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Existence of positive solutions for a class of fractional differential equations with the derivative term via a new fixed point theorem

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Abstract

In this paper, we firstly establish the existence and uniqueness of solutions of the operator equation $A(x,x) + B(x,x) + C(x) + e = x$, where A and B are two mixed monotone operators, C is a decreasing operator, and $e \in P$ with $\theta \le e \le h$. Then, using our abstract theorem, we prove a class of fractional boundary value problems with the derivative term to have a unique solution and construct the corresponding iterative sequences to approximate the unique solution.

Keywords: Existence and uniqueness; Mixed monotone operator; Decreasing operator; Derivative term; Fractional equation

1 Introduction

In this paper, we consider the following fractional order boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t), D_{0^+}^{\beta} u(t)) + g(t, u(t), (Hu)(t)) - b = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ [D_{0^+}^{\gamma} u(t)]_{t=1} = k(u(1)), \end{cases} \tag{1.1}$$

where $b > 0$ is a constant, $n - 1 < \alpha \le n$, $1 \le \beta < \gamma \le n - 2$, $n > 3$ $(n \in \mathbb{N})$. $f, g : [0,1] \times (-\infty, +\infty) \times (-\infty, +\infty) \to (-\infty, +\infty)$ are continuous functions, $k : [0, +\infty) \to [0, +\infty)$ is a continuous function, and $D_{0^+}^{\alpha}$ is the Riemann–Liouville fractional derivative of order α.

Fractional differential equations have been increasingly adopted to describe some physical phenomena in thermology, electromagnetic wave, electrochemistry, and other applications [1–8]. There are a great deal of results about the existence and uniqueness of positive solutions for fractional boundary value problems. For example, Zhao and Gong [9] studied the unique positive solution of a class of higher order fractional equations with a parameter by Banach fixed point theorem. In [10], Wang, Zhang, and Wang obtained fixed point theorems of nonlinear sum operators and applications in a fractional differential equation. On the other hand, much attention has been paid to fractional differential equations involving nonlinearities with the derivative term. In [11], Ji et al. investigated positive solutions for the nonlinear fractional differential equation with a derivative term.

Moreover, Yue and Zou [12] were concerned with a class of fractional Dirichlet boundary value problems with dependence on the first order derivative. Some sufficient conditions for the uniqueness of solutions for the above-mentioned problem were given. The main tool is also classical Banach's contraction mapping principle.

It is well known that problem (1.1) is the generalization of elastic beam equation [13]. In [14], Goodrich first studied the Green's function associated with problem (1.1) when $k \equiv 0$ and established the existence result on sublinear nonlinearity. Furthermore, Xu, Wei, and Dong [15] also considered sublinear problem (1.1) by using of the fixed point index theorem and spectral theory. Jleli and Samet [16] utilized a mixed monotone fixed point theorem to obtain a unique solution of problem (1.1) when $b = 0$. Moreover, Yang, Shen, and Xie [17] investigated the nonlinear term involving the first order derivative for problem (1.1).

We should mention the main results obtained in [18–22], which motivated us to consider problem (1.1). In [18], Wang and Zhang studied the operator equation $Ax + Bx + C(x,x) = x$, where A is an increasing α-concave operator, B is a decreasing operator, and C is a mixed monotone operator. Existence and uniqueness of the operator equation were established. Furthermore, Zhang and Tian [19] considered the following fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} x(t) + f(t, x(t), D_{0^+}^{\beta} x(t)) + g(t, x(t)) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ [D_{0^+}^{\gamma} x(t)]_{t=1} = k(x(1)), \end{cases} \tag{1.2}$$

where $n \geq 3$, $1 \leq \beta \leq \gamma \leq n-2$, $f : [0,1] \times [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$, $g : [0,1] \times [0,+\infty) \longrightarrow [0,+\infty)$, and $k : [0,+\infty) \longrightarrow [0,+\infty)$. The authors used the abstract theorem obtained in [18] to prove that problem (1.2) admits a unique positive solution. Subsequently, Wang [20] considered the following singular nonlinear fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + p(t) f(t, u(t), D_{0^+}^{\beta} u(t)) + q(t) g(t, u(t), (Hu)(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ [D_{0^+}^{\gamma} u(t)]_{t=1} = k(u(1)), \end{cases} \tag{1.3}$$

where $n-1 < \alpha \leq n$, $n > 3$, $1 \leq \beta \leq \gamma \leq n-2$, $p, q \in C((0,1),[0,+\infty))$, $p(t)$ and $q(t)$ are allowed to be singular at $t = 0$ or $t = 1$. $f : (0,1) \times (0,+\infty) \times (0,+\infty) \longrightarrow [0,+\infty)$ is continuous, $g : (0,1) \times [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ is continuous, and $k : [0,1) \rightarrow [0,+\infty)$ is also continuous. The author proved problem (1.3) to have a unique positive solution based on a new mixed monotone fixed theorem. Very recently, Sang and Ren [21] investigated the following fractional boundary value problem:

$$\begin{cases} -D_{0^+}^{\alpha} u(t) = f(t, u(t), u(t)) + g(t, u(t), u(t)) - b, \quad 0 < t < 1, n-1 < \alpha \leq n, \\ u^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ [D_{0^+}^{\beta} u(t)]_{t=1} = 0, \quad 1 \leq \beta \leq n-2, \end{cases} \tag{1.4}$$

where $n \geq 3$ ($n \in \mathbb{N}$), $b > 0$ is a constant, $f, g : [0,1] \times (-\infty, +\infty) \times (-\infty, +\infty) \to (-\infty, +\infty)$ are continuous functions. In fact, Zhai and Wang [22] considered the following problem:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = b, \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ u(1) = \beta \int_0^1 u(s)\, ds, \end{cases} \tag{1.5}$$

where $2 < \alpha \leq 3$, $0 < \beta < \alpha$, $b > 0$ is a constant, $f : [0,1] \times (-\infty, +\infty) \longrightarrow (-\infty, +\infty)$ is continuous. The authors introduced $\phi - (h, e)$ operators and used monotone iterative method to establish the existence and uniqueness of a nontrivial solution for problem (1.5).

Compared with problem (1.4), we add the derivative term $D_{0+}^{\beta} u(t)$, the operator term $(Hu)(t)$, and nonlinear boundary conditions $k(u(1))$ into problem (1.1). Furthermore, different from problems (1.2) and (1.3), we break through the restriction of positivity on nonlinearities f and g. The first goal of this paper is to establish the existence and uniqueness theorem of solution for the operator equation $A(x,x) + B(x,x) + C(x) + e = x$, where A and B are both mixed monotone, $C(x)$ is decreasing, and $e \in P$ with P is a cone in Banach space E. Our abstract theorem generalizes the result on the cone mappings (see Theorem 3.1 in [19]) to non-cone case. Some sufficient conditions under which problem (1.1) has a unique solution are provided. Moreover, we also construct two iterative sequences for approximating a unique solution.

