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On a nonlocal problem for parabolic equation with time dependent coefficients

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Abstract

This paper is devoted to the study of existence and uniqueness of a mild solution for a parabolic equation with conformable derivative. The nonlocal problem for parabolic equations appears in many various applications, such as physics, biology. The first part of this paper is to consider the well-posedness and regularity of the mild solution. The second one is to investigate the existence by using Banach fixed point theory.

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1 Introduction

Let T be a positive number and $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) be a bounded domain with the smooth boundary $\partial\Omega$. In this paper, we consider the nonlocal value problem for parabolic equation as follows:

$$\begin{cases} \frac{\mathcal{C}_{\partial}^{\beta}}{\partial t^{\beta}} u(x, t) - \psi(t) \Delta u = F(x, t, u(x, t)), & x \in \Omega, t \in (0, T), \\ u(x, t) = 0, & x \in \partial\Omega, t \in (0, T), \\ au(x, 0) + bu(x, T) = \varphi(x), & x \in \Omega, \end{cases} \quad (1)$$

where the symbol $\frac{\mathcal{C}_{\partial}^{\beta} v(t)}{\partial t^{\beta}}$ is called the conformable derivative which is defined clearly in Sect. 2. The function F represents external forces, and the function φ is the input datum which will be defined later. The function ψ is called time dependent coefficient.

There are applications of conformable derivative in various models, for example, the harmonic oscillator, the damped oscillator, and the forced oscillator (see [1]), electrical circuits (see [2]), chaotic systems in dynamics (see [3]), quantum mechanics [4]. Based on important notes in the article [5], we observe and think that studying the ODE problem with the compliance derivative is very different from studying the PDE problem with a suitable derivative. The positional results and methods for the ODE and PDEs models are not the same and completely different. In order for the reader to have more access to this kind of fractional diffusion equations with conformable derivative, we refer to [2, 6–17]. In addition, we can find the topics of initial and final problems, which are studied by many authors, in [18–27].

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Our paper is one of the first results on the nonlocal value problem given with parabolic equations with conformable derivative. In the linear part, we use the techniques of Hilbert scales space. In the nonlinear part, to establish the existence and uniqueness of the solution, we must use the Banach fixed mapping theorem combined with some techniques to evaluate inequality, some Sobolev embedding. One of the most difficult points is finding the right functional spaces for the solution. Another highlight in the results is to demonstrate the convergence of the mild solution as the parameter b approaches 0.

2 Preliminaries

Conformable derivative model: Let the function $v : [0, \infty) \rightarrow D$, where D is a Banach space. If the limitation

$$\frac{\mathcal{C}\partial^\beta v(t)}{\partial t^\beta} := \lim_{\varepsilon \rightarrow 0} \frac{v(t + \varepsilon t^{1-\beta}) - v(t)}{\varepsilon} \quad \text{in } D$$

for each $t > 0$ exists, then it is called be the conformable derivative of order $\beta \in (0, 1]$ of v . We can refer the reader to [11, 28].

Let \mathcal{A} be a linear, self-adjoint, unbounded, and positive definite operator. Assume that \mathcal{A} has the eigenvalues λ_n ($n \in \mathbb{N}^*$):

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{with } \lambda_n \rightarrow \infty \text{ for } n \rightarrow \infty$$

and the corresponding eigenelements e_n which form an orthonormal basis. For $\nu > 0$, we introduce fractional powers of \mathcal{A} as follows:

$$D(\mathcal{A}^\nu) = \left\{ g \in L^2(\Omega) : \sum_{n=1}^\infty |\langle g, e_n \rangle|^2 \lambda_n^{2\nu} < \infty \right\}. \tag{2}$$

The space $D(\mathcal{A}^\nu)$ is a Banach space in the following with the corresponding norm:

$$\|g\|_{D(\mathcal{A}^\nu)} := \left(\sum_{n=1}^\infty |\langle g, e_n \rangle|^2 \lambda_n^{2\nu} \right)^{\frac{1}{2}}, \quad g \in D(\mathcal{A}^\nu).$$

