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Existence and uniqueness of nonlocal boundary conditions for Hilfer–Hadamard-type fractional differential equations

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Abstract

In this paper, we use some fixed point theorems in Banach space for studying the existence and uniqueness results for Hilfer–Hadamard-type fractional differential equations

$$ {}_{\mathrm{H}}D^{\alpha,\beta}x(t) + f\bigl(t, x(t)\bigr) = 0 $$

on the interval $(1, e]$ with nonlinear boundary conditions

$$ x(1+\epsilon) = \sum_{i=1}^{n-2} \nu_i x(\zeta_i), \qquad {}_{\mathrm{H}}D^{1,1}x(e) = \sum_{i=1}^{n-2} \sigma_i\, {}_{\mathrm{H}}D^{1,1}x(\zeta_i). $$

MSC: 34A08; 35R11

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1 Introduction

In this paper, we discuss the existence and uniqueness of the solutions for the n-point nonlinear boundary value problems for Hilfer–Hadamard-type fractional differential equations of the form

$$ \begin{cases} {}_{\mathrm{H}}D^{\alpha,\beta}x(t) + f(t, x(t)) = 0, & t \in J := (1, e], \\ x(1+\epsilon) = \sum_{i=1}^{n-2} \nu_i x(\zeta_i), & {}_{\mathrm{H}}D^{1,1}x(e) = \sum_{i=1}^{n-2} \sigma_i\, {}_{\mathrm{H}}D^{1,1}x(\zeta_i), \end{cases} \tag{1.1} $$

where ${}_{\mathrm{H}}D^{\alpha,\beta}$ is the Hilfer–Hadamard fractional derivative of order $1 < \alpha \leq 2$ and type $\beta \in [0,1]$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $0 < \epsilon < 1$, $\zeta_i \in (1, e)$, $\nu_i, \sigma_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n-2$, $\zeta_1 < \zeta_2 < \cdots < \zeta_{n-2}$, and ${}_{\mathrm{H}}D^{1,1} = t\frac{d}{dt}$.

The fractional differential equations appear as more appropriate models for describing real world problems. Indeed, these problems cannot be described using the classical integer-order differential equations. In the past years, the theory of fractional differential equations has received much attention from the authors and has become an important field of investigation due to existing applications in engineering, biology, chemistry, economics, and numerous branches of physics [20, 27, 33, 40]. For example, the fractional differential equations are applied to describe the abundant phenomena such as flow in nonlinear electric circuits [15, 16, 20], properties of viscoelastic and dielectric materials [20, 21, 32], nonlinear oscillations of an earthquake [28], mechanics [35], aerodynamics, regular variations in thermodynamics [18], etc.

Fractional derivatives can be of several kinds, one of them is the Hadamard fractional derivative innovated by Hadamard in 1892 [17]. It differs from the preceding Riemann–Liouville- and Caputo-type fractional derivatives [33] in the sense that the kernel of the integral contains the logarithmic function of an arbitrary exponent. The properties of Hadamard fractional integral and derivative can be found in [26, 27]. Recently, scholars have studied the Hadamard-, Caputo–Hadamard- and Hilfer–Hadamard-type fractional derivatives by using the fixed point theorems with the boundary value problems and have given results of the existence and uniqueness of solutions, see [1–13, 22–25, 30, 31, 34, 36–39, 41, 43–45] and the references mentioned therein.

In this paper, we find a variety of results for the boundary value problem (1.1) by using traditional fixed point theorems. The first result is Theorem 3.2, which depends on Banach contraction mapping principle and presents the existence and uniqueness result for the solution of problem (1.1). In Theorem 3.3, we prove the second result of the existence and uniqueness through a fixed point theorem and for nonlinear contractions due to Boyd and Wong. In Theorem 3.4, we prove the third existence result by using Krasnoselskii's fixed point theorem. By using Leray–Schauder type of nonlinear alternative for single-valued maps, we prove the last result of existence, which is Theorem 3.5. Examples are included to illustrative our main results.

2 Preliminaries

In this section, we introduce some notations and definitions of Hilfer–Hadamard-type fractional calculus.

Definition 2.1 (Riemann–Liouville fractional integral, [27, 40]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $\varphi : [1,\infty) \to \mathbb{R}$ is defined by

$$\big(I^{\alpha}\varphi\big)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \frac{\varphi(\tau)\,d\tau}{(t-\tau)^{1-\alpha}} \quad (t > 1).$$

Here, $\Gamma(\alpha)$ is the Euler's Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} \tau^{\alpha-1}e^{-\tau}\,d\tau.$$

Definition 2.2 (Riemann–Liouville fractional derivative, [27, 40]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $\varphi : [1, \infty) \to \mathbb{R}$ is defined by

$$\begin{aligned}\left(D^{\alpha}\varphi\right)(t) &:= \left(\frac{d}{dt}\right)^{n}\left(I^{n-\alpha}\varphi\right)(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{1}^{t}\frac{\varphi(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}} \quad \left(n = [\alpha]+1; t > 1\right),\end{aligned}$$

where $[\alpha]$ is the integer part of α.

Definition 2.3 (Hadamard fractional integral, [27]) The Hadamard fractional integral of order $\alpha \in \mathbb{R}^{+}$ for a function $\varphi : [1, \infty) \to \mathbb{R}$ is defined as

$${}_{\mathrm{H}}I^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{\varphi(\tau)}{\tau}\,d\tau \quad (t > 1),$$

where $\log(\cdot) = \log_{e}(\cdot)$.

Definition 2.4 (Hadamard fractional derivative, [27]) The Hadamard fractional derivative of order α applied to the function $\varphi : [1, \infty) \to \mathbb{R}$ is defined as

$${}_{\mathrm{H}}D^{\alpha}\varphi(t) = \delta^{n}\left({}_{\mathrm{H}}I^{n-\alpha}\varphi(t)\right), \quad n-1 < \alpha < n, n = [\alpha]+1,$$

where $\delta^{n} = (t\frac{d}{dt})^{n}$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.5 (Caputo–Hadamard fractional derivative, [17]) The Caputo–Hadamard fractional derivative of order α applied to the function $\varphi \in AC_{\delta}^{n}[a,b]$ is defined as

$${}_{\mathrm{HC}}D^{\alpha}\varphi(t) = \left({}_{\mathrm{H}}I^{n-\alpha}\delta^{n}\varphi\right)(t), \quad n = [\alpha]+1,$$

where $\varphi \in AC_{\delta}^{n}[a,b] = \{\varphi : [a,b] \to \mathbb{C} : \delta^{(n-1)}\varphi \in AC[a,b], \delta = t\frac{d}{dt}\}$.

Definition 2.6 (Hilfer fractional derivative, [20, 22]) Let $n-1 < \alpha < n$, $0 \leq \beta \leq 1$, $\varphi \in L^{1}(a,b)$. The Hilfer fractional derivative $D^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}\left(D^{\alpha,\beta}\varphi\right)(t) &= \left(I^{\beta(n-\alpha)}\left(\frac{d}{dt}\right)^{n}I^{(n-\alpha)(1-\beta)}\varphi\right)(t) \\ &= \left(I^{\beta(n-\alpha)}\left(\frac{d}{dt}\right)^{n}I^{n-\gamma}\varphi\right)(t); \quad \gamma = \alpha + n\beta - \alpha\beta \\ &= \left(I^{\beta(n-\alpha)}D^{\gamma}\varphi\right)(t),\end{aligned}$$

where $I^{(\cdot)}$ and $D^{(\cdot)}$ are the Riemann–Liouville fractional integral and derivative defined by Definitions 2.1 and 2.2, respectively.

