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Note on norm of an m-linear integral-type operator between weighted-type spaces

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Abstract

We find a necessary and sufficient condition for the boundedness of an m-linear integral-type operator between weighted-type spaces of functions, and calculate norm of the operator, complementing some results by L. Grafakos and his collaborators. We also present an inequality which explains a detail in the proof of the boundedness of the linear integral-type operator on $L^p(\mathbb{R}^n)$ space.

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1 Introduction

Throughout this note the set of natural numbers is denoted by $\mathbb{N}$, the set of reals by $\mathbb{R}$, the set of positive reals by $\mathbb{R}_+$, the Euclidean n-dimensional space with the norm $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$, $x = (x_1, \dots, x_n)$, by $\mathbb{R}^n$, the unit sphere in $\mathbb{R}^n$ by $\mathbb{S}$, the $n-1$-dimensional surface measure by $d\sigma(\zeta)$, $\sigma_n = \sigma(\mathbb{S})$, the normalized surface measure $d\sigma(\zeta)/\sigma_n$ is denoted by $d\sigma_N(\zeta)$, the open unit ball in $\mathbb{R}^n$ by $\mathbb{B}$, the open unit ball in $\mathbb{R}^n$ centered at a and with radius r by $B(a,r)$, the Lebesgue volume measure on $\mathbb{R}^n$ by $dV(x)$, $v_n = V(\mathbb{B})$, the normalized Lebesgue volume measure $dV(x)/v_n$ is denoted by $dV_N(x)$, whereas $L^p_\alpha(\Omega)$ denotes the weighted Lebesgue space on a domain $\Omega \subseteq \mathbb{R}^n$ with the weight $w(x) = |x|^\alpha$, that is,

$$L^p_\alpha(\Omega) = \left\{ f : \|f\|_{L^p_\alpha(\Omega)} := \left(\int_\Omega \left|f(x)\right|^p dV_\alpha(x) \right)^{1/p} < +\infty \right\},$$

where $1 \le p < +\infty$, $\alpha > -n$, and $dV_\alpha(x) = |x|^\alpha dV(x)$ (for $\alpha = 0$, the space is reduced to the standard L^p space on the domain; see, e.g., [1]). If $k, l \in \mathbb{N}$ are such that $k \le l$, then the notation $j = \overline{k,l}$ stands for the set of all $j \in \mathbb{N}$ such that $k \le j \le l$.

The weighted-type space $H^\infty_\alpha(\mathbb{R}^n)$, $\alpha > 0$, consists of all measurable functions f such that

$$\|f\|_{H^\infty_\alpha} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |x|^\alpha \left|f(x)\right| < \infty.$$

The functional $\|\cdot\|_{H^\infty_\alpha}$ is a norm on the space, where, as usual, we identify functions which are dV almost everywhere equal. Weighted-type spaces on various domains frequently appear in the literature and are quite suitable for investigations (see, e.g., [2–7] and the references therein).

The following integral-type operator

$$L(f)(x)=\frac{1}{x}\int_0^x f(t)\,dt,\quad x\neq 0,\tag{1}$$

is a basic linear operator which has been studied on many spaces of functions. From the main result in [8] (see also [9]) we have that the operator is bounded on $L^p(\mathbb{R}_+)$ space when $p>1$. This result was later improved in [10] by proving the following formula:

$$\|L\|_{L^p_\alpha(\mathbb{R}_+)\to L^p_\alpha(\mathbb{R}_+)}=\frac{p}{p-\alpha-1}\tag{2}$$

for $p>\alpha+1$ in nowadays terminology.

Although there are many linear operators whose norms can be calculated (see, e.g., [1, 4–28] and the related references therein), they are, in fact, quite rare since for many more other operators the norms can be only estimated by some quantities. Some of these operators are integral-type ones, a topic of a considerable recent interest (see, e.g., [11, 13–21, 23–25, 28–34] and the related references therein).

Operator (1) was generalized in [35] by introducing the following n-dimensional integral-type operator:

$$\mathcal{H}(f)(x)=\frac{1}{V(B(0,|x|))}\int_{B(0,|x|)} f(y)\,dV(y),\quad x\in\mathbb{R}^n\setminus\{0\},\tag{3}$$

for nonnegative locally integrable functions on $\mathbb{R}^n$.