The information for negative fractional power $\mathcal{A}^{-\nu}$ can be defined by [29]. For any $m > 0$ and a Banach space \mathcal{B} , we introduce the following space:

$$C^m([0, T]; \mathcal{B}) = \left\{ v \in C([0, T]; \mathcal{B}) : \sup_{0 \leq s < t \leq T} \frac{\|v(\cdot, t) - v(\cdot, s)\|_{\mathcal{B}}}{|t - s|^m} < \infty \right\},$$

corresponding to the following norm:

$$\|v\|_{C^m([0, T]; \mathcal{B})} = \sup_{0 \leq s < t \leq T} \frac{\|v(\cdot, t) - v(\cdot, s)\|_{\mathcal{B}}}{|t - s|^m}.$$

Let $0 < m < 1$, and we recall the following space:

$$C^m((0, T]; \mathcal{B}) = \left\{ v \in C((0, T]; L^2(\Omega)) : \sup_{0 < t \leq T} t^m \|v(\cdot, t)\|_{\mathcal{B}} < \infty \right\},$$

with the norm $\|v\|_{C^m((0, T]; \mathcal{B})} := \sup_{0 < t \leq T} t^m \|v(\cdot, t)\|_{\mathcal{B}}$.

3 Inhomogeneous problem

In this section, we consider the nonlocal value problem for equation as follows:

$$\begin{cases} \frac{\partial^\beta}{\partial t^\beta} u(x, t) - \psi(t)\Delta u = F(x, t), & x \in \Omega, t \in (0, T), \\ u(x, t) = 0, & x \in \partial\Omega, t \in (0, T), \\ au(x, 0) + bu(x, T) = \varphi(x), & x \in \Omega, \end{cases} \tag{3}$$

where F is defined later.

3.1 Existence and uniqueness of the mild solution

In this subsection, we state the existence and uniqueness of the mild solution.

Theorem 3.1 *Let $a_0 \leq \psi(t) \leq b_0$, where a_0, b_0 are constants, $\varphi \in D(\mathcal{A}^{v+\theta})$ and $F \in L^\infty(0, T; D(\mathcal{A}^{v+\theta}))$, where $v > 0, 0 < \theta < 1$. Then we have the following regularity:*

$$\begin{aligned} \|u(\cdot, t)\|_{D(\mathcal{A}^v)} &\lesssim t^{-\beta\theta} (\|\varphi\|_{D(\mathcal{A}^{v+\theta})} + \|F\|_{L^{2r}(0, T; D(\mathcal{A}^{v+\theta}))}) \\ &\quad + \|F\|_{L^\infty(0, T; D(\mathcal{A}^{v+\theta}))}, \quad t > 0. \end{aligned} \tag{4}$$

Proof By a simple calculation, we get the following equality:

$$\begin{aligned} u_n(t) &= \langle u(\cdot, t), e_n \rangle \\ &= \exp\left(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds\right) \langle u(\cdot, 0), e_n \rangle \\ &\quad + \int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) F_n(u)(s) ds, \end{aligned} \tag{5}$$

where $F_n = \langle F, e_n \rangle$. Replacing t with T in the above expression, we get

$$\begin{aligned} u_n(T) &= \langle u(\cdot, T), e_n \rangle \\ &= \exp\left(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds\right) \langle u(\cdot, 0), e_n \rangle \\ &\quad + \int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) F_n(u)(s) ds. \end{aligned} \tag{6}$$

The condition $au(x, 0) + bu(x, T) = \varphi(x)$ gives the following result:

$$\begin{aligned} &\left(a + b \exp\left(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds\right)\right) u_n(0) \\ &\quad + b \int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) F_n(u)(s) ds = \varphi_n. \end{aligned} \tag{7}$$

By switching sides and combining them, we obtain the formula for $u_n(0)$ as follows:

$$u_n(0) = \frac{\varphi_n - b \int_0^T s^{\beta-1} \exp(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr) F_n(u)(s) ds}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)}. \tag{8}$$

Inserting (8) into the above formula (5) and after the reduced transformation, we arrive at

$$\begin{aligned} u_n(t) &= \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \varphi_n \\ &\quad - b \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) F_n(u)(s) ds \\ &\quad + \int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) F_n(u)(s) ds. \end{aligned} \tag{9}$$