In particular, if $0 < \alpha < 1$, then

$$\begin{aligned}\left(D^{\alpha,\beta}\varphi\right)(t) &= \left(I^{\beta(1-\alpha)}\frac{d}{dt}I^{(1-\alpha)(1-\beta)}\varphi\right)(t)\\ &= \left(I^{\beta(1-\alpha)}\frac{d}{dt}I^{1-\gamma}\varphi\right)(t); \quad \gamma = \alpha+\beta-\alpha\beta\\ &= \left(I^{\beta(1-\alpha)}D^{\gamma}\varphi\right)(t).\end{aligned}$$

Proposition 2.7 ([22, 34]) *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha+\beta-\alpha\beta$, and $\varphi \in L^1(a,b)$. If $D^{\gamma}\varphi$ exists and is in $L^1(a,b)$, then*

$$I^{\alpha}_{a+}\left(D^{\alpha,\beta}_{a+}\varphi\right)(t) = I^{\gamma}_{a+}\left(D^{\gamma}_{a+}\varphi\right)(t) = \varphi(t) - \frac{(I^{1-\gamma}_{a+}\varphi)(a)}{\Gamma(\gamma)}(t-a)^{\gamma-1}.$$

Definition 2.8 (Hilfer–Hadamard fractional derivative, [12, 21]) Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $\varphi \in L^1(a,b)$. The Hilfer–Hadamard fractional derivative ${}_{\mathrm{H}}D^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}\left({}_{\mathrm{H}}D^{\alpha,\beta}\varphi\right)(t) &= \left({}_{\mathrm{H}}I^{\beta(1-\alpha)}\delta\,{}_{\mathrm{H}}I^{(1-\alpha)(1-\beta)}\varphi\right)(t)\\ &= \left({}_{\mathrm{H}}I^{\beta(1-\alpha)}\delta\,{}_{\mathrm{H}}I^{1-\gamma}\varphi\right)(t); \quad \gamma = \alpha+\beta-\alpha\beta\\ &= \left({}_{\mathrm{H}}I^{\beta(1-\alpha)}\,{}_{\mathrm{H}}D^{\gamma}\varphi\right)(t),\end{aligned}$$

where ${}_{\mathrm{H}}I^{(\cdot)}$ and ${}_{\mathrm{H}}D^{(\cdot)}$ are the Hadamard fractional integral and derivative defined by Definitions 2.3 and 2.4, respectively.

Theorem 2.9 ([17, 27]) *Let $\Re(\alpha) > 0$, $n = [\Re(\alpha)] + 1$, and $0 < a < b < \infty$. If $\varphi \in L^1(a,b)$ and $({}_{\mathrm{H}}I^{n-\alpha}_{a+}\varphi)(t) \in AC^n_{\delta}[a,b]$, then*

$$\left({}_{\mathrm{H}}I^{\alpha}_{a+}\,{}_{\mathrm{H}}D^{\alpha}_{a+}\varphi\right)(t) = \varphi(t) - \sum_{j=0}^{n-1}\frac{(\delta^{(n-j-1)}({}_{\mathrm{H}}I^{n-\alpha}_{a+}\varphi))(a)}{\Gamma(\alpha-j)}\left(\log\frac{t}{a}\right)^{\alpha-j-1}.$$

Theorem 2.10 ([17]) *Let $\varphi(t) \in AC^n_{\delta}[a,b]$ or $\varphi(t) \in C^n_{\delta}[a,b]$, and $\alpha \in \mathbb{C}$, then*

$$\left({}_{\mathrm{H}}I^{\alpha}_{a+}\,{}_{\mathrm{HC}}D^{\alpha}_{a+}\varphi\right)(t) = \varphi(t) - \sum_{K=0}^{n-1}\frac{\delta^{K}\varphi(a)}{\Gamma(K+1)}\left(\log\frac{t}{a}\right)^{K}.$$

Definition 2.11 ([45]) Let E be a Banach space and let $F : E \to E$ be a mapping. Then F is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi(\phi) < \phi$ for all $\phi > 0$ with the property

$$\|Fx - Fy\| \le \psi\left(\|x-y\|\right), \quad x, y \in E.$$

Lemma 2.12 ([14]) *Let E be a Banach space and let $F : E \to E$ be a nonlinear contraction. Then, F has a unique fixed point in E.*

Theorem 2.13 (Krasnoselskii's fixed point theorem, [29]) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that*

(a) *$Ax + By \in M$, whenever $x, y \in M$;*
(b) *A is compact and continuous;*
(c) *B is a contraction mapping.*

Then, there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2.14 (Nonlinear alternative for single-valued maps, [19, 42]) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (i.e., $F(\overline{U})$ is a relatively compact subset of C) map. Then, either*
(i) *F has a fixed point in $\overline{U}$ or*
(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\bar{\lambda} \in (0,1)$ with $u = \bar{\lambda} F(u)$.*

Definition 2.15 (Hilfer–Hadamard fractional derivative, [38]) Let $n-1 < \alpha < n$, $0 \le \beta \le 1$, $\varphi \in L^1(a,b)$. The Hilfer–Hadamard fractional derivative ${}_{\mathrm{H}}D^{\alpha,\beta}$ of order α and type β of φ is defined as

$$\begin{aligned}
\left({}_{\mathrm{H}}D^{\alpha,\beta}\varphi\right)(t) &= \left({}_{\mathrm{H}}I^{\beta(n-\alpha)}(\delta)^n\, {}_{\mathrm{H}}I^{(n-\alpha)(1-\beta)}\varphi\right)(t) \\
&= \left({}_{\mathrm{H}}I^{\beta(n-\alpha)}(\delta)^n\, {}_{\mathrm{H}}I^{n-\gamma}\varphi\right)(t); \quad \gamma = \alpha + n\beta - \alpha\beta \\
&= \left({}_{\mathrm{H}}I^{\beta(n-\alpha)}\, {}_{\mathrm{H}}D^{\gamma}\varphi\right)(t),
\end{aligned}$$

where ${}_{\mathrm{H}}I^{(\cdot)}$ and ${}_{\mathrm{H}}D^{(\cdot)}$ are the Hadamard fractional integral and derivative defined by Definitions 2.3 and 2.4, respectively.

Lemma 2.16 ([38]) *Let $\Re(\alpha) > 0$, $0 \le \beta \le 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n-1 < \gamma \le n$, $n = [\Re(\alpha)] + 1$, and $0 < a < b < \infty$. If $\varphi \in L^1(a,b)$ and $({}_{\mathrm{H}}I^{n-\gamma}_{a+}\varphi)(t) \in AC^n_\delta[a,b]$, then*

$$\begin{aligned}
{}_{\mathrm{H}}I^{\alpha}_{a+}\left({}_{\mathrm{H}}D^{\alpha,\beta}_{a+}\varphi\right)(t) &= {}_{\mathrm{H}}I^{\gamma}_{a+}\left({}_{\mathrm{H}}D^{\gamma}_{a+}\varphi\right)(t) \\
&= \varphi(t) - \sum_{j=0}^{n-1} \frac{(\delta^{(n-j-1)}({}_{\mathrm{H}}I^{n-\gamma}_{a+}\varphi))(a)}{\Gamma(\gamma-j)}\left(\log\frac{t}{a}\right)^{\gamma-j-1}.
\end{aligned}$$

From this lemma, we notice that if $\beta = 0$ then the equation reduces to the equation in Theorem 2.9, and if the $\beta = 1$ then the equation reduces to the equation in Theorem 2.10.

