2\Gamma(\sigma-r+1)} + \frac{(r+2)(\Gamma(3-s)}{2(2-r)\Gamma(\sigma-s+1)}\right]\top^\sigma \\ &= \Delta. \end{aligned} \tag{13}$$

Proof 2 The proof of ($\mathscr{P}_1$) is obvious; to derive ($\mathscr{P}_2$), one has

$$\begin{aligned} &\max_{t\in\mathbf{J}} \int_0^\top \mathscr{K}(t,\ell)\,d\ell \\ &= \max_{t\in\mathbf{J}} \left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-\ell)^{\sigma-1}\,d\ell - \frac{1}{2\Gamma(\sigma)}\int_0^\top (\top-\ell)^{\sigma-1}\,d\ell\right. \\ &\quad + \frac{\Gamma(2-r)(\top-2t)}{2\Gamma(\sigma-r)\top^{1-r}}\int_0^\top (\top-\ell)^{\sigma-r-1}\,d\ell \\ &\quad \left. - \frac{[r\top^2-4\top t+2(2-r)t^2]\Gamma(3-s)}{4(2-r)\Gamma(\sigma-s)\top^{2-s}}\int_0^\top (\top-\ell)^{\sigma-s-1}\,d\ell\right) \\ &\le \max_{t\in\mathbf{J}} \left(\frac{\top^\sigma}{\Gamma(\sigma+1)} + \frac{\Gamma(2-r)\top^{\sigma-r+1}}{2\Gamma(\sigma-r+1)\top^{1-r}} - \frac{[r\top^2-4\top t+2(2-r)t^2]\Gamma(3-s)\top^{\sigma-s}}{4(2-r)\Gamma(\sigma-s+1)\top^{2-s}}\right) \\ &\le \frac{\top^\sigma}{\Gamma(\sigma+1)} + \frac{\Gamma(2-r)\top^\sigma}{2\Gamma(\sigma-r+1)} + \frac{(r+2)(\Gamma(3-s)\top^\sigma}{2(2-r)\Gamma(\sigma-s+1)}. \end{aligned}$$

We need the following assumptions to hold:
($\mathscr{F}_1$) Let there exist a constant $\mathbf{A}_{\mathcal{H}} > 0$, with

$$\begin{aligned} &\left|\mathcal{H}\big(t,\mathrm{w}(t),\mathrm{w}(\delta_1 t),\mathrm{w}(\delta_2 t),\ldots,\mathrm{w}(\delta_n t)\big) - \mathcal{H}\big(t,\bar{\mathrm{w}}(t),\bar{\mathrm{w}}(\delta_1 t),\bar{\mathrm{w}}(\delta_2 t),\ldots,\bar{\mathrm{w}}(\delta_m t)\big)\right| \\ &\quad \le \mathbf{A}_{\mathcal{H}}\left(\left|\mathrm{w}(t)-\bar{\mathrm{w}}(t)\right| + \sum_{j=1}^m \left|\mathrm{w}(\delta_j t)-\bar{\mathrm{w}}(\delta_j t)\right|\right), \end{aligned}$$

for $\mathrm{w},\bar{\mathrm{w}} \in \mathscr{X}$.

($\mathscr{F}_2$) There exist $\theta_0, \bar{\theta}, \theta_j \in \mathscr{X}$ for $j = 1, 2, \ldots, m$ with

$$\begin{aligned}&\bigl|\mathcal{H}\bigl(t, \mathrm{w}(t), \mathrm{w}(\delta_1 t), \mathrm{w}(\delta_2 t), \ldots, \mathrm{w}(\delta_n t)\bigr)\bigr| \\ &\quad \le \theta_0(t) + \bar{\theta}(t)\left[\bigl|\mathrm{w}(t)\bigr| + \sum_{j=1}^{m} \theta_j(t)\bigl|\mathrm{w}(\delta_j t)\bigr|\right], \quad \text{for } \mathrm{w} \in \mathscr{X},\end{aligned}$$

with $\theta_0^* = \sup_{t\in\mathbf{J}} |\theta_0(t)|$, $\bar{\theta}^* = \sup_{t\in\mathbf{J}} |\bar{\theta}(t)|$, $\theta_j^* = \sup_{t\in\mathbf{J}} |\theta_j(t)|$, $\forall j = 1, 2, \ldots, m$. Furthermore,

$$\theta^* = \max_{t\in\mathbf{J}} \bigl\{\bar{\theta}^*, \theta_1^*, \theta_2^*, \ldots, \theta_m^*\bigr\}.$$

Here we define $\mathbf{T} : \mathscr{X} \to \mathscr{X}$ as the operator

$$\mathbf{T}(\mathrm{w})(t) = \int_0^{\top} \mathscr{K}(t, \ell)\mathcal{H}\bigl(\ell, \mathrm{w}(\ell), \mathrm{w}(\delta_1\ell), \mathrm{w}(\delta_2\ell), \ldots, \mathrm{w}(\delta_n\ell)\bigr)\, d\ell. \tag{14}$$

Theorem 2 *The mapping* $\mathbf{T} : \mathscr{X} \to \mathscr{X}$ *given in* (14) *is completely continuous.*