By the properties of Fourier series, we get that

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} \underbrace{\frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \varphi_n e_n(x)}_{J_1(x,t)} \\ &\quad - b \underbrace{\sum_{n=1}^{\infty} \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \left(\int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) F_n(s) ds \right) e_n(x)}_{J_2(x,t)} \\ &\quad + \sum_{n=1}^{\infty} \left(\int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) F_n(s) ds \right) e_n(x). \end{aligned} \tag{10}$$

Using the inequality

$$\exp\left(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds\right) \leq \exp\left(-\lambda_n \frac{a_0 t^\beta}{\beta}\right) \leq C_\theta a_0^\mu \beta^{-\theta} \lambda_n^\theta t^{-\beta\theta},$$

we get

$$\begin{aligned} \|J_1(\cdot, t)\|_{D(A^v)}^2 &= \sum_{n=1}^{\infty} \lambda_n^{2v} \left(\frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \right)^2 \varphi_n^2 \\ &\lesssim t^{-2\beta\theta} \sum_{n=1}^{\infty} \lambda_n^{2v+2\theta} \varphi_n^2 = t^{-2\beta\theta} \|\varphi\|_{D(A^{v+\theta})}^2. \end{aligned} \tag{11}$$

Hence, we find that

$$\|J_1(\cdot, t)\|_{D(\mathcal{A}^v)} \lesssim t^{-\beta\theta} \|\varphi\|_{D(\mathcal{A}^{v+\theta})}. \tag{12}$$

We continue to estimate J_2 as follows:

$$\begin{aligned} & \|J_2(\cdot, t)\|_{D(\mathcal{A}^v)}^2 \\ &= b^2 \sum_{n=1}^{\infty} \lambda_n^{2v} \left(\frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \right)^2 \\ & \quad \times \left(\int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) F_n(s) ds \right)^2 \\ & \lesssim t^{-2\beta\theta} \sum_{n=1}^{\infty} \lambda_n^{2v+2\theta} \left(\int_0^T s^{\beta-1} |F_n(s)|^2 ds \right) = t^{-2\beta\theta} \left(\int_0^T s^{\beta-1} \|F(\cdot, s)\|_{D(\mathcal{A}^{v+\theta})}^2 ds \right). \end{aligned} \tag{13}$$

The Hölder inequality implies that

$$\int_0^T s^{\beta-1} \|F(\cdot, s)\|_{D(\mathcal{A}^{v+\theta})}^2 ds \leq \left(\int_0^T s^{(\beta-1)r^*} ds \right)^{\frac{1}{r^*}} \left(\int_0^T \|F(\cdot, s)\|_{D(\mathcal{A}^{v+\theta})}^{2r} ds \right)^{1/r}, \tag{14}$$

where $\frac{1}{r} + \frac{1}{r^*} = 1$. Let us choose $r > \frac{1}{\beta}$, we find that

$$\int_0^T s^{\beta-1} \|F(\cdot, s)\|_{D(\mathcal{A}^{v+\theta})}^2 ds \lesssim \|F\|_{L^{2r}(0, T; D(\mathcal{A}^{v+\theta}))}^2. \tag{15}$$

Inserting the latter estimate into (13), we arrive at

$$\|J_2(\cdot, t)\|_{D(\mathcal{A}^v)} \lesssim t^{-\beta\theta} \|F\|_{L^{2r}(0, T; D(\mathcal{A}^{v+\theta}))}. \tag{16}$$

The quantity J_3 is bounded by

$$\begin{aligned} & \|J_3(\cdot, t)\|_{D(\mathcal{A}^v)}^2 \\ &= \sum_{n=1}^{\infty} \lambda_n^{2v} \left(\int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) F_n(s) ds \right)^2. \end{aligned} \tag{17}$$

