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Investigation of the neutral fractional differential inclusions of Katugampola-type involving both retarded and advanced arguments via Kuratowski MNC technique

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Abstract

A class of the boundary value problem is investigated in this research work to prove the existence of solutions for the neutral fractional differential inclusions of Katugampola fractional derivative which involves retarded and advanced arguments. New results are obtained in this paper based on the Kuratowski measure of noncompactness for the suggested inclusion neutral system for the first time. On the one hand, this research concerns the set-valued analogue of Mönch fixed point theorem combined with the measure of noncompactness technique in which the right-hand side is convex valued. On the other hand, the nonconvex case is discussed via Covitz and Nadler fixed point theorem. An illustrative example is provided to apply and validate our obtained results.

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1 Introduction

The theory of fractional differential equations, particularly that of fractional boundary value problems, has motivated several mathematicians, physicists, engineers, ecologists, and biologists to study them due to the variety of their applications in multidisciplinary sciences [1–14]. One of the main reasons for this particular interest in studying these equations is the fact that fractional formulations provide a powerful tool for modeling various scientific phenomena that exhibit memory effects. For more information about this interesting research study, some recent research studies have been conducted on fractional differential equations (FrDEqs) in [15–29]. Various fractional definitions of derivatives and integrals have been proposed by mathematicians, and some of the most common ones are the fractional derivatives of Riemann–Liouville and Caputo. Along with these, many generalized formulations of fractional derivatives have been introduced by researchers such

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as the generalized Caputo fractional derivative [30, 31] which was proposed by Katugam-pola and Almeida with a purpose to define a fractional derivative that satisfies the property of semigroup, and it is capable of combining other fractional derivatives [32].

Neutral fractional equations have attracted the interests of many researchers due to the important applications of these equations in physics and engineering, and these equations share some physical characteristics of wave equation due to the physical explanation of these equations as damped waves that have constant propagation velocities [33]. Agarwal et al. [34] proved the fractional formulation's existence of neutral differential equations via Caputo fractional derivative. One of the most interesting applications of the fractional formulation of neutral systems is the controllability problem [35] which is a very important problem to study the theory of control systems. For other instances, see [36–41]. In addition, many suggested problems of neutral fractional equations still are open and need further research work. As a result, studying these equations is worthy and makes all of our results new.

In [42], Boumaaza, Benchohra, and Tunc investigated the following fractional boundary value problem (FBVP):

$$
\begin{aligned}
&{}^{\varrho}_{c}D^{\xi}_{n^{+}}\big(k(t) - q\big(t,k^{t}\big)\big) \in K\big(t,k^{t}\big), \quad t \in \mathcal{J} := [n,m], 1 < \xi \leq 2, \\
&k(t) = \chi(t), \quad t \in [n-r,n], r > 0, \\
&k(t) = \psi(t), \quad t \in [m,m+\gamma], \gamma > 0,
\end{aligned}
\tag{1}
$$

where ${}^{\varrho}_{c}D^{\xi}_{n^{+}}$ is a modified Caputo formulation of the Erdélyi–Kober fractional derivative of order $1 < \xi \leq 2$. In 2016, Agarwal et al. [43] extended their study to a set-valued version of the functional FBVP subject to retarded-advanced arguments as

$$
\begin{aligned}
&D^{\xi}k(t) \in K\big(t,k^{t}\big), \quad t \in \mathcal{J} := [1,e], 1 < \xi < 2, \\
&k(t) = \phi(t), \quad t \in [1-r,1], r > 0, \\
&k(t) = \psi(t), \quad t \in [e,e+\gamma], \gamma > 0,
\end{aligned}
\tag{2}
$$

where $K : \mathcal{J} \times C([-r,\gamma],\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a multifunction. Regarding the existence of solutions for this FBVP, they focused on some standard fixed point methods.

Stimulated by the aforesaid research, this research work investigates the existence of solutions for the neutral fractional functional differential inclusions via Katugampola fractional derivative (KaFrD) which involves retarded and advanced arguments as follows:

$$
{}^{\varrho}D^{\xi}_{n^{+}}\big(w(t) - q\big(t,w^{t}\big)\big) \in K\big(t,w^{t}\big), \quad t \in \mathcal{J} := [n,m], 1 < \xi \leq 2,
\tag{3}
$$

$$
w(t) = \chi(t), \quad t \in [n-s,n], s > 0,
\tag{4}
$$

$$
w(t) = \psi(t), \quad t \in [m,m+\gamma], \gamma > 0,
\tag{5}
$$

where a given function $K : \mathcal{J} \times C([-s,\gamma],\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ exists so that $\chi, \psi \in \mathcal{C}([n-s,m+\gamma],\mathbb{R})$ via $\chi(n) = 0$ and $\psi(m) = 0$, and a given mapping $q : \mathcal{J} \times \mathcal{C}([-s,\gamma],\mathbb{R}) \to \mathbb{R}$ exists such that $q(n,\chi^{n}) = 0$ and $q(m,\psi^{m}) = 0$. The element of $\mathcal{C}([-s,\gamma],\mathbb{R})$, denoted by w^{t}, is defined as follows:

$$
w^{t} := w(t+\Omega), \quad t \in [-s,\gamma].
$$

Unlike the previous research works, we here implement our theoretical techniques on a generalized inclusion version of the neutral functional system via generalized derivative attributed to Katugampola for the first time. Due to the importance of such neutral systems, we prefer to extract the existence of solutions with the help of a generalized operator which covers some previous results by assuming special kernels. In addition, we utilize a new technique relying on the Kuratowski measure of noncompactness (KMNC) along with some standard methods. Indeed, the main difference and the novelty of this research with respect to other works is that the KMNC technique is used for the generalized version of a neutral functional FBVP (3)–(5).

This research paper is divided into the following sections: Some important definitions, which are needed to obtain our results in the other sections, are discussed in Sect. 2. Two interesting results are obtained in Sect. 3 in relation to the set-valued analogue of the Mönch fixed point theorem (MFPThm) (Theorem 2.9) and the Covitz and Nadler fixed point theorem (CNFPThm) (Theorem 2.10). In Sect. 4, an application example is provided to validate and apply our obtained results. We conclude our research study in Sect. 5.

2 Preliminaries

This section introduces some important fundamental definitions that will be needed for obtaining our results in the next sections.

2.1 Fundamental definitions

Assume that $X_c^l(n, m)$ ($c \in \mathbb{R}$, $1 \le l \le \infty$) denotes the space of real-valued Lebesgue measurable functions k on $[n, m]$ via the norm

$$\|k\|_{X_c^l} := \left(\int_n^m \left| v^c k(v) \right|^l \frac{dv}{v} \right)^{\frac{1}{l}} < \infty \quad (1 \le l < \infty)$$

and

$$\|k\|_{X_c^\infty} := \operatorname*{ess\,sup}_{n \le t \le m} \left(\left| v^c k(v) \right| \right).$$