Proof 3 Continuity of $\mathbf{T}$ is dependent on $\mathcal{H}, \mathscr{K}(t, \ell)$. Let $\mathbb{B} \subset \mathscr{X}$ be a bounded set. Let $\mathrm{w} \in \mathbb{B}$, one has

$$\begin{aligned}\bigl|\mathbf{T}\mathrm{w}(t)\bigr| &= \left|\int_0^{\top} \mathscr{K}(t, \ell)\mathcal{H}\bigl(\ell, \mathrm{w}(\ell), \mathrm{w}(\delta_1\ell), \mathrm{w}(\delta_2\ell), \ldots, \mathrm{w}(\delta_n\ell)\bigr)\, d\ell\right| \\ &\le \int_0^{\top} \bigl|\mathscr{K}(t, \ell)\bigr|\bigl|\mathcal{H}\bigl(\ell, \mathrm{w}(\ell), \mathrm{w}(\delta_1\ell), \mathrm{w}(\delta_2\ell), \ldots, \mathrm{w}(\delta_n\ell)\bigr)\bigr|\, d\ell.\end{aligned} \tag{15}$$

By assumption ($\mathscr{F}_2$), on simplifying (15), we obtain

$$\|\mathbf{T}\mathrm{w}\|_{\mathscr{X}} \le \bigl[\theta_0^* + (m+1)\theta^*\|\mathrm{w}\|_{\mathscr{X}}\bigr]\Delta.$$

This yields the uniformly boundedness of $\mathbf{T}$. To derive discontinuity of $\mathbf{T}$, let $t_2 > t_1 \in \mathbf{J}$ such that

$$\begin{aligned}&\bigl|\mathbf{T}\mathrm{w}(t_1) - \mathbf{T}\mathrm{w}(t_2)\bigr| \\ &\quad \le \int_0^{t_2} \frac{(t_2 - \ell)^{\sigma-1}}{\Gamma(\sigma)}\bigl|\mathcal{H}\bigl(\ell, \mathrm{w}(\ell), \mathrm{w}(\delta_1\ell), \mathrm{w}(\delta_2\ell), \ldots, \mathrm{w}(\delta_n\ell)\bigr)\bigr|\, d\ell \\ &\qquad + \int_0^{t_2} \frac{(t_1 - \ell)^{\sigma-1}}{\Gamma(\sigma)}\bigl|\mathcal{H}\bigl(\ell, \mathrm{w}(\ell), \mathrm{w}(\delta_1\ell), \mathrm{w}(\delta_2\ell), \ldots, \mathrm{w}(\delta_n\ell)\bigr)\bigr|\, d\ell \\ &\qquad + \frac{2\Gamma(2-r)(t_2 - t_1)}{2\top^{1-r}\Gamma(\sigma - r)}\int_0^{\top} (\top - \ell)^{\sigma-r-1}\bigl|\mathcal{H}\bigl(\ell, \mathrm{w}(\ell), \mathrm{w}(\delta_1\ell), \mathrm{w}(\delta_2\ell), \ldots, \mathrm{w}(\delta_n\ell)\bigr)\bigr|\, d\ell \\ &\qquad + \frac{4\top(1 + 2\Gamma(2-r))(t_2 - t_1)}{4(2-r)\top^{2-s}\Gamma(\sigma - s)}\int_0^{\top} (\top - \ell)\bigl|\mathcal{H}\bigl(\ell, \mathrm{w}(\ell), \mathrm{w}(\delta_1\ell), \mathrm{w}(\delta_2\ell), \ldots, \mathrm{w}(\delta_n\ell)\bigr)\bigr|\, d\ell \\ &\quad \le \int_0^{t_2} \frac{(t_2 - \ell)^{\sigma-1}}{\Gamma(\sigma)}\bigl[\theta_0^* + (m+1)\theta^*\|\mathrm{w}\|_{\mathscr{X}}\bigr]\, d\ell \\ &\qquad + \int_0^{t_2} \frac{(t_1 - \ell)^{\sigma-1}}{\Gamma(\sigma)}\bigl[\theta_0^* + (m+1)\theta^*\|\mathrm{w}\|_{\mathscr{X}}\bigr]\, d\ell\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2\Gamma(2-r)(t_2-t_1)}{2\top^{1-r}\Gamma(\sigma-r)} \int_0^{\top} (\top-\ell)^{\sigma-r-1}\big[\theta_0^* + (m+1)\theta^*\|\mathrm{w}\|_{\mathscr{X}}\big]\,d\ell \\
&+ \frac{4\top(1+2\Gamma(2-r))(t_2-t_1)}{4(2-r)\top^{2-s}\Gamma(\sigma-s)} \int_0^{\top} (\top-\ell)^{\sigma-s}\big[\theta_0^* + (m+1)\theta^*\|\mathrm{w}\|_{\mathscr{X}}\big]\,d\ell \\
\leq{}& \left[\frac{(t_2^{\sigma}-t_2^{\sigma})}{\Gamma(\sigma+1)} + \frac{2\top^{\sigma-r}\Gamma(2-r)(t_2-t_1)}{2\top^{1-r}\Gamma(\sigma-r+1)} + \frac{4\top^{\sigma-s+1}(1+2\Gamma(2-r))(t_2-t_1)}{4(2-r)\top^{2-s}\Gamma(\sigma-s+1)}\right] \\
&\times \big[\theta_0^* + (m+1)\theta^*\|\mathrm{w}\|_{\mathscr{X}}\big]. \qquad (16)
\end{aligned}
$$

At $t_1 \to t_2$, (16) tends to zero in the right hand side. Thus equicontinuity of $\mathbf{T}$ is obtained, which also confirms uniform continuity. Analogously $\mathbf{T}(\mathscr{B}) \subset \mathscr{B}$. Therefore the operator $\mathbf{T}$ is completely continuous.