In [11] it was shown that the norm of the operator $\mathcal{H}:L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ can be calculated. Namely, the following formula holds:

$$\|\mathcal{H}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}=\frac{p}{p-1},\tag{4}$$

which matches the formula in (2) with $\alpha=0$.

Note that $\mathcal{H}(f)(x\zeta)=\mathcal{H}(f)(x)$ for every $\zeta\in\mathbb{S}$, that is, the function $\mathcal{H}(f)(x)$ is radial for every f, and

$$\mathcal{H}(f)(x)=\frac{1}{V(B(0,|x|))}\int_{\mathbb{B}} f\big(|x|z\big)|x|^n\,dV(z)=\int_{\mathbb{B}} f\big(|x|z\big)\,dV_N(z),$$

where we have used the change of variables $y=|x|z$ and the fact that $V(B(0,|x|))=|x|^nV(\mathbb{B})$.

In [13] the following m-linear extension of operator (3) was introduced:

$$\mathcal{H}^m(f_1,\dots,f_m)(x)=\frac{1}{v_{mn}|x|^{mn}}\int_{|(y_1,\dots,y_m)|<|x|}\prod_{j=1}^m f_j(y_j)\,dV(y_1)\cdots dV(y_m),\tag{5}$$

where $m\in\mathbb{N}$, $x\in\mathbb{R}^n\setminus\{0\}$, $y_1,\dots,y_m\in\mathbb{R}^n$, $y_j=(y_j^1,\dots,y_j^n)$, $j=\overline{1,m}$,

$$\big|(y_1,\dots,y_m)\big|=\Bigg(\sum_{j=1}^m|y_j|^2\Bigg)^{1/2}=\Bigg(\sum_{j=1}^m\sum_{l=1}^n\big(y_j^l\big)^2\Bigg)^{1/2},$$

f_j, $j = \overline{1,m}$, are nonnegative locally integrable functions on $\mathbb{R}^n$, and the norm of the m-linear operator was calculated from the product of weighted Lebesgue spaces $\prod_{j=1}^m L^{p_j}_{\alpha_j p_j/p}(\mathbb{R}^n)$ to $L^p_\alpha(\mathbb{R}^n)$, under some conditions posed on the parameters p, α, p_j, and α_j, $j = \overline{1,m}$.

For the definition of norm of an m-linear operator and some basic examples, see, for example, [36, pp. 51–55].

Note also that $\mathcal{H}^m(f)(x\zeta) = \mathcal{H}^m(f)(x)$ for every $\zeta \in \mathbb{S}$, that is, the function $\mathcal{H}^m(f)(x)$ is radial for every f, and

$$\mathcal{H}^m(f_1,\dots,f_m)(x) = \frac{1}{v_{mn}} \int_{|(z_1,\dots,z_m)|<1} \prod_{j=1}^m f_j\big(|x|z_j\big)\,dV(z_1)\cdots dV(z_m), \tag{6}$$

where we have used the change of variables $y_j = |x|z_j$, $j = \overline{1,m}$.

Our main aim here is to complement the results in [13] by calculating the norm of the operator $\mathcal{H}^m : \prod_{j=1}^m H^\infty_{\alpha_j}(\mathbb{R}^n) \to H^\infty_\alpha(\mathbb{R}^n)$ in the case $\alpha = \sum_{j=1}^m \alpha_j$. We also explain a detail appearing in the proof of the main result in [11].

The following known formula, which transforms integrals in the Descartes coordinates to the polar ones in $\mathbb{R}^n$, will be frequently used in the section that follows:

$$\int_{\mathbb{R}^n} f(x)\,dV(x) = \int_0^\infty \int_{\mathbb{S}} f(\rho\zeta)\,d\sigma(\zeta)\rho^{n-1}\,d\rho, \tag{7}$$

where f is a nonnegative measurable function (see, e.g., [1, pp. 149–150]).

2 Main results

This section presents and proves our main results in this note.