Specifically, if $c = \frac{1}{l}$, then the space $X_c^l(n, m)$ coincides with the space $\mathcal{L}^l[n, m]$. Suppose that $\mathcal{C}[n, m]$ is a Banach space of continuous functions as $u : [n, m] \to \mathbb{R}$ with the norm

$$\|u\|_{[n,m]} := \sup \left\{ \left| u(t) \right| : t \in [n, m] \right\}.$$

Let us define the following space:

$$AC^1[n, m] := \left\{ q : [n, m] \to \mathbb{R}, \delta\left(q(x) \right) \in AC[n, m] \right\},$$

where $AC[n, m]$ is a set of absolutely continuous functions from $[n, m]$ into $\mathbb{R}$ with $\delta := t \frac{d}{dt}$. Also, define the following space:

$$Q := \left\{ w : [n - s, m + \gamma] \to \mathbb{R}, w|_{[n-s,n]} \in \mathcal{C}[n - s, n], w|_{[n,m]} \in AC^1[n, m], \right.$$
$$\left. w|_{[m,m+\gamma]} \in \mathcal{C}[m, m + \gamma] \right\} \tag{6}$$

with the norm

$$\|u\|_Q := \sup\{|u(t)| : t \in [n - s, m + \gamma]\}.$$

Definition 2.1 ([30]) Let $\xi > 0$, $\varrho > 0$. The left-sided (Katugampola) generalized integral of fractional integral order ξ for a function $z \in X_c^p(a, b)$ is defined by

$$^\varrho I_{n^+}^\xi z(t) := \frac{1}{\Gamma(\xi)} \int_n^t \left(\frac{t^\varrho - n^\varrho}{\varrho}\right)^{\xi-1} z(v) v^{\varrho-1} \, dv,$$

where the Euler gamma function is represented by $\Gamma(.)$ which is expressed as follows:

$$\Gamma(\xi) := \int_0^\infty t^{\xi-1} e^{-t} \, dt, \quad \xi > 0.$$

Definition 2.2 ([31]) Let $\xi > 0$, $\varrho > 0$. The left-sided (Katugampola) generalized derivative of order ξ for a given function $z \in X_c^l(n, m)$ is defined by

$$^\varrho D_{n^+}^\xi z(t) = \delta_\varrho^k \, {}^\varrho I_{n^+}^{k-\xi} z(t)$$

$$= \frac{1}{\Gamma(k - \xi)} \left(t^{1-\varrho} \frac{d}{dt}\right)^k \int_n^t \left(\frac{t^\varrho - n^\varrho}{\varrho}\right)^{k-\xi-1} z(v) v^{\varrho-1} \, dv,$$

where $k = [\xi] + 1$ and $[\xi]$ is the integer part of ξ and $\delta_\varrho^k := (t^{1-\varrho} \frac{d}{dt})^k$.

Let us discuss some essential properties of the fractional derivatives and integrals as follows.

Lemma 2.3 ([44]) *Assume that $\xi > 0$ and $\varrho > 0$. Then we have*

$$\left({}^\varrho I_{n^+}^\xi \, {}^\varrho D_{n^+}^\xi z\right)(t) = z(t) + c_1 \left(\frac{t^\varrho - n^\varrho}{\varrho}\right)^{\xi-1} + c_2 \left(\frac{t^\varrho - n^\varrho}{\varrho}\right)^{\xi-2} + \cdots + c_k \left(\frac{t^\varrho - n^\varrho}{\varrho}\right)^{\xi-o}, \quad (7)$$

where $c_j \in \mathbb{R}$, $j = 1, 2, \ldots, k$, and $k = [\xi] + 1$.

Lemma 2.4 ([44]) *If $x > n$, then we have*
- $[{}^\varrho I_{n^+}^\xi (\frac{t^\varrho - n^\varrho}{\varrho})^{\gamma-1}](x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\xi)} (\frac{x^\varrho - n^\varrho}{\varrho})^{\gamma+\xi-1}$,
- $[{}^\varrho D_{n^+}^\xi (\frac{t^\varrho - n^\varrho}{\varrho})^{\xi-1}](x) = 0$.

Assume that X and Y are Banach spaces and $\mathcal{P}(X) = \{G \subset X : G \neq \emptyset\}$, $\mathcal{P}_{cp}(X) = \{G \in \mathcal{P}(X) : G \text{ is closed}\}$, $\mathcal{P}_{bo}(X) = \{G \in \mathcal{P}(X) : G \text{ is bounded}\}$, $\mathcal{P}_{cs}(X) = \{G \in \mathcal{P}(X) : G \text{ is convex}\}$, $\mathcal{P}_{cl}(X) = \{G \in \mathcal{P}(X) : G \text{ is compact}\}$, and $\mathcal{P}_{cl,c}(X) = \mathcal{P}_{cl}(X) \cap \mathcal{P}_{cs}(X)$.

At $x_0 \in X$, a multifunction F is upper semicontinuous if for every open set O that includes Fx_0 a neighborhood S of x_0 exists with $F(S) \subset O$ [45].

For every $x \in X$, $F : \mathcal{J} \to \mathcal{P}_{cp}(X)$ is named measurable if the function, which is expressed as

$$t \longmapsto d(x, F(t)) = \inf\{\|x - h\| : h \in F(t)\},$$

is measurable [45].

Definition 2.5 ([45]) $K : \mathcal{J} \times X \to \mathcal{P}(X)$ is $\mathcal{L}^1$-Carathéodory if

(CON1) $\forall x \in X, t \to K(t,x)$ is measurable;

(CON2) $\forall t \in \mathcal{J}$ (a.e.), $x \to K(t,x)$ is upper semicontinuous;

(CON3) for all $g > 0$, $\varphi_g \in \mathcal{L}^1(\mathcal{J}, \mathbb{R}_+)$ exists provided

$$\left\| K(t,x) \right\|_{\mathcal{P}} = \sup\left\{ \|k\| : k \in K(t,x) \right\} \le \varphi_g(t), \quad \|x\| \le g, \forall t \in \mathcal{J} \text{ (a.e.)}.$$

The multi-valued map K is Carathéodory whenever (CON1) and (CON2) are satisfied.

For every $w \in \mathcal{C}(\mathcal{J}, X]$, let us define $S_{K,w}$ as the following set of selections of K:

$$S_{K,w} = \left\{ k \in \mathcal{L}^1(\mathcal{J}, X) : k(t) \in K\big(t, w(t)\big), t \in \mathcal{J} \text{ (a.e.)} \right\}.$$

Regard a metric space, denoted by (X, d). By defining $\mathcal{H}_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ as follows:

$$\mathcal{H}_d(N, M) = \max\left\{ \sup_{n \in N} d(n, M), \sup_{m \in M} d(N, m) \right\},$$

where $d(N, m) = \inf_{n \in N} d(n, m)$ and $d(n, M) = \inf_{m \in M} d(n, m)$, we have $(\mathcal{P}_{cp}(X), \mathcal{H}_d)$, which is an extension of the metric space with the completeness property [46].

2.2 Measure of noncompactness

This subsection discusses some necessary background information about the measure of noncompactness in Banach spaces.

Definition 2.6 ([47]) By assuming X as a Banach space, for every $M \in \mathcal{P}_{bo}(X)$, the Kuratowski measure of noncompactness (KMNC) is a mapping $\zeta : \mathcal{P}_m(X) \to [0, +\infty]$ which is constructed as follows:

$$\zeta(M) = \inf\left\{ s > 0 : M \subseteq \bigcup_{j=1}^{k} M_j \text{ and } \operatorname{diam}(M_j) < s \right\}.$$

The following properties are valid for the Kuratowski measure of noncompactness ζ.

Proposition 2.7 ([47–49]) *For all bounded subsets N, M of X:*

1. $\zeta(M) = 0 \iff \overline{M}$ *is compact.*

2. $\zeta(\emptyset) = 0$.

3. $\zeta(M) = \zeta(\overline{M}) = \zeta(\operatorname{conv} M)$, *where* $\operatorname{conv} M$ *is a convex hull of M.*

4. *monotonicity*: $(N \subset M) \implies \zeta(N) \le \zeta(M)$.

5. *algebraic semi-additivity*: $\zeta(N + M) \le \zeta(N) + \zeta(M)$,

6. *semi-additivity*: $\zeta(N \cup M) = \max\{\zeta(N), \zeta(M)\}$.