The structure of this paper includes the following sections. In Sect. 2, we introduce some definitions and give preliminary results to be used in the proof of our main theorems. In Sect. 3, we establish the existence and uniqueness of solutions for problem (1.1) based on a new fixed point theorem.

2 Preliminaries

In this section, we give some definitions and preliminary results that are used in this paper [23–25].

In this paper, $(E, \|\cdot\|)$ is a real Banach space, which is partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$. θ is the zero element in E. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies: $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$ and $x \in P$, $-x \in P \Rightarrow x = \theta$. P is called to be normal if there exists $N > 0$ such that $\theta \leq x \leq y \Rightarrow \|x\| \leq N\|y\|$. Given $h > \theta$, we denote P_h by

$$P_h = \{x \in E \mid \text{there exist } \lambda > 0, \mu > 0 \text{ such that } \lambda h \leq x \leq \mu h\}.$$

Let $e \in P$ with $\theta \leq e \leq h$, we define

$$P_{h,e} = \{x \in E | x + e \in P_h\}.$$

Definition 2.1 ([23, 24]) Let an operator $A : P_{h,e} \times P_{h,e} \to E$ be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, i.e., for $u_i, v_i \in P_{h,e}$, $(i = 1,2)$, $u_1 \leq v_1$, $v_2 \leq u_2$ imply $A(u_1, u_2) \leq A(v_1, v_2)$. The element $x \in P_{h,e}$ is called a fixed point of A if $A(x,x) = x$.

Lemma 2.1 ([21]) *Let P be a normal cone of E and $T : P_{h,e} \times P_{h,e} \longrightarrow E$ be a mixed monotone operator with $T(h,h) \in P_{h,e}$, and the following condition is satisfied*:

(H) *There exists a mapping* $\varphi : (0,1) \longrightarrow (0,+\infty)$ *with* $\varphi(\lambda) > \lambda$ *such that*

$$T\big(\lambda u + (\lambda - 1)e, \lambda^{-1}v + \big(\lambda^{-1} - 1\big)e\big) \geq \varphi(\lambda)T(u,v) + \big(\varphi(\lambda) - 1\big)e$$

for all $u, v \in P_{h,e}$ *and* $\lambda \in (0,1)$. *Then*

(1) *There exist* $u_0, v_0 \in P_{h,e}$ *and* $s \in (0,1)$ *such that*

$$sv_0 \leq u_0 < v_0, \qquad u_0 \leq T(u_0,v_0) \leq T(v_0,u_0) \leq v_0;$$

(2) T *has a unique fixed point* x^* *in* $P_{h,e}$;

(3) *For any initial values* $x_0, y_0 \in P_{h,e}$, *by constructing successively the sequence as follows*:

$$x_n = T(x_{n-1}, y_{n-1}), \qquad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

Definition 2.2 ([3]) The Riemann–Liouville fractional derivative of order α of a function $y \in C[0,1]$ is defined by

$$D^{\alpha}_{0^+}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n-1}}\,ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.2 ([19, 26]) *Let* $h(t) \in C[0,1]$, *then the unique solution of the linear problem*

$$\begin{cases} D^{\alpha}_{0^+}u(t) + h(t) = 0, & 0 < t < 1, n-1 < \alpha \leq n, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ [D^{\gamma}_{0^+}u(t)]_{t=1} = k(u(1)), & 1 \leq \gamma \leq n-2, \end{cases}$$

is given by

$$u(t) = \int_0^1 G(t,s)h(s)\,ds + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\big(u(1)\big)t^{\alpha-1},$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-\gamma-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-\gamma-1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

is the Green's function.

Lemma 2.3 ([19]) *The Green's function* $G(t,s)$ *in Lemma* 2.2 *has the following properties*:

(1) $G(t,s) : [0,1] \times [0,1] \to [0,\infty)$ *is continuous*;

(2) *For all* $t, s \in [0,1]$, *we have*

$$0 \leq \bigl[1-(1-s)^{\gamma}\bigr](1-s)^{\alpha-\gamma-1}t^{\alpha-1} \leq \Gamma(\alpha)G(t,s) \leq (1-s)^{\alpha-\gamma-1}t^{\alpha-1};$$

(3) *For all* $t, s \in [0,1]$, *we have*

$$0 \leq \bigl[1-(1-s)^{\gamma-\beta}\bigr](1-s)^{\alpha-\gamma-1}t^{\alpha-\beta-1} \leq \Gamma(\alpha-\beta)D_{0^+}^{\beta}G(t,s) \leq (1-s)^{\alpha-\gamma-1}t^{\alpha-\beta-1}.$$

Theorem 2.1 *Let* P *be a normal cone in* E, *and let* $A, B : P_{h,e} \times P_{h,e} \longrightarrow E$ *be two mixed monotone operators*, $C : P \to P$ *be a decreasing operator satisfying the following conditions*:

(A1) *For all* $t \in (0,1)$ *and* $x, y \in P_{h,e}$, *there exists* $\psi(t) \in (t,1)$ *such that*

$$A\bigl(tx+(t-1)e, t^{-1}y+\bigl(t^{-1}-1\bigr)e\bigr) \geq \psi(t)A(x,y)+\bigl(\psi(t)-1\bigr)e;$$

(A2) *For all* $t \in (0,1)$ *and* $x, y \in P_{h,e}$,

$$B\bigl(tx+(t-1)e, t^{-1}y+\bigl(t^{-1}-1\bigr)e\bigr) \geq tB(x,y)+(t-1)e;$$

(A3) *For all* $t \in (0,1)$ *and* $y \in P$, *we have*

$$C\bigl(t^{-1}y+\bigl(t^{-1}-1\bigr)e\bigr) \geq tC(y);$$

(A4) $A(h,h) \in P_{h,e}$, $B(h,h) \in P_{h,e}$, *and* $C(h) \in P_h$;

(A5) *For all* $x, y \in P_{h,e}$, *there exists a constant* $\delta > 0$ *such that*

$$A(x,y) \geq \delta\bigl(B(x,y)+C(y)\bigr)+(\delta-1)e.$$

Then the operator equation $A(x,x)+B(x,x)+C(x)+e = x$ *has a unique solution* x^* *in* $P_{h,e}$, *and for any initial values* $x_0, y_0 \in P_{h,e}$, *by setting two iterative sequences* $\{x_n\}$ $\{y_n\}$ *as follows*:

$$x_n = A(x_{n-1}, y_{n-1})+B(x_{n-1}, y_{n-1})+C(y_{n-1})+e, \quad n = 1,2,\ldots,$$
$$y_n = A(y_{n-1}, x_{n-1})+B(y_{n-1}, x_{n-1})+C(x_{n-1})+e, \quad n = 1,2,\ldots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *in* E *as* $n \to \infty$.