2.1 Boundedness of operator (5) between weighted-type spaces

First we consider operator (3). We consider it separately since its proof explains the first main step in the proof of the general case and does not request complex calculation.

Theorem 1 *Let $\alpha \in (0,n)$. Then the operator $\mathcal{H}$ is bounded on $H^\infty_\alpha(\mathbb{R}^n)$. Moreover, the following formula holds*:

$$\|\mathcal{H}\|_{H^\infty_\alpha(\mathbb{R}^n)\to H^\infty_\alpha(\mathbb{R}^n)} = \frac{n}{n-\alpha}. \tag{8}$$

Proof Let

$$f_\alpha(x) = \begin{cases} 1/|x|^\alpha, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{9}$$

Then it is clear that

$$\|f_\alpha\|_{H^\infty_\alpha(\mathbb{R}^n)} = 1. \tag{10}$$

Further, we have

$$|x|^{\alpha}\left|\int_{\mathbb{B}} f_{\alpha}\left(|x|y\right)dV_N(y)\right| = |x|^{\alpha}\left|\int_{\mathbb{B}} \frac{dV_N(y)}{(|x||y|)^{\alpha}}\right|$$
$$= n\int_{\mathbb{S}} d\sigma_N(\zeta)\int_0^1 \rho^{n-1-\alpha}\,d\rho = \frac{n}{n-\alpha}$$

for each $x \neq 0$, from which it follows that

$$\left\|\mathcal{H}(f_{\alpha})\right\|_{H_{\alpha}^{\infty}(\mathbb{R}^n)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |x|^{\alpha}\left|\int_{\mathbb{B}} f_{\alpha}\left(|x|y\right)dV_N(y)\right| = \frac{n}{n-\alpha}. \tag{11}$$

Equalities (10) and (11) imply

$$\|\mathcal{H}\|_{H_{\alpha}^{\infty}(\mathbb{R}^n)\to H_{\alpha}^{\infty}(\mathbb{R}^n)} \geq \frac{n}{n-\alpha}. \tag{12}$$

On the other hand, we have

$$\begin{aligned}\left\|\mathcal{H}(f)\right\|_{H_{\alpha}^{\infty}(\mathbb{R}^n)} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |x|^{\alpha}\left|\int_{\mathbb{B}} f\left(|x|y\right)dV_N(y)\right| \\ &\leq \|f\|_{H_{\alpha}^{\infty}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n\setminus\{0\}} |x|^{\alpha}\left|\int_{\mathbb{B}} \frac{dV_N(y)}{(|x||y|)^{\alpha}}\right| \\ &= \frac{n}{n-\alpha}\|f\|_{H_{\alpha}^{\infty}(\mathbb{R}^n)}\end{aligned}$$

for every $f \in H_{\alpha}^{\infty}(\mathbb{R}^n)$, from which it follows that

$$\|\mathcal{H}\|_{H_{\alpha}^{\infty}(\mathbb{R}^n)\to H_{\alpha}^{\infty}(\mathbb{R}^n)} \leq \frac{n}{n-\alpha}, \tag{13}$$

and consequently the boundedness of the operator when $\alpha \in (0,n)$. From (12) and (13) the equality in (8) follows. □

Remark 1 Note that (10) holds for each $\alpha > 0$. However, if $\alpha \geq n$, then by using formula (7), we have

$$|x|^{\alpha}\left|\int_{\mathbb{B}} f_{\alpha}\left(|x|y\right)dV_N(y)\right| = n\int_0^1 \rho^{n-1-\alpha}\,d\rho = +\infty$$

for each $x \neq 0$, from which it follows that $\mathcal{H}(f_{\alpha}) \notin H_{\alpha}^{\infty}(\mathbb{R}^n)$. Hence, in this case the operator is not bounded on $H_{\alpha}^{\infty}(\mathbb{R}^n)$. From this and Theorem 1 we obtain the following corollary.

Corollary 1 *Let $\alpha > 0$. Then the operator $\mathcal{H}$ is bounded on $H_{\alpha}^{\infty}(\mathbb{R}^n)$ if and only if $\alpha < n$. Moreover, if $\alpha \in (0,n)$, then formula (8) holds.*

The following theorem deals with the boundedness of the m-linear operator defined in (5).

