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On a time fractional diffusion with nonlocal in time conditions

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Abstract

In this work, we consider a fractional diffusion equation with nonlocal integral condition. We give a form of the mild solution under the expression of Fourier series which contains some Mittag-Leffler functions. We present two new results. Firstly, we show the well-posedness and regularity for our problem. Secondly, we show the ill-posedness of our problem in the sense of Hadamard. Using the Fourier truncation method, we construct a regularized solution and present the convergence rate between the regularized and exact solutions.

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1 Introduction

Time-fractional partial differential equations are well known to describe modeling of anomalously slow transport processes. These models are often expressed in the form of fractional diffusion or subdiffusion equations which have many applications in various kinds of research areas, e.g., thermal diffusion in fractal domains [1] and protein dynamics [2], we can refer for more details to [3–6]. By replacing many differential operators of fractional order with different type of PDEs of integer order, we formulate various types of boundary value problems with fractional order. Let us refer to many papers [7–24] and the references therein.

Let T be a positive number. Let Ω be a bounded and smooth enough domain in $\mathbb{R}^N$. In our paper, we study the fractional diffusion equation with integral boundary conditions in the time variable on $[0,T]$ as follows:

$$\begin{cases} \partial_t^\alpha u(x,t) = \mathcal{A}u(x,t) + F(x,t), & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial n}(t,x) = 0, & x \in \partial\Omega, t \in (0,T], \\ au(x,T) + b\int_0^T u(t,x)\,dt = f(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\partial\Omega$ is a boundary of Ω and $a^2 + b^2 > 0$, $a \geq 0$, $b \geq 0$. Here, F is the source function, f is the given input data which is defined later, and $\mathcal{A}$ is a linear, unbounded, self-adjoint, and positive definite operator. The second equation in (1.1) is called the Neumann boundary condition. Our problem is to identify the function $u(\cdot,t)$ for $0 \le t < T$. Now, we explain the reason of the third equation which appears in our problem (1.1). As we know, most of some models on PDEs use an initial condition. However, in practice, some other models have to use nonlocal conditions, for example, including integrals over time intervals. Nonlocal conditions express and explain some full details about natural events because they consider additional information in the initial conditions. There are few papers on boundary conditions connecting the solution at different times, for instance, at initial time and at terminal time. The review of nonlocal initial conditions or nonlocal final conditions comes from real-life processes. For example, when the initial temperature or the final temperature for heat equation is not given immediately, but there is information regarding the temperature over a given period of time that can be described by a nonlocal initial condition. PDEs with nonlocal conditions were considered in many works, for example, see [25] for reaction-diffusion equations and [26–35] for some other PDEs. As we said before, there are not any results for considering our model (1.1) with the nonlocal final condition and the integral condition

$$au(x,T) + b\int_0^T u(x,t)\,dt = f(x). \tag{1.2}$$

The nonlocal integral condition (1.2) is motivated by the paper of Dokuchaev [36] where he investigated the well-posedness of Problem (1.1) in the integer order of derivative $\alpha = 1$. For more clarity, we can refer to some related works on our above models.

- If $a = 1$, $b = 0$, Problem (1.1) is called *backward in time* problem and is well known to be ill-posed in the sense of Hadamard. Some papers [9, 37] gave the ill-posedness and regularization results.
- If $a = 0$, $b = 1$, and $\alpha = 1$, Problem (1.1) and some similar models have been recently considered by Dokuchaev [36, 38], Volodymyr et al. [39], and Pulkina et al. [40] (see also the references therein).

It is obvious that our model in this paper is more complex than many previous models. In this paper, we consider the model with the nonlocal final and integral conditions for time fractional PDE. The main technique here is to use Fourier series in order to deal with the explicit solution for our problem. Now, we explain in more detail our studies that are the new and strong points of this article.

Two main results are described in this paper as follows:

- Our paper may be the first study on the existence and regularity of the solution of model (1.1) on Sobolev spaces.
- Our new second result is to give a regularization result when the noisy data $f^\delta \in L^q(\Omega)$ for $q \ge 1$. For case of L^2, we can use Parseval's equality directly to obtain stability estimates. However, Parseval's equality is impossible to apply for the L^p case with $p \ne 2$, which leads to the fact that some techniques here are more complex than the L^2 estimate. Our new techniques in this section are based on applying some Sobolev embedding. Based on the ideas of [7], we develop some similar techniques to deal with the L^p case. Let us emphasize that a regularized problem is not considered in [36].

This paper is organized as follows. In Sect. 2, we introduce some preliminaries and a mild solution. The main result in this section is to show the well-posedness of the mild solution. In Sect. 3, we show the ill-posedness and give a regularized solution.

2 Well-posedness and regularity

2.1 Preliminaries

Definition 2.1 (see [3]) The Mittag-Leffler function $E_{\alpha,\beta}(\cdot)$ is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta \in \mathbb{R}, z \in \mathbb{C}. \tag{2.1}$$

Lemma 2.1 (see [41]) *Let $0 < \beta < 1$. There exist two constants $M_1 > 0$, $M_2 > 0$ such that, for any $z > 0$,*

$$\frac{M_1}{1+z} \le E_{\beta,1}(-z) \le \frac{M_2}{1+z}. \tag{2.2}$$

Lemma 2.2 (see [41]) *Let $0 < \alpha < 1$ and $\lambda > 0$. Then*
(i) $\partial_t(E_\alpha(-\lambda t^\alpha)) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$ *for* $t > 0$;
(ii) $\partial_t(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)$ *for* $t > 0$.