Proof We prove Theorem 2.1 in view of Lemma 2.1. Firstly, by condition (A4), and combining with Lemma 2.2 in [18], we have that there exist constants $a_i > 0$ and $b_i > 0$ ($i = 1,2,3$) such that

$$a_1h+(a_1-1)e \leq A(h,h) \leq b_1h+(b_1-1)e, \tag{2.1}$$

$$a_2h+(a_2-1)e \leq B(h,h) \leq b_2h+(b_2-1)e, \tag{2.2}$$

$$a_3h \leq C(h) \leq b_3h. \tag{2.3}$$

Consequently, for all $x, y \in P_{h,e}$, by [21], we obtain

$$\psi(\mu)a_1h + \big(\psi(\mu)a_1 - 1\big)e \le A(x,y) \le \psi(\mu)^{-1}b_1h + \big(\psi(\mu)^{-1}b_1 - 1\big)e, \quad \mu \in (0,1).$$

Hence $A(x,y) \in P_{h,e}$, that is, $A : P_{h,e} \times P_{h,e} \to P_{h,e}$. Similarly, for all $x, y \in P_{h,e}$, we deduce that

$$\eta a_2h + (\eta a_2 - 1)e \le B(x,y) \le \eta^{-1}b_2h + \big(\eta^{-1}b_2 - 1\big)e, \quad \eta \in (0,1).$$

Therefore $B : P_{h,e} \times P_{h,e} \to P_{h,e}$.

For all $y \in P_h$, there exists $\sigma \in (0,1)$ such that $\sigma h \le y \le \sigma^{-1}h$. Since C is a decreasing operator, we have

$$C(y) \ge C\big(\sigma^{-1}h\big) \ge \sigma C(h) \ge \sigma a_3h,$$
$$C(y) \le C(\sigma h) \le \sigma^{-1}C(h) \le \sigma^{-1}b_3h.$$

Let $m_1 = \sigma a_3$, $m_2 = \sigma^{-1}b_3$, that is, $m_1h \le C(y) \le m_2h$. Hence $C(y) \in P_h$, that is, $C : P_h \to P_h$.

Now we define the operator $T = A + B + C + e : P_{h,e} \times P_{h,e} \to E$ by

$$T(x,y) = A(x,y) + B(x,y) + C(y) + e \quad \text{for all } x, y \in P_{h,e}. \tag{2.4}$$

Let $x_i, y_i \in P_{h,e}$ $(i = 1,2)$ with $x_1 \le x_2$, $y_1 \ge y_2$, we obtain

$$A(x_1,y_1) \le A(x_2,y_2), \qquad B(x_1,y_1) \le B(x_2,y_2), \qquad C(y_1) \le C(y_2).$$

Hence, $T(x_1,y_1) \le T(x_2,y_2)$, T is a mixed monotone operator.

From (2.4), we have

$$T(h,h) = A(h,h) + B(h,h) + C(h) + e.$$

By (2.1)–(2.3), we can deduce that

$$T(h,h) \ge (a_1 + a_2 + a_3)h + (a_1 + a_2 - 1)e \ge (a_1 + a_2)h + (a_1 + a_2 - 1)e$$

and

$$T(h,h) \le (b_1 + b_2 + b_3)h + (b_1 + b_2 - 1)e \le (b_1 + b_2 + b_3)h + (b_1 + b_2 + b_3 - 1)e.$$

Let $\varphi_1 = a_1 + a_2$ and $\varphi_2 = b_1 + b_2 + b_3$. Then

$$\varphi_1h + (\varphi_1 - 1)e \le T(h,h) \le \varphi_2h + (\varphi_2 - 1)e.$$

Hence $T(h,h) \in P_{h,e}$.

Finally, we prove that, for every $t \in (0,1)$, there exists $\varphi(t) \in (t,1]$ such that, for all $x, y \in P_{h,e}$,

$$T\big(tx + (t-1)e, t^{-1}y + (t^{-1}-1)e\big) \geq \varphi(t)T(x,y) + \big(\varphi(t)-1\big)e.$$

By condition (A5), we have that

$$A(x,y) + \delta A(x,y) \geq \delta\big(B(x,y) + C(y)\big) + (\delta - 1)e + \delta A(x,y),$$

that is,

$$A(x,y) \geq \frac{\delta}{1+\delta}T(x,y) - \frac{e}{1+\delta}. \tag{2.5}$$

Moreover, it follows from (2.4), (2.5) and conditions (A1)–(A3) that

$$\begin{aligned}
&T\big(tx + (t-1)e, t^{-1}y + (t^{-1}-1)e\big) - tT(x,y) \\
&\quad = A\big(tx + (t-1)e, t^{-1}y + (t^{-1}-1)e\big) + B\big(tx + (t-1)e, t^{-1}y + (t^{-1}-1)e\big) \\
&\qquad + C\big(t^{-1}y + (t^{-1}-1)e\big) + e - t\big(A(x,y) + B(x,y) + C(y) + e\big) \\
&\quad \geq \psi(t)A(x,y) + \big(\psi(t)-1\big)e + tB(x,y) + (t-1)e + tC(y) + e - tA(x,y) \\
&\qquad - tB(x,y) - tC(y) - te \\
&\quad = \big(\psi(t) - t\big)A(x,y) + \big(\psi(t)-1\big)e \\
&\quad \geq \big(\psi(t) - t\big)\left(\frac{\delta}{1+\delta}T(x,y) - \frac{e}{1+\delta}\right) + \big(\psi(t)-1\big)e \\
&\quad = \frac{\delta(\psi(t)-t)}{1+\delta}T(x,y) + \left(\psi(t) - 1 - \frac{\psi(t)-t}{1+\delta}\right)e \quad \text{for all } x, y \in P_{h,e}.
\end{aligned}$$

Thus

$$\begin{aligned}
&T\big(tx + (t-1)e, t^{-1}y + (t^{-1}-1)e\big) \\
&\quad \geq \left(\frac{\delta(\psi(t)-t)}{1+\delta} + t\right)T(x,y) + \left(\psi(t) - 1 - \frac{\psi(t)-t}{1+\delta}\right)e \\
&\quad = \frac{\delta\psi(t)+t}{1+\delta}T(x,y) + \left(\frac{\delta\psi(t)+t}{1+\delta} - 1\right)e \quad \text{for } x, y \in P_{h,e}.
\end{aligned} \tag{2.6}$$

Let $\varphi(t) = \frac{\delta\psi(t)+t}{1+\delta}$, then $\varphi(t) \in (t, \psi(t)) \subset (t,1]$, $t \in (0,1)$, by (2.6), we can conclude that

$$T\big(tx + (t-1)e, t^{-1}y + (t^{-1}-1)e\big) \geq \varphi(t)T(x,y) + \big(\varphi(t)-1\big)e, \quad \forall x, y \in P_{h,e}.$$

We derive the conclusion of Theorem 2.1 from Lemma 2.1. □

3 Main result

In the section, we use Theorem 2.1 to obtain the existence and uniqueness of a positive solution for problem (1.1).