Theorem 2 *Let $m \in \mathbb{N}$, $\alpha_j > 0$, $j = \overline{1,m}$, and*

$$\alpha = \sum_{j=1}^{m} \alpha_j. \tag{14}$$

Then the operator $\mathcal{H}^m : \prod_{j=1}^m H^\infty_{\alpha_j}(\mathbb{R}^n) \to H^\infty_\alpha(\mathbb{R}^n)$ *is bounded if and only if*

$$\int_{|\vec{y}|<1} \prod_{j=1}^m |y_j|^{-\alpha_j} \prod_{j=1}^m dV(y_j) < \infty, \tag{15}$$

where $\vec{y} = (y_1, y_2, \dots, y_m)$.

Moreover, if condition (15) *is satisfied, then the following formula for the norm of the operator holds*:

$$\|\mathcal{H}^m\|_{\prod_{j=1}^m H^\infty_{\alpha_j}(\mathbb{R}^n) \to H^\infty_\alpha(\mathbb{R}^n)} = \frac{1}{v_{nm}} \int_{|\vec{y}|<1} \prod_{j=1}^m |y_j|^{-\alpha_j} \prod_{j=1}^m dV(y_j), \tag{16}$$

where $\prod_{j=1}^m dV(y_j) := dV(y_1) \cdots dV(y_m)$.

Proof Let the family of functions f_α be defined in (9). Then clearly relation (10) holds. By using the condition in (14), after some simple calculation it follows that

$$\begin{aligned} |x|^\alpha \left| \int_{|\vec{y}|<1} \prod_{j=1}^m f_{\alpha_j}\big(|x|y_j\big) \prod_{j=1}^m dV(y_j) \right| &= |x|^\alpha \left| \int_{|\vec{y}|<1} \prod_{j=1}^m \big(|x||y_j|\big)^{-\alpha_j} \prod_{j=1}^m dV(y_j) \right| \\ &= \int_{|\vec{y}|<1} \prod_{j=1}^m |y_j|^{-\alpha_j} \prod_{j=1}^m dV(y_j) \end{aligned} \tag{17}$$

for each $x \neq 0$.

From (6), (17) and by using the definition of norm on the space $H^\infty_\alpha(\mathbb{R}^n)$, it follows that

$$\begin{aligned} \big\|\mathcal{H}(f_{\alpha_1}, \dots, f_{\alpha_m})\big\|_{H^\infty_\alpha(\mathbb{R}^n)} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{|x|^\alpha}{v_{mn}} \left| \int_{|\vec{y}|<1} \prod_{j=1}^m f_{\alpha_j}\big(|x|y_j\big) \prod_{j=1}^m dV(y_j) \right| \\ &= \frac{1}{v_{mn}} \int_{|\vec{y}|<1} \prod_{j=1}^m |y_j|^{-\alpha_j} \prod_{j=1}^m dV(y_j). \end{aligned} \tag{18}$$

Equality (10) implies

$$\prod_{j=1}^m \|f_{\alpha_j}\|_{H^\infty_{\alpha_j}} = 1,$$

from which along with (18) it follows that

$$\|\mathcal{H}^m\|_{\prod_{j=1}^m H^\infty_{\alpha_j}(\mathbb{R}^n) \to H^\infty_\alpha(\mathbb{R}^n)} \ge \frac{1}{v_{mn}} \int_{|\vec{y}|<1} \prod_{j=1}^m |y_j|^{-\alpha_j} \prod_{j=1}^m dV(y_j). \tag{19}$$