Theorem 3 *Under the complete continuity of the operator* $\mathbf{T}$ *and the Hypotheses* $(\mathscr{F}_1)$, $(\mathscr{F}_2)$, *the problem of delay FODEs* (2) *possesses at least one solution.*

Proof 4 Let $\mathscr{E}$ be the set

$$
\mathscr{E} = \big\{\mathrm{w} \in \mathscr{X} : \mathrm{w} = \rho\mathbf{T}(\mathrm{w}), 0 < \rho < 1\big\}.
$$

The operator $\mathbf{T} : \bar{\mathscr{E}} \to \mathscr{X}$ as provided in (14) is completely continuous by Theorem 2. Take $\mathrm{w} \in \mathscr{E}$ on using $(\mathscr{F}_2)$, one has

$$
\begin{aligned}
\|\mathrm{w}\|_{\mathscr{X}} &= \big\|\rho\mathbf{T}(\mathrm{w})\big\|_{\mathscr{X}} \\
&\leq \max_{t\in\mathbf{J}} \int_0^{\top} \big|\mathscr{K}(t,\ell)\big|\big|\mathcal{H}\big(\ell,\mathrm{w}(\ell),\mathrm{w}(\delta_1\ell),\mathrm{w}(\delta_2\ell),\ldots,\mathrm{w}(\delta_n\ell)\big)\big|\,d\ell \\
&\leq \max_{t\in\mathbf{J}} \int_0^{\top} \big|\mathscr{K}(t,\ell)\big|\big[\theta_0^* + (m+1)\theta^*\|\mathrm{w}\|_{\mathscr{X}}\big]\,d\ell.
\end{aligned}
$$

From this we have

$$
\|\mathrm{w}\|_{\mathscr{X}} \leq \frac{\theta_0^*\Delta}{1-(m+1)\theta^*\Delta)} = \mu. \qquad (17)
$$

Boundedness of $\mathscr{E}$ holds and so (2) has at least one solution.

Theorem 4 *Under assumption* $(\mathscr{F}_1)$ *and the condition* $[(m+1)\mathbf{A}_{\mathcal{H}}\Delta] < 1$, *where* Δ *is given in* (13), *the unique solution will be guaranteed for problem* (2) *in* $\mathscr{X}$.

Proof 5 On making use of the Banach principle, if $\mathrm{w}, \bar{\mathrm{w}} \in \mathscr{X}$, with $t \in \mathbf{J}$, one has

$$
\begin{aligned}
\big|\mathbf{T}\mathrm{w}(t) - \mathbf{T}\bar{\mathrm{w}}(t)\big| &= \bigg|\int_0^{\top} \mathscr{K}(t,\ell)\big[\mathcal{H}\big(\ell,\mathrm{w}(\ell),\mathrm{w}(\delta_1\ell),\mathrm{w}(\delta_2\ell),\ldots,\mathrm{w}(\delta_n\ell)\big) \\
&\qquad - \mathcal{H}\big(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\ldots,\bar{\mathrm{w}}(\delta_n\ell)\big)\big]\,d\ell\bigg| \\
&\leq \int_0^{\top} \big|\mathscr{K}(t,\ell)\big|\big|\mathcal{H}\big(\ell,\mathrm{w}(\ell),\mathrm{w}(\delta_1\ell),\mathrm{w}(\delta_2\ell),\ldots,\mathrm{w}(\delta_n\ell)\big) \\
&\qquad - \mathcal{H}\big(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\ldots,\bar{\mathrm{w}}(\delta_n\ell)\big)\big|\,d\ell. \qquad (18)
\end{aligned}
$$

In view of property $(\mathscr{P}_2)$, from (18), one has

$$\begin{aligned}\|\mathbf{T}\mathrm{w}-\mathbf{T}\bar{\mathrm{w}}\|_{\mathscr{X}} &\leq \max_{t\in\mathbf{J}}\int_0^{\top}\big|\mathscr{K}(t,\ell)\big|(m+1)\mathbf{A}_{\mathcal{H}}\|\mathrm{w}-\bar{\mathrm{w}}\|_{\mathscr{X}}\,d\ell\\ &\leq\big[(m+1)\mathbf{A}_{\mathcal{H}}\Delta\big]\|\mathrm{w}-\bar{\mathrm{w}}\|_{\mathscr{X}}.\end{aligned}$$

Since $[(m+1)\mathbf{A}_{\mathcal{H}}\Delta]<1$, the mapping $\mathbf{T}$ is a contraction which confirms that (2) has at most one solution.

4 Stability theory

Here we develop the required results for stability theory.

