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Some generalized Hermite–Hadamard–Fejér inequality for convex functions

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Abstract

In this paper, we have established some generalized inequalities of Hermite–Hadamard–Fejér type for generalized integrals. The results obtained are applied for fractional integrals of various type and therefore contain some previous results reported in the literature.

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1 Preliminaries

Consider a certain function $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$. This function is said to be convex if

$$f\big(tx + (1 - t)y\big) \leq tf(x) + (1 - t)f(y)$$

is satisfied for any $x, y \in [a,b]$, being $t \in [0,1]$. Likewise, f is called concave if $-f$ is convex.

One of the most important inequalities that have attracted many experts in the last few decades is the famous Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{2} \tag{1.1}$$

that holds for any function f convex on the interval $[a,b]$. This inequality was published by Hermite in 1883 and, independently, by Hadamard in 1893. It gives an estimation of the mean value of a convex function, and it is important to note that it also provides a refinement to the Jensen inequality. The interested reader is referred to [3, 4, 10, 13–16, 21, 22, 27–30] and the references therein for more information and other extensions of the Hermite–Hadamard inequality.

In this paper, following that path, we present a new integral operator (that we could call "integral of one function with respect to another" for reasons that we will see later) and apply it to the study of generalized inequalities, more general than (1.1), this time the so-called Hermite–Hadamard–Fejér inequality.

We assume that the reader is familiar with the classic definition of the Riemann integral, so we do not present it.

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As we know, basically there are two types of fractional differential operators, Riemann–Liouville and Caputo, respectively:

$$^{RL}D^\alpha f(t) = \frac{d}{dt}\{J^{1-\alpha}_{F,a}(f)(t)\},$$

$$^{C}D^\alpha f(t) = J^{1-\alpha}_{F,a}\left(\frac{df}{dt}\right)(t),$$

with certain kernel F. In particular, we will deal with real integral operators defined on $\mathbb{R}$ of the RL-type.

One of the first operators that can be called fractional is the Riemann–Liouville fractional derivative of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geq 0$ defined by [6].

Definition 1.1 Let $f \in L^1((a,b);\mathbb{R})$, $(a,b) \in \mathbb{R}^2$, $a < b$. The right- and left-hand side Riemann–Liouville fractional integrals of order $\alpha > 0$ are defined by

$$^{RL}J^\alpha_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\,ds, \quad t > a,$$

$$^{RL}J^\alpha_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)\,ds, \quad t < b,$$

and their corresponding differential operators are given by

$$D^\alpha_{a^+}f(t) = \frac{d}{dt}\left(^{RL}J^{1-\alpha}_{a^+}f(t)\right) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{f(t)}{(t-s)^\alpha}\,ds,$$

$$D^\alpha_{b^-}f(t) = -\frac{d}{dt}\left(^{RL}J^{1-\alpha}_{b^-}f(t)\right) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b \frac{f(t)}{(s-t)^\alpha}\,ds.$$

There have been many fractional integrals defined and investigated in the recent years, for example, so-called Hadamard, Katugampola, k-fractional integrals, etc. (see e.g. [1, 2, 11, 17, 23, 24, 26]).

Now, we give the definition of a general fractional integral. Throughout the work we consider that the integral operator kernel T defined below is an absolutely continuous function (see [8]).

Definition 1.2 Let I be an interval $I \subseteq \mathbb{R}$, $a,b,t \in I$, and $\alpha \in \mathbb{R}$. The integral operators J^α_T, right and left, are defined for every locally integrable function f on I as follows:

$$J^\alpha_{T,a^+}(f)(t) = \int_a^t \frac{f(s)}{T(t-s,\alpha)}\,ds, \quad t > a,$$

$$J^\alpha_{T,b^-}(f)(t) = \int_t^b \frac{f(s)}{T(s-t,\alpha)}\,ds, \quad b > t.$$