3 Main results

Lemma 3.1 *For $1 < \alpha \le 2$, $0 \le \beta \le 1$, $\gamma = \alpha + 2\beta - \alpha\beta$, $\gamma \in (1,2]$, and $\varphi \in C([1,e],\mathbb{R})$, the problem*

$$\begin{cases}
{}_{\mathrm{H}}D^{\alpha,\beta}x(t) + \varphi(t) = 0, & t \in J, 1 < \alpha \le 2, 0 \le \beta \le 1, \\
x(1+\epsilon) = \sum_{i=1}^{n-2} \nu_i x(\zeta_i), & {}_{\mathrm{H}}D^{1,1}x(e) = \sum_{i=1}^{n-2} \sigma_i\, {}_{\mathrm{H}}D^{1,1}x(\zeta_i),
\end{cases} \tag{3.1}$$

has a unique solution given by

$$\begin{aligned}
x(t) = -{}_{\mathrm{H}}I^{\alpha}\varphi(t) &+ \frac{(\gamma-1)\delta_1(\log t)^{\gamma-2} - (\gamma-2)\delta_2(\log t)^{\gamma-1}}{\lambda} \\
&\times \left[{}_{\mathrm{H}}I^{\alpha}\varphi(1+\epsilon) - \sum_{i=1}^{n-2} \nu_i\, {}_{\mathrm{H}}I^{\alpha}\varphi(\zeta_i)\right]
\end{aligned}$$

$$+\frac{\mu_2(\log t)^{\gamma-1}-\mu_1(\log t)^{\gamma-2}}{\lambda}\left[{}_{\mathrm{H}}I^{\alpha-1}\varphi(e)-\sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}\varphi(\zeta_i)\right],\quad t\in J,$$

where

$$\lambda=(\gamma-1)\delta_1\mu_2-(\gamma-2)\delta_2\mu_1,\quad \textit{with } \lambda\neq 0,$$

$$\mu_1=\bigl(\log(1+\epsilon)\bigr)^{\gamma-1}-\sum_{i=1}^{n-2}\nu_i\bigl(\log(\zeta_i)\bigr)^{\gamma-1},$$

$$\mu_2=\bigl(\log(1+\epsilon)\bigr)^{\gamma-2}-\sum_{i=1}^{n-2}\nu_i\bigl(\log(\zeta_i)\bigr)^{\gamma-2},$$

$$\delta_1=1-\sum_{i=1}^{n-2}\sigma_i\bigl(\log(\zeta_i)\bigr)^{\gamma-2},$$

$$\delta_2=1-\sum_{i=1}^{n-2}\sigma_i\bigl(\log(\zeta_i)\bigr)^{\gamma-3}.$$

Proof In view of Lemma 2.16, the solution of the Hilfer–Hadamard differential equation (3.1) can be written as

$$x(t)=-{}_{\mathrm{H}}I^{\alpha}\varphi(t)+c_0(\log t)^{\gamma-1}+c_1(\log t)^{\gamma-2}, \tag{3.2}$$

and

$${}_{\mathrm{H}}D^{1,1}x(t)=-{}_{\mathrm{H}}I^{\alpha-1}\varphi(t)+(\gamma-1)c_0(\log t)^{\gamma-2}+(\gamma-2)c_1(\log t)^{\gamma-3}. \tag{3.3}$$

The boundary condition $x(1+\epsilon)=\sum_{i=1}^{n-2}\nu_i x(\zeta_i)$ gives

$$c_1=\frac{1}{\mu_2}\left[{}_{\mathrm{H}}I^{\alpha}\varphi(1+\epsilon)-\sum_{i=1}^{n-2}\nu_i\,{}_{\mathrm{H}}I^{\alpha}\varphi(\zeta_i)-c_0\mu_1\right], \tag{3.4}$$

where

$$\mu_1=\bigl(\log(1+\epsilon)\bigr)^{\gamma-1}-\sum_{i=1}^{n-2}\nu_i\bigl(\log(\zeta_i)\bigr)^{\gamma-1},\qquad \mu_2=\bigl(\log(1+\epsilon)\bigr)^{\gamma-2}-\sum_{i=1}^{n-2}\nu_i\bigl(\log(\zeta_i)\bigr)^{\gamma-2}.$$

In view of the boundary condition ${}_{\mathrm{H}}D^{1,1}x(e)=\sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}D^{1,1}x(\zeta_i)$ and from equations (3.3) and (3.4), we have

$$c_0=\frac{1}{(\gamma-1)\delta_1}\left[-(\gamma-2)c_1\delta_2+{}_{\mathrm{H}}I^{\alpha-1}\varphi(e)-\sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}\varphi(\zeta_i)\right], \tag{3.5}$$

where

$$\delta_1=1-\sum_{i=1}^{n-2}\sigma_i\bigl(\log(\zeta_i)\bigr)^{\gamma-2},\qquad \delta_2=1-\sum_{i=1}^{n-2}\sigma_i\bigl(\log(\zeta_i)\bigr)^{\gamma-3}.$$

By using (3.5) in equation (3.4), we have

$$c_1 = \frac{1}{\lambda}\Bigg[(\gamma-1)\delta_1\Bigg[{}_{\mathrm{H}}I^{\alpha}\varphi(1+\epsilon) - \sum_{i=1}^{n-2}\nu_i\,{}_{\mathrm{H}}I^{\alpha}\varphi(\zeta_i)\Bigg] - \mu_1\Bigg[{}_{\mathrm{H}}I^{\alpha-1}\varphi(e) - \sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}\varphi(\zeta_i)\Bigg]\Bigg],$$

where

$$\lambda = (\gamma-1)\delta_1\mu_2 - (\gamma-2)\delta_2\mu_1, \quad \text{with } \lambda \neq 0.$$

By substituting the value of c_1 into (3.5), we have

$$c_0 = \frac{1}{\lambda}\Bigg[-(\gamma-2)\delta_2\Bigg[{}_{\mathrm{H}}I^{\alpha}\varphi(1+\epsilon) - \sum_{i=1}^{n-2}\nu_i\,{}_{\mathrm{H}}I^{\alpha}\varphi(\zeta_i)\Bigg] + \mu_2\Bigg[{}_{\mathrm{H}}I^{\alpha-1}\varphi(e) - \sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}\varphi(\zeta_i)\Bigg]\Bigg].$$

Now, substituting the values of c_0 and c_1 in (3.2), we obtain the solution of problem (3.1). □

Next, we present the existence and uniqueness of solutions for Hilfer–Hadamard-type fractional differential equation (1.1). For that, suppose that

$$K = C\big([1,e],\mathbb{R}\big) \tag{3.6}$$

is a Banach space of all continuous functions from $[1,e]$ into $\mathbb{R}$ equipped with the norm $\|x\| = \sup_{t\in J}|x(t)|$. From Lemma 3.1, we get an operator $\rho : K \to K$ defined as

$$\begin{aligned}(\rho x)(t) = {}&-{}_{\mathrm{H}}I^{\alpha}f\big(\tau,x(\tau)\big)(t)\\ &+ \frac{(\gamma-1)\delta_1(\log t)^{\gamma-2} - (\gamma-2)\delta_2(\log t)^{\gamma-1}}{\lambda}\Bigg[{}_{\mathrm{H}}I^{\alpha}f\big(\tau,x(\tau)\big)(1+\epsilon)\\ &- \sum_{i=1}^{n-2}\nu_i\,{}_{\mathrm{H}}I^{\alpha}f\big(\tau,x(\tau)\big)(\zeta_i)\Bigg]\\ &+ \frac{\mu_2(\log t)^{\gamma-1} - \mu_1(\log t)^{\gamma-2}}{\lambda}\Bigg[{}_{\mathrm{H}}I^{\alpha-1}f\big(\tau,x(\tau)\big)(e)\\ &- \sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}f\big(\tau,x(\tau)\big)(\zeta_i)\Bigg], \quad \text{with } \lambda \neq 0.\end{aligned} \tag{3.7}$$

It must be noticed that problem (1.1) has a solution if and only if operator ρ has a fixed point. The results of existence and uniqueness are based on the Banach contraction mapping principle.