Lemma 3 *The solution of the given equation with $t\in\mathbf{J}$*

$$\begin{cases}{}^{C}\mathrm{D}^{\sigma}_{+0}\bar{\mathrm{w}}(t)=\mathcal{H}(t,\bar{\mathrm{w}}(t),\bar{\mathrm{w}}(\delta_1 t),\bar{\mathrm{w}}(\delta_2 t),\dots,\bar{\mathrm{w}}(\delta_n t))+\psi(t), \quad 2<\sigma\leq 3,\\ \bar{\mathrm{w}}(0)=-\bar{\mathrm{w}}(\top), \qquad {}^{C}\mathrm{D}^{r}_{+0}\bar{\mathrm{w}}(0)=-{}^{C}\mathrm{D}^{r}_{+0}\bar{\mathrm{w}}(\top), \qquad {}^{C}\mathrm{D}^{s}_{+0}\bar{\mathrm{w}}(0)=-{}^{C}\mathrm{D}^{s}_{+0}\bar{\mathrm{w}}(\top),\end{cases} \tag{19}$$

obeys the given inequality

$$\left|\bar{\mathrm{w}}(t)-\int_0^{\top}\mathscr{K}(t,\ell)\mathcal{H}\big(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_n\ell)\big)\,d\ell\right|\leq\Delta\bar{\epsilon}. \tag{20}$$

Proof 6 Like Corollary 1, the solution of (19) can be provided as

$$\bar{\mathrm{w}}(t)=\int_0^{\top}\mathscr{K}(t,\ell)\mathcal{H}\big(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_n\ell)\big)\,d\ell+\int_0^{\top}\mathscr{K}(t,\ell)\psi(\ell)\,d\ell.$$

From this one has on using (i) of Remark 1 and property $(\mathscr{P}_2)$ of $\mathscr{K}$,

$$\begin{aligned}\left|\bar{\mathrm{w}}(t)-\int_0^{\top}\mathscr{K}(t,\ell)\mathcal{H}\big(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_n\ell)\big)\,d\ell\right| &\leq\int_0^{\top}\big|\mathscr{K}(t,\ell)\big|\big|\psi(\ell)\big|\,d\ell\\ &\leq\Delta\bar{\epsilon}, \quad t\in\mathbf{J}.\end{aligned}$$

Theorem 5 *If the conditions $(m+1)\mathbf{A}_{\mathcal{H}}\Delta<1$ hold, then the solution of (2) is HU and GUH stable.*

Proof 7 If $\mathrm{w}\in\mathscr{X}$ is at most one result of (2) and $\bar{\mathrm{w}}\in\mathscr{X}$ is any solution of the said problem, then we may consider with $t\in\mathbf{J}$

$$\begin{aligned}\|\bar{\mathrm{w}}-\mathrm{w}\|_{\mathscr{X}} &=\max_{t\in\mathbf{J}}\left|\bar{\mathrm{w}}-\int_0^{\top}\mathscr{K}(t,\ell)\mathcal{H}\big(\ell,\mathrm{w}(\ell),\mathrm{w}(\delta_\ell),\mathrm{w}(\delta_2\ell),\dots,\mathrm{w}(\delta_m\ell)\big)\,d\ell\right|\\ &\leq\max_{t\in\mathbf{J}}\left|\bar{\mathrm{w}}-\int_0^{\top}\mathscr{K}(t,\ell)\mathcal{H}\big(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_m\ell)\big)\,d\ell\right|\\ &\quad+\max_{t\in\mathbf{J}}\bigg|\int_0^{\top}\mathscr{K}(t,\ell)\mathcal{H}\big(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_m\ell)\big)\,d\ell\\ &\quad-\int_0^{\top}\mathscr{K}(t,\ell)\mathcal{H}\big(\ell,\mathrm{w}(\ell),\mathrm{w}(\delta_\ell),\mathrm{w}(\delta_2\ell),\dots,\mathrm{w}(\delta_m\ell)\big)\,d\ell\bigg|.\end{aligned}$$

By the application of assumption $(\mathscr{F}_1)$ and Lemma 3, we get

$$\|\bar{\mathrm{w}}-\mathrm{w}\|_{\mathscr{X}} \leq \Delta\bar{\epsilon} + \bigl[(m+1)\mathbf{A}_{\mathcal{H}}\Delta\bigr]\|\bar{\mathrm{w}}-\mathrm{w}\|_{\mathscr{X}}. \tag{21}$$

Upon simplification (21) yields

$$\|\bar{\mathrm{w}}-\mathrm{w}\| \leq \mathrm{C}_{\mathcal{H}}\bar{\epsilon}, \quad \mathrm{C}_{\mathcal{H}} = \frac{\Delta}{1-[(m+1)\mathbf{A}_{\mathcal{H}}\Delta]}. \tag{22}$$

Hence the problem (2) is HU stable. Let a nondecreasing function $\beta:(0,1)\to(0,\infty)$ be such that $\beta(\bar{\epsilon})=\bar{\epsilon}$ with $\beta(0)=0$, then from (21), we can write

$$\|\bar{\mathrm{w}}-\mathrm{w}\| \leq \mathrm{C}_{\mathcal{H}}\beta(\bar{\epsilon}). \tag{23}$$

Thus problem (2) is GHU stable.

