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On new generalized unified bounds via generalized exponentially harmonically s-convex functions on fractal sets

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Abstract

The visual beauty reflects the practicability and superiority of design dependent on the fractal theory. Based on the applicability in practice, it shows that it is the completely feasible, self-comparability and multifaceted nature of fractal sets that made it an appealing field of research. There is a strong correlation between fractal sets and convexity due to its intriguing nature in the mathematical sciences. This paper investigates the notions of generalized exponentially harmonically (*GEH*) convex and *GEH* s-convex functions on a real linear fractal sets $\mathbb{R}^{\alpha}$ $(0 < \alpha \leq 1)$. Based on these novel ideas, we derive the generalized Hermite–Hadamard inequality, generalized Fejér–Hermite–Hadamard type inequality and Pachpatte type inequalities for *GEH* s-convex functions. Taking into account the local fractal identity; we establish a certain generalized Hermite–Hadamard type inequalities for local differentiable *GEH* s-convex functions. Meanwhile, another auxiliary result is employed to obtain the generalized Ostrowski type inequalities for the proposed techniques. Several special cases of the proposed concept are presented in the light of generalized exponentially harmonically convex, generalized harmonically convex and generalized harmonically s-convex. Meanwhile, an illustrative example and some novel applications in generalized special means are obtained to ensure the correctness of the present results. This novel strategy captures several existing results in the corresponding literature. Finally, we suppose that the consequences of this paper can stimulate those who are interested in fractal analysis.

MSC: 26E60

Keywords: Harmonically convex function; Exponentially convex function; Exponentially harmonically s-convex function; Hermite–Hadamard-Fejér type inequality; Pachpatte type inequality; Ostrowski type inequality; Fractal sets

1 Introduction and preliminaries

The fractal sets in science have introduced some fascinating, complex graphs and picture compressions to computer graphics. Fractal is a Latin word, derived from the word "Fractus" which signifies "Broken." The expression "fractal" was first utilized by a young mathematician, Julia [1] when he was considering Cayley's problem identified with the conduct of Newton's method in a complex plane. A fractal is frequently utilized in real-world involving: fractal antennas, fractal transistors, and fractal heat ex-changers. It has an application in the music industry, creation of photography, soil mechanics, small-angle scattering theory and much more. It is to be emphasized that fractal theory assumes an essential job in the improvement of picturesque of fractal sets. The utilization of fractal sets is in cryptography and other useful areas of research has increased the interest of researchers to broaden the utilization in mathematical inequalities. Fractals are the elite, arbitrary examples abandoned by the erratic developments of the disorderly world at work. The most significant utilization of fractals in software engineering is fractal picture compression. This sort of compression utilizes the way that this present reality is very much portrayed by fractal geometry [2–4]. Interestingly, authors [5] investigated the local fractional functions on the fractal space deliberately, which comprises local fractional calculus and the monotonicity of functions. Numerous analysts contemplated the characteristics of functions on fractal space and built numerous sorts of fractional calculus by utilizing various strategies, [6–8].

Therefore, it is essential to create mathematical inequalities that inspect the fractal sets and their significance in different fields of mathematics and engineering problems [9–13]. On the other hand, the development of new concepts in convexity has enabled us to preserve more information on the evolutionary history of integral inequality to use it in predicting new outcomes. The word "convexity" is the most significant, natural, and fundamental notation in literature. Convex functions were introduced by Jensen over 100 years ago. Over the past few years, various speculations and expansions have been made for convexity. These expansions and speculations on the theory of inequalities have made valuable contributions to numerous branches of mathematics. The many novel ideas in this viewpoint concern exponentially convex, harmonically convex, Jensen convex, arithmetically-geometrically convex, $\hbar$-convex, Schur convex and strongly convex functions and many others. In the current situation, we intend to determine some novel generalized inequalities for differentiable functions in the frame of *GEH* s-convex functions via local fractional integrals.

The most distinguished inequality is the Hermite–Hadamard's type inequality [14, 15], which is stated as follows:

$$\mathcal{H}\left(\frac{m_1+m_2}{2}\right) \le \frac{1}{m_2-m_1}\int_{m_1}^{m_2}\mathcal{H}(x)\,dx \le \frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2}. \tag{1.1}$$

In [16], Fejér derived an important generalization which is the weighted generalization of the Hermite–Hadamard inequality.

Let $\Omega \subseteq \mathbb{R}$ and a function $\mathcal{H}:\Omega \to \mathbb{R}$ be a convex function. Then the inequalities

$$\begin{aligned}\mathcal{H}\left(\frac{m_1+m_2}{2}\right)\int_{m_1}^{m_2}\mathcal{W}(x)\,dx &\le \frac{1}{m_2-m_1}\int_{m_1}^{m_2}\mathcal{H}(x)\mathcal{W}(x)\,dx\\ &\le \frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2}\int_{m_1}^{m_2}\mathcal{W}(x)\,dx \end{aligned} \tag{1.2}$$

hold, where $\mathcal{W}:\Omega \to \mathbb{R}$ is nonnegative, integrable and symmetric with respect to $\frac{m_1+m_2}{2}$.

Inequalities (1.1) and (1.2) and their generalizations, refinements, extensions, and converses, etc. have many applications in different fields of science, for example electrical engineering, mathematical statistics, financial economics, information theory, guessing and coding [17–19].

In 1937, Ostrowski [20] established an interesting integral inequality associated with differentiable mappings in one dimension stipulates a bound between a function evaluated at an interior point x and the average of the function over an interval. That is,

$$\left|\mathcal{H}(x)-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}\mathcal{H}(x)\,dx\right|\leq\left[\frac{1}{4}-\frac{(x-\frac{m_1+m_2}{2})^2}{m_2-m_1}\right](m_2-m_1)\|\mathcal{H}\|_\infty \tag{1.3}$$

holds for all $x\in[m_1,m_2]$, where $x\in L^\infty(m_1,m_2)$ and $\mathcal{H}:[m_1,m_2]\to\mathbb{R}$ is a differentiable function on (m_1,m_2). The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one. Ostrowski inequalities have great importance while studying the error bounds of different numerical quadrature rules, for example, the midpoint rule, Simpson's rule, the trapezoidal rule; see [21–28].

In [29, 30], the author presented the concept of a harmonically convex function and harmonically s-convex function independently, as follows.

Definition 1.1 ([29]) Let $\Omega\subseteq\mathbb{R}\setminus\{0\}$ be a real interval and a function $\mathcal{H}:\Omega\to\mathbb{R}$ is said to be harmonically convex, if the inequality

$$\mathcal{H}\left(\frac{xy}{lx+(1-l)y}\right)\leq l\mathcal{H}(y)+(1-l)\mathcal{H}(x) \tag{1.4}$$

holds for all $x,y\in\Omega$ and $l\in[0,1]$.

Definition 1.2 ([30]) Let $\Omega\subseteq\mathbb{R}\setminus\{0\}$ be a real interval and a function $\mathcal{H}:\Omega\to\mathbb{R}$ is said to be harmonically s-convex, if the inequality

$$\mathcal{H}\left(\frac{xy}{lx+(1-l)y}\right)\leq l^s\mathcal{H}(y)+(1-l)^s\mathcal{H}(x) \tag{1.5}$$

holds for all $x,y\in\Omega$, $l\in[0,1]$ and for some fixed $s\in(0,1]$.

In [29], İşcan derived the celebrated Hermite–Hadamard inequality for harmonically s-convex functions as follows:

Let $\mathcal{H}:\Omega\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a harmonically convex function, if and only if it satisfies the inequalities

$$\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\leq\frac{m_1m_2}{m_2-m_1}\int_{m_1}^{m_2}\frac{\mathcal{H}(x)}{x^2}\,dx\leq\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2}. \tag{1.6}$$

Chen and Wu [31] presented another weighted generalization by employing harmonically convex functions. Let $\mathcal{H}:\Omega\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a harmonically convex function, if and only if it satisfies the inequality:

$$\begin{aligned}\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\int_{m_1}^{m_2}\frac{\mathcal{W}(x)}{x^2}\,dx&\leq\frac{m_1m_2}{m_2-m_1}\int_{m_1}^{m_2}\frac{\mathcal{H}(x)\mathcal{W}(x)}{x^2}\,dx\\&\leq\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2}\int_{m_1}^{m_2}\frac{\mathcal{W}(x)}{x^2}\,dx,\end{aligned} \tag{1.7}$$

where $\mathcal{W}:[m_1,m_2]\to\mathbb{R}$ is nonnegative, integrable and satisfies

$$\mathcal{W}\left(\frac{m_1m_2}{x}\right)=\mathcal{W}\left(\frac{m_1m_2}{m_1+m_2-x}\right).$$

For a generalization related to (1.6) and (1.7), and modification and refinements, we refer the reader to [29–33].

Now, we mention the preliminaries of the theory of local fractional calculus. These ideas and important consequences associated with the local fractional derivative and local fractional integral are mainly due to Yang [5].

Let a_1^α, a_2^α and a_3^α belong to the set $\mathbb{R}^\alpha$ $(0<\alpha\leq 1)$, then

(1) $a_1^\alpha+a_2^\alpha$ and $a_1^\alpha a_2^\alpha$ belongs to the set $\mathbb{R}^\alpha$;
(2) $a_1^\alpha+a_2^\alpha=a_2^\alpha+a_1^\alpha=(a_1+a_2)^\alpha=(a_2+a_1)^\alpha$;
(3) $a_1^\alpha+(a_2^\alpha+a_3^\alpha)=(a_1^\alpha+a_2^\alpha)+a_3^\alpha$;
(4) $a_1^\alpha a_2^\alpha=a_2^\alpha a_1^\alpha=(a_1a_2)^\alpha=(a_2a_1)^\alpha$;
(5) $a_1^\alpha(a_2^\alpha z^\alpha)=(a_1^\alpha a_2^\alpha)z^\alpha$;
(6) $a_1^\alpha(a_2^\alpha+a_3^\alpha)=a_1^\alpha a_2^\alpha+a_1^\alpha a_3^\alpha$;
(7) $a_1^\alpha+0^\alpha=0^\alpha+a_1^\alpha=a_1^\alpha$ and $a_1^\alpha 1^\alpha=1^\alpha a_1^\alpha=a_1^\alpha$.

Definition 1.3 A non-differentiable mapping $\mathcal{H}:\mathbb{R}\to\mathbb{R}^\alpha$, $\theta\to\mathcal{H}(\mu)$ is said to be local fractional continuous at $\mu_\circ$, if for any $\epsilon>0$, there exists $l>0$, satisfying

$$\left|\mathcal{H}(\mu)-\mathcal{H}(\mu_\circ)\right|<\epsilon^\alpha$$

for $|\mu-\mu_\circ|<\kappa$. If $\mathcal{H}(\mu)$ is local continuous on (m_1,m_2), then we denote it by $\mathcal{H}(\mu)\in\mathbb{C}_\alpha(m_1,m_2)$.

Definition 1.4 The local fractional derivative of $\mathcal{H}(\mu)$ of order α at $\mu=\mu_\circ$ is defined by the expression

$$\begin{aligned}\mathcal{H}^{(\alpha)}(\mu_\circ)={}_{\mu_\circ}\mathcal{D}_\mu^\alpha\mathcal{H}(\mu)&=\left.\frac{d^\alpha\mathcal{H}(\mu)}{d\mu^\alpha}\right|_{\mu=\mu_\circ}\\&=\lim_{\mu\to\mu_\circ}\frac{\Delta^\alpha(\mathcal{H}(\mu)-\mathcal{H}(\mu_\circ))}{(\mu-\mu_\circ)^\alpha},\end{aligned}$$

where $\Delta^\alpha(\mathcal{H}(\mu)-\mathcal{H}(\mu_\circ))=\Gamma(\alpha+1)(\mathcal{H}(\mu)-\mathcal{H}(\mu_\circ))$. Let $\mathcal{H}^{(\alpha)}(\mu)=\mathcal{D}_\mu^\alpha\mathcal{H}(\mu)$. If there exists $\mathcal{H}^{(k+1)\alpha}(\mu)=\overbrace{\mathcal{D}_\mu^\alpha\cdots\mathcal{D}_\mu^\alpha}^{(k+1)\text{ times}}\mathcal{H}(\mu)$ for any $\mu\in\Omega\subseteq\mathbb{R}$, then it is denoted by $\mathcal{H}\in\mathcal{D}_{(k+1)\alpha}(\mathcal{I})$, where $k=0,1,2,\ldots$.

Definition 1.5 Let $\mathcal{H}(\mu)\in\mathbb{C}_\alpha[m_1,m_2]$, and let $\Delta=\{\xi_0,\xi_1,\ldots,\xi_N\}$, $(N\in\mathbb{N})$ be a partition of $[m_1,m_2]$ which satisfies $m_1=\xi_0<\xi_1<\cdots<\xi_N=m_2$. Then the local fractional integral of $\mathcal{H}$ on $[m_1,m_2]$ of order α is defined as follows:

$${}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\mathcal{H}(\mu)=\frac{1}{\Gamma(1+\alpha)}\int_{m_1}^{m_2}\mathcal{H}(\xi)(d\xi)^\alpha:=\frac{1}{\Gamma(1+\alpha)}\lim_{\delta\xi\to 0}\sum_{j=0}^{N-1}\mathcal{H}(\xi_j)(\Delta\xi_j),$$

where $\delta\xi:=\max\{\Delta\xi_1,\Delta\xi_2,\ldots,\Delta\xi_{N-1}\}$ and $\Delta\xi_j:=\xi_{j+1}-\xi_j$, $j=0,\ldots,N-1$.

Here, it follows that ${}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\mathcal{H}(\mu) = 0$ if $m_1 = m_2$ and ${}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\mathcal{H}(\mu) = -{}_{m_2}\mathcal{I}_{m_1}^{(\alpha)}\mathcal{H}(\mu)$ if $m_1 < m_2$. For any $\mu \in [m_1, m_2]$, if there exists ${}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\mathcal{H}(\mu)$, then it is denoted by $\mathcal{H}(\mu) \in \mathcal{I}_{\mu}^{\alpha}[m_1, m_2]$.