7. *semi-homogeneity*: $\zeta(\lambda M) = |\lambda| \zeta(M)$, $\lambda \in \mathbb{R}$,

8. *invariance under translations*: $\zeta(M + x_0) = \zeta(M)$ *for any $x_0 \in X$.*

Let us state some necessary theorems.

Theorem 2.8 ([50]) *Regard the Banach space X and a countable set $\mathcal{C} \subset \mathcal{L}^1[n,m]$ subject to $|u(t)| \leq z(t)$ for any $u \in \mathcal{C}$ and almost all $t \in \mathcal{J}$, where $z \in \mathcal{L}^1[n,m]$. Then $\phi(t) := \zeta(\mathcal{C}(t))$ is contained in $\mathcal{L}^1([n,m], \mathbb{R}_+)$ and*

$$\zeta\left(\left\{ \int_n^m u(v)\,dv : u \in \mathcal{C} \right\}\right) \leq 2 \int_n^m \zeta\big(\mathcal{C}(v)\big)\,dv$$

is satisfied.

Theorem 2.9 ([51]) *Consider G as a closed set contained in the Banach space X which has the convexity property; $\mathcal{U} \subset G$ as a relatively open set, and $\mathcal{E} : \overline{\mathcal{U}} \longrightarrow \mathcal{P}_c(G)$. Suppose that $\mathrm{Graph}(\mathcal{E})$ is closed and $\mathcal{E}$ corresponds compact sets to ones with the relative compactness, and that for some $w_0 \in \mathcal{U}$, the following assertions occur:*

  ($\mathfrak{F}1$) *If $\mathcal{M} \subset \overline{\mathcal{U}}$ with $\mathcal{M} \subset \mathrm{conv}(\mathcal{E}(\mathcal{M}) \cup \{w_0\})$ and $\overline{\mathcal{M}} = \overline{\mathcal{C}}$, where $\mathcal{C} \subset \mathcal{M}$ is countable, then $\overline{\mathcal{M}}$ is compact.*

  ($\mathfrak{F}2$) *For all $w \in \overline{\mathcal{U}} \setminus \mathcal{U}$ and $\sigma \in (0,1)$, $x \notin (1-\sigma)w_0 + \sigma \mathcal{E}(w)$.*

Then there exists $x \in \overline{\mathcal{U}}$ such that $x \in \mathcal{E}(x)$.

Theorem 2.10 (Nadler–Covitz, [45]) *Let (X,d) be a complete metric space. For the contraction $\mathcal{E} : X \to \mathcal{P}_{cl}(X)$, we have $\mathrm{Fix}(\mathcal{E}) \neq \phi$.*

3 Existence of solutions

This section investigates the existence of solutions for (3)–(5) by considering the normed space (6).

Definition 3.1 A function $w \in Q$ is named a solution of (3)–(5) if $\upsilon \in \mathcal{L}^1([n,m], X)$ exists subject to $\upsilon(t) \in K(t, w^t)$ (a.e.) on $[n,m]$ so that $^{\varrho}D_{n^+}^{\xi}(w(t) - q(t, w^t)) = \upsilon(t)$ on $\mathcal{J}$, $w(t) = \chi(t)$ on $[n-s,n], w(n) = 0$ and $w(t) = \psi(t)$ on $[m, m+\gamma], w(m) = 0$.

In order to prove that solutions exist for FBVP (3)–(5), we need the following.

Lemma 3.2 *Assume that $z : \mathcal{J} \to \mathbb{R}$ is an integrable function. A function w is a solution for a fractional equation, expressed as follows:*

$$w(t) = \begin{cases} \chi(t), & t \in [n-s,n], s > 0, \\ q(t, w^t) + \int_n^m F(t,v)z(v)v^{\varrho-1}\,dv, & t \in \mathcal{J}, \\ \psi(t), & t \in [m, m+\gamma], \gamma > 0, \end{cases} \tag{8}$$

iff w is a solution of the following FBVP:

$$^{\rho}D_{n^+}^{\xi}\big(w(t) - q(t, w^t)\big) = z(t), \quad t \in \mathcal{J} := [n,m], 1 < \xi \leq 2, \tag{9}$$

$$w(t) = \chi(t), \quad t \in [n-s,n], s > 0, \tag{10}$$

$$w(t) = \psi(t), \quad t \in [m, m+\gamma], \gamma > 0, \tag{11}$$

where

$$F(t,v) = \frac{1}{\Gamma(\xi)} \begin{cases} (\frac{t^{\varrho}-v^{\varrho}}{\varrho})^{\xi-1} - (\frac{t^{\varrho}-n^{\varrho}}{m^{\varrho}-n^{\varrho}})^{\xi-1}(\frac{m^{\varrho}-v^{\varrho}}{\varrho})^{\xi-1}, & n \leq t \leq v \leq m, \\ -(\frac{t^{\varrho}-n^{\varrho}}{m^{\varrho}-n^{\varrho}})^{\xi-1}(\frac{m^{\varrho}-v^{\varrho}}{\varrho})^{\xi-1}, & n \leq v \leq t \leq m. \end{cases} \tag{12}$$

Proof From (7), we have

$$w(t) - q(t, w^t) = \frac{1}{\Gamma(\xi)} \int_n^t \left(\frac{t^\varrho - v^\varrho}{\varrho} \right)^{\xi-1} z(v) v^{\varrho-1} \, dv$$
$$+ c_1 \left(\frac{t^\varrho - n^\varrho}{\varrho} \right)^{\xi-1} + c_2 \left(\frac{t^\varrho - n^\varrho}{\varrho} \right)^{\xi-2}. \tag{13}$$

Using $w(n) = q(n, \chi^n) = w(m) = q(m, \chi^m) = 0$, we find that $c_2 = 0$ and

$$c_1 = -\left(\frac{m^\varrho - n^\varrho}{\varrho} \right)^{1-\xi} \frac{1}{\Gamma(\xi)} \int_n^m \left(\frac{m^\varrho - v^\varrho}{\varrho} \right)^{\xi-1} z(v) v^{\varrho-1} \, dv.$$

By substituting the value of c_1 and c_2 in (13), we obtain

$$w(t) = q(t, w^t) + \frac{1}{\Gamma(\xi)} \int_n^t \left(\frac{t^\varrho - v^\varrho}{\varrho} \right)^{\xi-1} z(v) v^{\varrho-1} \, dv$$
$$- \left(\frac{t^\varrho - n^\varrho}{\varrho} \right)^{\xi-1} \left(\frac{m^\varrho - n^\varrho}{\varrho} \right)^{1-\xi} \frac{1}{\Gamma(\xi)} \int_n^m \left(\frac{m^\varrho - v^\varrho}{\varrho} \right)^{\xi-1} z(v) v^{\varrho-1} \, dv$$
$$= q(t, w^t) + \frac{1}{\Gamma(\xi)} \int_n^t \left[\left(\frac{t^\varrho - v^\varrho}{\varrho} \right)^{\xi-1} - \left(\frac{t^\varrho - n^\varrho}{m^\varrho - n^\varrho} \right)^{\xi-1} \left(\frac{m^\varrho - v^\varrho}{\varrho} \right)^{\xi-1} \right] z(v) v^{\varrho-1} \, dv$$
$$\times \frac{1}{\Gamma(\xi)} \int_t^m \left[-\left(\frac{t^\varrho - n^\varrho}{m^\varrho - n^\varrho} \right)^{\xi-1} \left(\frac{m^\varrho - v^\varrho}{\varrho} \right)^{\xi-1} \right] z(v) v^{\varrho-1} \, dv$$
$$= q(t, w^t) + \int_n^m F(t,s) z(v) v^{\varrho-1} \, dv,$$

where $F(t, v)$ is given by (12). On the contrary, if w satisfies equation (8), then equations (9)–(11) hold obviously and the argument is ended. □