Theorem 3.2 *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption*

(Q_1) *there exists a constant $C > 0$ such that $|f(t,x) - f(t,y)| \leq C|x - y|$ for each $t \in J$ and $x, y \in \mathbb{R}$. If Φ is such that $C\Phi < 1$, where*

$$\begin{aligned}\Phi = \Bigg\{ &\frac{1}{\Gamma(\alpha+1)} + \frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|\Gamma(\alpha+1)} \\ &\times \Bigg[\big(\log(1+\epsilon)\big)^\alpha + \sum_{i=1}^{n-2} |\nu_i| \big(\log(\zeta_i)\big)^\alpha \Bigg] \\ &+ \frac{|\mu_2| + |\mu_1|}{|\lambda|\Gamma(\alpha)} \Bigg[1 + \sum_{i=1}^{n-2} |\sigma_i| \big(\log(\zeta_i)\big)^{\alpha-1} \Bigg] \Bigg\},\end{aligned} \tag{3.8}$$

then the boundary value problem (1.1) has a unique solution on J.

Proof We are using Banach contraction mapping principle to transform the boundary value problem (1.1) into a fixed point problem $x = \rho x$, where the operator ρ is defined by (3.7). We will show that ρ has a fixed point, which is a unique solution of problem (1.1).

We put $\sup|f(\tau,0)| = p < \infty$ and choose

$$r \geq \frac{\Phi P}{1 - C\Phi}. \tag{3.9}$$

Now, assume that $B_r = \{x \in K : |x| \leq r\}$. We will show that $\rho B_r \subset B_r$.

For any $x \in B_r$, we have

$$\begin{aligned}\|\rho x\| = \sup_{t \in J} \Bigg\{ \Bigg| &-{}_{\mathrm{H}}I^\alpha f\big(\tau, x(\tau)\big)(t) \\ &+ \frac{(\gamma-1)\delta_1(\log t)^{\gamma-2} - (\gamma-2)\delta_2(\log t)^{\gamma-1}}{\lambda} \Bigg[{}_{\mathrm{H}}I^\alpha f\big(\tau, x(\tau)\big)(1+\epsilon) \\ &- \sum_{i=1}^{n-2} \nu_i \, {}_{\mathrm{H}}I^\alpha f\big(\tau, x(\tau)\big)(\zeta_i) \Bigg] \\ &+ \frac{\mu_2(\log t)^{\gamma-1} - \mu_1(\log t)^{\gamma-2}}{\lambda} \Bigg[{}_{\mathrm{H}}I^{\alpha-1} f\big(\tau, x(\tau)\big)(e) \\ &- \sum_{i=1}^{n-2} \sigma_i \, {}_{\mathrm{H}}I^{\alpha-1} f\big(\tau, x(\tau)\big)(\zeta_i) \Bigg] \Bigg| \Bigg\} \\ \leq \sup_{t \in J} \Bigg\{ &{}_{\mathrm{H}}I^\alpha \big| f\big(\tau, x(\tau)\big) \big| (t) \\ &+ \frac{(|\gamma-1|)|\delta_1|(\log t)^{\gamma-2} + (|\gamma-2|)|\delta_2|(\log t)^{\gamma-1}}{|\lambda|} \Bigg[{}_{\mathrm{H}}I^\alpha \big| f\big(\tau, x(\tau)\big) \big| (1+\epsilon) \\ &+ \sum_{i=1}^{n-2} |\nu_i| \, {}_{\mathrm{H}}I^\alpha \big| f\big(\tau, x(\tau)\big) \big| (\zeta_i) \Bigg] \\ &+ \frac{|\mu_2|(\log t)^{\gamma-1} + |\mu_1|(\log t)^{\gamma-2}}{|\lambda|} \Bigg[{}_{\mathrm{H}}I^{\alpha-1} \big| f\big(\tau, x(\tau)\big) \big| (e)\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{n-2} |\sigma_i| \, {}_{\mathrm{H}}I^{\alpha-1} \bigl| f\bigl(\tau, x(\tau)\bigr) \bigr| (\zeta_i) \Biggr] \Biggr\} \\
&\leq {}_{\mathrm{H}}I^{\alpha} \bigl(\bigl| f\bigl(\tau, x(\tau)\bigr) - f(\tau,0) \bigr| + \bigl| f(\tau,0) \bigr| \bigr)(e) \\
&\quad + \frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|} \Biggl[{}_{\mathrm{H}}I^{\alpha} \bigl(\bigl| f\bigl(\tau, x(\tau)\bigr) - f(\tau,0) \bigr| + \bigl| f(\tau,0) \bigr| \bigr)(1+\epsilon) \\
&\quad + \sum_{i=1}^{n-2} |\nu_i| \, {}_{\mathrm{H}}I^{\alpha} \bigl(\bigl| f\bigl(\tau, x(\tau)\bigr) - f(\tau,0) \bigr| + \bigl| f(\tau,0) \bigr| \bigr)(\zeta_i) \Biggr] \\
&\quad + \frac{|\mu_2| + |\mu_1|}{|\lambda|} \Biggl[{}_{\mathrm{H}}I^{\alpha-1} \bigl(\bigl| f\bigl(\tau, x(\tau)\bigr) - f(\tau,0) \bigr| + \bigl| f(\tau,0) \bigr| \bigr)(e) \\
&\quad + \sum_{i=1}^{n-2} |\sigma_i| \, {}_{\mathrm{H}}I^{\alpha-1} \bigl(\bigl| f\bigl(\tau, x(\tau)\bigr) - f(\tau,0) \bigr| + \bigl| f(\tau,0) \bigr| \bigr)(\zeta_i) \Biggr] \\
&\leq (Cr+P) \Biggl\{ \frac{1}{\Gamma(\alpha+1)} + \frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|\Gamma(\alpha+1)} \\
&\quad \times \Biggl[\bigl(\log(1+\epsilon)\bigr)^{\alpha} + \sum_{i=1}^{n-2} |\nu_i| \bigl(\log(\zeta_i)\bigr)^{\alpha} \Biggr] \\
&\quad + \frac{|\mu_2| + |\mu_1|}{|\lambda|\Gamma(\alpha)} \Biggl[1 + \sum_{i=1}^{n-2} |\sigma_i| \bigl(\log(\zeta_i)\bigr)^{\alpha-1} \Biggr] \Biggr\} \\
&= (Cr+P)\Phi \leq r.
\end{aligned}
\tag{3.10}
$$

Thus, we have shown that $\rho B_r \subset B_r$.

Now, for $x, y \in K$ and $t \in J$, we have

$$
\begin{aligned}
&\bigl| (\rho x)(t) - (\rho y)(t) \bigr| \\
&\quad = \Biggl| -{}_{\mathrm{H}}I^{\alpha} \bigl(f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr) \bigr)(t) \\
&\qquad + \frac{(\gamma-1)\delta_1(\log t)^{\gamma-2} - (\gamma-2)\delta_2(\log t)^{\gamma-1}}{\lambda} \Biggl[{}_{\mathrm{H}}I^{\alpha} \bigl(f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr) \bigr)(1+\epsilon) \\
&\qquad - \sum_{i=1}^{n-2} \nu_i \, {}_{\mathrm{H}}I^{\alpha} \bigl(f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr) \bigr)(\zeta_i) \Biggr] \\
&\qquad + \frac{\mu_2(\log t)^{\gamma-1} - \mu_1(\log t)^{\gamma-2}}{\lambda} \Biggl[{}_{\mathrm{H}}I^{\alpha-1} \bigl(f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr) \bigr)(e) \\
&\qquad - \sum_{i=1}^{n-2} \sigma_i \, {}_{\mathrm{H}}I^{\alpha-1} \bigl(f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr) \bigr)(\zeta_i) \Biggr] \Biggr| \\
&\quad \leq {}_{\mathrm{H}}I^{\alpha} \bigl| f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr) \bigr| (t) \\
&\qquad + \frac{(|\gamma-1|)|\delta_1|(\log t)^{\gamma-2} + (|\gamma-2|)|\delta_2|(\log t)^{\gamma-1}}{|\lambda|}
\end{aligned}
$$