It is easy to see that the case of the J^α_T operators defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. However, we will see that it goes much further by containing the cases presented before. It is clear that from our definition new extensions and generalizations of known integral operators can be defined and many integral operators can be considered

particular cases of the integral operator defined above, we let the reader complete this analysis. For simplicity of notation, let us denote (see [14])

$$
{}_{(a,b)}J^{\alpha}_{T,a+}(f)(b) = \int_a^b \frac{f(s)}{T(\frac{b-s}{b-a},\alpha)}\,ds,
$$

$$
{}_{(a,b)}J^{\alpha}_{T,b-}(f)(a) = \int_a^b \frac{f(s)}{T(\frac{s-a}{b-a},\alpha)}\,ds.
$$

We also use the "central" integral operator defined by (see [31])

$$
J^{\alpha}_{T,a}(f)(b) = \int_a^b \frac{f(t)}{T(t,\alpha)}\,dt, \quad b > a.
$$

We present the definition of generalized derivative that was defined in [19].

Definition 1.3 For a function $f : [0,\infty) \to \mathbb{R}$, the N^{α}_T-derivative of f is defined by

$$
N^{\alpha}_T f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon T(t,\alpha)) - f(t)}{\varepsilon}
$$

for all $t > 0$, $\alpha \in (0,1)$, $T(t,\alpha)$ being some function.

We can also define the lateral derivative operators (right and left) in the case of our generalized derivative. For this, it is sufficient to consider them from the corresponding integral operator. To do this, just make use of the fact that if f is differentiable, then $N^{\alpha}_T f(t) = T(t,\alpha)f'(t)$, where $f'(t)$ is the ordinary derivative. For the right derivative we have $(N^{\alpha,b}_{T,a+}f)(t) = T(b-t,\alpha)f'(t)$, similar to the left one. The interested reader can consult [19] to study the properties of operator N^{α}_T and additional details.

Similar calculations as in [7] show that the following properties hold.

Theorem 1.4 *Let f and g be N^{α}_T-differentiable at $t > 0$ and $\alpha \in (0,1)$. Then*
   a) $N^{\alpha}_T(af + bg)(t) = aN^{\alpha}_T(f)(t) + bN^{\alpha}_T(g)(t)$;
   b) $N^{\alpha}_T(t^p) = T(t,\alpha)pt^{p-1}, p \in \mathbb{R}$;
   c) $N^{\alpha}_T(\lambda) = 0, \lambda \in \mathbb{R}$;
   d) $N^{\alpha}_T(fg)(t) = fN^{\alpha}_T(g)(t) + gN^{\alpha}_T(f)(t)$;
   e) $N^{\alpha}_T(\frac{f}{g})(t) = \frac{gN^{\alpha}_T(f)(t) - fN^{\alpha}_T(g)(t)}{g^2(t)}$;
   f) *If, in addition, f is differentiable, then $N^{\alpha}_T(f)(t) = T(t,\alpha)f'(t)$.*

Remark 1.5 The relations a), c), d), and e) are similar to the classical results mathematical analysis, and these relationships are not established (or do not occur) for fractional derivatives of global character (see [12] and [20] and the references cited there).

The following statement is analogous to the one known from the ordinary calculus (for a similar result, see [18]).

Theorem 1.6 *Let f be an N^{α}_T-differentiable function on (t_0,∞). Then, for all $t > t_0$, we have*
   a) *If f is differentiable, then $J^{\alpha}_{T,t_0}(N^{\alpha}_T f(t)) = f(t) - f(t_0)$.*
   b) $N^{\alpha}_T(J^{\alpha}_{T,t_0}f(t)) = f(t)$.

Proof a) By definition, one can calculate

$$J_{T,t_0}^{\alpha}\left(N_T^{\alpha}f(t)\right) = \int_{t_0}^{t} \frac{N_T^{\alpha}f(s)}{T(s,\alpha)}\,ds = \int_{t_0}^{t} \frac{T(s,\alpha)f'(s)}{T(s,\alpha)}\,ds = f(t) - f(t_0).$$

b) Analogously, we have

$$N_T^{\alpha}\left(J_{T,t_0}^{\alpha}f(t)\right) = T(t,\alpha)\frac{d}{dt}\left[\int_{t_0}^{t} \frac{f(s)}{T(s,\alpha)}\,ds\right] = f(t). \qquad \square$$

It is clear that many "classical" properties of integration theory can be proved without much difficulty, for example, the well-known mean value theorems for integral calculus or integration by parts.