Set $E = \{x | x \in C[0,1], D^{\beta}_{0^+}x \in C[0,1]\}$, then E is a Banach space with an order relation $u \leq v$ if $u(t) \leq v(t)$, $D^{\beta}_{0^+}u(t) \leq D^{\beta}_{0^+}v(t)$. Let $P \subset E$ be defined by $P = \{x \in E | x(t) \geq 0, D^{\beta}_{0^+}x(t) \geq 0\}$

for all $t \in [0,1]$. It is clear that P is a normal cone. Let

$$e(t) = \frac{b}{(\alpha-\gamma)\Gamma(\alpha)}\left(t^{\alpha-1} - \frac{\alpha-\gamma}{\alpha}t^{\alpha}\right), \quad t \in [0,1].$$

Theorem 3.1 *Assume that the following conditions are satisfied*:

(B1) $f,g : [0,1] \times [-e^*, +\infty) \times [-e^*, +\infty) \to (-\infty, +\infty)$ *are continuous and* $k : [0,+\infty) \to [0,+\infty)$ *is continuous. For all* $t \in [0,1]$, $g(t,0,H(L)) \geq 0$ *with* $g(t,0,H(L)) \not\equiv 0$, *where* $L \geq \frac{b}{(\alpha-\gamma)\Gamma(\alpha)}$ *and* $e^* = \max\{e(t) : t \in [0,1]\}$;

(B2) *For fixed* $t \in [0,1]$ *and* $y \in [-e^*, +\infty)$, $f(t,x,y)$, $g(t,x,y)$ *are increasing in* $x \in [-e^*, +\infty)$; *for fixed* $t \in [0,1]$ *and* $x \in [-e^*, +\infty)$, $f(t,x,y)$, $g(t,x,y)$ *are decreasing in* $y \in [-e^*, +\infty)$; $k(y)$ *is decreasing in* $y \in [0,+\infty)$, $k(L) \neq 0$;

(B3) *For all* $\lambda \in (0,1)$, *there exists* $\psi(\lambda) \in (\lambda, 1)$ *such that, for all* $t \in [0,1]$,
  (a) $f(t, \lambda x + (\lambda-1)\rho_1, \lambda^{-1}y + (\lambda^{-1}-1)\rho_2) \geq \psi(\lambda) f(t,x,y)$,
  (b) $g(t, \lambda x + (\lambda-1)\rho_1, \lambda^{-1}y + (\lambda^{-1}-1)\rho_2) \geq \lambda g(t,x,y)$,
  (c) $k(\lambda^{-1}y + (\lambda^{-1}-1)\rho_1) \geq \lambda k(y)$,
  where $x,y \in (-\infty,+\infty)$ *and* $\rho_1, \rho_2 \in [0,e^*]$;

(B4) *For all* $t \in [0,1]$, $x,y \in [-e^*, +\infty)$, *there exist two constants* $\delta_1, \delta_2 > 0$ *such that*
  (a) $f(t,x,y) \geq \delta_1 g(t,x,0)$,
  (b) $f(t,x,y) \geq \delta_2 k(y)$;

(B5) $H : C[0,1] \to C[0,1]$ *and satisfies the following conditions*:
  (a) $Hu \geq 0$ *for every* $u \in P_{h,e}$;
  (b) *for* $u,v \in P_{h,e}$, $u \leq v \Longrightarrow Hu \leq Hv$;
  (c) *for all* $\lambda \in (0,1)$ *and* $u \in P_{h,e}$ *such that*

$$H\big(\lambda u + (\lambda-1)\hat{e}\big) \geq \lambda H(u) + (\lambda-1)\hat{e}, \quad \hat{e} \in \big[0,e^*\big].$$

Then we have the following conclusions:

(1) *Problem* (1.1) *has a unique nontrivial solution* u^* *in* $P_{h,e}$, *where* $h(t) = Lt^{\alpha-1}$ *for all* $t \in [0,1]$;

(2) *We can construct the following two sequences*:

$$\begin{aligned}\omega_n(t) = {}& \int_0^1 G(t,s)\big[f\big(s,\omega_{n-1}(s), D_{0^+}^{\beta}\tau_{n-1}(s)\big) + g\big(s,\omega_{n-1}(s),(H\tau_{n-1})(s)\big)\big]\,ds \\ & + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\big(\tau_{n-1}(1)\big)t^{\alpha-1} - e(t), \quad n = 1,2,\ldots, \\ \tau_n(t) = {}& \int_0^1 G(t,s)\big[f\big(s,\tau_{n-1}(s), D_{0^+}^{\beta}\omega_{n-1}(s)\big) + g\big(s,\tau_{n-1}(s),(H\omega_{n-1})(s)\big)\big]\,ds \\ & + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\big(\omega_{n-1}(1)\big)t^{\alpha-1} - e(t), \quad n = 1,2,\ldots,\end{aligned}$$

for any initial values $\omega_0, \tau_0 \in P_{h,e}$, *and sequences* $\{\omega_n(t)\}$ *and* $\{\tau_n(t)\}$ *for approximating* $u^*(t)$, *we have* $\omega_n(t) \to u^*(t)$ *and* $\tau_n(t) \to u^*(t)$ *as* $n \to \infty$.

Proof We will use Theorem 2.1 to prove Theorem 3.1.

For $e \in P$, $t \in [0,1]$, we have

$$P_{h,e} = \big\{x \in C[0,1] | x + e \in P_h\big\}.$$

Furthermore, for $L \geq \frac{b}{(\alpha-\gamma)\Gamma(\alpha)}$ and $t \in [0,1]$, we have

$$e(t) = \frac{b}{(\alpha-\gamma)\Gamma(\alpha)}t^{\alpha-1} - \frac{b}{\alpha\Gamma(\alpha)}t^{\alpha} \leq \frac{b}{(\alpha-\gamma)\Gamma(\alpha)}t^{\alpha-1} \leq Lt^{\alpha-1} = h(t). \tag{3.1}$$

Hence $0 < e(t) \leq h(t)$.

From Lemma 2.2, problem (1.1) has the integral formulation

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)\bigl[f\bigl(s,u(s),D_{0^+}^{\beta}u(s)\bigr) + g\bigl(s,u(s),(Hu)(s)\bigr) - b\bigr]\,ds \\
&\quad + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\bigl(u(1)\bigr)t^{\alpha-1} \\
&= \int_0^1 G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}u(s)\bigr)\,ds + \int_0^1 G(t,s)g\bigl(s,u(s),(Hu)(s)\bigr)\,ds \\
&\quad + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\bigl(u(1)\bigr)t^{\alpha-1} - b\int_0^1 G(t,s)\,ds \\
&= \int_0^1 G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}u(s)\bigr)\,ds + \int_0^1 G(t,s)g\bigl(s,u(s),(Hu)(s)\bigr)\,ds \\
&\quad + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\bigl(u(1)\bigr)t^{\alpha-1} - \frac{b}{(\alpha-\gamma)\Gamma(\alpha)}\left(t^{\alpha-1} - \frac{\alpha-\gamma}{\alpha}t^{\alpha}\right) \\
&= \int_0^1 G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}u(s)\bigr)\,ds + \int_0^1 G(t,s)g\bigl(s,u(s),(Hu)(s)\bigr)\,ds \\
&\quad + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\bigl(u(1)\bigr)t^{\alpha-1} - e(t) \\
&= \int_0^1 G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}u(s)\bigr)\,ds - e(t) + \int_0^1 G(t,s)g\bigl(s,u(s),(Hu)(s)\bigr)\,ds \\
&\quad - e(t) + \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\bigl(u(1)\bigr)t^{\alpha-1} + e(t).
\end{aligned}$$