Next, let us recall that $\mathcal{A}$, see [3]. We have the following equality:

$$\mathcal{A}\phi_k(x) = -\lambda_k \phi_k(x), \quad x \in \Omega; \qquad \phi_k = 0, \quad x \in \partial\Omega, k \in \mathbb{N}, \tag{2.3}$$

where $\{\lambda_k\}_{k=1}^{\infty}$ satisfies that

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \tag{2.4}$$

and $\lim_{k\to\infty} \lambda_k = \infty$. For any $q \ge 0$, we also define the space

$$D^q(\Omega) = \left\{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2q} \left| \langle u, \phi_k \rangle \right|^2 < +\infty \right\}, \tag{2.5}$$

then $D^q(\Omega)$ is a Hilbert space endowed with the norm

$$\|u\|_{D^q(\Omega)} = \left(\sum_{k=1}^{\infty} \lambda_k^{2q} \left| \langle u, \phi_k \rangle \right|^2 \right)^{\frac{1}{2}}. \tag{2.6}$$

Lemma 2.3 *The following inclusions hold true*:

$$\left.\begin{aligned} &L^p(\Omega) \hookrightarrow D(\mathcal{A}^\sigma), \quad \textit{if} -\tfrac{d}{4} < \sigma \le 0, p \ge \tfrac{2d}{d-4\sigma}, \\ &D(\mathcal{A}^\sigma) \hookrightarrow L^p(\Omega), \quad \textit{if}\, 0 \le \sigma < \tfrac{d}{4}, p \le \tfrac{2d}{d-4\sigma}. \end{aligned}\right\} \tag{2.7}$$

2.2 The mild solution of our problem

Theorem 2.1 *Let* $0 < \beta < \min(1, \frac{1}{2\alpha})$ *and* $s \geq \beta + 1$, $\alpha > 0$ *and*

$$G \in L^\infty\big(0, T; D\big(\mathcal{A}^{s-\beta}\big)\big) \cap L^2\big(0, T; D\big(\mathcal{A}^{s-\beta-1}\big)\big).$$

Then Problem (1.1) *has a unique mild solution* $u \in L^p(0, T; D(\mathcal{A}^s))$ *such that*

$$\begin{aligned} \|u\|_{L^p(0,T;D(\mathcal{A}^s))} &\lesssim \|f\|_{D(\mathcal{A}^s)} + \|G\|_{L^\infty(0,T;D(\mathcal{A}^{s-\beta}))} \\ &\quad + \|G\|_{L^2(0,T;D(\mathcal{A}^{s-\beta-1}))} + \|G\|_{L^\infty(0,T;L^2(\Omega))} \end{aligned} \tag{2.8}$$

for $1 < p < \frac{1}{\alpha}$.

Proof Assume that Problem (1.1) has a unique solution u which is given by

$$u(x,t) = \sum_{k=1}^{\infty} \big\langle u(\cdot,t), \phi_k \big\rangle \phi_k(x). \tag{2.9}$$

From the results in Yamamoto [42], we know that

$$\begin{aligned} \big\langle u(\cdot,t), \phi_k(x) \big\rangle &= E_{\alpha,1}\big(-\lambda_k t^\alpha\big) \big\langle u(\cdot,0), \phi_k(x) \big\rangle \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) G_k(\tau)\, d\tau. \end{aligned} \tag{2.10}$$

The nonlocal condition $au(x,T) + b\int_0^T u(x,t)\,dt = f(x)$ gives that

$$\begin{aligned} & a \sum_{k=1}^{\infty} E_{\alpha,1}\big(-\lambda_k T^\alpha\big) \big\langle u(\cdot,0), \phi_k \big\rangle \phi_k(x) \\ &\quad + a \sum_{k=1}^{\infty} \bigg(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (T-\tau)^\alpha\big) G_k(\tau)\, d\tau \bigg) \phi_k(x) \\ &\quad + b \int_0^T \Bigg(\sum_{k=1}^{\infty} E_{\alpha,1}\big(-\lambda_k t^\alpha\big) \big\langle u(\cdot,0), \phi_k(x) \big\rangle \phi_k(x) \Bigg)\, dt \\ &\quad + b \int_0^T \Bigg(\sum_{k=1}^{\infty} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) G_k(\tau)\, d\tau \big\rangle \phi_k(x) \Bigg)\, dt \\ & = \sum_{k=1}^{\infty} \big\langle f(x), \phi_k(x) \big\rangle \phi_k(x). \end{aligned} \tag{2.11}$$

This implies that

$$\begin{aligned} & \bigg(a E_{\alpha,1}\big(-\lambda_k T^\alpha\big) + b \int_0^T E_{\alpha,1}\big(-\lambda_k t^\alpha\big)\, dt \bigg) \big\langle u(\cdot,0), \phi_k(x) \big\rangle \\ &\quad + a \bigg(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (T-\tau)^\alpha\big) G_k(\tau)\, d\tau \bigg) \\ &\quad + b \int_0^T \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) G_k(\tau)\, d\tau\, dt = \big\langle f(x), \phi_k(x) \big\rangle. \end{aligned} \tag{2.12}$$

Therefore, we get that

$$\begin{aligned}\langle u(\cdot,0),\phi_k(x)\rangle &= \frac{\langle f(x),\phi_k\rangle - a\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)G_k(\tau)\,d\tau}{aE_{\alpha,1}(-\lambda_k T^\alpha)+b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\,dt} \\ &\quad - \frac{b\int_0^T\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)G_k(\tau)\,d\tau\,dt}{aE_{\alpha,1}(-\lambda_k T^\alpha)+b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\,dt}.\end{aligned} \tag{2.13}$$

This together with (2.9) implies that

$$\begin{aligned} u(x,t) &= \underbrace{\sum_{k=1}^\infty \frac{E_{\alpha,1}(-\lambda_k t^\alpha)\langle f(x),\phi_k(x)\rangle}{aE_{\alpha,1}(-\lambda_k T^\alpha)+b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\,dt}\phi_k(x)}_{\mathcal{H}_1(x,t)} \\ &\quad - \underbrace{\sum_{k=1}^\infty \frac{aE_{\alpha,1}(-\lambda_k t^\alpha)\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)G_k(\tau)\,d\tau}{aE_{\alpha,1}(-\lambda_k T^\alpha)+b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\,dt}\phi_k(x)}_{\mathcal{H}_2(x,t)} \\ &\quad - \underbrace{\sum_{k=1}^\infty \frac{bE_{\alpha,1}(-\lambda_k t^\alpha)\int_0^T\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^\alpha)G_k(\tau)\,d\tau\,dt}{aE_{\alpha,1}(-\lambda_k T^\alpha)+b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\,dt}\phi_k(x)}_{\mathcal{H}_3(x,t)} \\ &\quad + \underbrace{\sum_{k=1}^\infty \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(t-\tau)^\alpha\big)G_k(\tau)\,d\tau\,\phi_k(x)}_{\mathcal{H}_4(x,t)}. \end{aligned} \tag{2.14}$$