Lemma 1.6 ([5])

(1) *Suppose that* $\mathcal{H}(x) = \mathcal{G}^{(\alpha)}(x) \in \mathbb{C}_\alpha[m_1, m_2]$, *then*

$$ {}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\mathcal{H}(x) = \mathcal{G}(m_2) - \mathcal{G}(m_1). $$

(2) *Suppose that* $\mathcal{H}(x), \mathcal{G}(x) \in \mathcal{D}_\alpha[m_1, m_2]$, *and* $\mathcal{H}^{(\alpha)}(x), \mathcal{G}^{(\alpha)}(x) \in \mathcal{C}_\alpha[m_1, m_2]$, *then*

$$ {}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\mathcal{H}(x)\mathcal{G}^{(\alpha)}(x) = \mathcal{H}(x)\mathcal{G}(x)|_{m_1}^{m_2} - {}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\mathcal{H}^{(\alpha)}(x)\mathcal{G}(x). $$

Lemma 1.7 ([5])

$$ \begin{aligned} &\frac{d^\alpha a_1^{k\alpha}}{da_1^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} a_1^{(k-1)\alpha}, \\ &\frac{1}{\Gamma(1+\alpha)} \int_{m_1}^{m_2} a_1^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \big(m_2^{(k+1)\alpha} - m_1^{(k+1)\alpha}\big), \quad k > 0. \end{aligned} $$

Lemma 1.8 ([34] Generalized Hölder inequality) *For* $s, q > 1$ *with* $s^{-1} + q^{-1} = 1$, *and let* $\mathcal{H}, \mathcal{G} \in \mathbb{C}_\alpha[m_1, m_2]$,

$$ \begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_{m_1}^{m_2} \big|\mathcal{H}(x)\mathcal{G}(x)\big|(n)^\alpha \\ &\quad \le \left(\frac{1}{\Gamma(1+\alpha)} \int_{m_1}^{m_2} \big|\mathcal{H}(x)\big|^s (dx)^\alpha\right)^{\frac{1}{s}} \left(\frac{1}{\Gamma(1+\alpha)} \int_{m_1}^{m_2} \big|\mathcal{G}(x)\big|^q (dx)^\alpha\right)^{\frac{1}{q}}. \end{aligned} $$

In [7], Mo et al. derived the generalized Hermite–Hadamard's inequality for generalized convex functions as follows:

$$ \mathcal{H}\left(\frac{m_1+m_2}{2}\right) \le \frac{\Gamma(1+\alpha)}{(m_2-m_1)^\alpha} {}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\mathcal{H}(x) \le \frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2^\alpha}. \tag{1.8} $$

In 1994, Hudzik and Maligranda [35] provided several generalizations linked with s-convex functions and some intriguing outcomes about Hermite–Hadamard's inequality for s-convex functions were elaborated. In 1915, Bernstein and Doetsch [36] established a variety of Hermite–Hadamard's inequality for s-convex functions in the second sense. Moreover, the investigation of some well-known integral inequalities for the local fractional integral has been studied by several researchers, for instance, Kilicman and Saleh [37, 38] derived generalized Hermite–Hadamard inequalities for generalized s-convex functions. In [39], Du et al. contemplated the certain inequalities for generalized m-convex functions on fractal sets with utilities. Also, Vivas et al. [40] explored generalized Jensen and Hermite–Hadamard inequalities for h-convex functions. For more results related to the local fractional inequalities, we refer the interested reader to [7, 41–43] and the references therein.

Adopting the above tendency, the key aim of this paper is to introduce a novel concept of GEH convex and GEH s-convex functions, then to discuss important properties for such functions. Additionally, we established some novel variants that interact between GHE s-convex functions and local fractional integrals. In fractal sets, a novel generalized identity has been carried out to investigate the local differentiability of GEH s-convex functions, GEH convex functions, and generalized harmonically convex functions. Generalized new special cases show the impressive performance of the local fractional integration. Some special cases are correlated with existing results in classical harmonically convexity, exponentially harmonically convexity and exponentially harmonically s-convexity and harmonically s-convexity.

2 Generalized exponentially harmonically convex functions

We now recall the concept of generalized exponentially harmonically convex functions on the fractal space as follows.

Definition 2.1 Let $\Omega \subset \mathbb{R} \setminus \{0\}$ be a real interval and a function $\mathcal{H} : \Omega \to \mathbb{R}^{\alpha}$ is said to be generalized exponentially harmonically convex functions if the inequality

$$\mathcal{H}\left(\frac{xy}{lx + (1-l)y}\right) \leq l^{\alpha} \frac{\mathcal{H}(y)}{e^{\theta y}} + (1-l)^{\alpha} \frac{\mathcal{H}(x)}{e^{\theta x}} \tag{2.1}$$

holds for all $x, y \in \Omega$, $\theta \in \mathbb{R}$ and $l \in [0, 1]$.

The generalized harmonically s-convex functions can be stated as follows.

Definition 2.2 Let $\Omega \subset \mathbb{R} \setminus \{0\}$ be a real interval and a function $\mathcal{H} : \Omega \to \mathbb{R}^{\alpha}$ is said to be generalized harmonically s-convex functions if the inequality

$$\mathcal{H}\left(\frac{xy}{lx + (1-l)y}\right) \leq l^{s\alpha} \mathcal{H}(y) + (1-l)^{s\alpha} \mathcal{H}(x) \tag{2.2}$$

holds for all $x, y \in \Omega$, $\theta \in \mathbb{R}$, $l \in [0, 1]$ and $s \in (0, 1]$.

Next, we present the idea of generalized exponentially harmonically s-convex function by connecting the Definitions 2.1 and 2.2 as follows.

Definition 2.3 Let $\Omega \subset \mathbb{R} \setminus \{0\}$ be a real interval and a function $\mathcal{H} : \Omega \to \mathbb{R}^{\alpha}$ is said to be *GEH* s-convex functions if the inequality

$$\mathcal{H}\left(\frac{xy}{lx + (1-l)y}\right) \leq l^{s\alpha} \frac{\mathcal{H}(y)}{e^{\theta y}} + (1-l)^{s\alpha} \frac{\mathcal{H}(x)}{e^{\theta x}} \tag{2.3}$$

holds for all $x, y \in \Omega$, $\theta \in \mathbb{R}$, $l \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Remark 2.4 In view of Definition 2.3:

1. If we take $s = 1$, then we get Definition 2.1.
2. If we take $\theta = 0$, then we get Definition 2.2.
3. If we take $\theta = 0$, along with $\alpha = 1$, then we get Definition 3 in [30].
4. If we take $\theta = 0$, and $s = 1$, then we get Definition 3.1 in [43].

Moreover, if we take $l = \frac{1}{2}$ in (2.3), then the *GEH s*-convex functions become Jensen-type generalized exponentially harmonically s-convex functions as follows:

$$\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right) \le \frac{1}{2^{s\alpha}}\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}} + \frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right] \tag{2.4}$$

holding for $m_1, m_2 \in \Omega$ and $s \in (0,1]$.

It is worth mentioning that *GEH s*-convex functions collapse to generalized harmonically convex, generalized harmonically s-convex functions and generalized exponentially harmonically convex functions as special cases. This shows that outcomes derived in the present paper continue to hold for these classes of convex functions and their variant forms.

Theorem 2.5 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ and if $\mathcal{H}, \mathcal{G} : \Omega \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}^\alpha$ is a GEH s-convex functions, then*
(1) *$\mathcal{H} + \mathcal{G}$ is GEH s-convex function*;
(2) *$\lambda^\alpha \mathcal{H}$ is GEH s-convex function.*

Proof (1) Since $\mathcal{H}$ and $\mathcal{G}$ are GEH s-convex functions on Ω, and $l \in [0,1]$, we have

$$\begin{aligned}(\mathcal{H}+\mathcal{G})\left(\frac{xy}{lx+(1-l)y}\right) &= \mathcal{H}\left(\frac{xy}{lx+(1-l)y}\right) + \mathcal{G}\left(\frac{xy}{lx+(1-l)y}\right) \\ &\le l^{s\alpha}\frac{\mathcal{H}(y)}{e^{\theta y}} + (1-l)^{s\alpha}\frac{\mathcal{H}(x)}{e^{\theta x}} + l^{s\alpha}\frac{\mathcal{G}(y)}{e^{\theta y}} + (1-l)^{s\alpha}\frac{\mathcal{G}(x)}{e^{\theta x}} \\ &= l^{s\alpha}\frac{(\mathcal{H}+\mathcal{G})(y)}{e^{2y\theta}} + (1-l)^{s\alpha}\frac{(\mathcal{H}+\mathcal{G})(x)}{e^{2x\theta}}.\end{aligned} \tag{2.5}$$

So, $\mathcal{H} + \mathcal{G}$ is a GEH s-convex function on Ω.

(2) Since $\mathcal{H}$ and $\mathcal{G}$ are GEH s-convex functions on $\Omega, l \in [0,1]$, and $\lambda \in \mathbb{R}_+$, we have

$$\begin{aligned}\left(\lambda^\alpha\mathcal{H}\left(\frac{xy}{lx+(1-l)y}\right)\right) &= \lambda^\alpha\mathcal{H}\left(\frac{xy}{lx+(1-l)y}\right) \\ &\le \lambda^\alpha\left[l^{s\alpha}\frac{\mathcal{H}(y)}{e^{\theta y}} + (1-l)^{s\alpha}\frac{\mathcal{H}(x)}{e^{\theta x}}\right] \\ &= l^{s\alpha}\frac{(\lambda^\alpha\mathcal{H})(x)}{e^{\theta x}} + (1-l)^{s\alpha}\frac{(\lambda^\alpha\mathcal{H})(y)}{e^{\theta y}},\end{aligned}$$

hence $\lambda^\alpha\mathcal{H}$ is a GEH s-convex function on Ω. □

Theorem 2.6 *For $n \in \mathbb{N}$ and let there is a sequence of GEH s-convex functions $\mathcal{H}_n : \Omega \to \mathbb{R}^\alpha$ converges pointwise to a function $\mathcal{H} : \Omega \to \mathbb{R}^\alpha$, then $\mathcal{H}$ is GEH s-convex function on Ω.*

Proof Let $x, y \in \Omega$, $l \in [0,1]$, and $\lim_{n\to\infty}\mathcal{H}_n(x) = \mathcal{H}(x)$, then

$$\begin{aligned}\mathcal{H}\left(\frac{xy}{lx+(1-l)y}\right) &= \lim_{n\to\infty}\mathcal{H}_n\left(\frac{xy}{lx+(1-l)y}\right) \\ &\le \lim_{n\to\infty}\left[l^{s\alpha}\frac{\mathcal{H}_n(y)}{e^{\theta y}} + (1-l)^{s\alpha}\frac{\mathcal{H}_n(x)}{e^{\theta x}}\right]\end{aligned}$$

$$= l^{s\alpha} \lim_{n\to\infty} \frac{\mathcal{H}_n(y)}{e^{\theta y}} + (1-l)^{s\alpha} \lim_{n\to\infty} \frac{\mathcal{H}_n(x)}{e^{\theta x}}$$
$$= l^{s\alpha} \frac{\mathcal{H}(y)}{e^{\theta y}} + (1-l)^{s\alpha} \frac{\mathcal{H}(x)}{e^{\theta x}},$$

that is, $\mathcal{H}$ is a GEH s-convex function on Ω. □

3 Generalized Hermite–Hadamard type inequalities

In this section, we present the generalized Hermite–Hadamard inequality for GEH s-convex functions via local fractional integrals.

Theorem 3.1 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ and letting $\mathcal{H} : \Omega \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}^\alpha$ be a GEH s-convex function on fractal space, $m_1, m_2 \in \Omega$ with $m_2 > m_1$, if $\mathcal{H}^{(\alpha)} \in \mathbb{C}_\alpha[m_1, m_2]$, then the following inequalities hold*:

$$\frac{2^{(s-1)\alpha}}{\Gamma(1+\alpha)} \mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right) \le \left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha {}_{m_1}\mathcal{I}_{m_2}^{(\alpha)} \frac{\mathcal{H}(x)}{(x^2e^{\theta x})^\alpha}$$
$$\le \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} \left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}} + \frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right]. \tag{3.1}$$

Proof Taking into account the inequality (2.4), for all $x, y \in \Omega$, we have

$$\mathcal{H}\left(\frac{2xy}{x+y}\right) \le \frac{1}{(2)^{s\alpha}} \left(\frac{\mathcal{H}(x)}{e^{\theta x}} + \frac{\mathcal{H}(y)}{e^{\theta y}}\right). \tag{3.2}$$

Substituting $x = \frac{m_1m_2}{lm_2+(1-l)m_1}$, $y = \frac{m_1m_2}{lm_1+(1-l)m_2}$, we have

$$\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right) \le \frac{1}{(2)^{s\alpha}} \left[\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right) e^{(\frac{-\theta m_1m_2}{lm_2+(1-l)m_1})}\right.$$
$$\left. + \mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right) e^{(\frac{-\theta m_1m_2}{lm_1+(1-l)m_2})}\right]. \tag{3.3}$$

Integrating the above inequality corresponding to l from 0 to 1, we have

$$\frac{1}{\Gamma(1+\alpha)} \mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)$$
$$\le \frac{1}{2^{s\alpha}} \left[\frac{1}{\Gamma(1+\alpha)} \int_0^1 \mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right) e^{(\frac{-\theta m_1m_2}{lm_2+(1-l)m_1})} (dl)^\alpha\right.$$
$$\left. + \frac{1}{\Gamma(1+\alpha)} \int_0^1 \mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right) e^{(\frac{-\theta m_1m_2}{lm_1+(1-l)m_2})} (dl)^\alpha\right]$$
$$= \left(\frac{1}{2}\right)^{(s-1)\alpha} \left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha$$
$$\times \left[\frac{1}{\Gamma(1+\alpha)} \int_{m_1}^{m_2} \frac{\mathcal{H}(x)}{a_1^{2\alpha} e^{\alpha\theta x}} (dx)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_{m_1}^{m_2} \frac{\mathcal{H}(y)}{y^{2\alpha} e^{\alpha\theta y}} (dy)^\alpha\right]$$
$$= \left(\frac{1}{2}\right)^{(s-1)\alpha} \left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha {}_{m_1}\mathcal{I}_{m_2}^{(\alpha)} \frac{\mathcal{H}(x)}{(x^2e^{\theta x})^\alpha},$$

using the fact that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 \mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)(dl)^\alpha = \frac{1}{\Gamma(1+\alpha)}\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right).$$

For the proof of the second inequality in (3.8), we note that $\mathcal{H}$ is a *GEH s*-convex function, for $l \in [0,1]$, we have

$$\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right) \le (l)^{s\alpha}\frac{\mathcal{H}(m_1)}{e^{\theta m_1}} + (1-l)^{s\alpha}\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}$$

and

$$\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right) \le (1-l)^{s\alpha}\frac{\mathcal{H}(m_1)}{e^{\theta m_1}} + (l)^{s\alpha}\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}.$$

Adding the above two inequalities, we get

$$\begin{aligned}&\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right) + \mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\\ &\quad\le \left[(l)^{s\alpha}+(1-l)^{s\alpha}\right]\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}} + \frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right].\end{aligned} \tag{3.4}$$

Integrating the above inequality corresponding to l from 0 to 1, we have

$$\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{H}(x)}{(x^2e^{\theta x})^\alpha} \le \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}} + \frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right],$$

where we have used the fact that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 (1-l)^{s\alpha}(dl)^\alpha = \frac{1}{\Gamma(\alpha+1)}\int_0^1 l^{s\alpha}(dl)^\alpha = \frac{\Gamma(1+\alpha)}{\Gamma(1+(s+1)\alpha)}.$$

This completes the proof. □

Some remarkable cases of Theorem 3.1 are presented in the form of corollaries and remarks.