It follows from the inequality

$$\exp\left(-2\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) \leq \exp\left(-\lambda_n \frac{a_0(t^\beta - s^\beta)}{\beta}\right) \leq C_\theta a_0^\mu \beta^{-\theta} \lambda_n^\theta (t^\beta - s^\beta)^{-\theta}$$

for any $0 \leq s \leq t$, that

$$\|J_3(\cdot, t)\|_{D(\mathcal{A}^v)}^2 \lesssim \sum_{n=1}^{\infty} \lambda_n^{2v+2\theta} \left(\int_0^t s^{\beta-1} (t^\beta - s^\beta)^{-\theta} F_n^2(s) ds \right)$$

$$\begin{aligned}
 &\lesssim \int_0^t s^{\beta-1} (t^\beta - s^\beta)^{-\theta} \|F(\cdot, s)\|_{D(\mathcal{A}^{v+\theta})}^2 ds \\
 &= \|F\|_{L^\infty(0, T; D(\mathcal{A}^{v+\theta}))}^2 \int_0^t s^{\beta-1} (t^\beta - s^\beta)^{-\theta} ds.
 \end{aligned} \tag{18}$$

Next, we continue to compute the integral term. Set the variable $\vartheta = s^\beta$. Then we get $d\vartheta = \beta s^{\beta-1} ds$. Then it follows from $0 < \theta < 1$ that

$$\int_0^t s^{\beta-1} (t^\beta - s^\beta)^{-\theta} ds = \frac{1}{\beta} \int_0^{t^\beta} (t^\beta - \vartheta)^{-\theta} d\vartheta = \frac{t^{\beta(1-\theta)}}{\beta(1-\theta)} \leq \frac{T^{\beta(1-\theta)}}{\beta(1-\theta)}. \tag{19}$$

This together with (18) yields that

$$\|J_3(\cdot, t)\|_{D(\mathcal{A}^v)} \lesssim \|F\|_{L^\infty(0, T; D(\mathcal{A}^{v+\theta}))}. \tag{20}$$

Combining (12), (16), and (20), we deduce that

$$\begin{aligned}
 &\|u(\cdot, t)\|_{D(\mathcal{A}^v)} \\
 &\leq \sum_{j=1}^3 \|J_j(\cdot, t)\|_{D(\mathcal{A}^v)} \\
 &\lesssim t^{-\beta\theta} (\|\varphi\|_{D(\mathcal{A}^{v+\theta})} + \|F\|_{L^{2r}(0, T; D(\mathcal{A}^{v+\theta}))}) + \|F\|_{L^\infty(0, T; D(\mathcal{A}^{v+\theta}))}.
 \end{aligned} \tag{21}$$

□

4 The mild solution for nonlinear problem

By the previous section, we define the following definition of a mild solution of the problem as follows.

Definition 4.1 The function u is called a mild solution of the problem if u belongs to the space $L^\infty(0, T; L^2(\Omega))$ and it also satisfies equality (1).

We recall that the formula of the solution u is performed as the following form:

$$\begin{aligned}
 u(x, t) &= \sum_{n=1}^\infty \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \varphi_n e_n(x) \\
 &\quad - b \sum_{n=1}^\infty \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \\
 &\quad \times \left(\int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) F_n(u)(s) ds \right) e_n(x) \\
 &\quad + \sum_{n=1}^\infty \left(\int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) F_n(u)(s) ds \right) e_n(x).
 \end{aligned} \tag{22}$$

Theorem 4.1 Let $\varphi \in L^2(\Omega)$, and there exists a constant $K_f \geq 0$ such that

$$\|F(v_1)(\cdot, t) - F(v_2)(\cdot, t)\|_{L^2(\Omega)} \leq K_f \|v_1(\cdot, t) - v_2(\cdot, t)\|_{L^2(\Omega)}. \tag{23}$$

If the condition $1 > (\frac{b}{a} + 1) \frac{K_f T^\beta}{\beta}$ is true, then problem (22) has a mild solution u which belongs to the space $L^\infty(0, T; L^2(\Omega))$.