First, we need to show that

$$\frac{\overline{C}_1}{\lambda_k} \le aE_{\alpha,1}\big(-\lambda_k T^\alpha\big) + b\int_0^T E_{\alpha,1}\big(-\lambda_k t^\alpha\big)\,dt \le \frac{\overline{C}_2}{\lambda_k}. \tag{2.15}$$

Indeed, from Lemma 2.1, we have the following estimate:

$$\int_0^T E_{\alpha,1}\big(-\lambda_k t^\alpha\big)\,dt \le D_2\int_0^T \frac{dt}{1+\lambda_k t^\alpha} \le \frac{D_2}{\lambda_k}\int_0^T \frac{dt}{t^\alpha} = \frac{D_2 T^{1-\alpha}}{(1-\alpha)\lambda_k}. \tag{2.16}$$

Due to the inequality

$$E_{\alpha,1}\big(-\lambda_k T^\alpha\big) \le \frac{D_2}{1+\lambda_k T^\alpha} \le \frac{D_2}{\lambda_k T^\alpha},$$

we find the following bound:

$$aE_{\alpha,1}\big(-\lambda_k T^\alpha\big) + b\int_0^T E_{\alpha,1}\big(-\lambda_k t^\alpha\big)\,dt \le \frac{aD_2}{\lambda_k T^\alpha} + \frac{bD_2}{\lambda_k T^\alpha} = \frac{\overline{C}_2}{\lambda_k}. \tag{2.17}$$

Due to the fact that $1 + \lambda_k t^\alpha \le \lambda_k(\frac{1}{\lambda_1} + t^\alpha)$, we also get

$$\begin{aligned} \int_0^T E_{\alpha,1}\big(-\lambda_k t^\alpha\big)\,dt &\ge D_1 \int_0^T \frac{dt}{1+\lambda_n t^\alpha} \ge \frac{D_1}{\lambda_k}\int_0^T \frac{dt}{\frac{1}{\lambda_1}+t^\alpha} \\ &\ge \frac{D_1}{\lambda_n}\int_0^T \frac{dt}{\frac{1}{\lambda_1}+T^\alpha} = \frac{D_1}{\lambda_k}\frac{T}{\frac{1}{\lambda_1}+T^\alpha}. \end{aligned} \tag{2.18}$$

Hence, we find that

$$aE_{\alpha,1}\big(-\lambda_k T^\alpha\big) + b\int_0^T E_{\alpha,1}\big(-\lambda_k t^\alpha\big)\,dt \ge \frac{\overline{C}_1}{\lambda_k}. \tag{2.19}$$

Step 1. Estimate $\|\mathcal{H}_1(\cdot,t)\|_{D(\mathcal{A}^s)}$. By using Paseval's equality and recalling that

$$E_{\alpha,1}\big(-\lambda_k t^\alpha\big) \le \frac{D_2}{1+\lambda_k t^\alpha} \le D_2\lambda_k^{-1}t^{-\alpha}, \tag{2.20}$$

we obtain

$$\begin{aligned} \big\|\mathcal{H}_1(\cdot,t)\big\|_{D(\mathcal{A}^s)}^2 &= \sum_{k=1}^\infty \lambda_k^{2s}\left[\frac{E_{\alpha,1}(-\lambda_k t^\alpha)\langle f(x),\phi_k(x)\rangle}{aE_{\alpha,1}(-\lambda_k T^\alpha) + b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\,dt}\right]^2 \\ &\le \sum_{k=1}^\infty \lambda_k^{2s} D_2^2 \lambda_k^{-2} t^{-2\alpha}\left(\frac{\lambda_k}{\overline{C}_1}\right)^2 \big\langle f(x),\phi_k(x)\big\rangle^2. \end{aligned} \tag{2.21}$$

Hence, there exists $D_3 = D_2(\overline{C}_1)^{-1}$ such that

$$\big\|\mathcal{H}_1(\cdot,t)\big\|_{D(\mathcal{A}^s)} \le D_3 t^{-\alpha}\|f\|_{D(\mathcal{A}^s)}. \tag{2.22}$$

Step 2. Estimate $\|\mathcal{H}_2(\cdot,t)\|_{D(\mathcal{A}^s)}$. First, we see that, for any $0 < \beta < 1$,

$$E_{\alpha,\alpha}\big(-\lambda_k(T-\tau)^\alpha\big) \le \frac{D_2}{1+\lambda_k(T-\tau)^\alpha} \le D_2\lambda_k^{-\beta}(T-\tau)^{-\alpha\beta}. \tag{2.23}$$

This implies that

$$\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n(T-\tau)^\alpha\big)G_k(\tau)\,d\tau \le D_2\lambda_k^{-\beta}\int_0^T (T-\tau)^{\alpha-1-\alpha\beta}G_k(\tau)\,d\tau. \tag{2.24}$$