For every $t \in [0,1]$ and $u, v \in P_{h,e}$, we consider the following operators:

$$A(u,v)(t) = \int_0^1 G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}v(s)\bigr)\,ds - e(t), \tag{3.2}$$

$$B(u,v)(t) = \int_0^1 G(t,s)g\bigl(s,u(s),(Hv)(s)\bigr)\,ds - e(t), \tag{3.3}$$

and

$$C(v)(t) = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\bigl(v(1)\bigr)t^{\alpha-1}. \tag{3.4}$$

It is clear that $u(t)$ is the solution of problem (1.1) if and only if u is the fixed point of the operator $A(u,u) + B(u,u) + C(u) + e$. Further, by (3.2)–(3.4), we can calculate that

$$D_{0^+}^{\beta}A(u,v)(t) = \int_0^1 D_{0^+}^{\beta}G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}v(s)\bigr)\,ds - D_{0^+}^{\beta}e(t), \tag{3.5}$$

$$D_{0^+}^{\beta}B(u,v)(t) = \int_0^1 D_{0^+}^{\beta}G(t,s)g\bigl(s,u(s),(Hv)(s)\bigr)\,ds - D_{0^+}^{\beta}e(t), \tag{3.6}$$

and

$$D^{\beta}_{0^+}C(v)(t)=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha-\beta)}k\bigl(v(1)\bigr)t^{\alpha-\beta-1}.$$

(1) Firstly, for all $u_i,v_i\in P_{h,e}$ $(i=1,2)$ with $u_1\geq u_2$, $v_1\leq v_2$, by (B5), we get that $H(v_1)\leq H(v_2)$. It follows from condition (B2), (3.2), and (3.5) that

$$\begin{aligned}A(u_1,v_1)(t)&=\int_0^1 G(t,s)f\bigl(s,u_1(s),D^{\beta}_{0^+}v_1(s)\bigr)\,ds-e(t)\\&\geq\int_0^1 G(t,s)f\bigl(s,u_2(s),D^{\beta}_{0^+}v_2(s)\bigr)\,ds-e(t)=A(u_2,v_2)(t),\end{aligned}$$

and

$$\begin{aligned}D^{\beta}_{0^+}A(u_1,v_1)(t)&=\int_0^1 D^{\beta}_{0^+}G(t,s)f\bigl(s,u_1(s),D^{\beta}_{0^+}v_1(s)\bigr)\,ds-D^{\beta}_{0^+}e(t)\\&\geq\int_0^1 D^{\beta}_{0^+}G(t,s)f\bigl(s,u_2(s),D^{\beta}_{0^+}v_2(s)\bigr)\,ds-D^{\beta}_{0^+}e(t)=D^{\beta}_{0^+}A(u_2,v_2)(t).\end{aligned}$$

Thus, A is a mixed monotone operator. Similarly, we have from (3.3) and (3.6) that

$$\begin{aligned}B(u_1,v_1)(t)&=\int_0^1 G(t,s)g\bigl(s,u_1(s),(Hv_1)(s)\bigr)\,ds-e(t)\\&\geq\int_0^1 G(t,s)g\bigl(s,u_2(s),(Hv_2)(s)\bigr)\,ds-e(t)=B(u_2,v_2)(t)\end{aligned}$$

and

$$\begin{aligned}D^{\beta}_{0^+}B(u_1,v_1)(t)&=\int_0^1 D^{\beta}_{0^+}G(t,s)g\bigl(s,u_1(s),(Hv_1)(s)\bigr)\,ds-D^{\beta}_{0^+}e(t)\\&\geq\int_0^1 D^{\beta}_{0^+}G(t,s)g\bigl(s,u_2(s),(Hv_2)(s)\bigr)\,ds-D^{\beta}_{0^+}e(t)=D^{\beta}_{0^+}B(u_2,v_2)(t).\end{aligned}$$

Hence, B is a mixed monotone operator. Since

$$C(v_1)(t)=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\bigl(v_1(1)\bigr)t^{\alpha-1}\geq\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\bigl(v_2(1)\bigr)t^{\alpha-1}=C(v_2)(t)$$

and

$$D^{\beta}_{0^+}C(v_1)(t)=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha-\beta)}k\bigl(v_1(1)\bigr)t^{\alpha-\beta-1}\geq\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha-\beta)}k\bigl(v_2(1)\bigr)t^{\alpha-\beta-1}=D^{\beta}_{0^+}C(v_2)(t),$$

Then C is a decreasing operator.

(2) In view of condition (B3)(a), for every $\lambda\in[0,1]$ and $t\in[0,1]$, there exists $\psi(\lambda)\in(\lambda,1)$ such that, for all $u,v\in P_{h,e}$, we have

$$\begin{aligned}&A\bigl(\lambda u+(\lambda-1)e,\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)e\bigr)(t)\\&\quad=\int_0^1 G(t,s)f\bigl(s,\lambda u(s)+(\lambda-1)e,D^{\beta}_{0^+}\bigl(\lambda^{-1}v(s)+\bigl(\lambda^{-1}-1\bigr)e\bigr)\bigr)\,ds-e(t)\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 G(t,s)f\bigl(s,\lambda u(s)+(\lambda-1)e,\lambda^{-1}D_{0^+}^{\beta}v(s)+\bigl(\lambda^{-1}-1\bigr)D_{0^+}^{\beta}e\bigr)\,ds-e(t)\\
&\geq \psi(\lambda)\int_0^1 G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}v(s)\bigr)\,ds-e(t)+\psi(\lambda)e(t)-\psi(\lambda)e(t)\\
&= \psi(\lambda)A(u,v)(t)+\bigl(\psi(\lambda)-1\bigr)e(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&D_{0^+}^{\beta}A\bigl(\lambda u+(\lambda-1)e,\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)e\bigr)(t)\\
&\quad= \int_0^1 D_{0^+}^{\beta}G(t,s)f\bigl(s,\lambda u(s)+(\lambda-1)e,D_{0^+}^{\beta}\bigl(\lambda^{-1}v(s)+\bigl(\lambda^{-1}-1\bigr)e\bigr)\bigr)\,ds-D_{0^+}^{\beta}e(t)\\
&\quad\geq \psi(\lambda)\int_0^1 D_{0^+}^{\beta}G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}v(s)\bigr)\,ds-D_{0^+}^{\beta}e(t)+\psi(\lambda)D_{0^+}^{\beta}e(t)-\psi(\lambda)D_{0^+}^{\beta}e(t)\\
&\quad= \psi(\lambda)D_{0^+}^{\beta}A(u,v)(t)+\bigl(\psi(\lambda)-1\bigr)D_{0^+}^{\beta}e(t).
\end{aligned}
$$