Remark 3.3 The function $K : t \mapsto \int_n^m |F(t,v)| v^{\varrho-1} \, dv$ is continuous on $[n, m]$, and hence is bounded. Thus, we assume

$$\widetilde{F} := \sup \left\{ \int_n^m |F(t,v)| v^{\varrho-1} \, dv, t \in [n, m] \right\}.$$

Let us assume the following:
(A1) $K : \mathcal{J} \times \mathcal{C}[-s, \gamma] \to \mathcal{P}(\mathbb{R})$ is Carathéodory;
(A2) There exist $l \in \mathcal{L}^\infty(\mathcal{J}, \mathbb{R}_+)$ and constants $0 < c_1$ and $c_2 \geq 0$ provided

$$\|K(t, u)\|_{\mathcal{P}} := \sup\{|v| : v \in K(t, u)\} \leq l(t)(c_1 \|u\|_{[-s,\gamma]} + c_2)$$

for any $u \in \mathcal{C}[-s, \gamma]$ and a.e. $t \in \mathcal{J}$;
(A3) For each $\mathcal{M} \subset \mathcal{C}[n, m]$ and $t \in \mathcal{J}$,

$$\zeta(K(t, \mathcal{M}^t)) \leq l(t) \zeta(\mathcal{M}(t)),$$

where $\mathcal{M}$ is bounded and $\mathcal{M}^t := \{w^t : w \in \mathcal{M}\}$;

(A4) $\phi \equiv 0$ is assumed to be a unique solution contained in $\mathcal{C}[n, m]$ satisfying

$$\phi(t) \leq 2l^* \int_n^m F(t, v)\phi(v)v^{\varrho-1}\, dv,$$

by setting $l^* := \operatorname{ess\,sup}_{t \in \mathcal{J}} l(t)$;

(A5) q is a map with the complete continuity, and for $\mathcal{M} \subset Q$ with the boundedness specification, $\{t \mapsto q(t, w^t) : w \in \mathcal{M}\}$ is equicontinuous in $\mathcal{C}[n, m]$;

(A6) There are $0 \leq d_1 < 1$ and $d_2 \geq 0$ with

$$\left| q(t, h) \right| \leq d_1 \|h\|_{[-s, \gamma]} + d_2, \quad t \in \mathcal{J}, h \in \mathcal{C}[-s, \gamma];$$

(A7) $\ell_1 > 0$ exists subject to

$$H_d\big(K(t, x), K(t, \overline{x})\big) \leq \ell_1 \|x - \overline{x}\|_{[-s, \gamma]}, \quad \forall x, \overline{x} \in \mathcal{C}[-s, \gamma];$$

(A8) $\mathcal{L} > 0$ exists subject to

$$\left| q(t, h) - q(t, \overline{h}) \right| \leq \mathcal{L} \|h - \overline{h}\|_{[-s, \gamma]}$$

for every $h, \overline{h} \in \mathcal{C}[-s, \gamma]$.

3.1 The convex case

Our first result is based on the set-valued analogue of MFPThm (Theorem 2.9) and the notion of KMNC. Here K is assumed to have convex and compact values.

Theorem 3.4 *Assume that our assumptions* (A1)–(A6) *are settled. If*

$$d_1 + l^* c_1 \widetilde{F} < 1, \tag{14}$$

then the neutral functional FBVP (3)–(5) *has at least one solution* $w \in Q$.

Proof Let the constant R be such that

$$R \geq \max\big\{ l^*(c_1 R + c_2)\widetilde{F}, \|\chi\|_{[n-s, n]}, \|\psi\|_{[m, m+\gamma]} \big\} \tag{15}$$

and

$$G := \big\{ w \in Q : \|w\|_Q \leq R \big\}.$$

Obviously, G is closed along with the convexity property and contained in the Banach space Q introduced in (6). Consider the following multi-valued operator: $\mathcal{E} : Q \longrightarrow \mathcal{P}(Q)$ which is expressed as follows:

$$\mathcal{E}(w) = \left\{ \hbar \in Q : \hbar(t) = \begin{cases} \chi(t), & t \in [n-s, n], \\ q\big(t, w^t\big) + \int_n^m F(t, v)\upsilon(v)v^{\varrho-1}\, dv, & t \in \mathcal{J}, \upsilon \in S_{K, w} \\ \psi(t), & t \in [m, m+\gamma] \end{cases} \right\}. \tag{16}$$

It is obvious from Lemma 3.2 that possible solutions to the suggested neutral functional FBVP (3)–(5) correspond to possible fixed points of $\mathcal{E}$. We shall verify that $\mathcal{E}$ possesses given assumptions of Theorem 2.9. The following is our step-by-step deductive proof.

STEP I: $\mathcal{E}(W)$ is a set with the convexity property for all $w \in Q$.

By assuming $\hbar_1, \hbar_2$ belonging to $\mathcal{E}(w)$, in this case, $\upsilon_1, \upsilon_2 \in S_{K,w}$ exist so that, for all $t \in \mathcal{J}$, we obtain

$$\hbar_j(t) = q(t, w^t) + \int_n^m F(t,v)\upsilon_j(v)v^{\varrho-1}\, dv, \quad j = 1, 2.$$

Suppose that $0 \le d \le 1$. Thus

$$\big(d\hbar_1 + (1-d)\hbar_2\big)(t) = q(t, w^t) + \int_n^m F(t,v)\big(d\upsilon_1(v) + (1-d)\upsilon_2(v)\big)v^{\varrho-1}\, dv.$$

Since $S_{K,w}$ is convex due to the convexity of K, we get

$$d\hbar_1 + (1-d)\hbar_2 \in \mathcal{E}(w),$$

and our first claim is verified.

STEP II: By assuming $\mathcal{M} \subset Q$ as a set with the compactness specification, $\mathcal{E}(\mathcal{M})$ is relatively compact.

In order to show that, we assume that $\mathcal{M} \subset Q$ is compact and suppose that $(\hbar_k)$ is a sequence of points belonging to $\mathcal{E}(\mathcal{M})$. $(\hbar_k)$ will be shown to be a convergent subsequence via the Arzèla–Ascoli criterion of noncompactness in Q. If $t \in [n-s, n]$, then

$$\big\{\hbar_k(t) : k \ge 1\big\} = \big\{\chi(t)\big\},$$

and thus it has the relative compactness for a.e. $t \in [n-s, n]$. If $t \in [m, m+\gamma]$, then

$$\big\{\hbar_k(t) : k \ge 1\big\} = \big\{\psi(t)\big\},$$

and thus similarly it has the relative compactness for a.e. $t \in [m, m+\gamma]$. Now, assume that $t \in \mathcal{J}$. Since $\hbar_k \in \mathcal{E}(\mathcal{M})$, there exist $w_k \in \mathcal{M}$ and $\upsilon_k \in S_{K,w_k}$ such that

$$\hbar_k(t) = q(t, w_k^t) + \int_n^m F(t,v)\upsilon_k(v)v^{\varrho-1}\, dv.$$

From Theorem 2.8 and with the help of the properties of the KMNC denoted by ζ (presented in 2.7), we get

$$\zeta\big(\big\{\hbar_k(t) : k \ge 1\big\}\big) = \zeta\left(\left\{q(t, w_k^t) + \int_n^m F(t,v)\upsilon_k(v)v^{\varrho-1}\, dv : k \ge 1\right\}\right)$$

$$\le \zeta\big(\big\{q(t, w_k^t) : k \ge 1\big\}\big) + \zeta\left(\left\{\int_n^m F(t,v)\upsilon_k(v)v^{\varrho-1}\, dv : k \ge 1\right\}\right)$$

$$\le \zeta\big(\big\{q(t, w_k^t) : k \ge 1\big\}\big) + 2\int_n^m \zeta\big(\big\{F(t,v)\upsilon_k(v)v^{\varrho-1} : k \ge 1\big\}\big)\, dv. \quad (17)$$