Lemma 4 *For the given problem* (19), *the following inequality holds*:

$$\left|\bar{\mathrm{w}}(t) - \int_0^{\top} \mathscr{K}(t,\ell)\mathcal{H}\bigl(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_m\ell)\bigr)\,d\ell\right| \leq \Delta\xi(t)\bar{\epsilon}, \quad t\in\mathbf{J}. \tag{24}$$

Proof 8 Keeping in mind Corollary 1 the solution of (19) is given by

$$\bar{\mathrm{w}}(t) = \int_0^{\top} \mathscr{K}(t,\ell)\mathcal{H}\bigl(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_m\ell)\bigr)\,d\ell + \int_0^{\top} \mathscr{K}(t,\ell)\psi(\ell)\,d\ell.$$

On applying Remark 2(i) and $(\mathscr{P}_2)$, one has

$$\begin{aligned}\left|\bar{\mathrm{w}}(t) - \int_0^{\top} \mathscr{K}(t,\ell)\mathcal{H}\bigl(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_m\ell)\bigr)\,d\ell\right| &\leq \int_0^{\top} \bigl|\mathscr{K}(t,\ell)\bigr|\bigl|\psi(\ell)\bigr|\,d\ell \\ &\leq \Delta\xi(t)\bar{\epsilon}, \quad t\in\mathbf{J}.\end{aligned}$$

Theorem 6 *Under assumption* $(\mathscr{F}_1)$ *and the condition* $(m+1)\mathbf{A}_{\mathcal{H}}\Delta < 1$, *the considered problem* (2) *is HUR stable.*

Proof 9 Let $\bar{\mathrm{w}}$ be any result of problem (2) with $\mathrm{w}\in\mathscr{X}$ at most one solution of inequality (2), then we have

$$\begin{aligned}\|\bar{\mathrm{w}}-\mathrm{w}\|_{\mathscr{X}} &= \max_{t\in\mathbf{J}}\left|\bar{\mathrm{w}} - \int_0^{\top} \mathscr{K}(t,\ell)\mathcal{H}\bigl(\ell,\mathrm{w}(\ell),\mathrm{w}(\delta_1\ell),\mathrm{w}(\delta_2\ell),\dots,\mathrm{w}(\delta_m\ell)\bigr)\,d\ell\right| \\ &\leq \max_{t\in\mathbf{J}}\left|\bar{\mathrm{w}} - \int_0^{\top} \mathscr{K}(t,\ell)\mathcal{H}\bigl(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_m\ell)\bigr)\,d\ell\right| \\ &\quad + \max_{t\in\mathbf{J}}\left|\int_0^{\top} \mathscr{K}(t,\ell)\mathcal{H}\bigl(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_m\ell)\bigr)\,d\ell\right. \\ &\quad \left. - \int_0^{\top} \mathscr{K}(t,\ell)\mathcal{H}\bigl(\ell,\mathrm{w}(\ell),\mathrm{w}(\delta_1\ell),\mathrm{w}(\delta_2\ell),\dots,\mathrm{w}(\delta_m\ell)\bigr)\,d\ell\right|.\end{aligned}$$

On using assumption $(\mathscr{F}_1)$ and Lemma 4, we get

$$\|\bar{\mathrm{w}}-\mathrm{w}\|_{\mathscr{X}} \le \Delta\xi(t)\bar{\epsilon} + \big[(m+1)\mathbf{A}_{\mathcal{H}}\Delta\big]\|\bar{\mathrm{w}}-\mathrm{w}\|_{\mathscr{X}}. \tag{25}$$

Upon simplification (25) gives

$$\|\bar{\mathrm{w}}-\mathrm{w}\|_{\mathscr{X}} \le \mathrm{C}_{\mathcal{H}}\xi(t)\bar{\epsilon}, \quad \mathrm{C}_{\mathcal{H}} = \frac{\Delta}{1-[(m+1)\mathbf{A}_{\mathcal{H}}\Delta]}. \tag{26}$$

Thus the solution of (2) is HUR stable.

Lemma 5 *The solution of the perturbed problem given in* (19) *produces the following relation*:

$$\left|\bar{\mathrm{w}}(t) - \int_0^{\top} \mathscr{K}(t,\ell)\mathcal{H}\big(\ell,\bar{\mathrm{w}}(\ell),\bar{\mathrm{w}}(\delta_1\ell),\bar{\mathrm{w}}(\delta_2\ell),\dots,\bar{\mathrm{w}}(\delta_m\ell)\big)\,d\ell\right| \le \Delta\xi(t), \quad t\in\mathbf{J}. \tag{27}$$

Proof 10 On using Lemma 3, the proof is simple.

Theorem 7 *Under the Hypothesis* $(\mathscr{F}_1)$ *and the inequalities* $(m+1)\mathbf{A}_{\mathcal{H}}\Delta < 1$ *holding, the solution of* (2) *is GHUR stable.*

Proof 11 Keeping in mind Theorem 6, one can write

$$\|\bar{\mathrm{w}}-\mathrm{w}\|_{\mathscr{X}} \le \mathrm{C}_{\mathcal{H}}\xi(t), \quad \mathrm{C}_{\mathcal{H}} = \frac{\Delta}{1-(m+1)\mathbf{A}_{\mathcal{H}}\Delta}. \tag{28}$$

Hence the solution of (2) is GHUR stable.