Hence, $A(\lambda x+(\lambda-1)e,\lambda^{-1}y+(\lambda^{-1}-1)e)\geq\psi(\lambda)A(x,y)+(\psi(\lambda)-1)e$. It follows from conditions (B3) and (B5) that

$$
\begin{aligned}
&H\bigl(\lambda u+(\lambda-1)e\bigr)\geq\lambda(Hu)+(\lambda-1)e,\\
&H\bigl(\lambda^{-1}u+\bigl(\lambda^{-1}-1\bigr)e\bigr)\leq\lambda^{-1}(Hu)+\bigl(\lambda^{-1}-1\bigr)e,\\
&B\bigl(\lambda u+(\lambda-1)e,\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)e\bigr)(t)\\
&\quad= \int_0^1 G(t,s)g\bigl(s,\lambda u(s)+(\lambda-1)e(s),H\bigl(\lambda^{-1}v(s)+\bigl(\lambda^{-1}-1\bigr)e(s)\bigr)\bigr)\,ds-e(t)\\
&\quad\geq \int_0^1 G(t,s)g\bigl(s,\lambda u(s)+(\lambda-1)e(s),\lambda^{-1}(Hv)(s)+\bigl(\lambda^{-1}-1\bigr)e(s)\bigr)\,ds-e(t)\\
&\quad\geq \lambda\int_0^1 G(t,s)g\bigl(s,u(s),(Hv)(s)\bigr)\,ds-e(t)+\lambda e(t)-\lambda e(t)\\
&\quad= \lambda B(u,v)(t)+(\lambda-1)e(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&D_{0^+}^{\beta}B\bigl(\lambda u+(\lambda-1)e,\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)e\bigr)(t)\\
&\quad= \int_0^1 D_{0^+}^{\beta}G(t,s)g\bigl(s,\lambda u(s)+(\lambda-1)e(s),H\bigl(\lambda^{-1}v(s)+\bigl(\lambda^{-1}-1\bigr)e(s)\bigr)\bigr)\,ds-D_{0^+}^{\beta}e(t)\\
&\quad\geq \int_0^1 D_{0^+}^{\beta}G(t,s)g\bigl(s,\lambda u(s)+(\lambda-1)e(s),\lambda^{-1}(Hv)(s)+\bigl(\lambda^{-1}-1\bigr)e(s)\bigr)\,ds-D_{0^+}^{\beta}e(t)\\
&\quad\geq \lambda\int_0^1 D_{0^+}^{\beta}G(t,s)g\bigl(s,u(s),(Hv)(s)\bigr)\,ds-D_{0^+}^{\beta}e(t)+\lambda D_{0^+}^{\beta}e(t)-\lambda D_{0^+}^{\beta}e(t)\\
&\quad= \lambda D_{0^+}^{\beta}B(u,v)(t)+(\lambda-1)D_{0^+}^{\beta}e(t).
\end{aligned}
$$

Thus, $B(\lambda u+(\lambda-1)e,\lambda^{-1}v+(\lambda^{-1}-1)e)\geq\lambda B(u,v)+(\lambda-1)e$. Moreover, by condition (B3)(c), we have

$$\begin{aligned}C\bigl(\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)e\bigr)(t)&=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k\bigl(\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)e\bigr)(1)t^{\alpha-1}\\&\geq\lambda\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}k(v)(1)t^{\alpha-1}=\lambda C(v)(t)\end{aligned}$$

and

$$\begin{aligned}D_{0^+}^{\beta}C\bigl(\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)e\bigr)(t)&=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha-\beta)}k\bigl(\lambda^{-1}v+\bigl(\lambda^{-1}-1\bigr)e\bigr)(1)t^{\alpha-\beta-1}\\&\geq\lambda\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha-\beta)}k(v)(1)t^{\alpha-\beta-1}=D_{0^+}^{\beta}\lambda C(v)(t).\end{aligned}$$

Thus, $C(\lambda^{-1}v+(\lambda^{-1}-1)e)\geq\lambda C(v)$. Consequently, conditions (A1)–(A3) of Theorem 2.1 are satisfied.

(3) By condition (B4), for every $u,v\in P_{h,e}$, $t\in[0,1]$, we have

$$\begin{aligned}A(u,v)(t)&=\int_0^1G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}v(s)\bigr)\,ds-e(t)\\&\geq\delta_1\int_0^1G(t,s)g\bigl(s,u(s),0\bigr)\,ds-e(t)-\delta_1e(t)+\delta_1e(t)\\&\geq\delta_1\biggl(\int_0^1G(t,s)g\bigl(s,u(s),(Hv)(s)\bigr)\,ds-e(t)\biggr)+(\delta-1)e(t)\\&=\delta_1B(u,v)(t)+(\delta_1-1)e(t)\end{aligned}$$

and

$$\begin{aligned}D_{0^+}^{\beta}A(u,v)(t)&=\int_0^1D_{0^+}^{\beta}G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}v(s)\bigr)\,ds-D_{0^+}^{\beta}e(t)\\&\geq\delta_1\int_0^1D_{0^+}^{\beta}G(t,s)g\bigl(s,u(s),0\bigr)\,ds-D_{0^+}^{\beta}e(t)-\delta_1D_{0^+}^{\beta}e(t)+\delta_1D_{0^+}^{\beta}e(t)\\&\geq\delta_1\biggl(\int_0^1D_{0^+}^{\beta}G(t,s)g\bigl(s,u(s),(Hv)(s)\bigr)\,ds-D_{0^+}^{\beta}e(t)\biggr)+(\delta_1-1)D_{0^+}^{\beta}e(t)\\&=\delta_1D_{0^+}^{\beta}B(u,v)(t)+(\delta_1-1)D_{0^+}^{\beta}e(t).\end{aligned}$$

Thus, $A(u,v)\geq\delta_1B(u,v)+(\delta_1-1)e$. Similarly, we get

$$\begin{aligned}A(u,v)(t)&=\int_0^1G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}v(s)\bigr)\,ds-e(t)\\&\geq\int_0^1G(t,s)\delta_2k\bigl(v(1)\bigr)\,ds-e(t)\\&\geq\delta_2k\bigl(v(1)\bigr)\frac{t^{\alpha-1}}{\Gamma(\alpha)}\biggl(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha-\beta}\biggr)-e(t)\\&=\frac{\delta_2}{\Gamma(\alpha-\gamma)}\biggl(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha-\beta}\biggr)C(v)(t)-e(t)\end{aligned}$$

and

$$\begin{aligned} D_{0^+}^{\beta}A(u,v)(t) &= \int_0^1 D_{0^+}^{\beta}G(t,s)f\bigl(s,u(s),D_{0^+}^{\beta}v(s)\bigr)\,ds - D_{0^+}^{\beta}e(t) \\ &\geq \int_0^1 D_{0^+}^{\beta}G(t,s)\delta_2 k\bigl(v(1)\bigr)\,ds - D_{0^+}^{\beta}e(t) \\ &\geq \delta_2 k\bigl(v(1)\bigr)\frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha-\beta}\right) - D_{0^+}^{\beta}e(t) \\ &= \frac{\delta_2}{\Gamma(\alpha-\gamma)}\left(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha-\beta}\right)D_{0^+}^{\beta}C(v)(t) - D_{0^+}^{\beta}e(t). \end{aligned}$$