Theorem 1.7 *Let f and g be differentiable functions on (a,b), $a < b$. Then*

$$J_{T,a}^{\alpha}\left\{\left[N_T^{\alpha}f(t)\right]g(t)\right\}(b) = \left[f(t)g(t)\right]_a^b - J_{T,a}^{\alpha}\left\{f(t)\left[N_T^{\alpha}g(t)\right]\right\}(b).$$

In [5], the so-called Hermite–Hadamard–Fejér inequality was established; it is the weighted generalization of the Hermite–Hadamard inequality (1.1).

Theorem 1.8 *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $g : [a,b] \to \mathbb{R}_0^+ = [0,\infty)$ be an integrable function symmetric about $\frac{a+b}{2}$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{1.2}$$

In this paper, we present a version of the Hermite–Hadamard–Fejér inequality for convex functions via generalized integral of Definition 1.2, while several remarks presented show the advantage of our framework.

2 Main results

Lemma 2.1 *Let $g : [a,b] \to \mathbb{R}$, $a < b$, be an integrable function symmetric with respect to $\frac{a+b}{2}$. Then we have*

$$J_{T,a+}^{\alpha}g(b) = J_{T,b-}^{\alpha}g(a) = \frac{J_{T,a+}^{\alpha}g(b) + J_{T,b-}^{\alpha}g(a)}{2}.$$

Proof Following [9] and taking into account the symmetry of g with respect to $\frac{a+b}{2}$, we have

$$J_{T,a+}^{\alpha}g(b) = \int_a^b \frac{g(t)}{T(b-t,\alpha)}\,dt = \int_a^b \frac{g(a+b-x)}{T(x-a,\alpha)}\,dx$$

$$= \int_a^b \frac{g(x)}{T(x-a,\alpha)}\,dx = J_{T,b-}^{\alpha}g(a).$$

This immediately implies the result needed. $\qquad \square$

Remark 2.2 In case of $T(t,\alpha) = t^{1-\alpha}\Gamma(\alpha)$, $\alpha > 0$, Lemma 2.1 becomes [9, Lemma 2.1].

Theorem 2.3 *Let $f : [a,b] \to \mathbb{R}$, $a < b$, be a convex function and $g : [a,b] \to \mathbb{R}_0^+$ be an integrable function symmetric about $\frac{a+b}{2}$. Then*

$$f\left(\frac{a+b}{2}\right)\left[{}_{(a,b)}J^\alpha_{T,a+}(g)(b) + {}_{(a,b)}J^\alpha_{T,b-}(g)(a)\right]$$

$$\leq {}_{(a,b)}J^\alpha_{T,a+}(fg)(b) + {}_{(a,b)}J^\alpha_{T,b-}(fg)(a)$$

$$\leq \frac{f(a)+f(b)}{2}\left[{}_{(a,b)}J^\alpha_{T,a+}(g)(b) + {}_{(a,b)}J^\alpha_{T,b-}(g)(a)\right]. \tag{2.1}$$

Proof From the convexity of f, we have

$$2f\left(\frac{a+b}{2}\right) = 2f\left(\frac{ta + (1-t)b + tb + (1-t)a}{2}\right)$$

$$\leq f\left(ta + (1-t)b\right) + f\left(tb + (1-t)a\right)$$

for any $t \in [0,1]$. Multiplying the first and the last members of this inequality by $\frac{g(tb+(1-t)a)}{T(t,\alpha)}$ and integrating on $[0,1]$, we obtain

$$2f\left(\frac{a+b}{2}\right)\int_0^1 \frac{g(tb+(1-t)a)}{T(t,\alpha)}\,dt$$

$$\leq \int_0^1 \frac{g(tb+(1-t)a)[f(ta+(1-t)b) + f(tb+(1-t)a)]}{T(t,\alpha)}\,dt$$