Proof Set the following:

$$\mathcal{H}(u)(t) = \mathcal{H}_0 + \mathcal{H}_1(u)(t) + \mathcal{H}_2(u)(t), \tag{24}$$

where

$$\mathcal{H}_0 = \sum_{n=1}^{\infty} \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \varphi_n e_n(x)$$

and

$$\begin{aligned} \mathcal{H}_1(u) &= -b \sum_{n=1}^{\infty} \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \\ &\quad \times \left(\int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) F_n(u)(s) ds \right) e_n(x) \end{aligned} \tag{25}$$

and

$$\mathcal{H}_{2,\beta}(u)(t) = \sum_{n=1}^{\infty} \left(\int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) F_n(u)(s) ds \right) e_n(x). \tag{26}$$

We have to show that the equation $\mathcal{H}u = u$ has a unique solution. Applying the Hölder inequality, we find that

$$\begin{aligned} &\| \mathcal{H}_{2,\beta}(u)(t) - \mathcal{H}_{2,\beta}(v)(t) \|_{L^2(\Omega)}^2 \\ &= \sum_{n=1}^{\infty} \left[\int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) (F_n(u)(s) - F_n(v)(s)) ds \right]^2 \\ &\leq \sum_{n=1}^{\infty} \left[\int_0^t s^{\beta-1} ds \right] \left[\int_0^t s^{\beta-1} \exp\left(-2\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) (F_n(u)(s) - F_n(v)(s))^2 ds \right] \\ &\leq \frac{T^\beta}{\beta} \left[\int_0^t s^{\beta-1} \|F(u)(\cdot, s) - F(v)(\cdot, s)\|_{L^2(\Omega)}^2 ds \right] \\ &\leq \frac{T^\beta}{\beta} K_f^2 \|u - v\|_{L^\infty(0, T; L^2(\Omega))}^2 \left[\int_0^t s^{\beta-1} ds \right] = \frac{K_f^2 T^{2\beta}}{\beta^2} \|u - v\|_{L^\infty(0, T; L^2(\Omega))}^2, \end{aligned} \tag{27}$$

and using the inequality

$$\left| -b \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \right| \leq \frac{b}{a},$$

we get that

$$\begin{aligned}
 & \|\mathcal{H}_{1,\beta}(u)(t) - \mathcal{H}_{1,\beta}(v)(t)\|_{L^2(\Omega)}^2 \\
 & \leq \frac{b^2}{a^2} \sum_{n=1}^{\infty} \left(\int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) (F_n(u)(s) - F_n(v)(s)) ds \right)^2 \\
 & \leq \frac{b^2}{a^2} \sum_{n=1}^{\infty} \left[\int_0^t s^{\beta-1} ds \right] \left(\int_0^T s^{\beta-1} \exp\left(-2\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) (F_n(u)(s) - F_n(v)(s))^2 ds \right) \\
 & \leq \frac{b^2}{a^2} \frac{T^\beta}{\beta} \left(\int_0^T s^{\beta-1} (F_n(u)(s) - F_n(v)(s))^2 ds \right) \leq \frac{b^2 K_f^2 T^{2\beta}}{a^2 \beta^2} \|u - v\|_{L^\infty(0,T;L^2(\Omega))}^2. \tag{28}
 \end{aligned}$$

Combining (27) and (28) leads to

$$\begin{aligned}
 & \|\mathcal{H}_\beta(u)(t) - \mathcal{H}_\beta(v)(t)\|_{L^2(\Omega)} \\
 & = \|\mathcal{H}_{1,\beta}(u)(t) - \mathcal{H}_{1,\beta}(v)(t)\|_{L^2(\Omega)} + \|\mathcal{H}_{2,\beta}(u)(t) - \mathcal{H}_{2,\beta}(v)(t)\|_{L^2(\Omega)} \\
 & \leq \frac{K_f T^\beta}{\beta} \left(1 + \frac{b}{a} \right) \|u - v\|_{L^\infty(0,T;L^2(\Omega))}. \tag{29}
 \end{aligned}$$

Since the left-hand side of the above observation is independent of t , we deduce that

$$\|\mathcal{H}_\beta(u) - \mathcal{H}_\beta(v)\|_{L^\infty(0,T;L^2(\Omega))} \leq \frac{K_f T^\beta}{\beta} \left(1 + \frac{b}{a} \right) \|u - v\|_{L^\infty(0,T;L^2(\Omega))}. \tag{30}$$