5 Illustrative problems

Here we address some illustrative problems.

Problem 1 Let us consider the FODEs under APBCs with proportional delay terms

$$\begin{cases} {}^{C}\mathrm{D}^{\frac{5}{2}}_{+0}\mathrm{w}(t) = \frac{1}{150}[t\cos|\mathrm{w}(t)| - \mathrm{w}(\frac{1}{2}t)\sin(t)] + \frac{\mathrm{w}(\frac{1}{3}t)}{100+\mathrm{w}(\frac{1}{3}t)}, \quad t\in\mathbf{J}=[0,1], \\ \mathrm{w}(0) = -\mathrm{w}(1), \qquad {}^{C}\mathrm{D}^{\frac{1}{2}}_{+0}\mathrm{w}(0) = -{}^{C}\mathrm{D}^{\frac{1}{2}}_{+0}\mathrm{w}(1), \qquad {}^{C}\mathrm{D}^{\frac{3}{2}}_{+0}\mathrm{w}(0) = -{}^{C}\mathrm{D}^{\frac{3}{2}}_{+0}\mathrm{w}(1). \end{cases} \tag{29}$$

Here one has $m=2$, $\sigma=\frac{5}{2}$, $r=\frac{1}{2}$, $s=\frac{3}{2}$, $\delta_1=\frac{1}{2}$, $\delta_2=\frac{1}{3}$, $\top=1$. Continuity of the function

$$\mathcal{H}\big(t,\mathrm{w}(t),\mathrm{w}(\delta_1 t),\mathrm{w}(\delta_2 t)\big) = \frac{1}{150}\left[t\cos\big|\mathrm{w}(t)\big| - \mathrm{w}\left(\frac{1}{2}t\right)\sin(t)\right] + \frac{\mathrm{w}(\frac{1}{3}t)}{100+\mathrm{w}(\frac{1}{3}t)}$$

is clear for $\mathrm{w}\in\mathscr{X}=C[0,1]$. Again by assumption $(\mathscr{F}_1)$, for any $\mathrm{w},\bar{\mathrm{w}}\in\mathbf{R}$, one has

$$\begin{aligned} &\big|\mathcal{H}\big(t,\mathrm{w}(t),\mathrm{w}(\delta_1 t),\mathrm{w}(\delta_2 t)\big) - \mathcal{H}\big(t,\bar{\mathrm{w}}(t),\bar{\mathrm{w}}(\delta_1 t),\bar{\mathrm{w}}(\delta_2 t)\big)\big| \\ &\quad = \left|\frac{1}{150}\left[t\cos\big|\mathrm{w}(t)\big| - \mathrm{w}\left(\frac{1}{2}t\right)\sin(t)\right] + \frac{\mathrm{w}(\frac{1}{3}t)}{100+\mathrm{w}(\frac{1}{3}t)}\right. \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{150}\left[t\cos\left|\bar{\mathrm{w}}(t)\right| - \bar{\mathrm{w}}\left(\frac{1}{2}t\right)\sin(t)\right] - \frac{\bar{\mathrm{w}}(\frac{1}{3}t)}{100+\bar{\mathrm{w}}(\frac{1}{3}t)}\Bigg| \\
&\le \frac{7}{300}\left|\mathrm{w}(t) - \bar{\mathrm{w}}(t)\right|.
\end{aligned}
$$

Hence we have $\mathbf{A}_{\mathcal{H}} = \frac{7}{300}$. On computation, we have $\Delta = 0.7809524$. On use of Theorem 4, one has

$$(m+1)\mathbf{A}_{\mathcal{H}}\Delta = 0.018222 < 1.$$

Thus the given problem of FODEs (1) has at most one solution. Furthermore, using Theorem 5, we see that

$$(m+1)\mathbf{A}_{\mathcal{H}}\Delta < 1.$$

Thus the concerned conditions for HU and GHU stability hold. Upon using Theorem 6 and taking the nondecreasing function $\xi(t) = t^2$ for $t \in (0,1)$, one has $\mathrm{C}_{\mathcal{H}} = \frac{\Delta}{1-[(m+1)\mathbf{A}_{\mathcal{H}}\Delta]} = 0.7954472$. Hence we see that with the results for the unique solution $\bar{\mathrm{w}} \in \mathscr{X}$ and any solution $\mathrm{w} \in \mathscr{X}$ the relation

$$\|\mathrm{w} - \bar{\mathrm{w}}\|_{\mathscr{X}} \le 0.7954472\bar{\epsilon}t^2, \quad \text{for all } t \in [0,1],$$

holds true. Thus the solution of (1) is HUR stable. Consequently it is GHUR stable on using Theorem 7.