$$= \int_0^1 \frac{g(tb+(1-t)a)f(ta+(1-t)b)}{T(t,\alpha)}\,dt$$

$$+ \int_0^1 \frac{g(tb+(1-t)a)f(tb+(1-t)a)}{T(t,\alpha)}\,dt.$$

Making the change of variables $x = tb + (1-t)a$), we obtain

$$\frac{2f(\frac{a+b}{2})}{b-a}\int_a^b \frac{g(x)}{T(\frac{x-a}{b-a},\alpha)}\,dx$$

$$\leq \frac{1}{b-a}\int_a^b \frac{g(x)f(a+b-x)}{T(\frac{x-a}{b-a},\alpha)}\,dx + \frac{1}{b-a}\int_a^b \frac{g(x)f(x)}{T(\frac{x-a}{b-a},\alpha)}\,dx$$

$$= \frac{1}{b-a}\int_a^b \frac{g(a+b-x)f(x)}{T(\frac{b-x}{b-a},\alpha)}\,dx + \frac{1}{b-a}\int_a^b \frac{g(x)f(x)}{T(\frac{x-a}{b-a},\alpha)}\,dx$$

$$= \frac{1}{b-a}\int_a^b \frac{g(x)f(x)}{T(\frac{b-x}{b-a},\alpha)}\,dx + \frac{1}{b-a}\int_a^b \frac{g(x)f(x)}{T(\frac{x-a}{b-a},\alpha)}\,dx.$$

This means

$$2f\left(\frac{a+b}{2}\right){}_{(a,b)}J^\alpha_{T,a+}(g)(b) \leq {}_{(a,b)}J^\alpha_{T,a+}(fg)(b) + {}_{(a,b)}J^\alpha_{T,b-}(fg)(a).$$

Using Lemma 2.1, we obtain the first inequality in (2.1).

To prove the second part of (2.1), we use the convexity of f. We have

$$f\left(ta + (1-t)b\right) + f\left(tb + (1-t)a\right) \leq f(a) + f(b)$$

for any $t \in [0, 1]$. Multiplying the previous inequality by $\frac{g(tb+(1-t)a)}{T(t,\alpha)}$ and integrating on $[0, 1]$, we obtain, after making the change of variables and considering the symmetry of g, the following inequality:

$$_{(a,b)}J^\alpha_{T,a+}(fg)(b) + {}_{(a,b)}J^\alpha_{T,b-}(fg)(a) \leq \big(f(a) + f(b)\big)_{(a,b)}J^\alpha_{T,a+}(g)(b).$$

Lemma 2.1 implies the second part of (2.1). This completes the proof of the theorem. $\square$

Remark 2.4 If we consider $T(t,\alpha) \equiv 1$, inequality (2.1) becomes the classic Hermite–Hadamard–Fejér inequality (1.2). In case of $T(t,\alpha) = t^{1-\alpha}\Gamma(\alpha)$, the previous result becomes [9, Theorem 2.2], and if $T(t,\alpha) = t^{1-\alpha}\Gamma(\alpha)$ and $g(x) \equiv 1$, then the above result becomes [25, Theorem 2].

Lemma 2.5 *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b), $a < b$, such that $N^\alpha_T f \in L(a, b)$. If in addition $g : [a, b] \to \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, then the following equality is valid*:

$$J^\alpha_{T,a}\left\{\left[N^\alpha_T f(t)\right]\int_a^t \frac{g(s)}{T(b-s,\alpha)}\,ds\right\}(b) + J^\alpha_{T,a}\left\{-\left[N^\alpha_T f(t)\right]\int_t^b \frac{g(s)}{T(s-a,\alpha)}\,ds\right\}(b)$$

$$= \frac{f(a)+f(b)}{2}\left[J^\alpha_{T,a+}(g)(b) + J^\alpha_{T,b-}(g)(a)\right] - \left[J^\alpha_{T,a+}(fg)(b) + J^\alpha_{T,b-}(fg)(a)\right]. \tag{2.2}$$