On the contrary, since $\mathcal{M}$ is compact in Q and q is completely continuous, then $\mathcal{M}^t$ is compact in $\mathcal{C}([-s, \gamma])$, and the set $\{q(t, w^t) : w \in \mathcal{M}\}$ involves the relative compactness specification in $\mathbb{R}$; hence we have

$$\zeta\left(\left\{q\left(t, w_k^t\right) : k \geq 1\right\}\right) \leq \zeta\left(\left\{q\left(t, w^t\right) : w \in \mathcal{M}\right\}\right) = 0.$$

As a result, $\zeta(\{q(t, w_k^t) : k \geq 1\}) = 0$ for a.e. $t \in \mathcal{J}$. In addition, a set $\mathcal{M}^t$ is compact in $\mathcal{C}([-s, \gamma], \mathbb{R})$ and $\upsilon_k(\nu) \in K(\nu, w_k^\nu) \subset K(\nu, \mathcal{M}^\nu)$, and so $K(\nu, \mathcal{M}^\nu)$ is compact, then the set $\{\upsilon_k(\nu) : k \geq 1\}$ is relatively compact for a.e. $\nu \in \mathcal{J}$. Hence, $\zeta(\{\upsilon_k(\nu) : k \geq 1\}) = 0$ for a.e. $\nu \in \mathcal{J}$. Consequently,

$$\zeta\left(\left\{F(t, \nu)\upsilon_k(\nu)\nu^{\varrho-1} : k \geq 1\right\}\right) = F(t, \nu)\nu^{\varrho-1}\zeta\left(\left\{\upsilon_k(\nu) : k \geq 1\right\}\right) = 0$$

for $t, \nu \in \mathcal{J}$ (a.e.). In consequence, (17) indicates that $\{\hbar_k(t) : k \geq 1\}$ is relatively compact in $\mathbb{R}$. At present, for arbitrary $t_*, t^* \in \mathcal{J}$, with $t_* < t^*$, we have

$$\left|\hbar_k\left(t^*\right) - \hbar_k(t_*)\right|$$

$$= \left|q\left(t^*, w_k^{t^*}\right) - q\left(t_*, w_k^{t_*}\right)\right| + \int_n^m \left|F\left(t^*, \nu\right) - F(t_*, \nu)\right|\left|\upsilon_k(\nu)\right|\nu^{\varrho-1}\, d\nu$$

$$\leq \left|q\left(t^*, w_k^{t^*}\right) - q\left(t_*, w_k^{t_*}\right)\right| + l^*(c_1 R + c_2)\int_n^m \left|F\left(t^*, \nu\right) - F(t_*, \nu)\right|\nu^{\varrho-1}\, d\nu. \qquad (18)$$

By (A5), we have $|q(t^*, w_k^{t^*}) - q(t_*, w_k^{t_*})| \to 0$, as $t_* \to t^*$. As a result, as $t_* \to t^*$, inequality (18) goes to zero, which proves that $\{\hbar_k : k \geq 1\}$ is equicontinuous. Hence, $\{\hbar_k : k \geq 1\}$ possesses the relative compactness in Q. Thus $(\hbar_k)$ has a convergent subsequence. Therefore, $\mathcal{E}(\mathcal{M})$ is relatively compact.

  STEP III: $\mathcal{E}$ has a closed graph.
  Assume that $w_k \to w_*$, $\hbar_k \in \mathcal{E}(w_k)$, and $\hbar_k \to \hbar_*$. We need to show that $\hbar_* \in \mathcal{E}(w_*)$. Now, $\hbar_k \in \mathcal{E}(w_k)$ implies that there exists $\upsilon_k \in S_{K, w_k}$ such that, for all $t \in \mathcal{J}$,

$$\hbar_k(t) = q\left(t, w_k^t\right) + \int_n^m F(t, \nu)\upsilon_k(\nu)\nu^{\varrho-1}\, d\nu.$$

Let us prove that some $\upsilon_*(t) \in S_{K, w_*}$ can be chosen so that

$$\hbar_*(t) = q\left(t, w_*^t\right) + \int_n^m F(t, \nu)\upsilon_*(\nu)\nu^{\varrho-1}\, d\nu$$

for all $t \in \mathcal{J}$. Since $K(t, \cdot)$ is upper semicontinuous, so for each $\epsilon > 0$ there exists $k_0(\epsilon) \geq 0$ such that, for all $k \geq k_0$, we get

$$\upsilon_k(t) \in K\left(t, w_k^t\right) \subset K\left(t, w_*^t\right) + \epsilon\mathbb{BALL}(0, 1) \quad \text{(a.e.)}\ t \in \mathcal{J}.$$

On the other hand, due to the compactness of K, there is a subsequence $\upsilon_{k_r}(\cdot)$ such that

$$\upsilon_{k_r}(t) \to \upsilon_*(t) \quad \text{as } r \to \infty, \text{a.e. } t \in \mathcal{J};$$

and thus $\upsilon_*(t) \in K(t, w_*^t)$ for almost all $t \in \mathcal{J}$. Further, it is clear that

$$\left|\upsilon_{k_r}(t)\right| \le l(t)(c_1 R + c_2).$$

According to the theorem of Lebesgue dominated convergence, it follows that $\upsilon_* \in \mathcal{L}^1(\mathcal{J})$, which yields $\upsilon_* \in S_{K,w_*}$. In conclusion,

$$\hbar_*(t) = q\left(t, w_*^t\right) + \int_n^m F(t, v)\upsilon_*(v)v^{\varrho-1}\,dv, \quad t \in \mathcal{J}.$$

So $\hbar_* \in \mathcal{E}(w_*)$.

 STEP IV: Condition ($\mathfrak{F}2$) in Theorem 2.9 holds.

Assume that $w \in Q$ such that $w \in \sigma\mathcal{E}(w)$ with $\sigma \in (0, 1)$. Then there is $\upsilon \in S_{K,w}$ such that, for all $t \in \mathcal{J}$, we have

$$w(t) = \sigma q\left(t, w^t\right) + \sigma \int_n^m F(t, v)\upsilon(v)v^{\varrho-1}\,dv. \tag{19}$$

From (A2) and (A6), we get

$$\begin{aligned}
\left|w(t)\right| &\le \left|q\left(t, w^t\right)\right| + \int_n^m \left|F(t, v)\right|\left|\upsilon(v)\right|v^{\varrho-1}\,dv \\
&\le d_1 \left\|w^t\right\|_{[-s,\gamma]} + d_2 + \int_n^m \left|F(t, v)\right|l(v)\left(c_1 \left\|w^v\right\|_{[-s,\gamma]} + c_2\right)v^{\varrho-1}\,dv \\
&\le d_1 \|w\|_Q + d_2 + \int_n^n \left|F(t, v)\right|l(v)\left(c_1 \|w\|_Q + c_2\right)v^{\varrho-1}\,dv \\
&\le d_1 \|w\|_Q + d_2 + l^*\left(c_1 \|w\|_Q + c_2\right)\int_n^m \left|F(t, v)\right|v^{\varrho-1}\,dv \\
&\le \left(d_1 + l^* c_1 \widetilde{F}\right)\|w\|_Q + d_2 + l^*\widetilde{F}c_2.
\end{aligned}$$

Then

$$\|w\|_Q \le \left(d_1 + l^* c_1 \widetilde{F}\right)\|w\|_Q + d_2 + l^*\widetilde{F}c_2,$$

i.e.,

$$\left(1 - d_1 - l^* c_1 \widetilde{F}\right)\|w\|_Q \le d_2 + l^*\widetilde{F}c_2.$$

Thus, by (14), we have

$$\|w\|_Q \le \frac{d_2 + l^*\widetilde{F}c_2}{1 - d_1 - l^* c_1 \widetilde{F}} := \mu.$$

Set

$$\mathcal{U} := \left\{w \in Q : \|w\|_Q < \mu + 1\right\}.$$

By choosing our open set as $\mathcal{U}$, condition ($\mathfrak{F}2$) is satisfied.