Thanks to inequality (2.20) we know that

$$\begin{aligned} &\big\|\mathcal{H}_2(\cdot,t)\big\|_{D(\mathcal{A}^s)}^2 \\ &\quad= \sum_{k=1}^\infty \lambda_k^{2s}\left[\frac{aE_{\alpha,1}(-\lambda_k t^\alpha)\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-\tau)^\alpha)G_k(\tau)\,d\tau}{aE_{\alpha,1}(-\lambda_k T^\alpha) + b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\,dt}\right]^2 \\ &\quad\le a^2\sum_{k=1}^\infty \lambda_k^{2s}D_2^2\lambda_k^{-2}t^{-2\alpha}\left(\frac{\lambda_k}{\overline{C}_1}\right)^2\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(T-\tau)^\alpha\big)G_k(\tau)\,d\tau\right)^2 \\ &\quad\le D_4^2 t^{-2\alpha}\sum_{k=1}^\infty \lambda_k^{2s-2\beta}\left(\int_0^T (T-\tau)^{\alpha-1-\alpha\beta}G_k(\tau)\,d\tau\right)^2. \end{aligned} \tag{2.25}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
&\left(\int_0^T (T-\tau)^{\alpha-1-\alpha\beta} G_k(\tau)\,d\tau\right)^2 \\
&\quad\le \left(\int_0^T (T-\tau)^{\alpha-1-\alpha\beta}\,d\tau\right)\left(\int_0^T (T-\tau)^{\alpha-1-\alpha\beta}\left|G_k(\tau)\right|^2 d\tau\right) \\
&\quad= \frac{T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}\left(\int_0^T (T-\tau)^{\alpha-1-\alpha\beta}\left|G_k(\tau)\right|^2 d\tau\right).
\end{aligned} \tag{2.26}$$

Combining (2.25) and (2.26), we find that

$$\begin{aligned}
\left\|\mathcal{H}_2(\cdot,t)\right\|^2_{D(\mathcal{A}^s)} &\le \frac{D_4^2 T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta} t^{-2\alpha}\left(\int_0^T (T-\tau)^{\alpha-1-\alpha\beta}\sum_{k=1}^{\infty}\lambda_k^{2s-2\beta}\left|G_k(\tau)\right|^2 d\tau\right) \\
&= \frac{D_4^2 T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta} t^{-2\alpha}\left(\int_0^T (T-\tau)^{\alpha-1-\alpha\beta}\left\|G(\tau)\right\|^2_{D(\mathcal{A}^{s-\beta})}\,d\tau\right) \\
&\le \frac{D_4^2 T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta} t^{-2\alpha}\|G\|^2_{L^\infty(0,T;D(\mathcal{A}^{s-\beta}))}\left(\int_0^T (T-\tau)^{\alpha-1-\alpha\beta}\,d\tau\right).
\end{aligned} \tag{2.27}$$

The latter estimate leads to

$$\left\|\mathcal{H}_2(\cdot,t)\right\|_{D(\mathcal{A}^s)} \le \frac{D_4 T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta} t^{-\alpha}\|G\|_{L^\infty(0,T;D(\mathcal{A}^{s-\beta}))}. \tag{2.28}$$

Step 3. Estimate $\|\mathcal{H}_3(\cdot,t)\|_{D(\mathcal{A}^s)}$. By using inequality (2.20), we obtain that

$$\begin{aligned}
&\left\|\mathcal{H}_3(\cdot,t)\right\|^2_{D(\mathcal{A}^s)} \\
&\quad= \sum_{k=1}^{\infty}\lambda_k^{2s}\left[\frac{aE_{\alpha,1}(-\lambda_k t^\alpha)\int_0^T\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\tau)^\alpha)G_k(\tau)\,d\tau\,dt}{aE_{\alpha,1}(-\lambda_k T^\alpha)+b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\,dt}\right]^2 \\
&\quad\le a^2\sum_{k=1}^{\infty}\lambda_k^{2s}D_2^2\lambda_k^{-2}t^{-2\alpha}\left(\int_0^T\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(t-\tau)^\alpha\big)G_k(\tau)\,d\tau\,dt\right)^2.
\end{aligned} \tag{2.29}$$

Next, by applying Hölder's inequality and (2.23) and noting that $0<\beta<\min(1,\frac{1}{2\alpha})$, we find that

$$\begin{aligned}
&\left|\int_0^T\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_k(t-\tau)^\alpha\big)G_k(\tau)\,d\tau\,dt\right| \\
&\quad\le D_2\lambda_k^{-\beta}\int_0^T\int_0^t (t-\tau)^{-\alpha\beta}G_k(\tau)\,d\tau\,dt \\
&\quad\le D_2\lambda_k^{-\beta}\int_0^T\left(\int_0^t (t-r)^{-2\alpha\beta}\,d\tau\right)^{1/2}\left(\int_0^t G_k^2(\tau)\,d\tau\right)^{1/2} dt \\
&\quad\le D_2\lambda_k^{-\beta}\sqrt{\frac{T^{1-2\alpha\beta}}{1-2\alpha\beta}}\int_0^T\left(\int_0^t G_k^2(\tau)\,d\tau\right)^{1/2} dt \\
&\quad\le D_2 T\lambda_k^{-\beta}\sqrt{\frac{T^{1-2\alpha\beta}}{1-2\alpha\beta}}\left(\int_0^T G_k^2(\tau)\,d\tau\right)^{1/2}.
\end{aligned} \tag{2.30}$$

By two above observations, we deduce that

$$\begin{aligned}\left\|\mathcal{H}_3(\cdot,t)\right\|^2_{D(\mathcal{A}^s)} &\le D_5^2 t^{-2\alpha}\sum_{k=1}^{\infty}\lambda_k^{2s-2-2\beta}\left(\int_0^T G_k^2(\tau)\,d\tau\right)\\ &\le D_5^2 t^{-2\alpha}\|G\|^2_{L^2(0,T;D(\mathcal{A}^{s-\beta-1}))}.\end{aligned}\tag{2.31}$$

Hence, we get immediately that

$$\left\|\mathcal{H}_3(\cdot,t)\right\|_{D(\mathcal{A}^s)} \le D_5 t^{-\alpha}\|G\|_{L^2(0,T;D(\mathcal{A}^{s-\beta-1}))}.\tag{2.32}$$

Step 4. Estimate $\|\mathcal{H}_4(\cdot,t)\|_{D(\mathcal{A}^s)}$. From inequality (2.23), we know that

$$\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_n(t-\tau)^\alpha\bigr)G_k(\tau)\,d\tau \le D_2\lambda_k^{-\beta}\int_0^t (t-\tau)^{\alpha-1-\alpha\beta}G_k(\tau)\,d\tau.\tag{2.33}$$