I. If one takes $\alpha = 1$, then we have a new result for exponentially harmonically s-convex functions.

Corollary 3.2 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ and letting $\mathcal{H}: \Omega \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be an exponentially harmonically s-convex function such that $L_1[m_1,m_2]$ with $m_1, m_2 \in \Omega$ and $m_2 > m_1$, then the following inequalities hold*:

$$2^{(s-1)}\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right) \le \left(\frac{m_1m_2}{m_2-m_1}\right)\int_{m_1}^{m_2}\frac{\mathcal{H}(x)}{x^2e^{\theta x}} \le \frac{1}{s+1}\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}} + \frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right]. \tag{3.5}$$

II. If one takes $s = \alpha = 1$, then we have a new result for exponentially harmonically convex functions.

Corollary 3.3 *Let $\mathcal{H}:\Omega\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be an exponentially harmonically convex function such that $L_1[m_1,m_2]$ with $m_1,m_2\in\Omega$ and $m_2>m_1$. Then the following inequalities hold*:

$$\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\leq\left(\frac{m_1m_2}{m_2-m_1}\right)\int_{m_1}^{m_2}\frac{\mathcal{H}(x)}{x^2e^{\theta x}}\leq\frac{1}{2}\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}}+\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right]. \tag{3.6}$$

III. If one takes $s=1$, then we have a new result for generalized exponentially harmonically convex functions.

Corollary 3.4 *For $\theta\in\mathbb{R}$ and letting $\mathcal{H}:\Omega\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^\alpha$ be a GEH convex function on fractal space, $m_1,m_2\in\Omega$ with $m_2>m_1$, if $\mathcal{H}^{(\alpha)}\in\mathbb{C}_\alpha[m_1,m_2]$, then the following inequalities hold*:

$$\begin{aligned}\frac{1}{\Gamma(1+\alpha)}\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)&\leq\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{H}(x)}{(x^2e^{\theta x})^\alpha}\\&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}}+\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right].\end{aligned} \tag{3.7}$$

IV. If one takes $\theta=0$, then we have a new result for generalized exponentially harmonically convex functions.

Corollary 3.5 *For $s\in(0,1]$ and letting $\mathcal{H}:\Omega\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^\alpha$ be a generalized exponentially harmonically s-convex function on fractal space, $m_1,m_2\in\Omega$ with $m_2>m_1$, if $\mathcal{H}^{(\alpha)}\in\mathbb{C}_\alpha[m_1,m_2]$, then the following inequalities hold*:

$$\begin{aligned}\frac{2^{(s-1)\alpha}}{\Gamma(1+\alpha)}\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)&\leq\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{H}(x)}{x^{2\alpha}}\\&\leq\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\mathcal{H}(m_1)+\mathcal{H}(m_2)\right].\end{aligned}$$

Remark 3.6 In Theorem 3.1:

(1) If we take $\theta=0$ and $\alpha=1$, then we get Theorem 3 in [30].
(2) If we take $\theta=0$ and $\alpha=s=1$, then we get Theorem 2.4 in [29].
(3) If we take $\theta=0$ and $s=1$, then we get Theorem 4.1 in [43].

The key aim of this section is to obtain novel bounds that refine generalized Hermite–Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is a generalized exponentially harmonically s-convex function. Sun [43] used the following lemma.

Lemma 3.7 ([43]) *Let $\mathcal{H}:\mathcal{I}^\circ\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^\alpha$ ($\mathcal{I}^\circ$ is the interior of $\mathcal{I}$) such that $\mathcal{H}\in\mathcal{D}_\alpha(\mathcal{I}^\circ)$ and $\mathcal{H}^{(\alpha)}\in\mathcal{C}_\alpha[m_1,m_2]$ for $m_1,m_2\in\Omega^\circ$ with $m_2>m_1$. Then the following equality holds*:

$$\begin{aligned}&\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2^\alpha}-\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\Gamma(1+\alpha)_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{H}(x)}{x^{2\alpha}}\\&=\left(\frac{m_1m_2(m_2-m_1)}{2}\right)^\alpha\frac{1}{\Gamma(1+\alpha)}\\&\quad\times\int_0^1\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\mathcal{H}^{(\alpha)}\left(\frac{m_1m_2}{m_2l+(1-l)m_1}\right)(dl)^\alpha.\end{aligned} \tag{3.8}$$

Theorem 3.8 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ with $p^{-1}+q^{-1}=1$ and letting $\mathcal{H}:\mathcal{I}^\circ \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}^\alpha$ be a differentiable function on Ω° ($\mathcal{I}^\circ$ is the interior of $\mathcal{I}$) such that $\mathcal{H}^{(\alpha)} \in \mathcal{C}_\alpha[m_1,m_2]$ for $m_1,m_2 \in \Omega^\circ$ with $m_2 > m_1$, if $|\mathcal{H}^{(\alpha)}|^q$ is GEH s-convex on Ω for $q \geq 1$, then the following inequality holds*:

$$\begin{aligned}&\left|\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2^\alpha} - \left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha \Gamma(1+\alpha)\,{}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(x)}{x^{2\alpha}}\right| \\ &\quad\leq \left(\frac{m_1m_2(m_2-m_1)}{2}\right)^\alpha \left[\mathcal{B}_3^{(\alpha)}\right]^{\frac{q-1}{q}} \left[\mathcal{B}_1^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}} + \mathcal{B}_2^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right]^{\frac{1}{q}},\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}\mathcal{B}_1^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right| l^{s\alpha}(dl)^\alpha \\ &= \frac{1}{(m_2-m_1)^{\alpha(s+2)}} \\ &\quad\times\left[\frac{(m_1+m_2)^\alpha\Gamma(1+\alpha(s-2))}{\Gamma(1+\alpha(s-1))}\left(2^\alpha\left(\frac{m_1+m_2}{2}\right)^{\alpha(s-1)} - m_1^{\alpha(s-1)} - m_2^{\alpha(s-1)}\right)\right. \\ &\quad - \frac{(m_1)^{s\alpha}((m_2^\alpha - m_1^\alpha)(m_2-m_1)^\alpha}{(m_1m_2)^\alpha\Gamma(\alpha+1)} \\ &\quad + \frac{2^\alpha\Gamma(1+(s-1)\alpha)}{\Gamma(1+\alpha s)}\left(-2^\alpha\left(\frac{m_1+m_2}{2}\right)^{\alpha s} + m_1^{s\alpha} + m_2^{s\alpha}\right) \\ &\quad \left.+ \left(2m_1^s\right)^\alpha\left(2^\alpha \ln_\alpha\left(\frac{m_1+m_2}{2}\right)^{\alpha s} - \ln_\alpha(m_1)^{s\alpha} - \ln_\alpha(m_2)^{s\alpha}\right)\right],\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathcal{B}_2^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right| (1-l)^{s\alpha}(dl)^\alpha \\ &= \frac{1}{(m_2-m_1)^{\alpha(s+2)}} \\ &\quad\times\left[\frac{(m_1+m_2)^\alpha\Gamma(1+\alpha(s-2))}{\Gamma(1+\alpha(s-1))}\left(-2^\alpha\left(\frac{m_1+m_2}{2}\right)^{\alpha(s-1)} + m_1^{\alpha(s-1)} + m_2^{\alpha(s-1)}\right)\right. \\ &\quad + \frac{(m_2)^{s\alpha}(m_2^\alpha - m_1^\alpha)(m_2-m_1)}{(m_1m_2)^\alpha\Gamma(\alpha+1)} \\ &\quad + \frac{2^\alpha\Gamma(1+(s-1)\alpha)}{\Gamma(1+\alpha s)}\left(2^\alpha\left(\frac{m_1+m_2}{2}\right)^{\alpha s} - m_1^{s\alpha} - m_2^{s\alpha}\right) \\ &\quad \left.+ \left(2m_2^s\right)^\alpha\left(-2^\alpha \ln_\alpha\left(\frac{m_1+m_2}{2}\right)^{\alpha s} + \ln_\alpha(m_1)^{s\alpha} + \ln_\alpha(m_2)^{s\alpha}\right)\right],\end{aligned} \tag{3.11}$$

$$\begin{aligned}\mathcal{B}_3^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right| (dl)^\alpha \\ &= \frac{(m_2^\alpha - m_1^\alpha)}{(m_2-m_1)^\alpha(m_1m_2)^\alpha\Gamma(1+\alpha)} \\ &\quad + \frac{2^\alpha}{(m_2-m_1)^\alpha}\left(-2^\alpha\ln_\alpha\left(\frac{m_1+m_2}{2}\right)^{\alpha s} + \ln_\alpha(m_1)^{s\alpha} + \ln_\alpha(m_2)^{s\alpha}\right),\end{aligned} \tag{3.12}$$

and $\ln_\alpha(x^\alpha)$ *symbolizes the inverse of the Mittag-Leffler function defined on fractal sets* $E_\alpha(x^\alpha) = \sum_{k=0}^\infty \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$; *see* [5].

Proof Let us estimate for $q = 1$ and $q > 1$.

Case I. $q = 1$.

Using Lemma 3.7, *GEH* s-convexity of $|\mathcal{H}^{(\alpha)}|$ and the modulus property, we have

$$\begin{aligned}
&\left|\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2^\alpha} - \left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha \Gamma(1+\alpha)\,{}_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{H}(x)}{x^{2\alpha}}\right| \\
&\leq \left(\frac{m_1m_2(m_2-m_1)}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \\
&\quad\times \int_0^1 \left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right| \left|\mathcal{H}^{(\alpha)}\left(\frac{m_1m_2}{m_2l+(1-l)m_1}\right)\right| (dl)^\alpha \\
&\leq \left(\frac{m_1m_2(m_2-m_1)}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \\
&\quad\times \int_0^1 \left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right| \left[l^{s\alpha}\frac{|\mathcal{H}^{(\alpha)}(m_1)|}{e^{\theta m_1}} + (1-l)^{s\alpha}\frac{|\mathcal{H}^{(\alpha)}(m_2)|}{e^{\theta m_2}}\right](dl)^\alpha \\
&= \left(\frac{m_1m_2(m_2-m_1)}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \left[\frac{|\mathcal{H}^{(\alpha)}(m_1)|}{e^{\theta m_1}} \int_0^1 \left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right| l^{s\alpha}(dl)^\alpha \right. \\
&\quad \left. + \frac{|\mathcal{H}^{(\alpha)}(m_2)|}{e^{\theta m_2}} \int_0^1 \left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right| (1-l)^{s\alpha}(dl)^\alpha\right] \\
&= \left(\frac{m_1m_2(m_2-m_1)}{2}\right)^\alpha \left[\mathcal{B}_1^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_1)|}{e^{\theta m_1}} + \mathcal{B}_2^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_2)|}{e^{\theta m_2}}\right].
\end{aligned} \tag{3.13}$$

Applying the change of variable technique $lm_2 + (1-l)m_1 = x$, we have

$$\begin{aligned}
\mathcal{B}_1^{(\alpha)} &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right| l^{s\alpha}(dl)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} \frac{(1-2l)^\alpha l^{s\alpha}}{(m_2l+(1-l)m_1)^{2\alpha}}(dl)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 \frac{(2l-1)^\alpha l^{s\alpha}}{(m_2l+(1-l)m_1)^{2\alpha}}(dl)^\alpha \\
&= \frac{1}{(m_2-m_1)^{\alpha(s+2)}} \frac{1}{\Gamma(1+\alpha)} \left[\int_{m_1}^{\frac{m_1+m_2}{2}} \left(\frac{(m_1+m_2)^\alpha}{m_1^{2\alpha}} - \frac{2^\alpha}{m_1^\alpha}\right)(x-m_1)^{s\alpha}(dx)^\alpha \right. \\
&\quad \left. + \int_{\frac{m_1+m_2}{2}}^{m_2} \left(\frac{2^\alpha}{m_1^\alpha} - \frac{(m_1+m_2)^\alpha}{m_1^{2\alpha}}\right)(x-m_1)^{s\alpha}(dx)^\alpha\right].
\end{aligned}$$

Again, applying the change of variable, $z = \frac{1}{x}$ if possible, and $\frac{1}{x^{2\alpha}}(dx)^\alpha = -(dz)^\alpha$, from Lemma 1.7, we get

$$\begin{aligned}
\mathcal{B}_1^{(\alpha)} &= \frac{1}{(m_2-m_1)^{\alpha(s+2)}} \\
&\quad\times \left[\frac{(m_1+m_2)^\alpha\Gamma(1+\alpha(s-2))}{\Gamma(1+\alpha(s-1))}\left(2^\alpha\left(\frac{m_1+m_2}{2}\right)^{\alpha(s-1)} - m_1^{\alpha(s-1)} - m_2^{\alpha(s-1)}\right)\right. \\
&\quad - \frac{(m_1)^{s\alpha}((m_2^\alpha - m_1^\alpha)(m_2-m_1)^\alpha}{(m_1m_2)^\alpha\Gamma(\alpha+1)}
\end{aligned}$$

$$+\frac{2^{\alpha}\Gamma(1+(s-1)\alpha)}{\Gamma(1+\alpha s)}\left(-2^{\alpha}\left(\frac{m_1+m_2}{2}\right)^{\alpha s}+m_1^{s\alpha}+m_2^{s\alpha}\right)$$
$$+\left(2m_1^{s}\right)^{\alpha}\left(2^{\alpha}\ln_{\alpha}\left(\frac{m_1+m_2}{2}\right)^{\alpha s}-\ln_{\alpha}(m_1)^{s\alpha}-\ln_{\alpha}(m_2)^{s\alpha}\right)\Bigg]. \tag{3.14}$$

Analogously, we have

$$\begin{aligned}\mathcal{B}_2^{(\alpha)} &= \frac{1}{(m_2-m_1)^{\alpha(s+2)}}\\ &\quad\times\left[\frac{(m_1+m_2)^{\alpha}\Gamma(1+\alpha(s-2))}{\Gamma(1+\alpha(s-1))}\right.\\ &\quad\times\left(-2^{\alpha}\left(\frac{m_1+m_2}{2}\right)^{\alpha(s-1)}+m_1^{\alpha(s-1)}+m_2^{\alpha(s-1)}\right)\\ &\quad+\frac{(m_2)^{s\alpha}(m_2^{\alpha}-m_1^{\alpha})(m_2-m_1)}{(m_1m_2)^{\alpha}\Gamma(\alpha+1)}\\ &\quad+\frac{2^{\alpha}\Gamma(1+(s-1)\alpha)}{\Gamma(1+\alpha s)}\left(2^{\alpha}\left(\frac{m_1+m_2}{2}\right)^{\alpha s}-m_1^{s\alpha}-m_2^{s\alpha}\right)\\ &\quad\left.+\left(2m_2^{s}\right)^{\alpha}\left(-2^{\alpha}\ln_{\alpha}\left(\frac{m_1+m_2}{2}\right)^{\alpha s}+\ln_{\alpha}(m_1)^{s\alpha}+\ln_{\alpha}(m_2)^{s\alpha}\right)\right].\end{aligned} \tag{3.15}$$

A combination of (3.13), (3.14) and (3.15), gives the desired inequality.