Proof Using Theorem 1.7, we calculate as follows:

$$J_1 = J^\alpha_{T,a}\left\{\left[N^\alpha_T f(t)\right]\int_a^t \frac{g(s)}{T(b-s,\alpha)}\,ds\right\}(b)$$

$$= \left[f(t)\int_a^t \frac{g(s)}{T(b-s,\alpha)}\,ds\right]_a^b - J^\alpha_{T,a}\left\{\frac{f(t)g(t)T(t,\alpha)}{T(b-t,\alpha)}\right\}(b)$$

$$= f(b)J^\alpha_{T,a+}(g)(b) - J^\alpha_{T,a+}(fg)(b)$$

and

$$J_2 = J^\alpha_{T,a}\left\{-\left[N^\alpha_T f(t)\right]\int_t^b \frac{g(s)}{T(s-a,\alpha)}\,ds\right\}(b)$$

$$= \left[-f(t)\int_t^b \frac{g(s)}{T(s-a,\alpha)}\,ds\right]_a^b - J^\alpha_{T,a}\left\{\frac{f(t)g(t)T(t,\alpha)}{T(t-a,\alpha)}\right\}(b)$$

$$= f(a)J^\alpha_{T,b-}(g)(a) - J^\alpha_{T,b-}(fg)(a).$$

According to Lemma 2.1, we have

$$J_1 = f(b)\left[\frac{J^\alpha_{T,a+}(g)(b) + J^\alpha_{T,b-}(g)(a)}{2}\right] - J^\alpha_{T,a+}(fg)(b),$$

$$J_2 = f(a)\left[\frac{J^\alpha_{T,a+}(g)(b) + J^\alpha_{T,b-}(g)(a)}{2}\right] - J^\alpha_{T,b-}(fg)(a),$$

from where the desired equality is obtained. $\square$

Remark 2.6 In case of $T(t,\alpha) = t^{1-\alpha}\Gamma(\alpha)$, Lemma 2.5 implies [9, Theorem 2.4], and if $T(t,\alpha) = t^{1-\alpha}\Gamma(\alpha)$ and $g(x) \equiv 1$, then Lemma 2.5 implies [25, Lemma 2].

Theorem 2.7 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b), $a < b$, such that $f' \in L(a,b)$, $|T^{-1}| \in L(0, b-a)$. If we also assume that $|f'|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} \left[J^{\alpha}_{T,a+}(g)(b) + J^{\alpha}_{T,b-}(g)(a) \right] - \left[J^{\alpha}_{T,a+}(fg)(b) + J^{\alpha}_{T,b-}(fg)(a) \right] \right|$$

$$\leq \|g\|_{\infty} \left(|f'(a)| + |f'(b)| \right) \left[\int_{\frac{b-a}{2}}^{b-a} \mathcal{T}(s)\, ds - \int_{0}^{\frac{b-a}{2}} \mathcal{T}(s)\, ds \right],$$

where $\mathcal{T}(s) = \int_{0}^{s} \frac{dt}{|T(t,\alpha)|}$.

Proof Let us see some preliminary details. First, note that (2.2) is equivalent to

$$\int_{a}^{b} \left\{ \int_{a}^{t} \frac{g(s)}{T(b-s,\alpha)}\, ds - \int_{t}^{b} \frac{g(s)}{T(s-a,\alpha)}\, ds \right\} f'(t)\, dt$$

$$= \frac{f(a) + f(b)}{2} \left[J^{\alpha}_{T,a+}(g)(b) + J^{\alpha}_{T,b-}(g)(a) \right] - \left[J^{\alpha}_{T,a+}(fg)(b) + J^{\alpha}_{T,b-}(fg)(a) \right]. \tag{2.3}$$

Since $|f'|$ is convex, we have

$$|f'(t)| = \left| f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b \right) \right| \leq \frac{1}{b-a} \left((b-t)|f'(a)| + (t-a)|f'(b)| \right) \tag{2.4}$$

for any $t \in [a,b]$. Using (2.3) and the symmetry of g, we get

$$\left| \frac{f(a) + f(b)}{2} \left[J^{\alpha}_{T,a+}(g)(b) + J^{\alpha}_{T,b-}(g)(a) \right] - \left[J^{\alpha}_{T,a+}(fg)(b) + J^{\alpha}_{T,b-}(fg)(a) \right] \right|$$