Hence, by applying Hölder's inequality, we get the following bound:

$$\begin{aligned}\left\|\mathcal{H}_4(\cdot,t)\right\|^2_{D(\mathcal{A}^s)} &= \sum_{k=1}^{\infty}\left[\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_k(t-\tau)^\alpha\bigr)G_k(\tau)\,d\tau\right]^2\\ &\le D_2^2\sum_{k=1}^{\infty}\lambda_k^{-2\beta}\left[\int_0^t (t-\tau)^{\alpha-1-\alpha\beta}G_k(\tau)\,d\tau\right]^2\\ &\le D_2^2\sum_{k=1}^{\infty}\lambda_k^{-2\beta}\left[\int_0^t (t-\tau)^{\alpha-1-\alpha\beta}\,d\tau\right]\left[\int_0^t (t-\tau)^{\alpha-1-\alpha\beta}\bigl|G_k(\tau)\bigr|^2\,d\tau\right]\\ &\le \frac{D_2^2T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}\sum_{k=1}^{\infty}\left(\int_0^t (t-\tau)^{\alpha-1-\alpha\beta}\sum_{k=1}^{\infty}\lambda_k^{-2\beta}\bigl|G_k(\tau)\bigr|^2\,d\tau\right)\\ &\le D_2^2\left(\frac{T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}\right)^2\|G\|^2_{L^\infty(0,T;D(\mathcal{A}^{-\beta}))}.\end{aligned}\tag{2.34}$$

Therefore, we obtain

$$\left\|\mathcal{H}_4(\cdot,t)\right\|_{D(\mathcal{A}^s)} \le D_2\left(\frac{T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}\right)\|G\|_{L^\infty(0,T;D(\mathcal{A}^{-\beta}))}.\tag{2.35}$$

Combining four steps as above, we deduce that

$$\begin{aligned}\left\|u(\cdot,t)\right\|_{D(\mathcal{A}^s)} &\le \sum_{j=1}^{4}\left\|\mathcal{H}_j(\cdot,t)\right\|_{D(\mathcal{A}^s)}\\ &\le D_3t^{-\alpha}\|f\|_{D(\mathcal{A}^s)} + \frac{D_4T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}t^{-\alpha}\|G\|_{L^\infty(0,T;D(\mathcal{A}^{s-\beta}))}\\ &\quad + D_5t^{-\alpha}\|G\|_{L^2(0,T;D(\mathcal{A}^{s-\beta-1}))} + D_2\left(\frac{T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}\right)\|G\|_{L^\infty(0,T;L^2(\Omega))}.\end{aligned}\tag{2.36}$$

Let us choose $1 < p < \frac{1}{\alpha}$. The latter estimate leads to

$$\begin{aligned}
\|u\|_{L^p(0,T;D(\mathcal{A}^s))} &= \left(\int_0^T \|u(\cdot,t)\|_{D(\mathcal{A}^s)}^p \,dt\right)^{1/p} \\
&\le D_3\|f\|_{D(\mathcal{A}^s)}\left(\int_0^T t^{-\alpha p}\,dt\right)^{1/p} \\
&\quad + \frac{D_4 T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}\|G\|_{L^\infty(0,T;D(\mathcal{A}^{s-\beta}))}\left(\int_0^T t^{-\alpha p}\,dt\right)^{1/p} \\
&\quad + D_5\|G\|_{L^2(0,T;D(\mathcal{A}^{s-\beta-1}))}\left(\int_0^T t^{-\alpha p}\,dt\right)^{1/p} \\
&\quad + D_2\left(\frac{T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}\right)\|G\|_{L^\infty(0,T;L^2(\Omega))}\left(\int_0^T dt\right)^{1/p}. \qquad (2.37)
\end{aligned}$$

Noting that the proper integral $\int_0^T t^{-\alpha p}\,dt$ is convergent, we deduce that

$$\begin{aligned}
\|u\|_{L^p(0,T;D(\mathcal{A}^s))} &\lesssim \|f\|_{D(\mathcal{A}^s)} + \|G\|_{L^\infty(0,T;D(\mathcal{A}^{s-\beta}))} \\
&\quad + \|G\|_{L^2(0,T;D(\mathcal{A}^{s-\beta-1}))} + \|G\|_{L^\infty(0,T;L^2(\Omega))}. \qquad (2.38)
\end{aligned}$$

□

3 Identification of the initial value in the case $G = 0$

In this section, we consider the problem of recovering the initial data $u(0,x)$ in the case $G = 0$.

3.1 The ill-posedness

Theorem 3.1 *The solution of Problem (1.1) is instability with respect to the L^2 norm in the case $t = 0$.*

Proof Let $u_0(x) = u(x,0)$. Let us consider the following operator $\mathcal{M} : L^2(\Omega) \to L^2(\Omega)$:

$$\begin{aligned}
\mathcal{M}u_0(x) &= \sum_{k=1}^{\infty}\left[aE_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b\int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right)dt\right]\langle u_0(x),\phi_k(x)\rangle\phi_k(x) \\
&= \int_\Omega p(x,\nu)u_0(\nu)\,d\nu, \qquad (3.1)
\end{aligned}$$

where we denote

$$p(x,\nu) = \sum_{k=1}^{\infty}\left[aE_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b\int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right)dt\right]\phi_k(x)\phi_k(\nu). \qquad (3.2)$$

It is obvious that $p(x,\nu) = p(\nu,x)$, we see that $\mathcal{M}$ is a self-adjoint operator. We will show that it is a compact operator. Consider the finite rank operator as follows:

$$\mathcal{M}_K u_0(x) = \sum_{k=1}^{K}\left[aE_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b\int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right)dt\right]\langle u_0,\phi_k\rangle\phi_k(x). \qquad (3.3)$$