$$\begin{aligned}
&\times \Bigl[{}_{\mathrm{H}}I^{\alpha} \bigl| f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr)\bigr| (1+\epsilon) \\
&+ \sum_{i=1}^{n-2} |\nu_i| \, {}_{\mathrm{H}}I^{\alpha} \bigl| f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr)\bigr| (\zeta_i) \Bigr] \\
&+ \frac{|\mu_2|(\log t)^{\gamma-1} + |\mu_1|(\log t)^{\gamma-2}}{|\lambda|} \Bigl[{}_{\mathrm{H}}I^{\alpha-1} \bigl| f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr)\bigr| (e) \\
&+ \sum_{i=1}^{n-2} |\sigma_i| \, {}_{\mathrm{H}}I^{\alpha-1} \bigl| f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr)\bigr| (\zeta_i) \Bigr] \\
\le{}& C\|x-y\| \biggl\{ \frac{1}{\Gamma(\alpha+1)} + \frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|\Gamma(\alpha+1)} \\
&\times \Bigl[\bigl(\log(1+\epsilon)\bigr)^{\alpha} + \sum_{i=1}^{n-2} |\nu_i| \bigl(\log(\zeta_i)\bigr)^{\alpha} \Bigr] \\
&+ \frac{|\mu_2| + |\mu_1|}{|\lambda|\Gamma(\alpha)} \Bigl[1 + \sum_{i=1}^{n-2} |\sigma_i| \bigl(\log(\zeta_i)\bigr)^{\alpha-1} \Bigr] \biggr\} \\
={}& C\|x-y\|\Phi.
\end{aligned} \tag{3.11}$$

Therefore, it has been shown that $\|(\rho x)(t) - (\rho y)(t)\| \le C\Phi\|x-y\|$, where $C\Phi < 1$. Hence, ρ is a contraction. Thus, by Banach contraction mapping principle, problem (1.1) has a unique solution. □

Theorem 3.3 *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption*

(Q_2) $|f(t,x) - f(t,y)| \le \varphi(t)(|x-y|/(P^* + |x-y|))$, $t \in J$, $x, y \ge 0$, *where $\varphi : J \to \mathbb{R}^+$ is continuous and a constant P^* is defined by*

$$\begin{aligned}
P^* ={}& {}_{\mathrm{H}}I^{\alpha}\varphi(e) + \frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|} \Bigl[{}_{\mathrm{H}}I^{\alpha}\varphi(1+\epsilon) + \sum_{i=1}^{n-2} |\nu_i| \, {}_{\mathrm{H}}I^{\alpha}\varphi(\zeta_i) \Bigr] \\
&+ \frac{|\mu_2| + |\mu_1|}{|\lambda|} \Bigl[{}_{\mathrm{H}}I^{\alpha-1}\varphi(e) + \sum_{i=1}^{n-2} |\sigma_i| \, {}_{\mathrm{H}}I^{\alpha-1}\varphi(\zeta_i) \Bigr].
\end{aligned} \tag{3.12}$$

Then, the boundary value problem (1.1) *has a unique solution on J.*

Proof We have the operator $\rho : K \to K$ defined by (3.7) and by applying Definition 2.11, we can define a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\phi) = \frac{P^*\phi}{P^* + \phi}, \quad \text{for } \phi \ge 0, \tag{3.13}$$

where the function Ψ satisfies $\Psi(0) = 0$ and $\Psi(\phi) < \phi$ for all $\phi > 0$.

For any $x, y \in K$ and for each $t \in J$, we have

$$\begin{aligned}
&\bigl|(\rho x)(t) - (\rho y)(t)\bigr| \\
&= \Bigl| -{}_{\mathrm{H}}I^{\alpha}\bigl(f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr)\bigr)(t)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\gamma-1)\delta_1(\log t)^{\gamma-2}-(\gamma-2)\delta_2(\log t)^{\gamma-1}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha}\bigl(f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr)(1+\epsilon)\\
&-\sum_{i=1}^{n-2}\nu_i\,{}_{\mathrm{H}}I^{\alpha}\bigl(f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr)(\zeta_i)\Biggr]\\
&+\frac{\mu_2(\log t)^{\gamma-1}-\mu_1(\log t)^{\gamma-2}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha-1}\bigl(f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr)(e)\\
&-\sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}\bigl(f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr)(\zeta_i)\Biggr]\Biggr|\\
\le{}&{}_{\mathrm{H}}I^{\alpha}\bigl|f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr|(t)\\
&+\frac{(|\gamma-1|)|\delta_1|(\log t)^{\gamma-2}+(|\gamma-2|)|\delta_2|(\log t)^{\gamma-1}}{|\lambda|}\\
&\times\Biggl[{}_{\mathrm{H}}I^{\alpha}\bigl|f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr|(1+\epsilon)\\
&+\sum_{i=1}^{n-2}|\nu_i|\,{}_{\mathrm{H}}I^{\alpha}\bigl|f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr|(\zeta_i)\Biggr]\\
&+\frac{|\mu_2|(\log t)^{\gamma-1}+|\mu_1|(\log t)^{\gamma-2}}{|\lambda|}\Biggl[{}_{\mathrm{H}}I^{\alpha-1}\bigl|f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr|(e)\\
&+\sum_{i=1}^{n-2}|\sigma_i|\,{}_{\mathrm{H}}I^{\alpha-1}\bigl|f\bigl(\tau,x(\tau)\bigr)-f\bigl(\tau,y(\tau)\bigr)\bigr|(\zeta_i)\Biggr]\\
\le{}&{}_{\mathrm{H}}I^{\alpha}\biggl(\varphi(\tau)\frac{|x(\tau)-y(\tau)|}{P^*+|x(\tau)-y(\tau)|}\biggr)(e)\\
&+\frac{(|\gamma-1|)|\delta_1|+(|\gamma-2|)|\delta_2|}{|\lambda|}\Biggl[{}_{\mathrm{H}}I^{\alpha}\biggl(\varphi(\tau)\frac{|x(\tau)-y(\tau)|}{P^*+|x(\tau)-y(\tau)|}\biggr)(1+\epsilon)\\
&+\sum_{i=1}^{n-2}|\nu_i|\,{}_{\mathrm{H}}I^{\alpha}\biggl(\varphi(\tau)\frac{|x(\tau)-y(\tau)|}{P^*+|x(\tau)-y(\tau)|}\biggr)(\zeta_i)\Biggr]\\
&+\frac{|\mu_2|+|\mu_1|}{|\lambda|}\Biggl[{}_{\mathrm{H}}I^{\alpha-1}\biggl(\varphi(\tau)\frac{|x(\tau)-y(\tau)|}{P^*+|x(\tau)-y(\tau)|}\biggr)(e)\\
&+\sum_{i=1}^{n-2}|\sigma_i|\,{}_{\mathrm{H}}I^{\alpha-1}\biggl(\varphi(\tau)\frac{|x(\tau)-y(\tau)|}{P^*+|x(\tau)-y(\tau)|}\biggr)(\zeta_i)\Biggr]\\
\le{}&\frac{\Psi(\|x-y\|)}{P^*}\Biggl\{{}_{\mathrm{H}}I^{\alpha}\varphi(e)+\frac{(|\gamma-1|)|\delta_1|+(|\gamma-2|)|\delta_2|}{|\lambda|}\\
&\times\Biggl[{}_{\mathrm{H}}I^{\alpha}\varphi(1+\epsilon)+\sum_{i=1}^{n-2}|\nu_i|\,{}_{\mathrm{H}}I^{\alpha}\varphi(\zeta_i)\Biggr]\\
&+\frac{|\mu_2|+|\mu_1|}{|\lambda|}\Biggl[{}_{\mathrm{H}}I^{\alpha-1}\varphi(e)+\sum_{i=1}^{n-2}|\sigma_i|\,{}_{\mathrm{H}}I^{\alpha-1}\varphi(\zeta_i)\Biggr]\Biggr\}\\
={}&\Psi\bigl(\|x-y\|\bigr),
\end{aligned}
\tag{3.14}
$$

which implies that $\|\rho x - \rho y\| \leq \Psi(\|x-y\|)$. Then, the operator ρ is a nonlinear contraction. Thus, by Lemma 2.12 (Banach contraction mapping principle) the operator ρ has a unique fixed point, which is the unique solution of problem (1.1). □

Next, we will give the existence result by using Theorem 2.13 (Krasnoselskii's fixed point theorem).