STEP V: $\overline{\mathcal{M}}$ is compact in Q.

Assume that $\mathcal{M} \subset \overline{\mathcal{U}}$ and $\mathcal{M} \subset \mathrm{conv}(\mathcal{E}(\mathcal{M}) \cup \{0\})$, and for some countable set $\mathcal{C} \subset \mathcal{M}$, we have $\overline{\mathcal{M}} = \overline{\mathcal{C}}$. In view of (18), we figure out that $\mathcal{E}(\mathcal{M})$ is equicontinuous. In that case, from the inclusion $\mathcal{M} \subset \mathrm{conv}(\mathcal{E}(\mathcal{M}) \cup \{0\})$, we arrive at the result that $\mathcal{M}$ is equicontinuous. With the help of Ascoli–Arzela theorem, let us prove that $\mathcal{M}(t)$ possesses the relative compactness in $\mathbb{R}$ for any $t \in \mathcal{J}$. Since $\mathcal{C} \subset \mathcal{M} \subset \mathrm{conv}(\mathcal{E}(\mathcal{M}) \cup \{0\})$ and $\mathcal{C}$ is countable, thus another countable set $Z := \{\hbar_k : k \geq 1\} \subset \mathcal{E}(\mathcal{M})$ exists with $\mathcal{C} \subset \mathrm{conv}(Z \cup \{0\})$. Then there are two members $w_k \in \mathcal{M}$ and $\upsilon_k \in S_{K,w_k}$ such that

$$\hbar_k(t) = q\big(t, w_k^t\big) + \int_n^m F(t, v)\upsilon_k(v)v^{\varrho-1}\, dv.$$

From $\mathcal{M} \subset \overline{\mathcal{M}} = \overline{\mathcal{C}} \subset \overline{\mathrm{conv}}(Z \cup \{0\})$ and Theorem 2.8, the following is obtained:

$$\zeta\big(\mathcal{M}(t)\big) \leq \zeta\big(\overline{\mathcal{C}}(t)\big) \leq \zeta\big(Z(t)\big) \leq \zeta\big(\{\hbar_k(t) : k \geq 1\}\big).$$

From (17), we get

$$\zeta\big(\mathcal{M}(t)\big) \leq \zeta\big(\{q(t, w_k^t) : k \geq 1\}\big) + 2\int_n^m \zeta\big(\{F(t, v)\upsilon_k(v)v^{\varrho-1} : k \geq 1\}\big)\, dv.$$

On the contrary, since $\mathcal{M} \subset \overline{\mathcal{U}} \subset D_R$, $t, v \in \mathcal{J}$, then $\mathcal{M}$ is bounded ($\mathcal{M}^t$ is bounded); since q is completely continuous and $\{t\} \times \mathcal{M}^t$ is bounded, consequently the set $\{q(t, w^t) : w \in \mathcal{M}\} = q(t, \mathcal{M}^t)$ is relatively compact in $\mathbb{R}$. Hence

$$\zeta\big(\{q\big(t, w_k^t\big) : k \geq 1\}\big) \leq \{q\big(t, w^t\big) : w \in \mathcal{M}\} = 0.$$

Also, since $\upsilon_k \in S_{K,w_k}$ and $w_k^v \in \mathcal{M}^v$, then from (A3) we have

$$\begin{aligned}
\zeta\big(\{F(t, v)\upsilon_k(v)v^{\varrho-1} : k \geq 1\}\big) &= F(t, v)v^{\varrho-1}\zeta\big(\{\upsilon_k(v) : k \geq 1\}\big) \\
&\leq F(t, v)v^{\varrho-1}\zeta\Big(\bigcup_{k \geq 1} K\big(v, w_k^v\big)\Big) \\
&\leq F(t, v)v^{\varrho-1}\zeta\big(K\big(v, \mathcal{M}^v\big)\big) \\
&\leq F(t, v)v^{\varrho-1}l(s)\zeta\big(\mathcal{M}(s)\big),
\end{aligned}$$

which implies that

$$\zeta\big(\mathcal{M}(t)\big) \leq 2l^* \int_n^m F(t, v)v^{\varrho-1}\zeta\big(\mathcal{M}(v)\big)\, dv.$$

As a result, by (A4), a function: $\varphi(t) := \zeta(\mathcal{M}(t))$ satisfies $\varphi \equiv 0$ provided that $\zeta(\mathcal{M}(t)) = 0$ for any $t \in \mathcal{J}$. Hence $\mathcal{M}(t)$ involves the relative compactness specification in $\mathbb{R}$ for each $t \in \mathcal{J}$. In consequence, the theorem of Ascoli–Arzela gives the relative compactness of $\mathcal{M}$ in Q. At last, from the above steps and Theorem 2.9, $\mathcal{E}$ includes a fixed point $w \in \overline{\mathcal{U}}$ which displays a solution of the suggested neutral functional FBVP (3)–(5), and the argument is ended.   □

3.2 The nonconvex case

The existence of solutions for the suggested neutral functional FBVP (3)–(5) with nonconvex values in the right-hand side is investigated in this subsection via the standard method based on the fixed point result proposed by Nadler and Covitz in [45]. Let us suppose that K has compact values.

Theorem 3.5 *Assume that our assumptions, i.e.,* (A1)–(A2) *and* (A5)–(A8), *are satisfied. If we have*

$$(\mathcal{L} + \ell_1 \widetilde{F}) < 1, \tag{20}$$

then the neutral functional FBVP (3)–(5) *has at least one solution $w \in Q$.*

Proof For each $w \in Q$, the set $S_{K,w}$ is nonempty since by (A1) and by this fact that $\mathbb{R}$ is separable, K has a measurable selection (refer to [52] and Theorem III.6). We shall prove that $\mathcal{E}$ given by (16) satisfies the assumptions of Theorem 2.10. The following steps will provide our proof.

STEP I: $\mathcal{E}(w) \in \mathcal{P}_{cl}(Q)$ for all $w \in Q$.

Assume that $(\hbar_k)_{k \geq 0} \subset \mathcal{E}(w)$ is such that $\hbar_k \to \hbar_*$ on Q. Then there exists $\upsilon_k \in S_{K,w}$ so that, for any $t \in \mathcal{J}$,

$$\hbar_k(t) = q(t, w^t) + \int_n^m F(t, v) \upsilon_k(v) v^{\varrho-1} \, dv.$$

From (A3) and by the fact that K has compact values, we need to move to a subsequence in order to deduce that $\upsilon_k \to \upsilon_*$ weakly in $\mathcal{L}_y^1(\mathcal{J}, X)$, which is a space furnished with the weak topology. As a result, by a simple approach, it is verified that υ_k converges strongly to υ_*, and so $\upsilon_* \in S_{K,w}$. Hence, for any $t \in \mathcal{J}$,

$$\hbar_k(t) \to \hbar_*(t) = q(t, w^t) + \int_n^m F(t, v) \upsilon_*(v) v^{\varrho-1} \, dv.$$

Thus $\hbar_* \in \mathcal{E}(w)$ and $\mathcal{E}(w) \in \mathcal{P}_{cl}(Q)$ for all $w \in Q$.