Case II. $q > 1$.

Using Lemma 3.7, *GEH s*-convexity of $|\mathcal{H}^{(\alpha)}|$ and the generalized Hölder inequality, we have

$$\begin{aligned}&\left|\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2^{\alpha}}-\left(\frac{m_1m_2}{m_2-m_1}\right)^{\alpha}\Gamma(1+\alpha)_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(x)}{x^{2\alpha}}\right|\\ &\quad\le\left(\frac{m_1m_2(m_2-m_1)}{2}\right)^{\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{(1-2l)^{\alpha}}{(m_2l+(1-l)m_1)^{2\alpha}}\right|\\ &\quad\quad\times\left|\mathcal{H}^{(\alpha)}\left(\frac{m_1m_2}{m_2l+(1-l)m_1}\right)\right|(dl)^{\alpha}\\ &\quad\le\left(\frac{m_1m_2(m_2-m_1)}{2}\right)^{\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{(1-2l)^{\alpha}}{(m_2l+(1-l)m_1)^{2\alpha}}\right|(dl)^{\alpha}\right]^{\frac{q-1}{q}}\\ &\quad\quad\times\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{(1-2l)^{\alpha}}{(m_2l+(1-l)m_1)^{2\alpha}}\right|\left|\mathcal{H}^{(\alpha)}\left(\frac{m_1m_2}{m_2l+(1-l)m_1}\right)\right|^{q}(dl)^{\alpha}\right]^{\frac{1}{q}}\\ &\quad\le\left(\frac{m_1m_2(m_2-m_1)}{2}\right)^{\alpha}\left[\mathcal{B}_3^{(\alpha)}\right]^{\frac{q-1}{q}}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{(1-2l)^{\alpha}}{(m_2l+(1-l)m_1)^{2\alpha}}\right|\right.\\ &\quad\quad\times\left.\left[l^{s\alpha}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^{q}}{e^{q\theta m_1}}+(1-l)^{s\alpha}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^{q}}{e^{q\theta m_2}}\right](dl)^{\alpha}\right]^{\frac{1}{q}}\\ &\quad=\left(\frac{m_1m_2(m_2-m_1)}{2}\right)^{\alpha}\left[\mathcal{B}_3^{(\alpha)}\right]^{\frac{q-1}{q}}\\ &\quad\quad\times\left[\frac{|\mathcal{H}^{(\alpha)}(m_1)|^{q}}{e^{q\theta m_1}}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{(1-2l)^{\alpha}}{(m_2l+(1-l)m_1)^{2\alpha}}\right|l^{s\alpha}(dl)^{\alpha}\right.\end{aligned}$$

$$+\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right|(1-l)^{s\alpha}(dl)^\alpha\Bigg]^{\frac{1}{q}}$$
$$=\left(\frac{m_1m_2(m_2-m_1)}{2}\right)^\alpha\big[\mathcal{B}_3^{(\alpha)}\big]^{\frac{q-1}{q}}\left[\mathcal{B}_1^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}+\mathcal{B}_2^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right]^{\frac{1}{q}}, \tag{3.16}$$

where applying the change of variable technique $lm_2+(1-l)m_1=x$, and Lemma 1.7, we have

$$\begin{aligned}
\mathcal{B}_3^{(\alpha)} &= \frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}\right|(dl)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\frac{(1-2l)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}(dl)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^1\frac{(2l-1)^\alpha}{(m_2l+(1-l)m_1)^{2\alpha}}(dl)^\alpha \\
&= \frac{1}{(m_2-m_1)^{2\alpha}}\frac{1}{\Gamma(1+\alpha)}\left[\int_{m_1}^{\frac{m_1+m_2}{2}}\left(\frac{(m_1+m_2)^\alpha}{m_1^{2\alpha}}-\frac{2^\alpha}{m_1^\alpha}\right)(dx)^\alpha\right. \\
&\quad \left.+\int_{\frac{m_1+m_2}{2}}^{m_2}\left(\frac{2^\alpha}{m_1^\alpha}-\frac{(m_1+m_2)^\alpha}{m_1^{2\alpha}}\right)(dx)^\alpha\right] \qquad (3.17)\\
&= \frac{(m_2^\alpha-m_1^\alpha)}{(m_2-m_1)^\alpha(m_1m_2)^\alpha\Gamma(1+\alpha)}+\frac{2^\alpha}{(m_2-m_1)^\alpha} \\
&\quad \times\left(-2^\alpha\ln_\alpha\left(\frac{m_1+m_2}{2}\right)^{\alpha s}+\ln_\alpha(m_1)^{s\alpha}+\ln_\alpha(m_2)^{s\alpha}\right).
\end{aligned}$$

After substituting (3.14), (3.15) and (3.17) in (3.16), we find the required inequality (3.9).

This completes the proof. □

Some special cases of Theorem 3.8 are presented as follows.

I. If we take $\alpha=1$, then we get a new result for exponentially harmonically s-convex functions.

Corollary 3.9 *For $\theta\in\mathbb{R}$, $s\in(0,1]$ with $p^{-1}+q^{-1}=1$ and letting $\mathcal{H}:\mathcal{I}^\circ\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on Ω° ($\mathcal{I}^\circ$ is the interior of $\mathcal{I}$) such that $\mathcal{H}'\in L_1[m_1,m_2]$ for $m_1,m_2\in\Omega^\circ$ with $m_2>m_1$, if $|\mathcal{H}'|^q$ is GEH s-convex on Ω for $q\geq 1$, then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2}-\left(\frac{m_1m_2}{m_2-m_1}\right)\int_{m_1}^{m_2}\frac{\mathcal{H}(x)}{x^2}\,dx\right| \\
&\quad \leq\left(\frac{m_1m_2(m_2-m_1)}{2}\right)[\mathcal{B}_3]^{\frac{q-1}{q}}\left[\mathcal{B}_1\frac{|\mathcal{H}'(m_1)|^q}{e^{q\theta m_1}}+\mathcal{B}_2\frac{|\mathcal{H}'(m_2)|^q}{e^{q\theta m_2}}\right]^{\frac{1}{q}},
\end{aligned}$$

where $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ can be gotten easily by replacing $\alpha=1$ in (3.10), (3.11) *and* (3.12), *respectively.*

Remark 3.10 In Theorem 3.8:

(1) If we take $\alpha=s=1$ and $\theta=0$, then we get Theorem 2.6 in [29].

(2) If we take $s=1$ and $\theta=0$, then we get Theorem 4.5 in [43].

(3) If we take $q=1$, then we get inequality (3.13).

Theorem 3.11 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ with $p^{-1} + q^{-1} = 1$ and letting $\mathcal{H} : \mathcal{I}^\circ \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}^\alpha$ be a differentiable function on Ω° ($\mathcal{I}^\circ$ is the interior of $\mathcal{I}$) such that $\mathcal{H}^{(\alpha)} \in \mathcal{C}_\alpha[m_1, m_2]$ for $m_1, m_2 \in \Omega^\circ$ with $m_2 > m_1$, if $|\mathcal{H}^{(\alpha)}|^q$ is GEH s-convex on Ω for $q \geq 1$, then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2^\alpha} - \left(\frac{m_1 m_2}{m_2 - m_1}\right)^\alpha \Gamma(1+\alpha)_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{H}(x)}{x^{2\alpha}}\right| \\
&\quad\leq \left(\frac{m_1 m_2(m_2 - m_1)}{2}\right)^\alpha \left[\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right]^{\frac{1}{p}} \\
&\quad\quad\times \left[\mathcal{C}_1^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}} + \mathcal{C}_2^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right]^{\frac{1}{q}},
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
\mathcal{C}_1^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{l^{s\alpha}}{(m_2 l + (1-l)m_1)^{2q\alpha}}(dl)^\alpha \\
&= \frac{1}{(m_2 - m_1)^{\alpha(s+1)}\Gamma(\alpha+1)} \\
&\quad\times \left[\frac{(m_2^{s+1-2q} - m_1^{s+1-2q})^\alpha}{(s+1-2q)^\alpha} - \frac{m_1^{\alpha s}(m_2^{1-2q} - m_1^{1-2q})^\alpha}{(1-2q)^\alpha}\right]
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
\mathcal{C}_2^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{(1-l)^{s\alpha}}{(m_2 l + (1-l)m_1)^{2q\alpha}}(dl)^\alpha \\
&= \frac{1}{(m_2 - m_1)^{\alpha(s+1)}\Gamma(\alpha+1)} \\
&\quad\times \left[\frac{m_2^{\alpha s}(m_2^{1-2q} - m_1^{1-2q})^\alpha}{(1-2q)^\alpha} - \frac{(m_2^{s+1-2q} - m_1^{s+1-2q})^\alpha}{(s+1-2q)^\alpha}\right].
\end{aligned} \tag{3.20}$$

Proof Using Lemma 3.7, *GEH* s-convexity of $|\mathcal{H}^{(\alpha)}|$ and the generalized Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2^\alpha} - \left(\frac{m_1 m_2}{m_2 - m_1}\right)^\alpha \Gamma(1+\alpha)_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{H}(x)}{x^{2\alpha}}\right| \\
&\quad\times \left(\frac{m_1 m_2(m_2 - m_1)}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \\
&\quad\times \int_0^1 \left|\frac{(1-2l)^\alpha}{(m_2 l + (1-l)m_1)^{2\alpha}}\right|\left|\mathcal{H}^{(\alpha)}\left(\frac{m_1 m_2}{m_2 l + (1-l)m_1}\right)\right|(dl)^\alpha \\
&\leq \left(\frac{m_1 m_2(m_2 - m_1)}{2}\right)^\alpha \left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 |1-2l|^{p\alpha}(dl)^\alpha\right]^{\frac{1}{p}} \\
&\quad\times \left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{1}{(m_2 l + (1-l)m_1)^{2q\alpha}}\left|\mathcal{H}^{(\alpha)}\left(\frac{m_1 m_2}{m_2 l + (1-l)m_1}\right)\right|^q (dl)^\alpha\right]^{\frac{1}{q}} \\
&\leq \left(\frac{m_1 m_2(m_2 - m_1)}{2}\right)^\alpha \left[\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right]^{\frac{1}{p}} \left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{1}{(m_2 l + (1-l)m_1)^{2q\alpha}}\right.
\end{aligned}$$

$$\begin{aligned}
&\times\left[l^{s\alpha}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}+(1-l)^{s\alpha}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right](dl)^\alpha\Bigg]^{\frac{1}{q}}\\
&=\left(\frac{m_1m_2(m_2-m_1)}{2}\right)^\alpha\left[\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right]^{\frac{1}{p}}\\
&\times\Bigg[\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{s\alpha}}{(m_2l+(1-l)m_1)^{2q\alpha}}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}(dl)^\alpha\\
&+\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{(1-l)^{s\alpha}}{(m_2l+(1-l)m_1)^{2q\alpha}}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\Bigg](dl)^\alpha]^{\frac{1}{q}}\\
&=\left(\frac{m_1m_2(m_2-m_1)}{2}\right)^\alpha\left[\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right]^{\frac{1}{p}}\\
&\times\left[\mathcal{C}_1^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}+\mathcal{C}_2^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right]^{\frac{1}{q}}.
\end{aligned}\tag{3.21}$$

By making the change of variable technique $lm_2+(1-l)m_1=x$, and Lemma 1.7, we have

$$\begin{aligned}
\mathcal{C}_1^{(\alpha)}&:=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{s\alpha}}{(m_2l+(1-l)m_1)^{2q\alpha}}(dl)^\alpha\\
&=\frac{1}{(m_2-m_1)^{\alpha(s+1)}}\frac{1}{\Gamma(1+\alpha)}\int_{m_1}^{m_2}\frac{(x-m_1)^\alpha}{m_1^{2\alpha}}(dx)^\alpha\\
&=\frac{1}{(m_2-m_1)^{\alpha(s+1)}\Gamma(\alpha+1)}\\
&\times\left[\frac{(m_2^{s+1-2q}-m_1^{s+1-2q})^\alpha}{(s+1-2q)^\alpha}-\frac{m_1^{\alpha s}(m_2^{1-2q}-m_1^{1-2q})^\alpha}{(1-2q)^\alpha}\right]
\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}
\mathcal{C}_2^{(\alpha)}&:=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{(1-l)^{s\alpha}}{(m_2l+(1-l)m_1)^{2q\alpha}}(dl)^\alpha\\
&=\frac{1}{(m_2-m_1)^{\alpha(s+1)}}\frac{1}{\Gamma(1+\alpha)}\int_{m_1}^{m_2}\frac{(m_2-x)^\alpha}{m_1^{2\alpha}}(dx)^\alpha\\
&=\frac{1}{(m_2-m_1)^{\alpha(s+1)}\Gamma(\alpha+1)}\\
&\times\left[\frac{m_2^{\alpha s}(m_2^{1-2q}-m_1^{1-2q})^\alpha}{(1-2q)^\alpha}-\frac{(m_2^{s+1-2q}-m_1^{s+1-2q})^\alpha}{(s+1-2q)^\alpha}\right].
\end{aligned}\tag{3.23}$$

A combination of (3.21), (3.22) and (3.23) gives the desired inequality (3.18). □

I. If we take $\alpha=1$, then we get a new result for exponentially harmonically s-convex functions.

Corollary 3.12 *For $\theta\in\mathbb{R}$, $s\in(0,1]$ with $p^{-1}+q^{-1}=1$ and letting $\mathcal{H}:\mathcal{I}^\circ\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on Ω° ($\mathcal{I}^\circ$ is the interior of $\mathcal{I}$) such that $\mathcal{H}'\in L_1[m_1,m_2]$ for $m_1,m_2\in\Omega^\circ$ with $m_2>m_1$, if $|\mathcal{H}'|^q$ is exponentially harmonically s-convex on Ω for $q\geq1$,* *then the following inequality holds*:

$$\left|\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2}-\left(\frac{m_1m_2}{m_2-m_1}\right)\int_{m_1}^{m_2}\frac{\mathcal{H}(x)}{x^2}\,dx\right|$$
$$\leq\left(\frac{m_1m_2(m_2-m_1)}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\mathcal{C}_1\frac{|\mathcal{H}'(m_1)|^q}{e^{q\theta m_1}}+\mathcal{C}_2\frac{|\mathcal{H}'(m_2)|^q}{e^{q\theta m_2}}\right]^{\frac{1}{q}},$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ can be obtained by replacing $\alpha = 1$ in (3.19) *and* (3.20), *respectively.*

Remark 3.13 In Theorem 3.11:

(1) If we take $\alpha = s = 1$ and $\theta = 0$, then we get Theorem 2.7 of [29].