On the other hand, we have

$$\begin{aligned}\left\|\mathcal{H}^m(f_1,\dots,f_m)\right\|_{H^\infty_\alpha(\mathbb{R}^n)} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\frac{|x|^\alpha}{v_{mn}}\left|\int_{|\vec{y}|<1}\prod_{j=1}^m f_j(|x|y_j)\prod_{j=1}^m dV(y_j)\right|\\ &\le \prod_{j=1}^m \|f_j\|_{H^\infty_{\alpha_j}}\sup_{x\in\mathbb{R}^n\setminus\{0\}}\frac{|x|^\alpha}{v_{mn}}\left|\int_{|\vec{y}|<1}\prod_{j=1}^m\big(|x||y_j|\big)^{-\alpha_j}\prod_{j=1}^m dV(y_j)\right|\\ &= \prod_{j=1}^m \|f_j\|_{H^\infty_{\alpha_j}}\frac{1}{v_{mn}}\int_{|\vec{y}|<1}\prod_{j=1}^m|y_j|^{-\alpha_j}\prod_{j=1}^m dV(y_j)\end{aligned}$$

for every $(f_1,\dots,f_m)\in\prod_{j=1}^m H^\infty_{\alpha_j}(\mathbb{R}^n)$, from which by taking the supremum over the unit balls in $H^\infty_{\alpha_j}(\mathbb{R}^n)$, $j=\overline{1,m}$, it follows that

$$\left\|\mathcal{H}^m\right\|_{\prod_{j=1}^m H^\infty_{\alpha_j}(\mathbb{R}^n)\to H^\infty_\alpha(\mathbb{R}^n)} \le \frac{1}{v_{mn}}\int_{|\vec{y}|<1}\prod_{j=1}^m|y_j|^{-\alpha_j}\prod_{j=1}^m dV(y_j), \tag{20}$$

and consequently the boundedness of the operator. From (19) and (20) the equality in (16) follows. □

Let

$$I_m := \frac{1}{v_{mn}}\int_{|\vec{y}|<1}\prod_{j=1}^m|y_j|^{-\alpha_j}\prod_{j=1}^m dV(y_j). \tag{21}$$

Employing the polar coordinates $y_j=\rho_j\zeta_j$, $j=\overline{1,m}$, and Fubini's theorem, we obtain

$$\begin{aligned} I_m &= \frac{1}{v_{mn}}\underbrace{\int_{\mathbb{S}}\cdots\int_{\mathbb{S}}}_{m\text{ times}}\int_{\sum_{j=1}^m\rho_j^2<1,\rho_j>0,j=\overline{1,m}}\prod_{j=1}^m\rho_j^{-\alpha_j}\prod_{j=1}^m\rho_j^{n-1}\,d\rho_1\cdots d\rho_m\,d\sigma(\zeta_1)\cdots d\sigma(\zeta_m)\\ &= \frac{\sigma_n^m}{v_{mn}}\int_{\sum_{j=1}^m\rho_j^2<1,\rho_j>0,j=\overline{1,m}}\prod_{j=1}^m\rho_j^{n-1-\alpha_j}\,d\rho_1\cdots d\rho_m. \end{aligned}\tag{22}$$

By using the m-dimensional spherical coordinates

$$\begin{aligned}\rho_1 &= r\cos\varphi_1,\\ \rho_2 &= r\sin\varphi_1\cos\varphi_2,\\ \rho_3 &= r\sin\varphi_1\sin\varphi_2\cos\varphi_3,\\ &\vdots\\ \rho_{m-1} &= r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{m-2}\cos\varphi_{m-1},\\ \rho_m &= r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{m-2}\sin\varphi_{m-1},\end{aligned}$$

where $r \geq 0$ is the radial coordinate and $\varphi_j, j = \overline{1, m-1}$, are angular coordinates, $\varphi_j \in [0, \pi]$, $j = \overline{1, m-2}$, $\varphi_{m-1} \in [0, 2\pi)$, and the known fact that the associated Jacobian is

$$|J_m| = r^{m-1} \sin^{m-2} \varphi_1 \sin^{m-3} \varphi_2 \cdots \sin \varphi_{m-2},$$