STEP II: There exists $\beta < 1$ such that $\mathcal{H}_d(\mathcal{E}(w), \mathcal{E}(\overline{w})) \leq \beta \|w - \overline{w}\|_Q$ for all $w, \overline{w} \in Q$.

To verify this step, let $t \in \mathcal{J}$, $w^t, \overline{w}^t \in \mathcal{C}[-s, \gamma]$ and $\hbar_1 \in \mathcal{E}(w)$. Then $\upsilon_1(t) \in K(t, w^t)$ exists so that, for any $t \in \mathcal{J}$,

$$\hbar_1(t) = q(t, w^t) + \int_n^m F(t, v) \upsilon_1(v) v^{\varrho-1} \, dv.$$

From (A7), we obtain the following:

$$\mathcal{H}_d\big(K(t, w^t), K(t, \overline{w}^t)\big) \leq \ell_1 \big\| w^t - \overline{w}^t \big\|_{[-s, \gamma]}.$$

Thus, there exists $\theta \in K(t, \overline{w}^t)$ such that

$$\big| \upsilon_1(t) - \theta \big| \leq \ell_1 \big\| w^t - \overline{w}^t \big\|_{[-s, \gamma]}, \quad t \in \mathcal{J}.$$

At this moment, consider $\mathcal{H} : \mathcal{J} \to \mathcal{P}_{cp}(\mathbb{R})$ which is expressed as

$$\mathcal{H}(t) = \left\{ \theta \in \mathbb{R} : \left| \upsilon_1(t) - \theta \right| \leq \ell_1 \left\| w^t - \overline{w}^t \right\|_{[-s,\gamma]} \right\}.$$

Since $\mathcal{U}(t) = \mathcal{H}(t) \cap K(t, \overline{w}^t)$ is measurable (see [52]), thus υ_2 exists, which is a selection for $\mathcal{U}$ via the measurability property. Hence, we have $\upsilon_2(t) \in K(t, \overline{w}^t)$ and

$$\left| \upsilon_1(t) - \upsilon_2(t) \right| \leq \ell_1 \left\| w^t - \overline{w}^t \right\|_{[-s,\gamma]} \quad \forall t \in \mathcal{J}.$$

Now, introduce

$$\hbar_2(t) = q(t, \overline{w}^t) + \int_n^m F(t, v) \upsilon_2(v) v^{\varrho - 1} \, dv.$$

In that case, for $t \in \mathcal{J}$,

$$\begin{aligned}
\left| \hbar_1(t) - \hbar_2(t) \right| &\leq \left| q(t, w^t) - q(t, \overline{w}^t) \right| + \int_n^m \left| F(t, v) \right| \left| \upsilon_1(v) - \upsilon_2(v) \right| v^{\varrho - 1} \, dv \\
&\leq \mathcal{L} \left\| w^t - \overline{w}^t \right\|_{[-s,\gamma]} + \ell_1 \left\| w^t - \overline{w}^t \right\|_{[-s,\gamma]} \int_n^m \left| F(t, v) \right| v^{\varrho - 1} \, dv \\
&\leq \mathcal{L} \left\| w^t - \overline{w}^t \right\|_{[-s,\gamma]} + \ell_1 \widetilde{F} \left\| w_t - \overline{w}^t \right\|_{[-s,\gamma]} \\
&\leq \mathcal{L} \| w - \overline{w} \|_Q + \ell_1 \widetilde{F} \| w - \overline{w} \|_Q \\
&\leq (\mathcal{L} + \ell_1 \widetilde{F}) \| w - \overline{w} \|_Q.
\end{aligned}$$

Therefore, we have

$$\| \hbar_1 - \hbar_2 \|_Q \leq (\mathcal{L} + \ell_1 \widetilde{F}) \| w - \overline{w} \|_Q.$$

According to the analogous relation and interchanging the roles of w and $\overline{w}$, we arrive at

$$\mathcal{H}_d \big(\mathcal{E}(w), \mathcal{E}(\overline{w}) \big) \leq (\mathcal{L} + \ell_1 \widetilde{F}) \| w - \overline{w} \|_Q.$$

Therefore, by (20), $\mathcal{E}$ is a contraction, and according to Theorem 2.10, $\mathcal{E}$ possesses a fixed point w that is a solution to given neutral functional FBVP (3)–(5). Thus, the argument is fully completed. $\qquad \square$

4 Application: neutral functional FBVP

This section provides an illustrative example based on the neutral fractional functional BVP in order to validate and apply our results in Theorem 3.4.

Example 4.1 This example deals with the neutral fractional functional BVP illustrated as follows:

$$\begin{cases}
\frac{1}{2} D_{n^+}^{\frac{3}{2}} (w(t) - q(t, w^t)) \in K(t, w^t), & t \in \mathcal{J} := [1, 2], \\
w(t) = \chi(t), & t \in [0, 1], \\
w(t) = \psi(t), & t \in [2, 3].
\end{cases} \tag{21}$$

Set $K(t, w^t) = [k_1(t, w^t), f_2(t, w^t)]$, where

$$k_1 : \mathcal{J} \times \mathcal{C}\big([-1, 1], \mathbb{R}\big) \to \mathbb{R}$$

is defined by $k_1(t, u) = 0$ and

$$k_2 : \mathcal{J} \times \mathcal{C}\big([-1, 1], \mathbb{R}\big) \to \mathbb{R}$$

is defined by $k_2(t, u) = \frac{1}{2(t+2)}(\|u\|_{[-s,\gamma]} + 1)$. Let

$$q(t, u) = \frac{\|u\|_{[-1,1]}}{2(1 + \|u\|_{[-1,1]})}.$$

It is obvious that K has compact and convex values. Also, $K(\cdot, u) : \mathcal{J} \to \mathcal{P}_{cl}(\mathbb{R})$ is measurable for any $u \in \mathcal{C}([-s, \gamma], \mathbb{R})$. For each $t \in [1, 2]$, $k_1(t, \cdot)$ is lower semi-continuous and $k_2(t, \cdot)$ is upper semi-continuous, and also we get

$$\big\|K(t, u)\big\|_{\mathcal{P}} := \sup\big\{|\upsilon| : \upsilon \in K(t, u)\big\} \leq \frac{1}{2(t+2)}\big(\|u\|_{[-s,\gamma]} + 1\big)$$

for $t \in \mathcal{J}$ (a.e.) and $u \in \mathcal{C}[-s, \gamma]$, with $l(t) = \frac{1}{2(t+2)}$ and $c_1 = c_2 = 1$. This expresses that $u \to K(t, u)$ is upper semicontinuous for almost all $t \in \mathcal{J}$. Thus, conditions (A1)–(A4) and (A2) hold via $l^* = \frac{1}{6}$. On the other hand, it is obvious that q satisfies (A5) and (A6), and

$$\big\|q\big(t, w^t\big)\big\| = \frac{\|u\|_{[-1,1]}}{2(1 + \|u\|_{[-1,1]})} \leq \frac{1}{2}.$$