(2) If we take $s = 1$ and $\theta = 0$, then we get Theorem 4.7 of [43].

4 Generalized Fejér type inequality

The generalized Fejér-type inequality for generalized exponentially harmonically s-convex functions can be presented in local fractional integral forms as follows.

Theorem 4.1 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ and letting $\mathcal{H} : \Omega \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}^\alpha$ be a GEH s-convex function on fractal space, $m_1, m_2 \in \Omega$ with $m_2 > m_1$, if $\mathcal{H}^{(\alpha)} \in \mathbb{C}_\alpha[m_1,m_2]$, and letting $\mathcal{W} : [m_1,m_2] \to \mathbb{R}^\alpha$ be positive, local fractional integrable and symmetric corresponding to $\frac{2m_1m_2}{m_1+m_2}$, then the following inequality holds*:

$$\begin{aligned}&2^{\alpha(s-1)}\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right){}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{W}(x)}{x^{2\alpha}}\\&\leq {}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(x)\mathcal{W}(x)}{(xe^{\theta x})^\alpha}\\&\leq\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}}+\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right]{}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{W}(x)}{x^{2\alpha}}.\end{aligned}\tag{4.1}$$

Proof Since $\mathcal{W}$ is nonnegative, integrable and symmetric with respect to $(\frac{2m_1m_2}{m_1+m_2})$,

$$\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)=\mathcal{W}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right).\tag{4.2}$$

Multiplying on both sides of (3.3) by $\mathcal{W}(\frac{m_1m_2}{lm_2+(1-l)m_1})$, then we have

$$\begin{aligned}&\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\\&\leq\frac{1}{(2)^{s\alpha}}\left[\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)e^{(\frac{-\theta m_1m_2}{lm_2+(1-l)m_1})}\right.\\&\quad\left.+\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)e^{(\frac{-\theta m_1m_2}{lm_1+(1-l)m_2})}\right].\end{aligned}\tag{4.3}$$

Integrating the above inequality corresponding to l from 0 to 1, we have

$$\begin{aligned}
&\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\frac{1}{\Gamma(1+\alpha)}\int_0^1\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)(dl)^\alpha\\
&\leq\frac{1}{(2)^{s\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)e^{(\frac{-\theta m_1m_2}{lm_2+(1-l)m_1})}(dl)^\alpha\right.\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\\
&\quad\left.\times\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)e^{(\frac{-\theta m_1m_2}{lm_1+(1-l)m_2})}(dl)^\alpha\right].
\end{aligned}\tag{4.4}$$

Using the fact that

$$\begin{aligned}
&\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\frac{1}{\Gamma(1+\alpha)}\int_0^1\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)(dl)^\alpha\\
&=\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_{m_1}^{m_2}\frac{\mathcal{W}(x)}{x^{2\alpha}}(dx)^\alpha\\
&=\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha{}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{W}(x)}{x^{2\alpha}}.
\end{aligned}\tag{4.5}$$

Also

$$\begin{aligned}
&\frac{1}{(2)^{s\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)e^{(-\theta\frac{m_1m_2}{lm_2+(1-l)m_1})}(dl)^\alpha\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\alpha)}\int_0^1\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)e^{(-\theta\frac{m_1m_2}{lm_1+(1-l)m_2})}(dl)^\alpha\right]\\
&\quad=\frac{1}{(2)^{s\alpha}}\frac{1}{\Gamma(1+\alpha)}\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\left[\int_{m_1}^{m_2}\frac{\mathcal{H}(x)\mathcal{W}(x)}{(x^2e^{\theta x})^\alpha}(dx)^\alpha+\int_{m_1}^{m_2}\frac{\mathcal{H}(x)\mathcal{W}(x)}{(x^2e^{\theta x})^\alpha}(dx)^\alpha\right]\\
&=\left(\frac{1}{2}\right)^{\alpha(s-1)}\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha{}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(x)\mathcal{W}(x)}{(x^2e^{\theta x})^\alpha}.
\end{aligned}\tag{4.6}$$

From (4.5) and (4.6), then we conclude

$$2^{\alpha(s-1)}\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right){}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{W}(x)}{x^{2\alpha}}\leq{}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(x)\mathcal{W}(x)}{(x^2e^{\theta x})^\alpha}.\tag{4.7}$$

Since $\mathcal{W}$ is nonnegative, integrable and symmetric with respect to $(\frac{2m_1m_2}{m_1+m_2})$, and multiplying on both sides of (3.4) by $\mathcal{W}(\frac{m_1m_2}{lm_2+(1-l)m_1})$, we have

$$\begin{aligned}
&\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha{}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(x)\mathcal{W}(x)}{(x^2e^{\theta x})^\alpha}\\
&=\frac{1}{\Gamma(1+\alpha)}\int_{m_1}^{m_2}\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\left[\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)e^{(\frac{-\theta m_1m_2}{lm_2+(1-l)m_1})}\right.\\
&\quad\left.+\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)e^{(\frac{-\theta m_1m_2}{lm_1+(1-l)m_2})}\right](dl)^\alpha
\end{aligned}$$

$$\leq \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}}+\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right]\int_0^1 \frac{1}{\Gamma(1+\alpha)}\mathcal{W}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)(dl)^\alpha$$
$$= \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}}+\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right]\frac{1}{\Gamma(1+\alpha)}\int_{m_1}^{m_2}\frac{\mathcal{W}(x)}{x^{2\alpha}}(dx)^\alpha$$
$$= \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}}+\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right]{}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{W}(x)}{x^{2\alpha}}. \tag{4.8}$$

From (4.7) and (4.8), we get the desired inequality (4.1). □

I. If we take $\alpha = 1$, then we get a new result for exponentially harmonically s-convex function.

Corollary 4.2 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ and let $\mathcal{H} : \Omega \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a GEH s-convex function with $m_2 > m_1$ and $m_1, m_2 \in \Omega$. If $\mathcal{H}' \in L_1[m_1,m_2]$, and let $\mathcal{W} : [m_1,m_2] \to \mathbb{R}$ be positive, integrable and symmetric corresponding to $\frac{2m_1m_2}{m_1+m_2}$, then the following inequality holds*:

$$2^{\alpha(s-1)}\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\int_{m_1}^{m_2}\frac{\mathcal{W}(x)}{x^2}\,dx \leq \int_{m_1}^{m_2}\frac{\mathcal{H}(x)\mathcal{W}(x)}{(x^2e^{\theta x})}$$
$$\leq \frac{1}{s+1}\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}}+\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right]\int_{m_1}^{m_2}\frac{\mathcal{W}(x)}{x^2}\,dx.$$

II. If we take $s = \alpha = 1$, then we get a new result for exponentially harmonically convex functions.

Corollary 4.3 *For $\theta \in \mathbb{R}$, and letting $\mathcal{H} : \Omega \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a GEH s-convex function with $m_2 > m_1$ and $m_1, m_2 \in \Omega$, if $\mathcal{H}' \in L_1[m_1,m_2]$, and letting $\mathcal{W} : [m_1,m_2] \to \mathbb{R}$ be positive, integrable and symmetric corresponding to $\frac{2m_1m_2}{m_1+m_2}$, then the following inequality holds*:

$$\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\int_{m_1}^{m_2}\frac{\mathcal{W}(x)}{x^2}\,dx \leq \int_{m_1}^{m_2}\frac{\mathcal{H}(x)\mathcal{W}(x)}{(x^2e^{\theta x})}$$
$$\leq \frac{1}{2}\left[\frac{\mathcal{H}(m_1)}{e^{\theta m_1}}+\frac{\mathcal{H}(m_2)}{e^{\theta m_2}}\right]\int_{m_1}^{m_2}\frac{\mathcal{W}(x)}{x^2}\,dx.$$

Remark 4.4 In Theorem 4.1:

(1) If we take $\mathcal{W}(x) = 1$, then we get Theorem 3.1.

(2) If we take $\alpha = s = 1$, and $\theta = 0$, then we get Theorem 8 of [31].

5 Generalized Pachpatte type inequalities

Theorem 5.1 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ and letting $\mathcal{H},\mathcal{U} : \Omega \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}^\alpha$ be GEH s-convex functions on fractal space, $m_1, m_2 \in \Omega$ with $m_2 > m_1$, if $\mathcal{H},\mathcal{U} \in \mathbb{C}_\alpha[m_1,m_2]$, then the following inequalities hold*:

$$\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha {}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{U}(x)\mathcal{H}(x)}{x^{2\alpha}}$$
$$\leq \frac{\Gamma(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right]$$

$$+\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{\Gamma(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\right]$$
$$\times\left[\frac{\mathcal{H}(m_1)\mathcal{U}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{H}(m_2)\mathcal{U}(m_1)}{e^{\theta(m_1+m_2)}}\right] \tag{5.1}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\mathcal{U}\left(\frac{2m_1m_2}{m_1+m_2}\right)\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\\
&\leq\left(\frac{1}{2}\right)^{\alpha}\left[\left(\frac{m_1m_2}{m_2-m_1}\right)^{\alpha}_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{U}(x)\mathcal{H}(x)}{x^{2\alpha}}\right.\\
&\quad+\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{\Gamma(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\right]\\
&\quad\times\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right]\\
&\quad\left.+\frac{(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_1)}{e^{\theta(m_1+m_2)}}\right]\right].
\end{aligned} \tag{5.2}$$

Proof Since $\mathcal{U}$ and $\mathcal{H}$ are *GEH* s-convex on Ω, then, for $l\in[0,1]$, it follows from Definition 2.3 that

$$\begin{aligned}
&\mathcal{U}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\\
&\leq(l)^{2s\alpha}\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+(1-l)^{2s\alpha}\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\\
&\quad+l^{s\alpha}(1-l)^{s\alpha}\left[\frac{\mathcal{H}(m_1)\mathcal{U}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{H}(m_2)\mathcal{U}(m_1)}{e^{\theta(m_1+m_2)}}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{U}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\\
&\leq(l)^{2s\alpha}\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}+(1-l)^{2s\alpha}\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}\\
&\quad+l^{s\alpha}(1-l)^{s\alpha}\left[\frac{\mathcal{H}(m_1)\mathcal{U}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{H}(m_2)\mathcal{U}(m_1)}{e^{\theta(m_1+m_2)}}\right].
\end{aligned}$$

Adding the above inequalities, we have

$$\begin{aligned}
&\mathcal{U}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\\
&\quad+\mathcal{U}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\\
&\leq\left[(l)^{2\alpha s}+(1-l)^{2\alpha s}\right]\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right]\\
&\quad+2^{\alpha}l^{s\alpha}(1-l)^{s\alpha}\left[\frac{\mathcal{H}(m_1)\mathcal{U}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{H}(m_2)\mathcal{U}(m_1)}{e^{\theta(m_1+m_2)}}\right].
\end{aligned}$$

Integrating the above inequality corresponding to l from 0 to 1, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 \mathcal{U}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)(dl)^\alpha \\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \mathcal{U}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)(dl)^\alpha \\
&\leq \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left[(l)^{2\alpha s}+(1-l)^{2\alpha s}\right](dl)^\alpha\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right] \\
&\quad+\frac{2^\alpha}{\Gamma(1+\alpha)}\int_0^1 l^{s\alpha}(1-l)^{s\alpha}(dl)^\alpha\left[\frac{\mathcal{H}(m_1)\mathcal{U}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{H}(m_2)\mathcal{U}(m_1)}{e^{\theta(m_1+m_2)}}\right].
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\frac{1}{\Gamma(1+\alpha)}\left[\int_{m_1}^{m_2}\frac{\mathcal{U}(x)\mathcal{H}(x)}{x^{2\alpha}}(dx)^\alpha+\int_{m_1}^{m_2}\frac{\mathcal{U}(y)\mathcal{H}(y)}{y^{2\alpha}}(dy)^\alpha\right] \\
&\leq \frac{2^\alpha\Gamma(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right] \\
&\quad+2^\alpha\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{\Gamma(1+2s\alpha)}{\Gamma(1+()2s+1)\alpha}\right]\left[\frac{\mathcal{H}(m_1)\mathcal{U}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{H}(m_2)\mathcal{U}(m_1)}{e^{\theta(m_1+m_2)}}\right].
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha {}_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{U}(x)\mathcal{H}(x)}{x^{2\alpha}} \\
&\leq \frac{\Gamma(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right] \\
&\quad+\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{\Gamma(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha}\right]\left[\frac{\mathcal{H}(m_1)\mathcal{U}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{H}(m_2)\mathcal{U}(m_1)}{e^{\theta(m_1+m_2)}}\right].
\end{aligned}$$

Next, we establish the inequality (5.2). Again using the *GEH* s-convexity of $\mathcal{U}$ and $\mathcal{H}$ on Ω, we have

$$\begin{aligned}
&\mathcal{U}\left(\frac{2m_1m_2}{m_1+m_2}\right)\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right) \\
&=\mathcal{U}\left(\frac{1}{2}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}+\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\right) \\
&\quad\times\mathcal{H}\left(\frac{1}{2}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}+\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\right) \\
&\leq\left(\frac{1}{4}\right)^\alpha\left[\mathcal{U}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)+\mathcal{U}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\right] \\
&\quad\times\left[\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)+\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\right] \\
&\leq\left(\frac{1}{4}\right)^\alpha\left[\mathcal{U}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\right] \\
&\quad+\left(\frac{1}{4}\right)^\alpha\left[\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)+\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\right]
\end{aligned}$$

$$+\frac{l^{s\alpha}(1-l)^{s\alpha}}{2^{\alpha}}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right]$$
$$+\frac{l^{2s\alpha}+(1-l)^{2s\alpha}}{4^{\alpha}}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_1)}{e^{\theta(m_1+m_2)}}\right]. \quad (5.3)$$