Let us choose T such that

$$T \leq \left(\frac{\beta a}{(a+b)K_f} \right)^{1/\beta},$$

we know that \mathcal{H} is a contraction mapping. Next, we continue to show that if $v \in L^\infty(0, T; L^2(\Omega))$, then $\mathcal{H}_\beta(v) \in L^\infty(0, T; L^2(\Omega))$. We only check that if $v = 0$, then

$$\mathcal{H}_\beta(v)(\cdot, t) = \sum_{n=1}^{\infty} \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \varphi_n e_n(x), \tag{31}$$

which allows us to obtain that

$$\begin{aligned}
 \|\mathcal{H}_\beta(v)(\cdot, t)\|_{L^2(\Omega)}^2 & = \sum_{n=1}^{\infty} \left(\frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \right)^2 \varphi_n^2 \\
 & \leq \frac{1}{a^2} \sum_{n=1}^{\infty} \varphi_n^2 \\
 & = \frac{1}{a^2} \|\varphi\|_{L^2(\Omega)}^2. \tag{32}
 \end{aligned}$$

The latter estimate leads to $\mathcal{H}_\beta : L^\infty(0, T; L^2(\Omega)) \rightarrow L^\infty(0, T; L^2(\Omega))$. By applying the Banach fixed point theory, we can deduce that our problem (1) has a unique solution $u \in L^\infty(0, T; L^2(\Omega))$. \square

Theorem 4.2 *Let w_a be the solution of the initial value problem*

$$\begin{cases} \frac{\partial^\beta}{\partial t^\beta} u(x, t) - \psi(t)\Delta u = F(x, t), & x \in \Omega, t \in (0, T), \\ u(x, t) = 0, & x \in \partial\Omega, t \in (0, T), \\ au(x, 0) = \varphi(x), & x \in \Omega. \end{cases} \tag{33}$$

Then we have

$$\lim_{b \rightarrow 0} \|w_{a,\beta} - u_{a,b,\beta}\|_{L^\infty(0,T;L^2(\Omega))} = 0 \tag{34}$$

and

$$\|w_{a,\beta} - u_{a,b,\beta}\|_{L^\infty(0,T;L^2(\Omega))} \leq \frac{b\|\varphi\|_{L^2(\Omega)}}{a^2(1 - (\frac{b}{a} + 1)\frac{K_f T^\beta}{\beta})}. \tag{35}$$

Proof It is easy to see that the mild solution of problem (33) is given by

$$\begin{aligned} &w_{a,\beta}(x, t) \\ &= \sum_{n=1}^{\infty} \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a} \varphi_n e_n(x) \\ &\quad - b \sum_{n=1}^{\infty} \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a} \\ &\quad \times \left(\int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) F_n(w_{a,\beta})(s) ds \right) e_n(x) \\ &\quad + \sum_{n=1}^{\infty} \left(\int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) F_n(w_{a,\beta})(s) ds \right) e_n(x). \end{aligned} \tag{36}$$

This together with (22) yields

$$\begin{aligned} &u_{a,b,\beta}(x, t) - w_{a,\beta}(x, t) \\ &= \sum_{n=1}^{\infty} \left(\frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a} - \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \right) \varphi_n e_n(x) \\ &\quad - b \sum_{n=1}^{\infty} \frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds)}{a} \\ &\quad \times \left[\int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) \left(F_n(w_{a,\beta})(s) - F_n(u_{a,b,\beta})(s) \right) ds \right] e_n(x) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{n=1}^{\infty} \left[\int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) \left(F_n(w_{a,\beta})(s) - F_n(u_{a,b,\beta})(s)\right) ds \right] e_n(x) \\
 & = A_1 + A_2 + A_3.
 \end{aligned} \tag{37}$$

The term A_1 is bounded by

$$\begin{aligned}
 \|A_1\|_{L^2(\Omega)}^2 & = \frac{b^2}{a^2} \sum_{n=1}^{\infty} \left(\frac{\exp(-\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds) \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)}{a + b \exp(-\lambda_n \int_0^{\frac{T^\beta}{\beta}} \psi(s) ds)} \right)^2 \varphi_n^2 \\
 & \leq \frac{b^2}{a^4} \sum_{n=1}^{\infty} \varphi_n^2 = \frac{b^2}{a^4} \|\varphi\|_{L^2(\Omega)}^2.
 \end{aligned} \tag{38}$$