in (22), we have

$$\begin{aligned} I_m &= \frac{\sigma_n^m}{v_{mn}} \int_0^1 r^{mn-1-\sum_{j=1}^m \alpha_j} \, dr \\ &\quad \times \int_0^{\pi/2} \cdots \int_0^{\pi/2} \prod_{j=1}^{m-1} (\sin \varphi_j)^{n(m-j)-1-\sum_{i=j+1}^m \alpha_i} (\cos \varphi_j)^{n-1-\alpha_j} \, d\varphi_1 \cdots d\varphi_{m-1} \\ &= \frac{\sigma_n^m}{v_{mn}(mn-\alpha)} \prod_{j=1}^{m-1} \int_0^1 t_j^{n(m-j)-1-\sum_{i=j+1}^m \alpha_i} \left(1-t_j^2\right)^{\frac{n-2-\alpha_j}{2}} \, dt_j \\ &= \frac{\sigma_n^m 2^{1-m}}{v_{mn}(mn-\alpha)} \prod_{j=1}^{m-1} \int_0^1 s_j^{\frac{n(m-j)-\sum_{i=j+1}^m \alpha_i}{2}-1} (1-s_j)^{\frac{n-\alpha_j}{2}-1} \, ds_j \\ &= \frac{\sigma_n^m 2^{1-m}}{v_{mn}(mn-\alpha)} \prod_{j=1}^{m-1} B\left(\frac{n(m-j)-\sum_{i=j+1}^m \alpha_i}{2}, \frac{n-\alpha_j}{2}\right), \end{aligned} \tag{23}$$

where we have also used the fact that $\varphi_j \in (0, \pi/2)$, $j = \overline{1, m-1}$, which is a consequence of integrating over a set in the first orthant, the changes of variables $t_j = \sin \varphi_j$, $j = \overline{1, m-1}$, and $s_j = t_j^2$, $j = \overline{1, m-1}$, as well as the definition of the beta function (see, e.g., [36, p. 437]).

By using the following well-known relation between Euler's beta and gamma functions:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

(see, for example, [36]), after some simple calculations, we see that the following relations hold:

$$\begin{aligned} \prod_{j=1}^{m-1} B\left(\frac{n(m-j)-\sum_{i=j+1}^m \alpha_i}{2}, \frac{n-\alpha_j}{2}\right) &= \prod_{j=1}^{m-1} \frac{\Gamma(\frac{n(m-j)-\sum_{i=j+1}^m \alpha_i}{2})\Gamma(\frac{n-\alpha_j}{2})}{\Gamma(\frac{n(m-(j-1))-\sum_{i=j}^m \alpha_i}{2})} \\ &= \frac{\prod_{j=1}^m \Gamma(\frac{n-\alpha_j}{2})}{\Gamma(\frac{nm-\alpha}{2})}. \end{aligned} \tag{24}$$

Combining relations (16), (23), and (24), it follows that the following corollary holds.

Corollary 2 *Let $m \in \mathbb{N}$, $\alpha_j > 0$, $j = \overline{1, m}$, and $\alpha = \sum_{j=1}^m \alpha_j$. Then the operator $\mathcal{H}^m : \prod_{j=1}^m H_{\alpha_j}^\infty(\mathbb{R}^n) \to H_\alpha^\infty(\mathbb{R}^n)$ is bounded if and only if (15) holds. Moreover, if (15) holds, then the following formulas hold*:

$$\begin{aligned} \left\| \mathcal{H}^m \right\|_{\prod_{j=1}^m H_{\alpha_j}^\infty(\mathbb{R}^n) \to H_\alpha^\infty(\mathbb{R}^n)} &= \frac{\sigma_n^m 2^{1-m}}{v_{mn}(mn-\alpha)} \frac{\prod_{j=1}^m \Gamma(\frac{n-\alpha_j}{2})}{\Gamma(\frac{nm-\alpha}{2})} \\ &= \frac{\sigma_n^m 2^{1-m}}{v_{mn}(mn-\alpha)} \prod_{j=1}^{m-1} B\left(\frac{n(m-j)-\sum_{i=j+1}^m \alpha_i}{2}, \frac{n-\alpha_j}{2}\right). \end{aligned}$$

Remark 2 From the above consideration and Corollary 2 we see that when $\alpha_j > 0$, $j = \overline{1,m}$, condition (15) holds if and only if $\alpha_j \in (0,n)$, $j = \overline{1,m}$.