$$\leq \int_{a}^{b} \left| \int_{a}^{t} \frac{g(s)}{T(b-s,\alpha)}\, ds - \int_{t}^{b} \frac{g(s)}{T(s-a,\alpha)}\, ds \right| |f'(t)|\, dt$$

$$= \int_{a}^{b} \left| \int_{a}^{t} \frac{g(s)}{T(b-s,\alpha)}\, ds - \int_{a}^{a+b-t} \frac{g(s)}{T(b-s,\alpha)}\, ds \right| |f'(t)|\, dt$$

$$= \int_{a}^{b} |f'(t)| \left| \int_{t}^{a+b-t} \frac{g(s)}{T(b-s,\alpha)}\, ds \right| dt$$

$$\leq \|g\|_{\infty} \int_{a}^{\frac{a+b}{2}} |f'(t)| \left\{ \int_{t}^{a+b-t} \frac{ds}{|T(b-s,\alpha)|} \right\} dt$$

$$\quad + \|g\|_{\infty} \int_{\frac{a+b}{2}}^{b} |f'(t)| \left\{ \int_{a+b-t}^{t} \frac{ds}{|T(b-s,\alpha)|} \right\} dt$$

$$= \|g\|_{\infty} \int_{a}^{\frac{a+b}{2}} |f'(t)| \left\{ \int_{t-a}^{b-t} \frac{ds}{|T(s,\alpha)|} \right\} dt$$

$$\quad + \|g\|_{\infty} \int_{\frac{a+b}{2}}^{b} |f'(t)| \left\{ \int_{b-t}^{t-a} \frac{ds}{|T(s,\alpha)|} \right\} dt,$$

from where (2.4) implies

$$
\left| \frac{f(a) + f(b)}{2} \left[J_{T,a+}^{\alpha}(g)(b) + J_{T,b-}^{\alpha}(g)(a) \right] - \left[J_{T,a+}^{\alpha}(fg)(b) + J_{T,b-}^{\alpha}(fg)(a) \right] \right|
$$

$$
\leq \frac{\|g\|_{\infty}}{b-a} \int_{a}^{\frac{a+b}{2}} \left((b-t)|f'(a)| + (t-a)|f'(b)| \right) \left(\mathcal{T}(b-t) - \mathcal{T}(t-a) \right) dt
$$

$$
+ \frac{\|g\|_{\infty}}{b-a} \int_{\frac{a+b}{2}}^{b} \left((b-t)|f'(a)| + (t-a)|f'(b)| \right) \left(\mathcal{T}(t-a) - \mathcal{T}(b-t) \right) dt
$$

$$
= \frac{\|g\|_{\infty}}{b-a} |f'(a)| \left[\int_{a}^{\frac{a+b}{2}} (b-t)\mathcal{T}(b-t)\, dt + \int_{\frac{a+b}{2}}^{b} (b-t)\mathcal{T}(t-a)\, dt \right]
$$

$$
- \frac{\|g\|_{\infty}}{b-a} |f'(a)| \left[\int_{a}^{\frac{a+b}{2}} (b-t)\mathcal{T}(t-a)\, dt + \int_{\frac{a+b}{2}}^{b} (b-t)\mathcal{T}(b-t)\, dt \right]
$$

$$
+ \frac{\|g\|_{\infty}}{b-a} |f'(b)| \left[\int_{a}^{\frac{a+b}{2}} (t-a)\mathcal{T}(b-t)\, dt + \int_{\frac{a+b}{2}}^{b} (t-a)\mathcal{T}(t-a)\, dt \right]
$$

$$
- \frac{\|g\|_{\infty}}{b-a} |f'(b)| \left[\int_{a}^{\frac{a+b}{2}} (t-a)\mathcal{T}(t-a)\, dt + \int_{\frac{a+b}{2}}^{b} (t-a)\mathcal{T}(b-t)\, dt \right]
$$

$$
= \frac{\|g\|_{\infty}}{b-a} |f'(a)| \left[\int_{\frac{b-a}{2}}^{b-a} s\mathcal{T}(s)\, ds + \int_{\frac{b-a}{2}}^{b-a} (b-a-s)\mathcal{T}(s)\, ds \right]
$$