Problem 2 Consider another problem of FODEs involving delay terms:

$$
\begin{cases}
{}^{C}\mathrm{D}_{+0}^{\frac{5}{2}}\mathrm{w}(t) = \frac{\exp(-4\pi t)}{50} \\
\qquad\qquad + \frac{\exp(-10t)}{100+(1-t)^2}[\sin(|\mathrm{w}(t)|) + \mathrm{w}(\frac{1}{4}t) + \mathrm{w}(\frac{1}{5}t) + \mathrm{w}(\frac{1}{6}t)], \quad t \in [0,1], \\
\mathrm{w}(0) = -\mathrm{w}(1), \qquad {}^{C}\mathrm{D}_{+0}^{\frac{1}{2}}\mathrm{w}(0) = -{}^{C}\mathrm{D}_{+0}^{\frac{1}{2}}\mathrm{w}(1), \qquad {}^{C}\mathrm{D}_{+0}^{\frac{3}{2}}\mathrm{w}(0) = -{}^{C}\mathrm{D}_{+0}^{\frac{3}{2}}\mathrm{w}(1).
\end{cases}
\tag{30}
$$

Here $\sigma = \frac{5}{2}$, $r = \frac{1}{2}$, $s = \frac{3}{2}$, $\delta = \frac{1}{4}$, $\top = 1$, $m = 3$ and

$$
\begin{aligned}
&\mathcal{H}\big(t, \mathrm{w}(t), \mathrm{w}(\delta_1 t), \mathrm{w}(\delta_2 t), \mathrm{w}(\delta_3 t)\big) \\
&\quad = \frac{\exp(-4\pi t)}{50} + \frac{\exp(-10t)}{100+(1-t)^2}\left[\sin\big(|\mathrm{w}(t)|\big) + \mathrm{w}\left(\frac{1}{4}t\right) + \mathrm{w}\left(\frac{1}{5}t\right) + \mathrm{w}\left(\frac{1}{6}t\right)\right].
\end{aligned}
$$

Clearly $\mathcal{H}$ is continuous.

Now for any $\mathrm{w}, \bar{\mathrm{w}} \in \mathscr{X}$, one has

$$
\begin{aligned}
&\left|\mathcal{H}\big(t, \mathrm{w}(t), \mathrm{w}(\delta_1 t), \mathrm{w}(\delta_2 t), \mathrm{w}(\delta_3 t)\big) - \mathcal{H}\big(t, \bar{\mathrm{w}}(t), \bar{\mathrm{w}}(\delta_1 t), \bar{\mathrm{w}}(\delta_2 t), \bar{\mathrm{w}}(\delta_3 t)\big)\right| \\
&\quad \le \frac{1}{100}\left[4\left|\mathrm{w}(t) - \bar{\mathrm{w}}(t)\right|\right] \\
&\quad = \frac{1}{25}\left[\left|\mathrm{w}(t) - \bar{\mathrm{w}}(t)\right|\right].
\end{aligned}
$$

Hence $\mathcal{H}$ satisfies the Hypothesis $(\mathscr{F}_1)$ with $\mathbf{A}_{\mathcal{H}} = \frac{1}{25}$. The function $\mathcal{H}$ also satisfies the Hypothesis $(\mathscr{F}_2)$ with $\theta_0(t) = \frac{\exp(-4\pi t)}{50}$, $\bar{\theta}(t) = \frac{\exp(-t)}{100+(1-t)^2}$ and $\theta_1(t) = \frac{\exp(-t)}{100+(1-t)^2}$, $\theta_2(t) = \frac{\exp(-t)}{100+(1-t)^2}$, where $\theta_0^*(t) = \frac{1}{50}$, $\bar{\theta}^* = \frac{1}{101}$, $\theta_1^* = \theta_2^* = \frac{1}{101}$. Furthermore, $\theta^* = \frac{1}{101}$. Also one has

$$\begin{aligned}\Delta &= \frac{1}{\Gamma(\frac{5}{2}+1)} + \frac{\Gamma(2-\frac{1}{2})}{2\Gamma(\frac{5}{2}-\frac{1}{2}+1)} + \frac{(\frac{1}{2}+2)(\Gamma(3-\frac{3}{2}))}{2(2-\frac{1}{2})\Gamma(\frac{5}{2}-\frac{3}{2}+1)} \\ &= 1.26098028.\end{aligned}$$

By Theorem 3, one has $\mu = \frac{\theta_0^*\Delta}{(1-4\theta^*\Delta)} = 0.2654527$. Thus the given problem of FODEs (2) has at least one solution. Furthermore, using Theorem 4, we see that

$$4\mathbf{A}_{\mathcal{H}}\Delta = 0.0201757 < 1.$$

Hence the solution is unique. Analogously by Theorem 5, we have

$$4\mathbf{A}_{\mathcal{H}}\Delta = 0.0201757 < 1.$$

Hence the solution is HU stable. Furthermore, it is also GHU stable. For HUR stability, in view of Theorem 6 and by considering the nondecreasing function $\xi(t) = t$ for $t \in (0,1)$, one has $\mathrm{C}_{\mathcal{H}} = \frac{\Delta}{1-(4\mathbf{A}_{\mathcal{H}}\Delta)} = 1.2741035$. Hence, we see that with the results for any solution $\bar{\mathrm{w}} \in \mathscr{X}$ and unique solution $\mathrm{w} \in \mathscr{X}$ the relation

$$\|\bar{\mathrm{w}} - \mathrm{w}\|_{\mathscr{X}} \leq 1.2741035\bar{\epsilon}t, \quad \text{for all } t \in [0,1],$$

holds true. Hence the solution of (1) is HUR stable. Consequently it is obviously GHUR stable on using Theorem 7.