Integrating the above inequality corresponding to l from 0 to 1, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1\mathcal{U}\left(\frac{2m_1m_2}{m_1+m_2}\right)\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)(dl)^{\alpha}\\
&\le\left(\frac{1}{4}\right)^{\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[\mathcal{U}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\mathcal{H}\left(\frac{m_1m_2}{lm_2+(1-l)m_1}\right)\right](dl)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{1}{4}\right)^{\alpha}\left[\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)+\mathcal{H}\left(\frac{m_1m_2}{lm_1+(1-l)m_2}\right)\right](dl)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{s\alpha}(1-l)^{s\alpha}}{2^{\alpha}}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right](dl)^{\alpha}\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{2s\alpha}+(1-l)^{2s\alpha}}{4^{\alpha}}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_1)}{e^{\theta(m_1+m_2)}}\right](dl)^{\alpha}.
\end{aligned} \quad (5.4)$$

It follows that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\mathcal{U}\left(\frac{2m_1m_2}{m_1+m_2}\right)\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\\
&\le\left(\frac{1}{4}\right)^{\alpha}\left(\frac{m_1m_2}{m_2-m_1}\right)^{\alpha}\frac{1}{\Gamma(1+\alpha)}\left[\int_{m_1}^{m_2}\frac{\mathcal{U}(x)\mathcal{H}(x)}{x^{2\alpha}}(dx)^{\alpha}+\int_{m_1}^{m_2}\frac{\mathcal{U}(y)\mathcal{H}(y)}{y^{2\alpha}}(dy)^{\alpha}\right]\\
&\quad+\left(\frac{1}{2}\right)^{\alpha}\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{\Gamma(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\right]\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right]\\
&\quad+\left(\frac{1}{2}\right)^{\alpha}\frac{(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_1)}{e^{\theta(m_1+m_2)}}\right].
\end{aligned} \quad (5.5)$$

Consequently, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\mathcal{U}\left(\frac{2m_1m_2}{m_1+m_2}\right)\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right)\\
&\le\left(\frac{1}{2}\right)^{\alpha}\left(\frac{m_1m_2}{m_2-m_1}\right)^{\alpha}_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{U}(x)\mathcal{H}(x)}{x^{2\alpha}}\\
&\quad+\left(\frac{1}{2}\right)^{\alpha}\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-\frac{\Gamma(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\right]\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right]\\
&\quad+\left(\frac{1}{2}\right)^{\alpha}\frac{(1+2s\alpha)}{\Gamma(1+(2s+1)\alpha)}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_1)}{e^{\theta(m_1+m_2)}}\right].
\end{aligned} \quad (5.6)$$

This completes the proof. □

I. If we take $\alpha = 1$, then we get a new result for exponentially harmonically s-convex functions.

Corollary 5.2 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ and letting $\mathcal{H},\mathcal{U} : \Omega \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be exponentially harmonically s-convex functions on Ω, $m_1, m_2 \in \Omega$ with $m_2 > m_1$, if $\mathcal{H},\mathcal{U} \in L_1[m_1, m_2]$,* *then the following inequalities hold*:

$$\left(\frac{m_1m_2}{m_2-m_1}\right)\int_{m_1}^{m_2}\frac{\mathcal{U}(x)\mathcal{H}(x)}{x^2}\,dx \le \frac{1}{1+2s}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right] + \frac{2\Gamma(s+1)}{\Gamma(2s+2)}\left[\frac{\mathcal{H}(m_1)\mathcal{U}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{H}(m_2)\mathcal{U}(m_1)}{e^{\theta(m_1+m_2)}}\right]$$

and

$$2^{2s-1}\mathcal{U}\left(\frac{2m_1m_2}{m_1+m_2}\right)\mathcal{H}\left(\frac{2m_1m_2}{m_1+m_2}\right) \le \left(\frac{m_1m_2}{m_2-m_1}\right)\int_{m_1}^{m_2}\frac{\mathcal{U}(x)\mathcal{H}(x)}{x^2}\,dx + \frac{2\Gamma(s+1)}{\Gamma(2s+2)}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_1)}{e^{2\theta m_1}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_2)}{e^{2\theta m_2}}\right] + \frac{1}{(2s+1)}\left[\frac{\mathcal{U}(m_1)\mathcal{H}(m_2)}{e^{\theta(m_1+m_2)}}+\frac{\mathcal{U}(m_2)\mathcal{H}(m_1)}{e^{\theta(m_1+m_2)}}\right].$$

6 Generalized Ostrowski type inequalities

Lemma 6.1 *Let $\mathcal{H}:\mathcal{I}^\circ\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^\alpha$ ($\mathcal{I}^\circ$ is the interior of $\mathcal{I}$) such that $\mathcal{H}\in\mathcal{D}_\alpha(\mathcal{I}^\circ)$ and $\mathcal{H}^{(\alpha)}\in\mathcal{C}_\alpha[m_1,m_2]$ for $m_1,m_2\in\Omega^\circ$ with $m_2>m_1$. Then the following equality holds*:

$$\mathcal{H}(x)-\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\Gamma(1+\alpha)_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(x)}{x^{2\alpha}} \tag{6.1}$$

$$=\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\left[\frac{(x-m_1)^{2\alpha}}{\Gamma(1+\alpha)}\int_0^1\frac{l^\alpha}{(lm_1+(1-l)x)^{2\alpha}}\mathcal{H}^{(\alpha)}\left(\frac{m_1x}{lm_1+(1-l)x}\right)(dl)^\alpha - \frac{(m_2-x)^{2\alpha}}{\Gamma(1+\alpha)}\int_0^1\frac{l^\alpha}{(lm_2+(1-l)x)^{2\alpha}}\mathcal{H}^{(\alpha)}\left(\frac{m_2x}{lm_2+(1-l)x}\right)(dl)^\alpha\right]. \tag{6.2}$$

Proof Using local fractional integration by parts and changing variables yield

$$\begin{aligned}
&\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\left[\frac{(x-m_1)^{2\alpha}}{\Gamma(1+\alpha)}\int_0^1\frac{l^\alpha}{(lm_1+(1-l)x)^{2\alpha}}\mathcal{H}^{(\alpha)}\left(\frac{m_1x}{lm_1+(1-l)x}\right)(dl)^\alpha\right.\\
&\quad\left.-\frac{(m_2-x)^{2\alpha}}{\Gamma(1+\alpha)}\int_0^1\frac{l^\alpha}{(lm_2+(1-l)x)^{2\alpha}}\mathcal{H}^{(\alpha)}\left(\frac{m_2x}{lm_2+(1-l)x}\right)(dl)^\alpha\right]\\
&=\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\left[(x-m_1)^{2\alpha}\left[\frac{l^\alpha}{(m_1x)^\alpha(x-m_1)^\alpha}\mathcal{H}\left(\frac{m_1x}{lm_1+(1-l)x}\right)\Big|_0^1\right.\right.\\
&\quad\left.-\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)(x-m_1)^{2\alpha}}\int_{m_1}^x\frac{\mathcal{H}(u)}{u^{2\alpha}}(du)^\alpha\right]\\
&\quad+(m_2-x)^{2\alpha}\left[\frac{l^\alpha}{(m_2x)^\alpha(m_2-x)^\alpha}\mathcal{H}\left(\frac{m_2x}{lm_2+(1-l)x}\right)\Big|_0^1\right.\\
&\quad\left.\left.-\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)(m_2-x)^{2\alpha}}\int_x^{m_2}\frac{\mathcal{H}(u)}{u^{2\alpha}}(du)^\alpha\right]\right]\\
&=\frac{(m_2(x-m_1)+(m_2-x)m_1)^\alpha}{a_1^\alpha(m_2-m_1)^\alpha}\mathcal{H}(x)-\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\Gamma(1+\alpha)_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(u)}{u^{2\alpha}}\\
&=\mathcal{H}(x)-\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\Gamma(1+\alpha)_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(u)}{u^{2\alpha}},
\end{aligned} \tag{6.3}$$

the required result. □

Remark 6.2 If we take $\alpha = 1$, then we get Lemma 1 of [30].

Theorem 6.3 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ with $p^{-1} + q^{-1} = 1$ and letting $\mathcal{H} : \mathcal{I}^\circ \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}^\alpha$ be a differentiable function on Ω° ($\mathcal{I}^\circ$ is the interior of $\mathcal{I}$) such that $\mathcal{H}^{(\alpha)} \in \mathcal{C}_\alpha[m_1, m_2]$ for $m_1, m_2 \in \Omega^\circ$ with $m_2 > m_1$, if $|\mathcal{H}^{(\alpha)}|^q$ is GEH s-convex on Ω for $q \geq 1$, then the following equality holds*:

$$\begin{aligned}
&\left|\mathcal{H}(x) - \left(\frac{m_1 m_2}{m_2 - m_1}\right)^\alpha \Gamma(1+\alpha)_{m_1}\mathcal{I}^{(\alpha)}_{m_2} \frac{\mathcal{H}(x)}{x^{2\alpha}}\right| \\
&\quad \leq \left(\frac{m_1^\alpha m_2^\alpha (x - m_1)^{2\alpha}}{(m_2 - m_1)^\alpha}\right)\left(\mathcal{S}_1^{(\alpha)}\left(m_1^\alpha, x^\alpha\right)\right)^{\frac{q-1}{q}} \left(\left[\mathcal{S}_2^{(\alpha)} \frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}} + \mathcal{S}_3^{(\alpha)} \frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}\right]\right)^{\frac{1}{q}} \\
&\qquad + \left(\frac{m_1^\alpha m_2^\alpha (m_2 - x)^{2\alpha}}{(m_2 - m_1)^\alpha}\right)\left(\mathcal{S}_1^{(\alpha)}\left(m_2^\alpha, x^\alpha\right)\right)^{\frac{q-1}{q}} \\
&\qquad \times \left(\left[\mathcal{S}_4^{(\alpha)} \frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}} + \mathcal{S}_5^{(\alpha)} \frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right]\right)^{\frac{1}{q}},
\end{aligned} \tag{6.4}$$

where

$$\begin{aligned}
\mathcal{S}_1^{(\alpha)}\left(m_1^\alpha, x^\alpha\right) &:= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{l^\alpha}{(lm_2 + (1-l)x)^{2\alpha}} (dl)^\alpha \\
&= \frac{1}{(m_1 - x)^{2\alpha}} \left(\ln_\alpha (m_1)^\alpha - \ln_\alpha (x)^\alpha\right) - \frac{1}{m_1^\alpha \Gamma(1+\alpha)(m_1 - x)^\alpha},
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
\mathcal{S}_2^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{l^{\alpha(s+1)}}{(lm_1 + (1-l)x)^{2\alpha}} (dl)^\alpha \\
&= \frac{1}{(x - m_1)^{\alpha(s+2)}} \left[\frac{\Gamma(1+\alpha(s-1))}{\Gamma(1+\alpha s)} \left(m_1^{\alpha s} - m_1^{\alpha s}\right) - \frac{m_1^{\alpha(s+1)}}{\Gamma(1+\alpha)} \left(\frac{1}{m_1} - \frac{1}{x}\right)^\alpha\right],
\end{aligned} \tag{6.6}$$

$$\begin{aligned}
\mathcal{S}_3^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{l^\alpha (1-l)^{\alpha s}}{(lm_1 + (1-l)x)^{2\alpha}} (dl)^\alpha \\
&= \frac{(1-x)^\alpha}{(m_1 - x)^{2\alpha}} \frac{\Gamma(1+\alpha(s+1))}{\Gamma(1+\alpha s)} - \frac{m_1^{\alpha(s+1)}}{(m_1 - x)^{\alpha(s+1)}} - \frac{m_1^{\alpha s}}{(m_1 - x)^{\alpha(s+1)}},
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
\mathcal{S}_1^{(\alpha)}\left(m_2^\alpha, x^\alpha\right) &:= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{l^\alpha}{(lm_2 + (1-l)x)^{2\alpha}} (dl)^\alpha \\
&= \frac{1}{(m_2 - x)^{2\alpha}} \left(\ln_\alpha (m_2)^\alpha - \ln_\alpha (x)^\alpha\right) - \frac{1}{m_2^\alpha \Gamma(1+\alpha)(m_2 - x)^\alpha},
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
\mathcal{S}_4^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{l^{\alpha(s+1)}}{(lm_2 + (1-l)x)^{2\alpha}} (dl)^\alpha \\
&= \frac{1}{(x - m_2)^{\alpha(s+2)}} \left[\frac{\Gamma(1+\alpha(s-1))}{\Gamma(1+\alpha s)} \left(m_1^{\alpha s} - m_2^{\alpha s}\right) - \frac{a_1^{\alpha(s+1)}}{\Gamma(1+\alpha)} \left(\frac{1}{m_2} - \frac{1}{x}\right)^\alpha\right],
\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}
\mathcal{S}_5^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{l^\alpha (1-l)^{\alpha s}}{(lm_2 + (1-l)x)^{2\alpha}} (dl)^\alpha \\
&= \frac{(1-x)^\alpha}{(m_2 - x)^{2\alpha}} \frac{\Gamma(1+\alpha(s+1))}{\Gamma(1+\alpha s)} - \frac{m_2^{\alpha(s+1)}}{(m_2 - x)^{\alpha(s+1)}} - \frac{m_2^{\alpha s}}{(m_2 - x)^{\alpha(s+1)}}.
\end{aligned} \tag{6.10}$$