Theorem 3.4 *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption (Q_1). In addition, assume that*

$$(Q_3) \quad \bigl|f(t,x)\bigr| \leq g(t), \quad \text{for } (t,x) \in J \times \mathbb{R} \text{ and } g \in C\bigl([1,e],\mathbb{R}^+\bigr).$$

If

$$C\Gamma(\alpha+1) < 1, \tag{3.15}$$

then the boundary value problem (1.1) has at least one solution on J.

Proof We put $\sup_{t\in J}|g(t)| = \|g\|$ and choose a suitable constant $\hat{r}$ such that

$$\hat{r} \geq \|g\|\Phi, \tag{3.16}$$

where Φ is defined by (3.8). Moreover, we consider the operators $\mathscr{F}$ and $\mathscr{G}$ on $B_{\hat{r}} = \{x \in K : \|x\| \leq \hat{r}\}$ defined as

$$\begin{aligned}
(\mathscr{F}x)(t) ={}& \frac{(\gamma-1)\delta_1(\log t)^{\gamma-2} - (\gamma-2)\delta_2(\log t)^{\gamma-1}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha} f\bigl(\tau,x(\tau)\bigr)(1+\epsilon) \\
&- \sum_{i=1}^{n-2} \nu_i\, {}_{\mathrm{H}}I^{\alpha} f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr] \\
&+ \frac{\mu_2(\log t)^{\gamma-1} - \mu_1(\log t)^{\gamma-2}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha-1} f\bigl(\tau,x(\tau)\bigr)(e) \\
&- \sum_{i=1}^{n-2} \sigma_i\, {}_{\mathrm{H}}I^{\alpha-1} f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr], \quad t \in J; \\
(\mathscr{G}x)(t) ={}& -{}_{\mathrm{H}}I^{\alpha} f\bigl(\tau,x(\tau)\bigr)(t), \quad t \in J.
\end{aligned} \tag{3.17}$$

For any $x, y \in B_{\hat{r}}$, we have

$$\begin{aligned}
\|\mathscr{F}x + \mathscr{G}x\| \leq \|g\|\Biggl(&\frac{1}{\Gamma(\alpha+1)} + \frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|\Gamma(\alpha+1)} \\
&\times \Biggl[\bigl(\log(1+\epsilon)\bigr)^{\alpha} + \sum_{i=1}^{n-2}|\nu_i|\bigl(\log(\zeta_i)\bigr)^{\alpha}\Biggr] \\
&+ \frac{|\mu_2| + |\mu_1|}{|\lambda|\Gamma(\alpha)}\Biggl[1 + \sum_{i=1}^{n-2}|\sigma_i|\bigl(\log(\zeta_i)\bigr)^{\alpha-1}\Biggr]\Biggr)
\end{aligned}$$

$$= \|g\|\Phi \le \hat{r}, \tag{3.18}$$

which implies that $\mathscr{F}x + \mathscr{G}x \in B_{\hat{r}}$. It follows from assumption (Q_1), together with (3.15), that $\mathscr{G}$ is a contraction. Furthermore, it is easy to show that the operator $\mathscr{F}$ is continuous. Moreover,

$$\begin{aligned}\|\mathscr{F}x\| \le \|g\|\Biggl(&\frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|\Gamma(\alpha+1)}\Biggl[\bigl(\log(1+\epsilon)\bigr)^{\alpha} + \sum_{i=1}^{n-2}|\nu_i|\bigl(\log(\zeta_i)\bigr)^{\alpha}\Biggr] \\ &+ \frac{|\mu_2| + |\mu_1|}{|\lambda|\Gamma(\alpha)}\Biggl[1 + \sum_{i=1}^{n-2}|\sigma_i|\bigl(\log(\zeta_i)\bigr)^{\alpha-1}\Biggr]\Biggr).\end{aligned} \tag{3.19}$$

Hence, $\mathscr{F}$ is uniformly bounded on $B_{\hat{r}}$.

Next, we prove that the operator $\mathscr{F}$ is compact. For that, we put $\sup_{(t,x)\in J\times B_{\hat{r}}}|f(t,x)| = \bar{p} < \infty$.

Consequently, for $t_1, t_2 \in J$, we get

$$\begin{aligned}&\bigl|(\mathscr{F}x)(t_1) - (\mathscr{F}x)(t_2)\bigr| \\ &= \Biggl|\Biggl\{\frac{(\gamma-1)\delta_1(\log t_1)^{\gamma-2} - (\gamma-2)\delta_2(\log t_1)^{\gamma-1}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(1+\epsilon) \\ &\quad - \sum_{i=1}^{n-2}\nu_i\,{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr] \\ &\quad + \frac{\mu_2(\log t_1)^{\gamma-1} - \mu_1(\log t_1)^{\gamma-2}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)(e) \\ &\quad - \sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr]\Biggr\} \\ &\quad - \Biggl\{\frac{(\gamma-1)\delta_1(\log t_2)^{\gamma-2} - (\gamma-2)\delta_2(\log t_2)^{\gamma-1}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(1+\epsilon) \\ &\quad - \sum_{i=1}^{n-2}\nu_i\,{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr] \\ &\quad + \frac{\mu_2(\log t_2)^{\gamma-1} - \mu_1(\log t_2)^{\gamma-2}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)(e) \\ &\quad - \sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr]\Biggr\}\Biggr| \\ &\le \bar{p}\frac{(|\gamma-1|)|\delta_1||(\log t_2)^{\gamma-2} - \log t_1)^{\gamma-2}| + (|\gamma-2|)|\delta_2||(\log t_2)^{\gamma-1} - (\log t_1)^{\gamma-1}|}{|\lambda|\Gamma(\alpha+1)} \\ &\quad \times \Biggl[\bigl(\log(1+\epsilon)\bigr)^{\alpha} + \sum_{i=1}^{n-2}|\nu_i|\bigl(\log(\zeta_i)\bigr)^{\alpha}\Biggr] \\ &\quad + \bar{p}\frac{|\mu_2||(\log t_2)^{\gamma-1} - \log t_1)^{\gamma-1}| + |\mu_1||(\log t_2)^{\gamma-2} - (\log t_1)^{\gamma-2}|}{|\lambda|\Gamma(\alpha)}\end{aligned}$$

$$\times \left[1 + \sum_{i=1}^{n-2} |\sigma_i| \left(\log(\zeta_i) \right)^{\alpha-1} \right],$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathscr{F}$ is equicontinuous. Hence, $\mathscr{F}$ is relatively compact on $B_{\hat{r}}$. Therefore, by the Arzelà–Ascoli theorem, $\mathscr{F}$ is compact on $B_{\hat{r}}$. Thus, by Theorem 2.13, the boundary value problem (1.1) has at least one solution on J. □

Now, the final existence result is based on Theorem 2.14 (nonlinear alternative for single-valued maps).

Theorem 3.5 *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and assume that*:

(Q_4) *there exists a continuous nondecreasing function $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+ \backslash \{0\}$ such that*

$$\left| f(t,x) \right| \le q(t) \vartheta \left(|x| \right) \quad \text{for each } (t,x) \in J \times \mathbb{R}, \tag{3.20}$$

where $q \in C([1,e], \mathbb{R}^+)$ is a function;

(Q_5) *there exists a constant $L > 0$ such that*

$$\frac{L}{\|q\| \vartheta(L) \Phi} > 1, \tag{3.21}$$

where Φ is defined by (3.8). Then, the boundary value problem (1.1) has at least one solution on J.