Hence, we obtain $d_1 = 0$ and $d_2 = \frac{1}{2}$. Moreover, for any $t \in \mathcal{J}$, we obtain

$$\int_n^m \big|F(t, v)\big| v^{\varrho-1}\, dv \leq \frac{1}{\Gamma(\xi)} \int_n^t \left(\frac{t^\varrho - v^\varrho}{\varrho}\right)^{\xi-1} v^{\varrho-1}\, dv$$

$$+ \left(\frac{t^\varrho - n^\varrho}{m^\varrho - n^\varrho}\right)^{\xi-1} \frac{1}{\Gamma(\xi)} \int_n^m \left(\frac{m^\varrho - v^\varrho}{\varrho}\right)^{\xi-1} v^{\varrho-1}\, dv$$

$$\leq \frac{1}{\Gamma(\xi)} \int_n^t \left(\frac{t^\varrho - v^\varrho}{\varrho}\right)^{\xi-1} v^{\varrho-1}\, dv$$

$$+ \frac{1}{\Gamma(\xi)} \int_n^m \left(\frac{m^\varrho - v^\varrho}{\varrho}\right)^{\xi-1} v^{\varrho-1}\, dv$$

$$\leq \frac{1}{\Gamma(\xi+1)} \left(\frac{t^\varrho - n^\varrho}{\varrho}\right)^\xi + \frac{1}{\Gamma(\xi+1)} \left(\frac{m^\varrho - n^\varrho}{\varrho}\right)^\xi$$

$$\leq \frac{2}{\Gamma(\xi+1)} \left(\frac{m^\varrho - n^\varrho}{\varrho}\right)^\xi.$$

Therefore, we get

$$\widetilde{F} \leq \frac{2}{\Gamma(\xi+1)} \left(\frac{m^\varrho - n^\varrho}{\varrho}\right)^\xi.$$

All of the hypotheses of Theorem 3.4 hold. So, when we have the following condition

$$
d_1 + l^* c_1 \widetilde{F} \le 0 + \frac{1}{6} \times 1 \times \frac{2}{\Gamma(\frac{5}{2})} \left(\frac{2^{\frac{1}{2}} - 1}{\frac{1}{2}} \right)^{\frac{3}{2}}
$$

$$
\le \frac{1}{3\Gamma(\frac{5}{2})} \left(\frac{2^{\frac{1}{2}} - 1}{\frac{1}{2}} \right)^{\frac{3}{2}} \simeq 0.189070603 < 1,
$$

we have used Theorem 3.4, and it implies that the simulative neutral functional inclusion FBVP (21) has at least one solution $w \in Q$.

Example 4.2　This example deals with the neutral fractional functional BVP illustrated as follows:

$$
\begin{cases}
\frac{1}{2} D_{n^+}^{\frac{3}{2}} (w(t) - q(t, w^t)) \in K(t, w^t), & t \in \mathcal{J} := [1, 2], \\
w(t) = \chi(t), & t \in [0, 1], \\
w(t) = \psi(t), & t \in [2, 3].
\end{cases}
\tag{22}
$$

Set $K(t, w^t) = K_1(t, w^t) \cup K_2(t, w^t)$, where $K_1(t, w^t) = [k_1(t, w^t), k_2(t, w^t)]$ and $K_2(t, w^t) = [k_3(t, w^t), k_4(t, w^t)]$, so that

$$
k_1 : \mathcal{J} \times \mathcal{C}\big([-1, 1], \mathbb{R}\big) \to \mathbb{R}
$$

is defined by $k_1(t, u) = 0$ and

$$
k_2 : \mathcal{J} \times \mathcal{C}\big([-1, 1], \mathbb{R}\big) \to \mathbb{R}
$$

is defined by $k_2(t, u) = \frac{1}{2(t+4)} (\|u\|_{[-s,\gamma]} + 1)$.

$$
k_3 : \mathcal{J} \times \mathcal{C}\big([-1, 1], \mathbb{R}\big) \to \mathbb{R}
$$

is defined by $k_3(t, u) = \frac{1}{2(t+3)} (\|u\|_{[-s,\gamma]} + 1)$.

$$
k_4 : \mathcal{J} \times \mathcal{C}\big([-1, 1], \mathbb{R}\big) \to \mathbb{R}
$$

is defined by $k_4(t, u) = \frac{1}{2(t+2)} (\|u\|_{[-s,\gamma]} + 1)$. Let

$$
q(t, u) = \frac{\|u\|_{[-1,1]}}{2(1 + \|u\|_{[-1,1]})}.
$$

It is obvious that K has compact and nonconvex values. Also, $K(\cdot, u) : \mathcal{J} \to \mathcal{P}_{cl}(\mathbb{R})$ is measurable for any $u \in \mathcal{C}([-s, \gamma], \mathbb{R})$. We get

$$
\big\| K(t, u) \big\|_{\mathcal{P}} := \sup\{ |\upsilon| : \upsilon \in K(t, u) \} \le \frac{1}{2(t+2)} \big(\|u\|_{[-s,\gamma]} + 1 \big)
$$

for $t \in \mathcal{J}$ (a.e.) and $u \in \mathcal{C}[-s, \gamma]$, with $l(t) = \frac{1}{2(t+2)}$ and $c_1 = c_2 = 1$. This expresses that $u \to K(t, u)$ is upper semi-continuous for almost all $t \in \mathcal{J}$. Thus, conditions (A1)–(A2) hold via

$l^* = \frac{1}{6}$. On the other hand, it is obvious that q satisfies (A5) and (A6), and

$$\|q(t, w^t)\| = \frac{\|u\|_{[-1,1]}}{2(1 + \|u\|_{[-1,1]})} \leq \frac{1}{2}.$$

Hence, we arrive at $d_1 = 0$ and $d_2 = \frac{1}{2}$. Along with these, for any $t \in \mathcal{J}$, we obtain

$$H_d\big(K(t,x), K(t,\overline{x})\big) \leq \frac{1}{2(t+2)} \|x - \overline{x}\|_{[-s,\gamma]}, \quad \forall x, \overline{x} \in \mathcal{C}[-s, \gamma];$$

$$H_d\big(K(t,x), K(t,\overline{x})\big) \leq \frac{1}{4} \|x - \overline{x}\|_{[-s,\gamma]}, \quad \forall x, \overline{x} \in \mathcal{C}[-s, \gamma];$$

$$H_d\big(K(t,x), K(t,\overline{x})\big) \leq \ell_1 \|x - \overline{x}\|_{[-s,\gamma]}, \quad \forall x, \overline{x} \in \mathcal{C}[-s, \gamma].$$

Therefore, (A7) holds. We have

$$\|q(t, w^t) - q(t, w_1^t)\| \leq \frac{1}{2}\big(\|w^t\|_{[-s,\gamma]} - \|w_1^t\|_{[-s,\gamma]}\big)$$
$$\leq \frac{1}{2}\|w^t - w_1^t\|_{[-s,\gamma]}$$

and

$$\widetilde{F} \leq \frac{2}{\Gamma(\xi + 1)}\left(\frac{m^\varrho - n^\varrho}{\varrho}\right)^\xi \quad \text{(see Example 4.1)}.$$

All of the hypotheses of Theorem 3.5 are verified. So, when we have the following condition:

$$(\mathcal{L} + \ell_1\widetilde{F}) \leq \frac{1}{2} + \frac{1}{4} \times 1 \times \frac{2}{\Gamma(\frac{5}{2})}\left(\frac{2^{\frac{1}{2}} - 1}{\frac{1}{2}}\right)^{\frac{3}{2}} \simeq 0.5354507382 < 1,$$

we have used Theorem 3.5, and it implies that the simulative neutral functional inclusion FBVP (22) has at least one solution $w \in Q$.

5 Conclusion

The existence of solutions for our proposed fractional boundary value problem has been successfully investigated for the neutral fractional differential inclusions of Katugampola fractional derivative which involves retarded and advanced arguments. Two cases have been discussed throughout our investigation via fixed point theorems for convex and non-convex multifunctions. The notion of the KMNC is utilized for the convex case, and the Nadler–Covitz result is implemented for the nonconvex case. An application in the format of a simulative example of the neutral functional FBVP has been provided to validate all our obtained results. This research work sheds the light on the importance of studying neutral fractional problem with its application in science and engineering. Indeed, one can extend this work to complicated structures involving generalized fractional operators via nonsingular kernels which cover mathematical models of real phenomena appropriately.

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