Let $\delta_3 = \frac{\delta_2}{\Gamma(\alpha-\gamma)}(\frac{1}{\alpha-\gamma}-\frac{1}{\alpha-\beta})$, then $A(u,v) \geq \delta_3 C(v) - e$. Choose $\delta_4 = \min\{\delta_1,\delta_3\}$ and $\delta_5 = \frac{1}{2}\delta_4$, then

$$A(u,v) \geq \delta_5\bigl(B(u,v)+C(v)\bigr) + (\delta_5 - 1)e.$$

Therefore, condition (A5) of Theorem 2.1 is satisfied.

(4) Finally, we prove that condition (A4) is satisfied. By (3.1) and equality $D_{0^+}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}$, we have

$$D_{0^+}^{\beta}h(t) = D_{0^+}^{\beta}\bigl(Lt^{\alpha-1}\bigr) = L\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}.$$

In view of (B1), (B2), (B4), and Lemma 2.3, we have

$$\begin{aligned} A(h,h)(t)+e(t) &= \int_0^1 G(t,s)f\bigl(s,h(s),D_{0^+}^{\beta}h(s)\bigr)\,ds \\ &\geq \int_0^1 \frac{[1-(1-s)^{\gamma}](1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}t^{\alpha-1}f\bigl(s,Ls^{\alpha-1},D_{0^+}^{\beta}Ls^{\alpha-1}\bigr)\,ds \\ &\geq \int_0^1 \frac{[1-(1-s)^{\gamma}](1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}t^{\alpha-1}f\left(s,0,L\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}s^{\alpha-\beta-1}\right)ds \\ &\geq h(t)\int_0^1 \frac{[1-(1-s)^{\gamma}](1-s)^{\alpha-\gamma-1}}{L\Gamma(\alpha)}f\left(s,0,L\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right)ds \\ &\geq h(t)\int_0^1 \frac{[1-(1-s)^{\gamma-\beta}](1-s)^{\alpha-\gamma-1}}{L\Gamma(\alpha)}f\left(s,0,L\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right)ds, \\ A(h,h)(t)+e(t) &= \int_0^1 G(t,s)f\bigl(s,h(s),D_{0^+}^{\beta}h(s)\bigr)\,ds \\ &\leq \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}t^{\alpha-1}f\bigl(s,Ls^{\alpha-1},0\bigr)\,ds \\ &\leq h(t)\frac{1}{L\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\gamma-1}f(s,L,0)\,ds, \end{aligned}$$

and

$$\begin{aligned} &D_{0^+}^{\beta}A(h,h)(t) + D_{0^+}^{\beta}e(t) \\ &\quad= \int_0^1 D_{0^+}^{\beta}G(t,s)f\bigl(s,h(s),D_{0^+}^{\beta}h(s)\bigr)\,ds \end{aligned}$$

$$\begin{aligned}
&\geq \int_0^1 \frac{[1-(1-s)^{\gamma-\beta}](1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} f\bigl(s, Ls^{\alpha-1}, D_{0^+}^{\beta} Ls^{\alpha-1}\bigr)\,ds \\
&\geq \int_0^1 \frac{[1-(1-s)^{\gamma-\beta}](1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} f\left(s, 0, L\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} s^{\alpha-\beta-1}\right) ds \\
&\geq D_{0^+}^{\beta} h(t) \int_0^1 \frac{[1-(1-s)^{\gamma-\beta}](1-s)^{\alpha-\gamma-1}}{L\Gamma(\alpha)} f\left(s, 0, L\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right) ds,
\end{aligned}$$

$$\begin{aligned}
&D_{0^+}^{\beta} A(h,h)(t) + D_{0^+}^{\beta} e(t) \\
&\quad= \int_0^1 D_{0^+}^{\beta} G(t,s) f\bigl(s, h(s), D_{0^+}^{\beta} h(s)\bigr)\,ds \\
&\quad\leq \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} f\bigl(s, Ls^{\alpha-1}, 0\bigr)\,ds \\
&\quad\leq D_{0^+}^{\beta} h(t) \frac{1}{L\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} f(s, L, 0)\,ds.
\end{aligned}$$

Let

$$\begin{aligned}
l_1 &= \frac{1}{L\Gamma(\alpha)} \int_0^1 \bigl[1-(1-s)^{\gamma-\beta}\bigr](1-s)^{\alpha-\gamma-1} f\left(s, 0, L\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\right) ds, \\
l_2 &= \frac{1}{L\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} f(s, L, 0)\,ds.
\end{aligned}$$

Then $l_1 h \leq A(h,h) + e \leq l_2 h$. Thus, $A(h,h) + e \in P_h$, that is, $A(h,h) \in P_{h,e}$.
On the other hand, we have

$$\begin{aligned}
B(h,h)(t) + e(t) &= \int_0^1 G(t,s) g\bigl(s, h(s), (Hh)(s)\bigr)\,ds \\
&\geq \int_0^1 \frac{[1-(1-s)^{\gamma}](1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)} t^{\alpha-1} g\bigl(s, 0, H\bigl(Ls^{\alpha-1}\bigr)\bigr)\,ds \\
&\geq \int_0^1 \frac{[1-(1-s)^{\gamma}](1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)} t^{\alpha-1} g\bigl(s, 0, H(L)\bigr)\,ds \\
&= h(t) \frac{1}{L\Gamma(\alpha)} \int_0^1 \bigl[1-(1-s)^{\gamma}\bigr](1-s)^{\alpha-\gamma-1} g\bigl(s, 0, H(L)\bigr)\,ds \\
&\geq h(t) \frac{1}{L\Gamma(\alpha)} \int_0^1 \bigl[1-(1-s)^{\gamma-\beta}\bigr](1-s)^{\alpha-\gamma-1} g\bigl(s, 0, H(L)\bigr)\,ds, \\
B(h,h)(t) + e(t) &= \int_0^1 G(t,s) g\bigl(s, h(s), (Hh)(s)\bigr)\,ds \\
&\leq \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)} t^{\alpha-1} g\bigl(s, Ls^{\alpha-1}, 0\bigr)\,ds \\
&\leq h(t) \frac{1}{L\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} g(s, L, 0)\,ds,
\end{aligned}$$