Proof We have the operator ρ defined by (3.7). Firstly, we will show that ρ maps bounded sets (balls) into bounded sets in K. For that, let $\bar{r}$ be a positive number, and $B_{\bar{r}} = \{x \in K : \|x\| \le \bar{r}\}$ be a bounded ball in K, where K is defined by (3.6). For $t \in J$, we have

$$\begin{aligned}
\left| \rho x(t) \right| &\le {}_{\mathrm{H}}I^{\alpha} \left| f\left(\tau, x(\tau)\right) \right| (e) \\
&\quad + \frac{(|\gamma - 1|)|\delta_1| + (|\gamma - 2|)|\delta_2|}{|\lambda|} \Bigg[{}_{\mathrm{H}}I^{\alpha} \left| f\left(\tau, x(\tau)\right) \right| (1+\epsilon) \\
&\quad + \sum_{i=1}^{n-2} |\nu_i| \, {}_{\mathrm{H}}I^{\alpha} \left| f\left(\tau, x(\tau)\right) \right| (\zeta_i) \Bigg] \\
&\quad + \frac{|\mu_2| + |\mu_1|}{|\lambda|} \Bigg[{}_{\mathrm{H}}I^{\alpha-1} \left| f\left(\tau, x(\tau)\right) \right| (e) \\
&\quad + \sum_{i=1}^{n-2} |\sigma_i| \, {}_{\mathrm{H}}I^{\alpha-1} \left| f\left(\tau, x(\tau)\right) \right| (\zeta_i) \Bigg] \\
&\le \|q\| \vartheta\left(\|x\|\right) \frac{1}{\Gamma(\alpha+1)} \\
&\quad + \|q\| \vartheta\left(\|x\|\right) \frac{(|\gamma - 1|)|\delta_1| + (|\gamma - 2|)|\delta_2|}{|\lambda| \Gamma(\alpha+1)} \left[\left(\log(1+\epsilon) \right)^{\alpha} + \sum_{i=1}^{n-2} |\nu_i| \left(\log(\zeta_i) \right)^{\alpha} \right] \\
&\quad + \|q\| \vartheta\left(\|x\|\right) \frac{|\mu_2| + |\mu_1|}{|\lambda| \Gamma(\alpha)} \left[1 + \sum_{i=1}^{n-2} |\sigma_i| \left(\log(\zeta_i) \right)^{\alpha-1} \right]
\end{aligned}$$

$$
\begin{aligned}
&\le \|q\|\vartheta(\bar{r})\Biggl\{\frac{1}{\Gamma(\alpha+1)}+\frac{(|\gamma-1|)|\delta_1|+(|\gamma-2|)|\delta_2|}{|\lambda|\Gamma(\alpha+1)}\\
&\quad\times\Biggl[\bigl(\log(1+\epsilon)\bigr)^{\alpha}+\sum_{i=1}^{n-2}|\nu_i|\bigl(\log(\zeta_i)\bigr)^{\alpha}\Biggr]\\
&\quad+\frac{|\mu_2|+|\mu_1|}{|\lambda|\Gamma(\alpha)}\Biggl[1+\sum_{i=1}^{n-2}|\sigma_i|\bigl(\log(\zeta_i)\bigr)^{\alpha-1}\Biggr]\Biggr\}\\
&:= C_1,
\end{aligned}
\tag{3.22}
$$

which implies that $\|\rho x\| \le C_1$.

Now, we will show that ρ maps bounded sets into equicontinuous sets of K. For that, let $\sup_{(t,x)\in J\times B_{\bar{r}}}|f(t,x)| = p^{\star} < \infty$, where $\omega_1,\omega_2 \in J$, with $\omega_1 < \omega_2$ and $x \in B_{\bar{r}}$. Hence, we have

$$
\begin{aligned}
&\bigl|(\rho x)(\omega_1)-(\rho x)(\omega_2)\bigr|\\
&\quad=\Biggl|\Biggl\{-{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(\omega_1)+\frac{(\gamma-1)\delta_1(\log\omega_1)^{\gamma-2}-(\gamma-2)\delta_2(\log\omega_1)^{\gamma-1}}{\lambda}\\
&\qquad\times\Biggl[{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(1+\epsilon)-\sum_{i=1}^{n-2}\nu_i\,{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr]\\
&\qquad+\frac{\mu_2(\log\omega_1)^{\gamma-1}-\mu_1(\log\omega_1)^{\gamma-2}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)(e)\\
&\qquad-\sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr]\Biggr\}\\
&\qquad-\Biggl\{-{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(\omega_2)+\frac{(\gamma-1)\delta_1(\log\omega_2)^{\gamma-2}-(\gamma-2)\delta_2(\log\omega_2)^{\gamma-1}}{\lambda}\\
&\qquad\times\Biggl[{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(1+\epsilon)-\sum_{i=1}^{n-2}\nu_i\,{}_{\mathrm{H}}I^{\alpha}f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr]\\
&\qquad+\frac{\mu_2(\log t_2)^{\gamma-1}-\mu_1(\log t_2)^{\gamma-2}}{\lambda}\Biggl[{}_{\mathrm{H}}I^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)(e)\\
&\qquad-\sum_{i=1}^{n-2}\sigma_i\,{}_{\mathrm{H}}I^{\alpha-1}f\bigl(\tau,x(\tau)\bigr)(\zeta_i)\Biggr]\Biggr\}\Biggr|\\
&\quad\le p^{\star}\frac{|(\log\omega_2)^{\alpha}-\log\omega_1)^{\alpha}|}{\Gamma(\alpha+1)}\\
&\qquad+p^{\star}\frac{(|\gamma-1|)|\delta_1||(\log\omega_2)^{\gamma-2}-\log\omega_1)^{\gamma-2}|+(|\gamma-2|)|\delta_2||(\log\omega_2)^{\gamma-1}-(\log\omega_1)^{\gamma-1}|}{|\lambda|\Gamma(\alpha+1)}\\
&\qquad\times\Biggl[\bigl(\log(1+\epsilon)\bigr)^{\alpha}+\sum_{i=1}^{n-2}|\nu_i|\bigl(\log(\zeta_i)\bigr)^{\alpha}\Biggr]\\
&\qquad+p^{\star}\frac{|\mu_2||(\log\omega_2)^{\gamma-1}-\log\omega_1)^{\gamma-1}|+|\mu_1||(\log\omega_2)^{\gamma-2}-(\log\omega_1)^{\gamma-2}|}{|\lambda|\Gamma(\alpha)}\\
&\qquad\times\Biggl[1+\sum_{i=1}^{n-2}|\sigma_i|\bigl(\log(\zeta_i)\bigr)^{\alpha-1}\Biggr].
\end{aligned}
$$

Clearly, as $\omega_2 \to \omega_1$, the right-hand side of the latter inequality tends to zero, which happens independently of $x \in B_{\bar{r}}$. Thus, by the Arzelà–Ascoli theorem, it follows that $\rho : K \to K$ is completely continuous.