It is obvious that $\mathcal{M}_K$ is a finite rank operator. It follows from (3.1) and (3.3) that

$$\begin{aligned}
&\|\mathcal{M}_K u_0 - \mathcal{M} u_0\|^2_{L^2(\Omega)} \\
&\quad = \sum_{k=K+1}^{\infty} \left[aE_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b\int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right) dt \right] \left\langle u_0(x), \phi_k(x)\right\rangle^2 \\
&\quad = |\overline{C}_2|^2 \sum_{j=K+1}^{\infty} \frac{1}{\lambda_k^2} \left\langle f(x), \varphi_n(x)\right\rangle^2 \le \frac{|\overline{C}_2|^2}{\lambda_K^2} \sum_{j=K+1}^{\infty} \left\langle u_0(x), \phi_k(x)\right\rangle^2,
\end{aligned} \tag{3.4}$$

where we note that

$$aE_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b\int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right) dt \le \frac{\overline{C}_2}{\lambda_k}.$$

Hence, we obtain that

$$\|\mathcal{M}_K u_0 - \mathcal{M} u_0\|_{L^2(\Omega)} \le \frac{\overline{C}_2}{\lambda_K}.\|u_0\|_{L^2(\Omega)}. \tag{3.5}$$

This leads to $\|\mathcal{M}_K - \mathcal{M}\|_{L^2(\Omega)} \to 0$ in the sense of operator norm in $L(L^2(\Omega);L^2(\Omega))$ as $K \to \infty$. Moreover, $\mathcal{M}$ is a compact operator. From (3.1), we get an operator equation as follows:

$$\mathcal{M} u_0(x) = f(x), \tag{3.6}$$

and by Kirsch [43] we conclude that the problem is ill-posed. Next, we continue to give an example for ill-posedness. Taking the input final data $f^j(x) = \frac{\phi_j(x)}{\sqrt{\lambda_j}}$. Then the initial data with respect to f^j is

$$\begin{aligned}
u_0^j(x) &= \sum_{k=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_k t^\alpha)\langle f^j(x), \phi_k(x)\rangle}{aE_{\alpha,1}(-\lambda_k T^\alpha) + b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\, dt} \phi_k(x) \\
&= \frac{\phi_j(x)}{\sqrt{\lambda_j}(aE_{\alpha,1}(-\lambda_k T^\alpha) + b\int_0^T E_{\alpha,1}(-\lambda_k t^\alpha)\, dt)}.
\end{aligned} \tag{3.7}$$

It is obvious that

$$\lim_{j\to+\infty} \|f^j\|_{L^2(\Omega)} = \lim_{j\to+\infty} \frac{1}{\sqrt{\lambda_j}} = 0. \tag{3.8}$$

An error in the L^2 norm of the initial data is as follows:

$$\|u_0^j\|_{L^2(\Omega)} = \left\| \frac{\phi_j(x)}{\sqrt{\lambda_j}(aE_{\alpha,1}(-\lambda_j T^\alpha) + b\int_0^T E_{\alpha,1}(-\lambda_j t^\alpha)\, dt)} \right\|_{L^2(\Omega)} \ge \frac{\sqrt{\lambda_j}}{\overline{C}_2}. \tag{3.9}$$

It is easy to see that

$$\lim_{j\to+\infty} \|u_0^j\|_{L^2(\Omega)} > \lim_{j\to+\infty} \frac{\sqrt{\lambda_j}}{\overline{C}_2} = +\infty. \tag{3.10}$$

Combining (3.8) and (3.10), we conclude that the solution is instability. □

3.2 Regularization and L^p error estimate

In this subsection, we construct a regularized solution and investigate the error between the regularized solution and the exact solution. Let us assume that f_δ is noisy data and satisfies that

$$\|f_\delta - f\|_{L^q(\Omega)} \le \delta \tag{3.11}$$

for any $q \ge 1$.

Theorem 3.2 *Let f_δ be as in (3.11). Let u_0 be the function which belongs to $D(\mathcal{A}^\sigma)$ for any $\sigma > 0$. Let us give a regularized solution as follows*:

$$u_0^\delta(x) = \sum_{k=1}^{\lambda_k \le M_\delta} \left[a E_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b \int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right) dt \right]^{-1} \left\langle f^\delta, \phi_k \right\rangle \phi_k(x). \tag{3.12}$$

Let us choose

$$M_\delta = \left(\frac{1}{\delta}\right)^{\frac{1-\mu}{m-h+1}}, \quad 0 < \mu < 1,$$

where

$$-\frac{d}{4} < h \le \min\left(0, \frac{(q-2)d}{4q}\right), \quad 0 \le m < \frac{d}{4}. \tag{3.13}$$

Then we get the following estimate:

$$\left\| u_0^\delta - u_0 \right\|_{L^{\frac{2d}{d-4m}}(\Omega)} \lesssim C_{h,q} \delta^\mu + \delta^{\frac{\sigma(1-\mu)}{m-h+1}} \|u_0\|_{D(\mathcal{A}^\sigma)}. \tag{3.14}$$

Proof Since the Sobolev embedding $L^q(\Omega) \hookrightarrow D(\mathcal{A}^h)$, we find that there exists a positive constant $C := C_{h,q}$

$$\left\|f^\delta - f\right\|_{D(\mathcal{A}^h)} \le C_{h,q} \left\|f^\delta - f\right\|_{L^q(\Omega)} \le C_{h,q} \delta. \tag{3.15}$$

For $m > 0$, we consider the term $\|u_0^\epsilon - u_0\|_{D(\mathcal{A}^m)}$. Using the triangle inequality, we obtain

$$\left\| u_0^\delta - u_0 \right\|_{D(\mathcal{A}^m)} \le \left\| u_0^\delta - \overline{u}_0^\delta \right\|_{D(\mathcal{A}^m)} + \left\| \overline{u}_0^\delta - u_0 \right\|_{D(\mathcal{A}^m)}, \tag{3.16}$$

where

$$\overline{u}_0^\delta(x) = \sum_{k=1}^{\lambda_k \le M_\delta} \left[a E_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b \int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right) dt \right]^{-1} \langle f, \phi_k \rangle \phi_k(x).$$