Problem 3 Here we take another proportional delay problem of FODEs:

$$\begin{cases} {}^{C}\mathrm{D}_{+0}^{\frac{7}{3}}\mathrm{w}(t) = \frac{t}{50} + \frac{t^2+3}{500}\sqrt{|\mathrm{w}(t)|} + \frac{(t+3)^2}{500}\sqrt{|\mathrm{w}(\frac{1}{2}t)|} \\ \qquad\qquad + \frac{(t+2)^2}{500}\sqrt{|\mathrm{w}(\frac{1}{3}t)|} + \frac{(t+4)^2}{500}\sqrt{|\mathrm{w}(\frac{1}{4}t)|}, \quad t \in [0,1], \\ \mathrm{w}(0) = -\mathrm{w}(1), \qquad {}^{C}\mathrm{D}_{+0}^{\frac{1}{3}}\mathrm{w}(0) = -{}^{C}\mathrm{D}_{+0}^{\frac{1}{3}}\mathrm{w}(1), \qquad {}^{C}\mathrm{D}_{+0}^{\frac{4}{3}}\mathrm{w}(0) = -{}^{C}\mathrm{D}_{+0}^{\frac{4}{3}}\mathrm{w}(1). \end{cases} \tag{31}$$

Here $\sigma = \frac{7}{3}$, $r = \frac{1}{3}$, $s = \frac{4}{3}$, $\top = 1$ and

$$\begin{aligned}&\mathcal{H}\big(t, \mathrm{w}(t), \mathrm{w}(\delta_1 t), \mathrm{w}(\delta_2 t), \mathrm{w}(\delta_3 t), \mathrm{w}(\delta_4 t)\big) \\ &\quad = \frac{t}{50} + \frac{(t^2+3)}{1000}\sqrt{|\mathrm{w}(t)|} + \frac{(t+3)^2}{1000}\sqrt{\left|\mathrm{w}\left(\frac{1}{2}t\right)\right|} \\ &\qquad + \frac{(t+2)^2}{1000}\sqrt{\left|\mathrm{w}\left(\frac{1}{3}t\right)\right|} + \frac{(t+4)^2}{1000}\sqrt{\left|\mathrm{w}\left(\frac{1}{4}t\right)\right|}.\end{aligned}$$

Now we take $\mathrm{w}, \bar{\mathrm{w}} \in \mathbf{R}$, such that

$$\begin{aligned}
&\left|\mathcal{H}\big(t, \mathrm{w}(t), \mathrm{w}(\delta_1 t), \mathrm{w}(\delta_2 t), \mathrm{w}(\delta_3 t)\big) - \mathcal{H}\big(t, \bar{\mathrm{w}}(t), \bar{\mathrm{w}}(\delta_1 t), \bar{\mathrm{w}}(\delta_2 t), \bar{\mathrm{w}}(\delta_3 t)\big)\right| \\
&\quad \le \frac{36}{1000}\big[4\big|\mathrm{w}(t) - \bar{\mathrm{w}}(t)\big|\big] \\
&\quad = \frac{18}{125}\big[\big|\mathrm{w}(t) - \bar{\mathrm{w}}(t)\big|\big].
\end{aligned}$$

Hence $\mathcal{H}$ satisfies the Hypothesis $(\mathscr{F}_1)$ with $\mathbf{A}_{\mathcal{H}} = \frac{18}{125}$. The function $\mathcal{H}$ also satisfies the Hypothesis $(\mathscr{F}_2)$ with $m = 3$ and $\Delta = 1.2176$ $\theta_0(t) = \frac{t}{50}$, $\bar{\theta}(t) = \frac{t^2+3}{500}$, $\theta_1(t) = \frac{(t+3)^2}{1000}$, $\theta_2(t) = \frac{(t+2)^2}{1000}$, $\theta_3(t) = \frac{(t+4)^2}{1000}$, $\theta_4(t) = \frac{(t+5)^2}{1000}$ where $\theta_0^*(t) = \frac{1}{25}$, $\bar{\theta}^*(t) = \frac{1}{125}$, $\theta_1^*(t) = \frac{2}{125}$, $\theta_2^*(t) = \frac{9}{1000}$, $\theta_3^*(t) = \frac{1}{40}$, $\theta_4^*(t) = \frac{9}{250}$. Upon computation, $(m+1)\mathbf{A}_{\mathcal{H}}\Delta = 0.7013376 < 1$. Thus on using Theorem 4, the Problem 3 has at most one solution. Moreover, it also satisfies the condition of HU stability and consequently GHU stability by using Theorem 5. Taking a nondecreasing function $\xi(t) = 1 + \frac{t^2}{2}$, one can prove that the result of (3) is HUR stable and hence GHUR stable upon the application of Theorem 6 and Theorem 7, respectively.

6 Conclusion

A comprehensive analysis corresponding to the existence theory of solution and stability results has been established for a multi-term pantograph FODEs under APBCs. This type of problem has been investigated with respect to the subject conditions for the first time in terms of delay differential equations of any positive real order. The considered delay type differential equations have important applications in various scientific fields, like electro-locomotive dynamics. The whole analysis has been demonstrated by some proper examples.

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Authors' contributions

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