$$
- \frac{\|g\|_{\infty}}{b-a} |f'(a)| \left[\int_{0}^{\frac{b-a}{2}} (b-a-s)\mathcal{T}(s)\, ds + \int_{0}^{\frac{b-a}{2}} s\mathcal{T}(s)\, ds \right]
$$

$$
+ \frac{\|g\|_{\infty}}{b-a} |f'(b)| \left[\int_{\frac{b-a}{2}}^{b-a} (b-a-s)\mathcal{T}(s)\, ds + \int_{\frac{b-a}{2}}^{b-a} s\mathcal{T}(s)\, ds \right]
$$

$$
- \frac{\|g\|_{\infty}}{b-a} |f'(b)| \left[\int_{0}^{\frac{b-a}{2}} s\mathcal{T}(s)\, ds + \int_{0}^{\frac{b-a}{2}} (b-a-s)\mathcal{T}(s)\, ds \right].
$$

Adding up the members on the right-hand side of the last inequality immediately gives the desired inequality. □

Remark 2.8 In case of $T(t,\alpha) = t^{1-\alpha}\Gamma(\alpha)$, $\alpha > 0$, Theorem 2.7 implies [9, Theorem 2.4].

In a similar way the following result can be proved.

Theorem 2.9 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b), $a < b$, such that $|f'|^q \in L(a,b)$, $q > 1$, $|T^{-1}| \in L(0, b-a)$. If we also assume that $|f'|^q$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, then*

$$
\left| \frac{f(a) + f(b)}{2} \left[J_{T,a+}^{\alpha}(g)(b) + J_{T,b-}^{\alpha}(g)(a) \right] - \left[J_{T,a+}^{\alpha}(fg)(b) + J_{T,b-}^{\alpha}(fg)(a) \right] \right|
$$

$$
\leq 2^{1-1/q} \|g\|_{\infty} \left(|f'(a)|^q + |f'(b)|^q \right)^{1/q} \left[\int_{\frac{b-a}{2}}^{b-a} \mathcal{T}(s)\, ds - \int_{0}^{\frac{b-a}{2}} \mathcal{T}(s)\, ds \right],
$$

where $\mathcal{T}(s) = \int_{0}^{s} \frac{dt}{|T(t,\alpha)|}$.

Proof Using (2.3) and Hölder's inequality, we get

$$\left| \frac{f(a)+f(b)}{2} \left[J_{T,a^+}^\alpha (g)(b) + J_{T,b^-}^\alpha (g)(a) \right] - \left[J_{T,a^+}^\alpha (fg)(b) + J_{T,b^-}^\alpha (fg)(a) \right] \right|$$

$$\leq \left[\int_a^b \left| \int_t^{a+b-t} \frac{g(s)}{T(b-s,\alpha)} \, ds \right| dt \right]^{1-1/q} \cdot \left[\int_a^b |f'(t)|^q \left| \int_t^{a+b-t} \frac{g(s)}{T(b-s,\alpha)} \, ds \right| dt \right]^{1/q}$$

$$= I_1 \cdot I_2.$$

Analogously as in the proof of Theorem 2.7, we obtain

$$I_2 \leq \|g\|_\infty^{1/q} \left(|f'(a)|^q + |f'(b)|^q \right)^{1/q} \left[\int_{\frac{b-a}{2}}^{b-a} \mathcal{T}(s) \, ds - \int_0^{\frac{b-a}{2}} \mathcal{T}(s) \, ds \right]^{1/q}.$$