In the following, we first consider the term $\|u_0^\delta - \overline{u}_0^\delta\|_{D(\mathcal{A}^m)}$ for any $0 < m < \frac{d}{4}$. Indeed, we get

$$\begin{aligned}
&\left\| u_0^\delta - \overline{u}_0^\delta \right\|^2_{D(\mathcal{A}^m)} \\
&\quad = \sum_{k=1}^{\lambda_k \le \mathcal{M}_\delta} \lambda_k^{2m} \left[aE_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b \int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right) dt \right]^{-2} \left\langle f^\delta - f, \phi_k \right\rangle^2 \\
&\quad \le \sum_{k=1}^{\lambda_k \le \mathcal{M}_\delta} \lambda_k^{2m-2h} \left[aE_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b \int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right) dt \right]^{-2} \lambda_k^{2h} \left\langle f^\delta - f, \phi_k \right\rangle^2. \qquad (3.17)
\end{aligned}$$

From the fact that

$$aE_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b \int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right) dt \ge \frac{\overline{C}_1}{\lambda_k},$$

we get that

$$\left\| u_0^\delta - \overline{u}_0^\delta \right\|^2_{D(\mathcal{A}^m)} \le (\mathcal{M}_\delta)^{2m-2h+2} \left\| f^\delta - f \right\|^2_{D(\mathcal{A}^h)} \le C_{h,q}^2 \delta^2 (\mathcal{M}_\delta)^{2m-2h+2}. \qquad (3.18)$$

Next, we continue to get the following estimate:

$$\begin{aligned}
\left\| \overline{u}_0^\delta - u_0 \right\|^2_{D(\mathcal{A}^m)} &= \sum_{k=1}^{\lambda_k \rangle M_\delta} \left[aE_{\alpha,1}\left(-\lambda_k T^\alpha\right) + b \int_0^T E_{\alpha,1}\left(-\lambda_k t^\alpha\right) dt \right]^{-2} \langle f, \phi_k \rangle^2 \\
&= \sum_{k=1}^{\lambda_k \rangle M_\delta} \langle u_0, \phi_k \rangle^2 = \sum_{k=1}^{\lambda_k \rangle M_\delta} \lambda_k^{-2\sigma} \lambda_k^{2\sigma} \langle u_0, \phi_k \rangle^2 \\
&\le (\mathcal{M}_\delta)^{-2\sigma} \sum_{k=1}^{\lambda_k \rangle M_\delta} \lambda_k^{2\sigma} \langle u_0, \phi_k \rangle^2 \le (\mathcal{M}_\delta)^{-2\sigma} \|u_0\|^2_{D(\mathcal{A}^\sigma)}. \qquad (3.19)
\end{aligned}$$

From the Sobolev embedding $D(\mathcal{A}^m) \hookrightarrow L^{\frac{2d}{d-4m}}(\Omega)$ and combining (3.18) and (3.19), we get that

$$\begin{aligned}
\left\| u_0^\delta - u_0 \right\|_{L^{\frac{2d}{d-4m}}(\Omega)} &\le C \left\| u_0^\delta - u_0 \right\|_{D(\mathcal{A}^m)} \le \left\| u_0^\delta - \overline{u}_0^\delta \right\|_{D(\mathcal{A}^m)} + \left\| \overline{u}_0^\delta - u_0 \right\|_{D(\mathcal{A}^m)} \\
&\le C_{h,q} \delta (\mathcal{M}_\delta)^{m-h+1} + (\mathcal{M}_\delta)^{-\sigma} \|u_0\|_{D(\mathcal{A}^\sigma)}. \qquad (3.20)
\end{aligned}$$

The proof of Theorem 3.2 is completed. □

4 Conclusions

In this paper, we focus on the fractional diffusion equation with nonlocal integral condition. By using the mild solution in a Fourier series form and the Mittag-Leffler function, we show two results as follows. First of all, we show the properties of the well-posedness and regularity of the mild solution to this problem. Next, we present that our problem is ill-posed (in the sense of Hadamard). In addition, we construct a regularized solution and present the convergence rate between the regularized and exact solutions by the Fourier truncation method.

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References

1. Nigmatulin, R.R.: The realization of the generalized transfer equation in a medium with fractal geometry. Phys. Status Solidi B **133**, 425–430 (1986)
2. Odibat, Z., Baleanu, D.: Numerical simulation of initial value problems with generalized Caputo-type fractional derivatives. Appl. Numer. Math. **156**, 94–105 (2020)
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, New York (2006)
4. Nguyen, H.T., Nguyen, H.C., Wang, R., Zhou, Y.: Initial value problem for fractional Volterra integro-differential equations with Caputo derivative. Discr. Contin. Dyn. Syst., Ser. B **22**(11) (2017)
5. Caraballo, T., Guo, B., Tuan, N.H., Wang, R.: Asymptotically autonomous robustness of random attractors for a class of weakly dissipative stochastic wave equations on unbounded domains. Proc. R. Soc. Edinb., Sect. A, Math. (2020). https://doi.org/10.1017/prm.2020.77
6. Tuan, N.H., Van Au, V., Xu, R., Wang, R.: On the initial and terminal value problem for a class of semilinear strongly material damped plate equations. J. Math. Anal. Appl. **492**(2), 124481 (2020)
7. Tuan, N.H., Caraballo, T.: On initial and terminal value problems for fractional nonclassical diffusion equations. Proc. Am. Math. Soc. **149**(1), 143–161 (2021)
8. Tuan, N.H., Kirane, M., Hoan, L.V.C., Mohsin, B.B.: A regularization method for time-fractional linear inverse diffusion problems. Electron. J. Differ. Equ. **2016**, 290 (2016)
9. Tuan, N.H., Long, L.D., Thinh, N.V., Thanh, T.: On a final value problem for the time-fractional diffusion equation with inhomogeneous source. Inverse Probl. Sci. Eng. **25**, 1367–1395 (2017)
10. Tuan, N.H., Kirane, M., Hoan, L.V.C., Long, L.D.: Identification and regularization for unknown source for a time-fractional diffusion equation. Comput. Math. Appl. **73**, 931–950 (2017)
11. Tuan, N.H., Hoan, L.V.C., Tarta, S.: An inverse problem for an inhomogeneous time-fractional diffusion equation: a regularization method and error estimate. Comput. Appl. Math. **38**, 32 (2019)
12. Triet, N.A., Au, V.V., Long, L.D., Baleanu, D., Tuan, N.H.: Regularization of a terminal value problem for time fractional diffusion equation. Math. Methods Appl. Sci. **43**(6), 3850–3878 (2020)
13. Shiri, B., Wu, G.C., Baleanu, D.: Collocation methods for terminal value problems of tempered fractional differential equations. Appl. Numer. Math. **156**, 385–395 (2020)
14. Tuan, N.H., Long, L.D., Thinh, N.V.: Regularized solution of an inverse source problem for a time fractional diffusion equation. Appl. Math. Model. **40**(19–20), 8244–8264 (2016)
15. De Andrade, B., Cuevas, C., Soto, H.: On fractional heat equations with non-local initial conditions. Proc. Edinb. Math. Soc. **59**(1), 65–76 (2016)
16. Azevedo, J., Cuevas, C., Henriquez, E.: Existence and asymptotic behaviour for the time-fractional Keller–Segel model for chemotaxis. Math. Nachr. **292**(3), 462–480 (2019)
17. Dwivedi, K.D., Das, S., Baleanu, D.: Numerical solution of nonlinear space–time fractional-order advection–reaction–diffusion equation. J. Comput. Nonlinear Dyn. **15**(6), 061005 (2020)
18. Kumar, S., Baleanu, D.: Numerical solution of two-dimensional time fractional cable equation with Mittag-Leffler kernel. Math. Methods Appl. Sci. **43**(15), 8348–8362 (2020)
19. Afshari, H., Kalantari, S., Karapinar, E.: Solution of fractional differential equations via coupled fixed point. Electron. J. Differ. Equ. **2015**, 286 (2015)
20. Afshari, H., Karapınar, E.: A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces. Adv. Differ. Equ. **2020**, 616 (2020)
21. Afshari, H., Atapour, M., Karapınar, E.: A discussion on a generalized Geraghty multi-valued mappings and applications. Adv. Differ. Equ. **2020**, 356 (2020)

22. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solution of a boundary value problem associated with a fractional differential equation. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6652
23. Baitiche, Z., Derbazi, C., Benchohra, M.: ψ-Caputo fractional differential equations with multi-point boundary conditions by topological degree theory. Res. Nonlinear Anal. **3**, 167–178 (2020)
24. Benchohra, M., Slimane, M.: Fractional differential inclusions with non instantaneous impulses in Banach spaces. Res. Nonlinear Anal. **2**, 36–47 (2019)
25. Pao, C.V.: Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions. J. Math. Anal. Appl. **195**, 702–718 (1995)
26. Rassias, J.M., Karimov, E.T.: Boundary-value problems with non-local initial condition for degenerate parabolic equations. Contemp. Anal. Appl. Math. **1**(1), 42–48 (2013)
27. Rassias, J.M., Karimov, E.T.: Boundary-value problems with non-local initial condition for parabolic equations with parameter. Eur. J. Pure Appl. Math. **3**(6), 948–957 (2010)
28. Ashyralyev, A.: A note on the Bitsadze–Samarskii type nonlocal boundary value problem in a Banach space. J. Math. Anal. Appl. **344**(1), 557–573 (2008)
29. Ashyralyev, A., Ozturk, E.: On Bitsadze–Samarskii type nonlocal boundary value problems for elliptic differential and difference equations: well-posedness. Appl. Math. Comput. **219**(3), 1093–1107 (2012)
30. Patil, J., Chaudhari, A., Abdo, M.S., Hardan, B.: Upper and lower solution method for positive solution of generalized Caputo fractional differential equations. Adv. Theory Nonlinear Anal. Appl. **4**(4), 279–291 (2020)
31. Angelov, V.: Spin three-body problem of classical electrodynamics with radiation terms—(I) derivation of spin equations. Res. Nonlinear Anal. **4**(1), 1–20 (2021)
32. Afshari, H.: Solution of fractional differential equations in quasi-*b*-metric and *b*-metric-like spaces. Adv. Differ. Equ. **2019**, 285 (2019)
33. Afshari, H., Aydi, H., Karapinar, E.: On generalized Geraghty contractions on *b*-metric spaces. Georgian Math. J. **27**, 9–21 (2020)
34. Binh, T.T., Luc, N.H., O'Regan, D., Can, N.H.: On an initial inverse problem for a diffusion equation with a conformable derivative. Adv. Differ. Equ. **2019**, 481 (2019)
35. Tuan, N.H., Zhou, Y., Long, L.D., et al.: Identifying inverse source for fractional diffusion equation with Riemann–Liouville derivative. Comput. Appl. Math. **39**, 75 (2020)
36. Dokuchaev, N.: On recovering parabolic diffusions from their time-averages. Calc. Var. Partial Differ. Equ. **58**(1), Paper No. 27 (2019)
37. Tuan, N.H., Huynh, L.N., Ngoc, T.B., Zhou, Y.: On a backward problem for nonlinear fractional diffusion equations. Appl. Math. Lett. **92**, 76–84 (2019)
38. Dokuchaev, N.: Regularity of complexified hyperbolic wave equations with integral conditions (2019). https://arxiv.org/abs/1907.03527
39. Il'kiv, V.S., Nytrebych, Z.M., Pukach, P.Y.: Boundary-value problems with integral conditions for a system of Lamé equations in the space of almost periodic functions. Electron. J. Differ. Equ. **2016**, 304 (2016)
40. Pulkina, S.L., Savenkova, A.E.: A problem with a nonlocal, with respect to time, condition for multidimensional hyperbolic equations. Russ. Math. **60**(10), 33–43 (2016)
41. Podlubny, I.: Fractional Differential Equations. Academic Press, London (1999)
42. Sakamoto, K., Yamamoto, M.: Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. J. Math. Anal. Appl. **382**, 426–447 (2011)
43. Kirsch, A.: An Introduction to the Mathematical Theory of Inverse Problems. Applied Mathematical Sciences, vol. 120. Springer, New York (2011)