Using $\exp(-2\lambda_n \int_0^{\frac{t^\beta}{\beta}} \psi(s) ds) \leq 1$, the term A_2 is bounded by

$$\begin{aligned}
 & \|A_2\|_{L^2(\Omega)}^2 \\
 & = \frac{b^2}{a^2} \sum_{n=1}^{\infty} \left[\int_0^T s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) \left(F_n(w_{a,\beta})(s) - F_n(u_{a,b,\beta})(s)\right) ds \right]^2 \\
 & \leq \frac{b^2}{a^2} \sum_{n=1}^{\infty} \left(\int_0^T s^{\beta-1} ds \right) \\
 & \quad \times \int_0^T s^{\beta-1} \exp\left(-2\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{T^\beta}{\beta}} \psi(r) dr\right) \left(F_n(w_{a,\beta})(s) - F_n(u_{a,b,\beta})(s)\right)^2 ds \\
 & \leq \frac{b^2}{a^2} \frac{T^\beta}{\beta} \left[\int_0^t s^{\beta-1} \|F(w_{a,\beta})(\cdot, s) - F(u_{a,b,\beta})(\cdot, s)\|_{L^2(\Omega)}^2 ds \right] \\
 & \leq \frac{b^2}{a^2} \frac{K_f^2 T^{2\beta}}{\beta^2} \|w_{a,\beta} - u_{a,b,\beta}\|_{L^\infty(0,T;L^2(\Omega))}^2.
 \end{aligned} \tag{39}$$

The term A_3 can be estimated as follows:

$$\begin{aligned}
 \|A_3\|_{L^2(\Omega)}^2 & = \sum_{n=1}^{\infty} \left[\int_0^t s^{\beta-1} \exp\left(-\lambda_n \int_{\frac{s^\beta}{\beta}}^{\frac{t^\beta}{\beta}} \psi(r) dr\right) \left(F_n(w_{a,\beta})(s) - F_n(u_{a,b,\beta})(s)\right) ds \right]^2 \\
 & \leq \sum_{n=1}^{\infty} \left[\int_0^T s^{\beta-1} ds \right] \left[\int_0^t s^{\beta-1} \|F(w_{a,\beta})(\cdot, s) - F(u_{a,b,\beta})(\cdot, s)\|_{L^2(\Omega)}^2 ds \right] \\
 & \leq \frac{K_f^2 T^{2\beta}}{\beta^2} \|w_{a,\beta} - u_{a,b,\beta}\|_{L^\infty(0,T;L^2(\Omega))}^2.
 \end{aligned} \tag{40}$$

Therefore, we find that

$$\|A_3\|_{L^2(\Omega)} \leq \frac{K_f T^\beta}{\beta} \|w_{a,\beta} - u_{a,b,\beta}\|_{L^\infty(0,T;L^2(\Omega))}. \tag{41}$$

Some above observations lead to

$$\begin{aligned} & \|w_{a,\beta} - u_{a,b,\beta}\|_{L^\infty(0,T;L^2(\Omega))} \\ & \leq \|A_1\|_{L^2(\Omega)} + \|A_2\|_{L^2(\Omega)} + \|A_3\|_{L^2(\Omega)} \\ & \leq \frac{b}{a^2} \|\varphi\|_{L^2(\Omega)} + \left(\frac{b}{a} + 1\right) \frac{K_f T^\beta}{\beta} \|w_{a,\beta} - u_{a,b,\beta}\|_{L^\infty(0,T;L^2(\Omega))}. \end{aligned} \quad (42)$$

Hence, we find that

$$\|w_{a,\beta} - u_{a,b,\beta}\|_{L^\infty(0,T;L^2(\Omega))} \leq \frac{b\|\varphi\|_{L^2(\Omega)}}{a^2\left(1 - \left(\frac{b}{a} + 1\right)\frac{K_f T^\beta}{\beta}\right)}. \quad (43)$$

It is easy to see that

$$\lim_{b \rightarrow 0} \|w_{a,\beta} - u_{a,b,\beta}\|_{L^\infty(0,T;L^2(\Omega))} = 0. \quad (44)$$

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