and

$$\begin{aligned}
&D_{0^+}^{\beta} B(h,h)(t) + D_{0^+}^{\beta} e(t) \\
&\quad= \int_0^1 D_{0^+}^{\beta} G(t,s) g\bigl(s, h(s), (Hh)(s)\bigr)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^1 \frac{[1-(1-s)^{\gamma-\beta}](1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} g\bigl(s, Ls^{\alpha-1}, H\bigl(Ls^{\alpha-1}\bigr)\bigr)\,ds \\
&\geq D_{0^+}^{\beta} h(t) \frac{1}{L\Gamma(\alpha)} \int_0^1 \bigl[1-(1-s)^{\gamma-\beta}\bigr](1-s)^{\alpha-\gamma-1} g\bigl(s, 0, H(L)\bigr)\,ds, \\
&D_{0^+}^{\beta} B(h,h)(t) + D_{0^+}^{\beta} e(t) \\
&= \int_0^1 D_{0^+}^{\beta} G(t,s) g\bigl(s, h(s), (Hh)(s)\bigr)\,ds \\
&\leq \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} g\bigl(s, Ls^{\alpha-1}, 0\bigr)\,ds \\
&\leq D_{0^+}^{\beta} h(t) \frac{1}{L\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} g(s, L, 0)\,ds.
\end{aligned}
$$

Let

$$
\begin{aligned}
l_3 &= \frac{1}{L\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\gamma-1} g(s, L, 0)\,ds, \\
l_4 &= \frac{1}{L\Gamma(\alpha)} \int_0^1 \bigl[1-(1-s)^{\gamma-\beta}\bigr](1-s)^{\alpha-\gamma-1} g\bigl(s, 0, H(L)\bigr)\,ds.
\end{aligned}
$$

Then $l_4 h \leq B(h,h) + e \leq l_3 h$, thus $B(h,h) \in P_{h,e}$. In addition, we have

$$
C(h)(t) = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)} k\bigl(h(1)\bigr) t^{\alpha-1} = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)} k(L) t^{\alpha-1}.
$$

Then $C(h) \in P_h$. Consequently, (A4) is proved. Therefore, all the conditions of Theorem 2.1 are satisfied. The conclusions of Theorem 3.1 hold. □

Now, we give an example to illustrate our main result.

Example 3.1 For problem (1.1), we choose $n = 5$, $\alpha = \frac{9}{2}$, $\beta = \frac{3}{2}$, $\gamma = \frac{5}{2}$, and $b = 2$. Consider the following boundary value problem:

$$
\begin{cases}
D_{0^+}^{\frac{9}{2}} u(t) + [\frac{e(t)}{e^*} u(t) + e(t)]^{\frac{1}{2}} + [\frac{e(t)}{e^*} D_{0^+}^{\frac{3}{2}} u(t) + e(t) + 1]^{-\frac{1}{2}} \\
\quad + [\int_0^t (u(s) + e^*)\,ds + e(t) + 2]^{-\frac{1}{3}} - 2 = 0, \quad t \in (0,1), \\
u(0) = u'(0) = u''(0) = u'''(0) = 0, \\
[D_{0^+}^{\frac{5}{2}} u(t)]_{t=1} = \frac{1}{\sqrt[3]{u(1)+2}},
\end{cases}
\tag{3.7}
$$

where $e(t) = \frac{t^{\frac{7}{2}} - \frac{4}{9} t^{\frac{9}{2}}}{\Gamma(\frac{9}{2})}$, $t \in [0,1]$, and $e^* = \max\{e(t) : t \in [0,1]\} = \frac{5}{9\Gamma(\frac{9}{2})}$. Clearly, $e(t) \leq \frac{t^{\frac{7}{2}}}{\Gamma(\frac{9}{2})} \leq h(t)$. Let $f(t,u,v) = [\frac{e(t)}{e^*} u + e(t)]^{\frac{1}{2}} + [\frac{e(t)}{e^*} v + e(t) + 1]^{-\frac{1}{2}}$, $g(t,u,v) = [v + e(t) + 2]^{-\frac{1}{3}}$, $k(u(1)) = \frac{1}{\sqrt[3]{u(1)+2}}$, and $(Hu)(t) = \int_0^t (u(s) + e^*)\,ds$. It is easy to check that all the conditions in Theorem 3.1 are satisfied. By Theorem 3.1, problem (3.7) has a unique nontrivial solution u^* in

$P_{h,e}$. Furthermore, we can set up the following sequences:

$$\begin{aligned}\omega_n(t) = \int_0^1 G(t,s)\Bigg\{&\left[\frac{e(s)}{e^*}\omega_{n-1}(s) + e(s)\right]^{\frac{1}{2}} + \left[\frac{e(s)}{e^*}D_{0^+}^{\frac{3}{2}}\tau_{n-1}(s) + e(s) + 1\right]^{-\frac{1}{2}} \\ &+ \left[\int_0^t \left(\tau_{n-1}(s) + e^*\right)ds + e(s) + 2\right]^{-\frac{1}{3}} + \frac{\Gamma(2)}{\Gamma(\frac{9}{2})}\frac{1}{\sqrt[3]{\tau_{n-1}(1) + 2}}t^{\frac{7}{2}} \\ &- \frac{1}{\Gamma(\frac{9}{2})}t^{\frac{7}{2}} + \frac{4}{9\Gamma(\frac{9}{2})}t^{\frac{9}{2}}\Bigg\}\,ds\end{aligned}$$

and

$$\begin{aligned}\tau_n(t) = \int_0^1 G(t,s)\Bigg\{&\left[\frac{e(s)}{e^*}\tau_{n-1}(s) + e(s)\right]^{\frac{1}{2}} + \left[\frac{e(s)}{e^*}D_{0^+}^{\frac{3}{2}}\omega_{n-1}(s) + e(s) + 1\right]^{-\frac{1}{2}} \\ &+ \left[\int_0^t \left(\omega_{n-1}(s) + e^*\right)ds + e(s) + 2\right]^{-\frac{1}{3}} + \frac{\Gamma(2)}{\Gamma(\frac{9}{2})}\frac{1}{\sqrt[3]{\omega_{n-1}(1) + 2}}t^{\frac{7}{2}} \\ &- \frac{1}{\Gamma(\frac{9}{2})}t^{\frac{7}{2}} + \frac{4}{9\Gamma(\frac{9}{2})}t^{\frac{9}{2}}\Bigg\}\,ds,\end{aligned}$$

for any given $\omega_0, \tau_0 \in P_{h,e}$, we have $\{\omega_n(t)\}$ and $\{\tau_n(t)\}$ both converge to $u^*(t)$ uniformly for all $t \in [0,1]$.

4 Conclusions

In this paper, we establish the existence and uniqueness theorem of a solution for the operator equation $A(x,x) + B(x,x) + C(x) + e = x$, where A and B are both mixed monotone, $C(x)$ is decreasing, and $e \in P$ with P is a cone in Banach space E. Using the abstract result, we give some sufficient conditions under which problem (1.1) has a unique solution. Furthermore, we also construct two iterative sequences for approximating the unique solution.

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