2.2 A comment on the proof of the main result in [11]

As we have already mentioned, in [11] the norm of the operator $\mathcal{H} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ was calculated in a nice way by proving formula (4). In the proof of the result the authors applied the convolution inequality $\|g * L\|_{L^p} \le \|g\|_{L^p}\|L\|_{L^1}$ in the group $(\mathbb{R}_+, \frac{dt}{t})$. However, the operator appearing there is not a convolution in the standard sense. On the other hand, the used inequality really holds. Hence, it needs some explanations related to the inequality, which might be useful in other similar situations. Namely, the following theorem holds and its proof is analogous to the one for the convolution operator. For some information on locally compact groups and related topics, see, e.g., [15, 37].

Theorem 3 *Let $p \in [1,\infty]$, $f \in L^p(G)$, $g \in L^1(G)$, where G is a locally compact group G, and μ be a right-invariant Haar measure on G. Then*

$$(f \odot g)(x) := \int_G f(xy)g(y)\, d\mu(y)$$

exists μ a.e. and

$$\|f \odot g\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^1}. \tag{25}$$

Proof If $p = 1$, then the result follows from Fubini's theorem and the right-invariance of measure μ. If $p = \infty$, then we have

$$\begin{aligned}
\|f \odot g\|_{L^\infty} &= \operatorname*{ess\,sup}_{x\in G}\left|\int_G f(xy)g(y)\, d\mu(y)\right| \le \operatorname*{ess\,sup}_{x\in G}\int_G \left|f(xy)\right|\left|g(y)\right| d\mu(y) \\
&\le \int_G \|f\|_{L^\infty}\left|g(y)\right| d\mu(y) = \|f\|_{L^\infty}\|g\|_{L^1}.
\end{aligned}$$

If $p \in (1,\infty)$ and if we use the notation $p' = p/(p-1)$, then by using the Hölder inequality, Fubini's theorem, a change of variables, and the right-invariance of measure μ, we have

$$\begin{aligned}
\|f \odot g\|_{L^p}^p &= \int_G \left|\int_G f(xy)g(y)\, d\mu(y)\right|^p d\mu(x) \le \int_G \left(\int_G \left|f(xy)\right|\left|g(y)\right| d\mu(y)\right)^p d\mu(x) \\
&= \int_G \left(\int_G \left|f(xy)\right|\left|g(y)\right|^{1/p}\left|g(y)\right|^{1/p'} d\mu(y)\right)^p d\mu(x) \\
&\le \int_G \int_G \left|f(xy)\right|^p \left|g(y)\right| d\mu(y)\left(\int_G \left|g(y)\right| d\mu(y)\right)^{p/p'} d\mu(x) \\
&\le \|g\|_{L^1}^{p/p'} \int_G \left|g(y)\right| \int_G \left|f(xy)\right|^p d\mu(x)\, d\mu(y) \\
&= \|g\|_{L^1}^{p-1} \int_G \left|g(y)\right| \int_G \left|f(z)\right|^p d\mu(z)\, d\mu(y) \\
&= \|f\|_{L^p}^p \|g\|_{L^1}^p,
\end{aligned}$$

from which it follows that (25) holds, and since $f \odot g \in L^p(G)$ it follows that $(f \odot g)(x)$ exists μ a.e. □

Remark 3 Inequality (25) can be also obtained by using Minkowski's inequality. Indeed, by using the inequality and the right-invariance of measure μ, we have

$$\begin{aligned}
\|f \odot g\|_{L^p} &= \left(\int_G \left| \int_G f(xy)g(y)\,d\mu(y) \right|^p d\mu(x) \right)^{1/p} \\
&\le \left(\int_G \left(\int_G \left|f(xy)\right| \left|g(y)\right| d\mu(y) \right)^p d\mu(x) \right)^{1/p} \\
&\le \int_G \left(\int_G \left|f(xy)\right|^p d\mu(x) \right)^{1/p} \left|g(y)\right| d\mu(y) \\
&= \int_G \left(\int_G \left|f(z)\right|^p d\mu(z) \right)^{1/p} \left|g(y)\right| d\mu(y) = \|f\|_{L^p} \|g\|_{L^1}.
\end{aligned}$$

Remark 4 By using inequality (25) instead of the corresponding one for the convolution operator, the proof of the main result in [11] is clear and complete.

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