Proof Using Lemma 6.1, the generalized power mean inequality and the *GEH* *s*-convexity of $|\mathcal{H}^{(\alpha)}|^q$ on Ω, yield

$$\begin{aligned}
&\left|\mathcal{H}(x)-\left(\frac{m_1m_2}{m_2-m_1}\right)^{\alpha}\Gamma(1+\alpha)_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{H}(x)}{x^{2\alpha}}\right|\\
&\quad\le\left(\frac{m_1m_2}{m_2-m_1}\right)^{\alpha}\left[\frac{(x-m_1)^{2\alpha}}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_1+(1-l)x)^{2\alpha}}\left|\mathcal{H}^{(\alpha)}\left(\frac{m_1x}{lm_1+(1-l)x}\right)\right|(dl)^{\alpha}\right.\\
&\quad\quad\left.+\frac{(m_2-x)^{2\alpha}}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_2+(1-l)x)^{2\alpha}}\left|\mathcal{H}^{(\alpha)}\left(\frac{m_2x}{lm_2+(1-l)x}\right)\right|(dl)^{\alpha}\right]\\
&\quad\le\left(\frac{m_1^{\alpha}m_2^{\alpha}(x-m_1)^{2\alpha}}{(m_2-m_1)^{\alpha}}\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_1+(1-l)x)^{2\alpha}}(dl)^{\alpha}\right)^{\frac{q-1}{q}}\\
&\quad\quad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_1+(1-l)x)^{2\alpha}}\right.\\
&\quad\quad\left.\times\left[l^{\alpha s}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+(1-l)^{\alpha s}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}\right](dl)^{\alpha}\right)^{\frac{1}{q}}\\
&\quad\quad+\left(\frac{m_1^{\alpha}m_2^{\alpha}(m_2-x)^{2\alpha}}{(m_2-m_1)^{\alpha}}\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_2+(1-l)x)^{2\alpha}}(dl)^{\alpha}\right)^{\frac{q-1}{q}}\\
&\quad\quad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_2+(1-l)x)^{2\alpha}}\right.\\
&\quad\quad\left.\times\left[l^{\alpha s}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+(1-l)^{\alpha s}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right](dl)^{\alpha}\right)^{\frac{1}{q}}\\
&\quad=\left(\frac{m_1^{\alpha}m_2^{\alpha}(x-m_1)^{2\alpha}}{(m_2-m_1)^{\alpha}}\right)\left(\mathcal{S}_1^{(\alpha)}\left(m_1^{\alpha},x^{\alpha}\right)\right)^{\frac{q-1}{q}}\left(\mathcal{S}_2^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+\mathcal{S}_3^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}\right)^{\frac{1}{q}}\\
&\quad\quad+\left(\frac{m_1^{\alpha}m_2^{\alpha}(m_2-x)^{2\alpha}}{(m_2-m_1)^{\alpha}}\right)\left(\mathcal{S}_1^{(\alpha)}\left(m_2^{\alpha},x^{\alpha}\right)\right)^{\frac{q-1}{q}}\\
&\quad\quad\times\left(\mathcal{S}_4^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+\mathcal{S}_5^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right)^{\frac{1}{q}}.
\end{aligned}\tag{6.11}$$

Applying the change of variable technique $lm_1+(1-l)x=z$, and Lemma 1.7, we have

$$\begin{aligned}
\mathcal{S}_1^{(\alpha)}\left(m_1^{\alpha},x^{\alpha}\right)&:=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_1+(1-l)x)^{2\alpha}}(dl)^{\alpha}\\
&=\frac{1}{(m_1-x)^{2\alpha}}\int_x^a\left(\frac{1}{z^{\alpha}}-\frac{a_1^{\alpha}}{z^{2\alpha}}\right)(dz)^{\alpha}\\
&=\frac{1}{(m_1-x)^{2\alpha}}\left(\ln_{\alpha}(m_1)^{\alpha}-\ln_{\alpha}(x)^{\alpha}\right)-\frac{1}{m_1^{\alpha}\Gamma(1+\alpha)(m_1-x)^{\alpha}},
\end{aligned}\tag{6.12}$$

$$\begin{aligned}
\mathcal{S}_2^{(\alpha)}&:=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha(s+1)}}{(lm_1+(1-l)x)^{2\alpha}}(dl)^{\alpha}\\
&=\frac{1}{(x-m_1)^{\alpha(s+2)}}\int_{m_1}^x\left(z^{\alpha(s-1)}-\frac{a_1^{\alpha(s+1)}}{z^{2\alpha}}\right)(du)^{\alpha}\\
&=\frac{1}{(x-m_1)^{\alpha(s+2)}}\left[\frac{\Gamma(1+\alpha(s-1))}{\Gamma(1+\alpha s)}\left(m_1^{\alpha s}-m_1^{\alpha s}\right)-\frac{a_1^{\alpha(s+1)}}{\Gamma(1+\alpha)}\left(\frac{1}{m_1}-\frac{1}{x}\right)^{\alpha}\right],
\end{aligned}\tag{6.13}$$

and

$$\begin{aligned}\mathcal{S}_3^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{l^\alpha(1-l)^{\alpha s}}{(lm_1+(1-l)x)^{2\alpha}}(dl)^\alpha \\ &= \frac{1}{(x-m_1)^{\alpha(s+2)}}\int_{m_1}^{x}\left(\frac{m_1^\alpha}{z^{2\alpha}}-\frac{1}{z^\alpha}\right)(m_1-z)^{\alpha s}(du)^\alpha \\ &= \frac{(1-x)^\alpha}{(m_1-x)^{2\alpha}}\frac{\Gamma(1+\alpha(s+1))}{\Gamma(1+\alpha s)}-\frac{m_1^{\alpha(s+1)}}{(m_1-x)^{\alpha(s+1)}}-\frac{m_1^{\alpha s}}{(m_1-x)^{\alpha(s+1)}}. \end{aligned} \tag{6.14}$$

Again, applying the change of variable technique $lm_2+(1-l)x=z$, and Lemma 1.7, we have

$$\begin{aligned}\mathcal{S}_1^{(\alpha)}\left(m_2^\alpha,x^\alpha\right) &:= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{l^\alpha}{(lm_2+(1-l)x)^{2\alpha}}(dl)^\alpha \\ &= \frac{1}{(m_2-x)^{2\alpha}}\int_x^{m_1}\left(\frac{1}{z^\alpha}-\frac{m_1^\alpha}{z^{2\alpha}}\right)(dz)^\alpha \\ &= \frac{1}{(m_2-x)^{2\alpha}}\left(\ln_\alpha(m_2)^\alpha-\ln_\alpha(x)^\alpha\right)-\frac{1}{m_2^\alpha\Gamma(1+\alpha)(m_2-x)^\alpha}, \end{aligned} \tag{6.15}$$

$$\begin{aligned}\mathcal{S}_4^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{l^{\alpha(s+1)}}{(lm_2+(1-l)x)^{2\alpha}}(dl)^\alpha \\ &= \frac{1}{(x-m_2)^{\alpha(s+2)}}\int_{m_2}^{x}\left(z^{\alpha(s-1)}-\frac{m_1^{\alpha(s+1)}}{z^{2\alpha}}\right)(du)^\alpha \\ &= \frac{1}{(x-m_2)^{\alpha(s+2)}}\left[\frac{\Gamma(1+\alpha(s-1))}{\Gamma(1+\alpha s)}\left(m_1^{\alpha s}-m_2^{\alpha s}\right)-\frac{m_1^{\alpha(s+1)}}{\Gamma(1+\alpha)}\left(\frac{1}{m_2}-\frac{1}{x}\right)^\alpha\right], \end{aligned} \tag{6.16}$$

$$\begin{aligned}\mathcal{S}_5^{(\alpha)} &:= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \frac{l^\alpha(1-l)^{\alpha s}}{(lm_2+(1-l)x)^{2\alpha}}(dl)^\alpha \\ &= \frac{1}{(x-m_2)^{\alpha(s+2)}}\int_{m_2}^{x}\left(\frac{m_1^\alpha}{z^{2\alpha}}-\frac{1}{z^\alpha}\right)(m_2-z)^{\alpha s}(du)^\alpha \\ &= \frac{(1-x)^\alpha}{(m_2-x)^{2\alpha}}\frac{\Gamma(1+\alpha(s+1))}{\Gamma(1+\alpha s)}-\frac{m_2^{\alpha(s+1)}}{(m_2-x)^{\alpha(s+1)}}-\frac{m_2^{\alpha s}}{(m_2-x)^{\alpha(s+1)}}. \end{aligned} \tag{6.17}$$

A combination of (6.11)–(6.16) and (6.17) gives the inequality (6.4). □

Some special cases of Theorem 6.3 are presented as follows.

I. If we take $\alpha=1$, then we get a new result for exponentially harmonically s-convex functions.

Corollary 6.4 *For $\theta\in\mathbb{R}$, $s\in(0,1]$ with $p^{-1}+q^{-1}=1$ and letting $\mathcal{H}:\mathcal{I}^\circ\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on Ω° ($\mathcal{I}^\circ$ is the interior of $\mathcal{I}$) such that $\mathcal{H}'\in L_1[m_1,m_2]$ for $m_1,m_2\in\Omega^\circ$ with $m_2>m_1$, if $|\mathcal{H}'|^q$ is exponentially harmonically s-convex on Ω for $q\geq 1$, then the following equality holds*:

$$\begin{aligned}&\left|\mathcal{H}(x)-\frac{m_1m_2}{m_2-m_1}\int_{m_1}^{m_2}\frac{\mathcal{H}(x)}{x^{2\alpha}}\,dx\right| \\ &\quad\leq\left(\frac{m_1m_2(x-m_1)^2}{(m_2-m_1)}\right)\left(S_1(m_1,x)\right)^{\frac{q-1}{q}}\left(\left[\mathcal{S}_2\frac{|\mathcal{H}'(x)|^q}{e^{q\theta x}}+\mathcal{S}_3\frac{|\mathcal{H}'(m_1)|^q}{e^{q\theta m_1}}\right]\right)^{\frac{1}{q}}\end{aligned}$$

$$+\left(\frac{m_1m_2(m_2-x)^2}{(m_2-m_1)}\right)\big(S_1(m_2,x)\big)^{\frac{q-1}{q}}$$
$$\times\left(\left[\mathcal{S}_4\frac{|\mathcal{H}'(x)|^q}{e^{q\theta x}}+\mathcal{S}_5\frac{|\mathcal{H}'(m_2)|^q}{e^{q\theta m_2}}\right]\right)^{\frac{1}{q}},$$

where $\mathcal{S}_1(m_1,x)$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_1(m_2,x)$, $\mathcal{S}_4$ *and* $\mathcal{S}_5$ *can be obtained by replacing* $\alpha=1$ *in* (6.5)–(6.10), *respectively.*

Remark 6.5 If we take $\alpha=1$ and $\theta=0$ in Theorem 6.3, then we get Theorem 6 of [30].

Theorem 6.6 *For* $\theta\in\mathbb{R}$, $s\in(0,1]$ *with* $p^{-1}+q^{-1}=1$ *and letting* $\mathcal{H}:\mathcal{I}^\circ\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^\alpha$ *be a differentiable function on* Ω° ($\mathcal{I}^\circ$ *is the interior of* $\mathcal{I}$) *such that* $\mathcal{H}^{(\alpha)}\in\mathcal{C}_\alpha[m_1,m_2]$ *for* $m_1,m_2\in\Omega^\circ$ *with* $m_2>m_1$, *if* $|\mathcal{H}^{(\alpha)}|^q$ *is GEH s-convex on* Ω *for* $q\geq1$, *then the following equality holds*:

$$\left|\mathcal{H}(x)-\left(\frac{m_1m_2}{m_2-m_1}\right)^\alpha\Gamma(1+\alpha)_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(x)}{x^{2\alpha}}\right|$$
$$\leq\left(\frac{m_1^\alpha m_2^\alpha(x-m_1)^{2\alpha}}{(m_2-m_1)^\alpha}\right)\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{q-1}{q}}\left(\left[\mathcal{S}_6^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+\mathcal{S}_7^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}\right]\right)^{\frac{1}{q}}$$
$$+\left(\frac{m_1^\alpha m_2^\alpha(m_2-x)^{2\alpha}}{(m_2-m_1)^\alpha}\right)\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{q-1}{q}}$$
$$\times\left(\left[\mathcal{S}_8^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+\mathcal{S}_9^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right]\right)^{\frac{1}{q}},\tag{6.18}$$

where

$$\mathcal{S}_6^{(\alpha)}:=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha(s+1)}}{(lm_1+(1-l)x)^{2q\alpha}}(dl)^\alpha\tag{6.19}$$
$$=\frac{1}{(m_1-x)^{\alpha(s+2)}}\left[\frac{\Gamma(1+(1+s-2q)\alpha)}{\Gamma(1+(2+s-2q)\alpha)}\big(m_1^{(2+s-2q)\alpha}-x^{(2+s-2q)\alpha}\big)\right.$$
$$\left.-\frac{m_1^{\alpha(1+s)}(m_1^{1-2q}-x^{1-2q})^\alpha}{(1-2q)^\alpha\Gamma(1+\alpha)}\right]$$

and

$$\mathcal{S}_7^{(\alpha)}:=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{(1-l)^{\alpha(s+1)}}{(lm_1+(1-l)x)^{2q\alpha}}(dl)^\alpha$$
$$=\frac{1}{(m_1-x)^{\alpha(s+2)}}\left[\frac{m_1^\alpha\Gamma(1+(1-2q)\alpha)}{\Gamma(1+(2-2q)\alpha)}\big(m_1^{\alpha(2-2q)}-x^{\alpha(2-2q)}\big)-\frac{\Gamma(1+(1-2q+s)\alpha)}{\Gamma(1+(2-2q+s)\alpha)}\right.$$
$$\times\big(m_1^{\alpha(2-2q+s)}-x^{\alpha(2-2q+s)}\big)+\frac{x_1^\alpha\Gamma(1+(s-2q)\alpha)}{\Gamma(1+(s-2q)\alpha)}\big(a_1^{\alpha(1+s-2q)}-m_1^{\alpha(1+s-2q)}\big)$$
$$\left.-\frac{(xm_1)^\alpha(m_1^{1-2q}-x^{1-2q})}{(1-2q)^\alpha\Gamma(1+\alpha)}\right].\tag{6.20}$$

Proof Using Lemma 6.1, the generalized power mean inequality and the *GEH* s-convexity of $|\mathcal{H}^{(\alpha)}|^q$ on Ω, yield

$$\begin{aligned}
&\left|\mathcal{H}(x)-\left(\frac{m_1m_2}{m_2-m_1}\right)^{\alpha}\Gamma(1+\alpha)_{m_1}\mathcal{I}^{(\alpha)}_{m_2}\frac{\mathcal{H}(x)}{x^{2\alpha}}\right|\\
&\le\left(\frac{m_1m_2}{m_2-m_1}\right)^{\alpha}\left[\frac{(x-m_1)^{2\alpha}}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_1+(1-l)x)^{2\alpha}}\left|\mathcal{H}^{(\alpha)}\left(\frac{m_1x}{lm_1+(1-l)x}\right)\right|(dl)^{\alpha}\right.\\
&\quad\left.+\frac{(m_2-x)^{2\alpha}}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_2+(1-l)x)^{2\alpha}}\left|\mathcal{H}^{(\alpha)}\left(\frac{m_2x}{lm_2+(1-l)x}\right)\right|(dl)^{\alpha}\right]\\
&\le\left(\frac{m_1^{\alpha}m_2^{\alpha}(x-m_1)^{2\alpha}}{(m_2-m_1)^{\alpha}}\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 l^{\alpha}(dl)^{\alpha}\right)^{\frac{q-1}{q}}\\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_1+(1-l)x)^{2q\alpha}}\right.\\
&\quad\left.\times\left[l^{\alpha s}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+(1-l)^{\alpha s}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}\right](dl)^{\alpha}\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{m_1^{\alpha}m_2^{\alpha}(m_2-x)^{2\alpha}}{(m_2-m_1)^{\alpha}}\right)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 l^{\alpha}(dl)^{\alpha}\right)^{\frac{q-1}{q}}\\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha}}{(lm_2+(1-l)x)^{2q\alpha}}\right.\\
&\quad\left.\times\left[l^{\alpha s}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+(1-l)^{\alpha s}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right](dl)^{\alpha}\right)^{\frac{1}{q}}\\
&=\left(\frac{m_1^{\alpha}m_2^{\alpha}(x-m_1)^{2\alpha}}{(m_2-m_1)^{\alpha}}\right)\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{q-1}{q}}\left(\left[\mathcal{S}_6^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+\mathcal{S}_7^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}\right]\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{m_1^{\alpha}m_2^{\alpha}(m_2-x)^{2\alpha}}{(m_2-m_1)^{\alpha}}\right)\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{q-1}{q}}\\
&\quad\times\left(\left[\mathcal{S}_8^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(x)|^q}{e^{q\theta x}}+\mathcal{S}_9^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right]\right)^{\frac{1}{q}}.
\end{aligned}\tag{6.21}$$