Finally, let x be a solution. So, for $t \in J$, following similar computations as in the first step, we have

$$\begin{aligned}
\|x\| \le{}& \|q\|\vartheta\big(\|x\|\big)\frac{1}{\Gamma(\alpha+1)} \\
&+ \|q\|\vartheta\big(\|x\|\big)\frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|\Gamma(\alpha+1)}\left[\big(\log(1+\epsilon)\big)^{\alpha} + \sum_{i=1}^{n-2}|\nu_i|\big(\log(\zeta_i)\big)^{\alpha}\right] \\
&+ \|q\|\vartheta\big(\|x\|\big)\frac{|\mu_2| + |\mu_1|}{|\lambda|\Gamma(\alpha)}\left[1 + \sum_{i=1}^{n-2}|\sigma_i|\big(\log(\zeta_i)\big)^{\alpha-1}\right] \\
={}& \|q\|\vartheta\big(\|x\|\big)\Phi.
\end{aligned}$$

Thus, we have

$$\frac{\|x\|}{\|q\|\vartheta(\|x\|)\Phi} \le 1.$$

In view of (Q_5), there exists L such that $\|x\| \ne L$. Let us set

$$V = \big\{x \in K : \|x\| < L\big\}.$$

Note that the operator $\rho : \overline{V} \to K$ is continuous and completely continuous. From the choice of V, there is no $x \in \partial V$ such that $x = \bar{\lambda}\rho x$ for some $\bar{\lambda} \in (0,1)$. Thus, by Theorem 2.14, the operator ρ has a fixed point in $\overline{V}$, which is a solution of the boundary value problem (1.1). □

4 Example

Example 4.1 Consider the following boundary value problem for Hilfer–Hadamard-type fractional differential equation:

$$\begin{cases}
{}_{\mathrm{H}}D^{3/2,1/2}x(t) + f(t,x(t)) = 0, \quad t \in J := (1,e], \\
x(1.3) = \frac{1}{2}x(3/2) - \frac{3}{4}x(7/4), \\
{}_{\mathrm{H}}D^{1,1}x(e) = \frac{2}{3}\,{}_{\mathrm{H}}D^{1,1}x(3/2) + \frac{4}{3}\,{}_{\mathrm{H}}D^{1,1}x(7/4).
\end{cases} \tag{4.1}$$

Here, $\alpha = 3/2$, $\beta = 1/2$, $\gamma = 7/4$, $\nu_1 = 1/2$, $\nu_2 = -3/4$, $\sigma_1 = 2/3$, $\sigma_2 = 4/3$, $\zeta_1 = 3/2$, $\zeta_2 = 7/4$, $\epsilon = 0.3$, $1+\epsilon = 1.3$, and

$$f\big(t,x(t)\big) = \frac{(\sqrt{t} + \log t^2)}{2e^t(3+t)^2}\left(\frac{|x(t)|}{2+|x(t)|}\right).$$

Clearly,

$$\big|f(t,x) - f(t,y)\big| \le \frac{3}{64e}\big(|x-y|\big).$$

Hence, (Q_1) is satisfied with $C = \frac{3}{64e}$. We can show that

$$
\begin{aligned}
\mu_1 &= \big(\log(1+\epsilon)\big)^{\gamma-1} - \sum_{i=1}^{n-2} \nu_i \big(\log(\zeta_i)\big)^{\gamma-1} \approx 0.59779, \\
\mu_2 &= \big(\log(1+\epsilon)\big)^{\gamma-2} - \sum_{i=1}^{n-2} \nu_i \big(\log(\zeta_i)\big)^{\gamma-2} \approx 1.63780, \\
\delta_1 &= 1 - \sum_{i=1}^{n-2} \sigma_i \big(\log(\zeta_i)\big)^{\gamma-2} \approx -1.37703, \\
\delta_2 &= 1 - \sum_{i=1}^{n-2} \sigma_i \big(\log(\zeta_i)\big)^{\gamma-3} \approx -3.81518, \\
\lambda &= (\gamma-1)\delta_1\mu_2 - (\gamma-2)\delta_2\mu_1 \approx -2.26164, \\
\Phi &= \frac{1}{\Gamma(\alpha+1)} + \frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|\Gamma(\alpha+1)} \left[\big(\log(1+\epsilon)\big)^{\alpha} + \sum_{i=1}^{n-2} |\nu_i| \big(\log(\zeta_i)\big)^{\alpha} \right] \\
&\quad + \frac{|\mu_2| + |\mu_1|}{|\lambda|\Gamma(\alpha)} \left[1 + \sum_{i=1}^{n-2} |\sigma_i| \big(\log(\zeta_i)\big)^{\alpha-1} \right] \\
&\approx 3.835201, \\
C\Phi &= \frac{3}{64e}(3.835201) \approx 0.06613554378 < 1.
\end{aligned}
$$

Therefore, by Theorem 3.2, the boundary value problem (4.1) has a unique solution on J.

Example 4.2 Consider the following boundary value problem for Hilfer–Hadamard-type fractional differential equation:

$$
\begin{cases}
{}_{\mathrm{H}}D^{3/2,2/3}x(t) + f(t,x(t)) = 0, \quad t \in J := (1,e], \\
x(1.5) = 2x(4/3) - \frac{1}{2}x(2) + \frac{5}{3}x(9/7), \\
{}_{\mathrm{H}}D^{1,1}x(e) = -{}_{\mathrm{H}}D^{1,1}x(4/3) + 3D^{1,1}x(2) - \frac{11}{3}\, {}_{\mathrm{H}}D^{1,1}x(9/7).
\end{cases}
\tag{4.2}
$$

Here, $\alpha = 3/2$, $\beta = 2/3$, $\gamma = 11/6$, $\nu_1 = 2$, $\nu_2 = -1/2$, $\nu_3 = 5/3$, $\sigma_1 = -1$, $\sigma_2 = 3$, $\sigma_3 = -11/3$, $\zeta_1 = 4/3$, $\zeta_2 = 2$, $\zeta_2 = 9/7$, $\epsilon = 0.5$, $1+\epsilon = 1.5$, and

$$
f\big(t,x(t)\big) = \frac{(1+\log t)}{(t+1)^2}\left(\frac{|x(t)|+1}{3+|x(t)|}\right).
$$

Clearly,

$$
\begin{aligned}
\big|f(t,x)\big| &\le \left|\frac{(1+\log t)}{(t+1)^2}\left(\frac{|x(t)|+1}{3+|x(t)|}\right)\right| \\
&\le \left|(1+\log t)\left(\frac{|x(t)|+1}{12}\right)\right|.
\end{aligned}
$$

We choose $q(t) = 1 + \log t$ and $\vartheta(|x|) = (|x(t)| + 1)/12$. Then, we can show that

$$\mu_1 = \bigl(\log(1+\epsilon)\bigr)^{\gamma-1} - \sum_{i=1}^{n-2} \nu_i \bigl(\log(\zeta_i)\bigr)^{\gamma-1} \approx -0.395713,$$

$$\mu_2 = \bigl(\log(1+\epsilon)\bigr)^{\gamma-2} - \sum_{i=1}^{n-2} \nu_i \bigl(\log(\zeta_i)\bigr)^{\gamma-2} \approx -2.865742,$$

$$\delta_1 = 1 - \sum_{i=1}^{n-2} \sigma_i \bigl(\log(\zeta_i)\bigr)^{\gamma-2} \approx 3.65750,$$

$$\delta_2 = 1 - \sum_{i=1}^{n-2} \sigma_i \bigl(\log(\zeta_i)\bigr)^{\gamma-3} \approx 19.04369,$$

$$\lambda = (\gamma-1)\delta_1\mu_2 - (\gamma-2)\delta_2\mu_1 \approx -9.990516,$$

$$\begin{aligned}\Phi &= \frac{1}{\Gamma(\alpha+1)} + \frac{(|\gamma-1|)|\delta_1| + (|\gamma-2|)|\delta_2|}{|\lambda|\Gamma(\alpha+1)}\Biggl[\bigl(\log(1+\epsilon)\bigr)^{\alpha} + \sum_{i=1}^{n-2} |\nu_i| \bigl(\log(\zeta_i)\bigr)^{\alpha}\Biggr] \\ &\quad + \frac{|\mu_2| + |\mu_1|}{|\lambda|\Gamma(\alpha)}\Biggl[1 + \sum_{i=1}^{n-2} |\sigma_i| \bigl(\log(\zeta_i)\bigr)^{\alpha-1}\Biggr] \\ &\approx 3.414437455.\end{aligned}$$

Now, by (Q_5) we have

$$\frac{L}{(2)((L+1)/12)(3.414437455)} > 1.$$

Hence, $L > 1.320578171$. Therefore, by Theorem 3.5, the boundary value problem (4.2) has at least one solution on J.

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