By similar calculation,

$$I_1 \leq \|g\|_\infty^{1-1/q} \left[\int_a^{\frac{a+b}{2}} \left\{ \int_{t-a}^{b-t} \frac{ds}{|T(s,\alpha)|} \right\} dt + \int_{\frac{a+b}{2}}^b \left\{ \int_{b-t}^{t-a} \frac{ds}{|T(s,\alpha)|} \right\} dt \right]^{1-1/q}$$

$$= \|g\|_\infty^{1-1/q} \left[\int_a^{\frac{a+b}{2}} \mathcal{T}(b-t) - \mathcal{T}(t-a) \, dt + \int_{\frac{a+b}{2}}^b \mathcal{T}(t-a) - \mathcal{T}(b-t) \, dt \right]^{1-1/q}$$

$$= \|g\|_\infty^{1-1/q} \left[\int_{\frac{b-a}{2}}^{b-a} \mathcal{T}(s) \, ds - \int_0^{\frac{b-a}{2}} \mathcal{T}(s) \, ds + \int_{\frac{b-a}{2}}^{b-a} \mathcal{T}(s) \, ds - \int_0^{\frac{b-a}{2}} \mathcal{T}(s) \, ds \right]^{1-1/q},$$

from where a multiplication gives the desired inequality.   □

Remark 2.10 In case of $T(t,\alpha) = t^{1-\alpha}\Gamma(\alpha)$, $\alpha > 0$, under the conditions of Theorem 2.7, we have

$$\left| \frac{f(a)+f(b)}{2} \left[{}^{RL}J_{a^+}^\alpha (g)(b) + {}^{RL}J_{b^-}^\alpha (g)(a) \right] - \left[{}^{RL}J_{a^+}^\alpha (fg)(b) + {}^{RL}J_{b^-}^\alpha (fg)(a) \right] \right|$$

$$\leq \frac{2(b-a)^{\alpha+1}\|g\|_\infty}{\Gamma(\alpha+2)} \left(1 - \frac{1}{2^\alpha} \right) \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}$$

that corrects the result of [9, Theorem 2.8].

Remark 2.11 If we consider $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we can obtain other inequalities.

3 Conclusions

In this article we study new inequalities of the Hermite–Hadamard–Fejér type for functions with derivatives of special type via convexity. The results obtained allow to extend known results in the literature with an appropriate choice of the kernel, since we use general fractional integrals. For example, let us take $T(t,\alpha) = k\Gamma_k(\alpha)t^{1-\alpha/k}$, $\alpha > 0$, and therefore consider the Riemann–Liouville k-fractional integrals of order $\alpha > 0$ defined by (see [1, 17])

$$^{RL}J_{a^+}^{\alpha,k}f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (t-s)^{\alpha/k-1}f(s) \, ds, \quad t > a,$$

$$
{}^{RL}J_{b^-}^{\alpha,k}f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_t^b (s-t)^{\alpha/k-1}f(s)\,ds, \quad t < b.
$$

Under the conditions of Theorem 2.7, we have

$$
\left| \frac{f(a)+f(b)}{2}\left[{}^{RL}J_{a^+}^{\alpha,k}(g)(b) + {}^{RL}J_{b^-}^{\alpha,k}(g)(a)\right] - \left[{}^{RL}J_{a^+}^{\alpha,k}(fg)(b) + {}^{RL}J_{b^-}^{\alpha,k}(fg)(a)\right] \right|
$$
$$
\leq \frac{(b-a)^{\alpha/k+1}\|g\|_\infty}{\Gamma_k(\alpha+k)(\frac{\alpha}{k}+1)}\left(1 - \frac{1}{2^{\alpha/k}}\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right),
$$

while under the conditions of Theorem 2.9,

$$
\left| \frac{f(a)+f(b)}{2}\left[{}^{RL}J_{a^+}^{\alpha,k}(g)(b) + {}^{RL}J_{b^-}^{\alpha,k}(g)(a)\right] - \left[{}^{RL}J_{a^+}^{\alpha,k}(fg)(b) + {}^{RL}J_{b^-}^{\alpha,k}(fg)(a)\right] \right|
$$
$$
\leq \frac{2(b-a)^{\alpha/k+1}\|g\|_\infty}{\Gamma_k(\alpha+k)(\frac{\alpha}{k}+1)}\left(1 - \frac{1}{2^{\alpha/k}}\right)\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}.
$$

The results obtained also open the door to the generalization study with other types of integral inequalities such as Ostrowski, Grüss, Jensen among others, as well as for functions with other types of generalized convexity such as strongly convex, *m*-convex, *s*-convex, relative convex among others. Finally, our results can also be extended to functions of various variables using convexity in coordinates.

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