By applying the change of variable technique $lm_1+(1-l)x=z$, and Lemma 1.7, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 l^{\alpha}(dl)^{\alpha}:=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)},$$

$$\begin{aligned}
\mathcal{S}_6^{(\alpha)}&:=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{l^{\alpha(s+1)}}{(lm_1+(1-l)x)^{2q\alpha}}(dl)^{\alpha}\\
&=\frac{1}{(m_1-x)^{\alpha(s+2)}}\left[\frac{\Gamma(1+(1+s-2q)\alpha)}{\Gamma(1+(2+s-2q)\alpha)}\left(x^{(2+s-2q)\alpha}-m_1^{(2+s-2q)\alpha}\right)\right.\\
&\quad\left.-\frac{m_1^{\alpha(1+s)}(x^{1-2q}-m_1^{1-2q})^{\alpha}}{(1-2q)^{\alpha}\Gamma(1+\alpha)}\right],
\end{aligned}\tag{6.22}$$

$$\begin{aligned}
\mathcal{S}_7^{(\alpha)}&:=\frac{1}{\Gamma(1+\alpha)}\int_0^1\frac{(1-l)^{\alpha(s+1)}}{(lm_1+(1-l)x)^{2q\alpha}}(dl)^{\alpha}\\
&=\frac{1}{(m_1-x)^{\alpha(s+2)}}\int_{m_1}^{x}\left(z^{\alpha(1-2q)}-\frac{m_1^{\alpha}}{z^{2\alpha q}}\right)(m_1-z)^{\alpha s}(dz)^{\alpha}
\end{aligned}\tag{6.23}$$

$$= \frac{1}{(m_1-x)^{\alpha(s+2)}}\left[\frac{m_1^{\alpha}\Gamma(1+(1-2q)\alpha)}{\Gamma(1+(2-2q)\alpha)}\left(m_1^{\alpha(2-2q)}-x^{\alpha(2-2q)}\right)-\frac{\Gamma(1+(1-2q+s)\alpha)}{\Gamma(1+(2-2q+s)\alpha)}\right.$$
$$\times\left(x^{\alpha(2-2q+s)}-m_1^{\alpha(2-2q+s)}\right)+\frac{x_1^{\alpha}\Gamma(1+(s-2q)\alpha)}{\Gamma(1+(s-2q)\alpha)}\left(m_1^{\alpha(1+s-2q)}-x^{\alpha(1+s-2q)}\right)$$
$$\left.-\frac{(xm_1)^{\alpha}(m_1^{1-2q}-x^{1-2q})}{(1-2q)^{\alpha}\Gamma(1+\alpha)}\right].$$

$\mathcal{S}_8^{(\alpha)}$ and $\mathcal{S}_9^{(\alpha)}$ can be found by replacing m_1 by m_2 in (6.22) and (6.23).

Therefore, by combination of (6.21), (6.22) and (6.23), one concludes to the inequality (6.18).

This completes the proof. □

I. If we take $\alpha = 1$, then we get a new result for exponentially harmonically s-convex functions.

Corollary 6.7 *For $\theta \in \mathbb{R}$, $s \in (0,1]$ with $p^{-1}+q^{-1}=1$ and letting $\mathcal{H}:\mathcal{I}^{\circ}\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}^{\alpha}$ be a differentiable function on Ω° ($\mathcal{I}^{\circ}$ is the interior of $\mathcal{I}$) such that $\mathcal{H}^{(\alpha)}\in\mathcal{C}_{\alpha}[m_1,m_2]$ for $m_1,m_2\in\Omega^{\circ}$ with $m_2>m_1$, if $|\mathcal{H}^{(\alpha)}|^q$ is GEH s-convex on Ω for $q\geq 1$, then the following equality holds*:

$$\left|\mathcal{H}(x)-\frac{m_1m_2}{m_2-m_1}\int_{m_1}^{m_2}\frac{\mathcal{H}(x)}{x^{2\alpha}}\,dx\right|$$
$$\leq\left(\frac{m_1m_2(x-m_1)^2}{(m_2-m_1)}\right)\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\left[\mathcal{S}_6^{(\alpha)}\frac{|\mathcal{H}'(x)|^q}{e^{q\theta x}}+\mathcal{S}_7^{(\alpha)}\frac{|\mathcal{H}'(m_1)|^q}{e^{q\theta m_1}}\right]\right)^{\frac{1}{q}}$$
$$+\left(\frac{m_1m_2(x-m_1)^2}{(m_2-m_1)}\right)\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\left[\mathcal{S}_8^{(\alpha)}\frac{|\mathcal{H}'(x)|^q}{e^{q\theta x}}+\mathcal{S}_9^{(\alpha)}\frac{|\mathcal{H}'(m_2)|^q}{e^{q\theta m_2}}\right]\right)^{\frac{1}{q}},$$

where $\mathcal{S}_6,\mathcal{S}_7,\mathcal{S}_8$ and $\mathcal{S}_9$ can be obtained by replacing $\alpha=1$ in (6.22) and (6.23), respectively.

Remark 6.8 If we take $\alpha=1$ and $\theta=0$ in Theorem 6.6, then we get Theorem 5 of [30].

7 Example

In this section, we present an example to illustrate our main contribution.

Example 7.1 Let $\mathcal{H}(x)=x^2\ln x$, for $x\in(0,\infty)$. Then $\mathcal{H}$ is a *GEH s*-convex function with $\alpha\in(0,1]$. If we take $\alpha=1$, $m_1=1$, $m_2=2$, then all assumptions in Theorem 3.8 are satisfied.

The left hand side term of (3.9) is

$$\left|\frac{\mathcal{H}(m_1)+\mathcal{H}(m_2)}{2^{\alpha}}-\left(\frac{m_1m_2}{m_2-m_1}\right)^{\alpha}\Gamma(1+\alpha)_{m_1}\mathcal{I}_{m_2}^{(\alpha)}\frac{\mathcal{H}(x)}{x^{2\alpha}}\right|$$
$$=\left|\frac{1\ln 1+2^2\ln 2}{2}-\left(\frac{1.2}{2-1}\right)_1\mathcal{I}_2^{(1)}\frac{x^2\ln x}{x^2}\right|$$
$$=|2\ln 2-4\ln 2+2|\approx 0.6137.$$

The right hand side term of (3.9) is

$$\begin{aligned}&\left(\frac{m_1m_2(m_2-m_1)}{2}\right)^{\alpha}\left[\mathcal{B}_3^{(\alpha)}\right]^{\frac{q-1}{q}}\left[\mathcal{B}_1^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_1)|^q}{e^{q\theta m_1}}+\mathcal{B}_2^{(\alpha)}\frac{|\mathcal{H}^{(\alpha)}(m_2)|^q}{e^{q\theta m_2}}\right]^{\frac{1}{q}}\\&=\left(\frac{1.2(2-1)}{2}\right)(0.0794)^{\frac{1}{3}}\left[-0.0457(0)+0.1454(2\ln 2)^{1.5}\right]^{\frac{1}{1.5}}\\&\approx 7.6474.\end{aligned}$$

It is clear that $0.6137 < 7.6474$, which demonstrates the result described in Theorem 3.8.

8 Application to special means

In this section, we recall the following α-type special means for two positive real numbers m_1^{α}, m_2^{α} where $m_1 < m_2$:

(1) The arithmetic mean

$$\mathbb{A}_{\alpha}(m_1,m_2)=\frac{m_1^{\alpha}+m_2^{\alpha}}{2^{\alpha}}.$$

(2) The geometric mean

$$\mathbb{G}_{\alpha}(m_1,m_2)=\sqrt{m_1^{\alpha}m_2^{\alpha}}.$$

(3) The harmonic mean

$$\mathbb{H}_{\alpha}(m_1,m_2)=\frac{(2m_1m_2)^{\alpha}}{m_1^{\alpha}+m_2^{\alpha}}.$$

(4) The r-logarithmic mean

$$\mathbb{L}_{r\alpha}(m_1,m_2)=\left(\frac{\Gamma(1+r\alpha)}{\Gamma(1+(r+1)\alpha)}\frac{m_2^{(r+1)\alpha}-m_1^{(r+1)\alpha}}{(m_2-m_1)^{\alpha}}\right)^{\frac{1}{r}},\quad r\in\mathbb{R}\setminus\{-1,0\}.$$

These means have a lot of applications in areas and different types of numerical approximations. However, the following simple relationships are known in the literature:

$$\mathbb{H}_{\alpha}(m_1,m_2)\le\mathbb{G}_{\alpha}(m_1,m_2)\le\mathbb{A}_{\alpha}(m_1,m_2).$$

Assume the mapping $\mathcal{V}:(0,\infty)\mapsto\mathbb{R}^{\alpha}$, $\mathcal{V}(z)=\frac{\Gamma(1+u\alpha)}{\Gamma(1+(u+1)\alpha)}z^{(q+1)\alpha}$, $z>0$, $q\ge 1$ and $r\ge 1$. Then $|\mathcal{V}^{(\alpha)}(z)|^r=z^{qr\alpha}$ is GEH s-convex on $(0,\infty)$. Therefore, we can obtain the following results for $\Phi(z)=\frac{\Gamma(1+u\alpha)}{\Gamma(1+(u+1)\alpha)}x^{(u+1)\alpha}$.

Proposition 8.1 *For* $0<m_1<m_2$, $u,q>1$ *and* $\alpha\in(0,1]$, *we have the following inequality*:

$$\begin{aligned}&\left|\mathbb{A}_{\alpha}\left(m_1^{u+1},m_2^{u+1}\right)-\Gamma(1+\alpha)\mathbb{G}_{\alpha}^2(m_1,m_2)\mathbb{L}_{(u-1)\alpha}^{u-1}(m_1,m_2)\right|\\&\le\frac{\Gamma(1+(u+1)\alpha)}{\Gamma(1+u\alpha)}\frac{(m_2-m_1)^{\alpha}}{\mathbb{G}_{\alpha}^2(2,m_1,m_2)}\left[\mathcal{B}_3^{(\alpha)}\right]^{\frac{q-1}{q}}\left[\mathcal{B}_1^{(\alpha)}\frac{m_1^{qu\alpha}}{e^{q\theta m_1}}+\mathcal{B}_2^{(\alpha)}\frac{m_2^{qu\alpha}}{e^{q\theta m_2}}\right]^{\frac{1}{q}},\end{aligned}$$

where $\mathcal{B}_1^{(\alpha)}$, $\mathcal{B}_2^{(\alpha)}$ *and* $\mathcal{B}_3^{(\alpha)}$ *are given in* (3.10)–(3.12), *respectively.*

Proof Taking $\mathcal{H}(z) = \frac{\Gamma(1+u\alpha)}{\Gamma(1+(u+1)\alpha)} z^{(s+1)\alpha}$, $u \geq 1$ for $z > 0$ in Theorem 3.8, then we get the immediate consequence. □

Proposition 8.2 *For* $0 < m_1 < m_2, p, q > 1$ *and* $\alpha \in (0,1]$, *we have the following inequality*:

$$\begin{aligned} &\left|\mathbb{A}_\alpha\left(m_1^{u+1}, m_2^{u+1}\right) - \Gamma(1+\alpha)\mathbb{G}_\alpha^2(m_1,m_2)\mathbb{L}_{(u-1)\alpha}^{u-1}(m_1,m_2)\right| \\ &\quad \leq \frac{\Gamma(1+(u+1)\alpha)}{\Gamma(1+u\alpha)} \frac{(m_2-m_1)^\alpha}{\mathbb{G}_\alpha^2(2,m_1,m_2)} \left[\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right]^{\frac{1}{p}} \left[\mathcal{C}_1^{(\alpha)} \frac{m_1^{qu\alpha}}{e^{q\theta m_1}} + \mathcal{C}_2^{(\alpha)} \frac{m_2^{qu\alpha}}{e^{q\theta m_2}}\right]^{\frac{1}{q}}, \end{aligned}$$

where $\mathcal{C}_1^{(\alpha)}$ *and* $\mathcal{C}_2^{(\alpha)}$ *are given in* (3.19) *and* (3.20), *respectively.*

Proof Taking $\mathcal{H}(z) = \frac{\Gamma(1+u\alpha)}{\Gamma(1+(u+1)\alpha)} z^{(u+1)\alpha}$, $u \geq 1$ for $z > 0$ in Theorem 3.11, then we get the immediate consequence. □

9 Conclusions

In this study, we have investigated two new classes of convex functions known as *GEH* convex functions and *GEH* s-convex functions on the fractal domain and presented new properties for the more general class *GEH* s-convex functions. The new concept takes into account the several generalizations that have been derived in the framework of local fractional integrals for generalized differentiable functions. We have derived a new version of the generalized Hermite–Hadamard inequality and Hermite–Hadamard–Fejér type inequalities. We have established an integral identity involving first order differentiability, and we obtained more refinements of trapezium type inequality, generalized Pachpatte type, and generalized Ostrowski type inequalities for GEH s-convex functions. We discussed some new special cases of the obtained results which showed that the results obtained are quite unifying and capture the results for classical harmonically convex and exponentially harmonically convex functions at the same moment by changing the parameter values of θ and s. The outcomes acquired by the future plan are all the more invigorating as contrasted with results accessible in the literature given by [29, 30, 32] and [43]. Finally, our consequences have a potential connection in fractal theory and machine learning [2, 3]. This new concept will be opening new doors of investigation toward fractal differentiations and integrations in convexity, preinvexity, fractal image processing and camouflage in the garment industry. It is hoped that the main results of this paper will inspire interested readers and will stimulate further research in this field.

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