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# A new generalization of some quantum integral inequalities for quantum differentiable convex functions

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## Abstract

In this paper, we offer a new quantum integral identity, the result is then used to obtain some new estimates of Hermite–Hadamard inequalities for quantum integrals. The results presented in this paper are generalizations of the comparable results in the literature on Hermite–Hadamard inequalities. Several inequalities, such as the midpoint-like integral inequality, the Simpson-like integral inequality, the averaged midpoint–trapezoid-like integral inequality, and the trapezoid-like integral inequality, are obtained as special cases of our main results.

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**Keywords:** Hermite–Hadamard inequality; Trapezoid inequalities; Midpoint inequalities; Quantum calculus; Convex functions

## 1 Introduction

The study of  $q$ -calculus was initiated in the early 20th century after the work of Jackson (1910) who defined an integral later known as the  $q$ -Jackson integral (see [16, 22, 23, 27, 28]). In  $q$ -calculus, the classical derivative is replaced by the  $q$ -difference operator to deal with nondifferentiable functions. For more discussion on this subject, we refer to [8, 21]. Applications of  $q$ -calculus can be found in various disciplines of mathematics and physics (see [13, 26, 36, 45]).

Many well-known integral inequalities, such as Hölder inequality, Hermite–Hadamard inequalities and Ostrowski inequality, Cauchy–Bunyakovsky–Schwarz inequality, Grüss inequality, Grüss–Chebysev inequality, and other integral inequalities, have been studied in the setup of  $q$ -calculus using the concept of classical convexity. For more results in this direction, we refer to [1–7, 10, 11, 14, 18–20, 24, 29–32, 34, 35, 37, 39–43, 46, 47].

The purpose of this paper is to prove several new quantum integral inequalities by applying the newly defined concept of a  $q^b$ -integral. We also discuss the relation of our results with comparable results existing in the literature.

The organization of this paper is as follows: In Sect. 2, a brief description of the concepts of  $q$ -calculus and some related works in this direction are given. In Sect. 3, the bounds of Hermite–Hadamard-type inequalities for the  $q^b$ -integrals are presented. In Sect. 4, several

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special cases of our main results are discussed. The relationship between the results presented herein and comparable results in the literature is also studied. Section 5 contains some conclusions and further directions for future research. We believe that the study initiated in this paper may inspire new research in this area.

## 2 Preliminaries of $q$ -calculus and some inequalities

In this section, we first present some known definitions and related inequalities in the  $q$ -calculus. Set the following notation (see [28]):

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}, \quad q \in (0, 1).$$

Jackson [27] defined the  $q$ -Jackson integral of a given function  $f$  from 0 to  $b$  as follows:

$$\int_0^b f(x) d_q x = (1 - q)b \sum_{n=0}^{\infty} q^n f(bq^n), \quad \text{where } 0 < q < 1, \tag{2.1}$$

provided that the sum converges absolutely.

Jackson [27] defined the  $q$ -Jackson integral of a given function over the interval  $[a, b]$  as follows:

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

**Definition 1** ([38]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function. The  $q_a$ -derivative of  $f$  at  $x \in [a, b]$  is identified by the the following expression:

$${}_a D_q f(x) = \frac{f(x) - f(qx + (1 - q)a)}{(1 - q)(x - a)}, \quad x \neq a. \tag{2.2}$$

If  $x = a$ , we define  ${}_a D_q f(a) = \lim_{x \rightarrow a} {}_a D_q f(x)$  if it exists and is finite.

**Definition 2** ([15]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function. The  $q^b$ -derivative of  $f$  at  $x \in [a, b]$  is given by

$${}^b D_q f(x) = \frac{f(qx + (1 - q)b) - f(x)}{(1 - q)(b - x)}, \quad x \neq b.$$

If  $x = b$ , we define  ${}^b D_q f(b) = \lim_{x \rightarrow b} {}^b D_q f(x)$  if it exists and is finite.

**Definition 3** ([38]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function. Then, the  $q_a$ -definite integral on  $[a, b]$  is defined by

$$\begin{aligned} \int_a^b f(x)_a d_q x &= (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n b + (1 - q^n)a) \\ &= (b - a) \int_0^1 f((1 - t)a + tb) d_q t. \end{aligned}$$

Alp et al. [11] proved the following  $q_a$ -Hermite–Hadamard inequalities for convex functions in the setting of quantum calculus:

**Theorem 1** *If  $f : [a, b] \rightarrow \mathbb{R}$  is a convex differentiable function on  $[a, b]$  and  $0 < q < 1$ . Then we have*

$$f\left(\frac{qa + b}{[2]_q}\right) \leq \frac{1}{b - a} \int_a^b f(x)_a d_q x \leq \frac{qf(a) + f(b)}{[2]_q}. \tag{2.3}$$

In [11] and [33], the authors established some bounds for the left- and right-hand sides of the inequality (2.3).

On the other hand, Bermudo et al. [15] gave the following definition and obtained the related Hermite–Hadamard-type inequalities:

**Definition 4** ([15]) *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function. Then, the  $q^b$ -definite integral on  $[a, b]$  is given by*

$$\begin{aligned} \int_a^b f(x)^b d_{q^b} x &= (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n a + (1 - q^n)b) \\ &= (b - a) \int_0^1 f(ta + (1 - t)b) d_q t. \end{aligned}$$

**Theorem 2** ([15]) *If  $f : [a, b] \rightarrow \mathbb{R}$  is a convex, differentiable function on  $[a, b]$  and  $0 < q < 1$ , then the  $q$ -Hermite–Hadamard inequalities are given as follows:*

$$f\left(\frac{a + qb}{[2]_q}\right) \leq \frac{1}{b - a} \int_a^b f(x)^b d_q x \leq \frac{f(a) + qf(b)}{[2]_q}. \tag{2.4}$$

From Theorems 1 and 2, one can obtain the following inequalities:

**Corollary 1** ([15]) *For any convex function  $f : [a, b] \rightarrow \mathbb{R}$  and  $0 < q < 1$ , we have*

$$\begin{aligned} f\left(\frac{qa + b}{[2]_q}\right) + f\left(\frac{a + qb}{[2]_q}\right) &\leq \frac{1}{b - a} \left\{ \int_a^b f(x)_a d_q x + \int_a^b f(x)^b d_q x \right\} \\ &\leq f(a) + f(b) \end{aligned} \tag{2.5}$$

and

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{2(b - a)} \left\{ \int_a^b f(x)_a d_q x + \int_a^b f(x)^b d_q x \right\} \leq \frac{f(a) + f(b)}{2}. \tag{2.6}$$

We recall the following well-known inequality:

**Theorem 3** (Hölder’s inequality, [12, p. 604]) *Let  $x > 0$ ,  $0 < q < 1$ ,  $p_1 > 1$  be such that  $\frac{1}{p_1} + \frac{1}{r_1} = 1$ . Then*

$$\int_0^x |f(x)g(x)| d_q x \leq \left( \int_0^x |f(x)|^{p_1} d_q x \right)^{\frac{1}{p_1}} \left( \int_0^x |g(x)|^{r_1} d_q x \right)^{\frac{1}{r_1}}.$$

### 3 Main results

In this section, we give some new estimates of Hermite–Hadamard-type inequalities for functions whose first  $q^b$ -derivatives in absolute value are convex.

Let's start with the following useful lemma.

**Lemma 1** *If  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  is a  $q^b$ -differentiable function on  $(a, b)$  such that  ${}^bD_q f$  is continuous and integrable on  $[a, b]$ , then we have*

$$\begin{aligned}
 (b-a) & \left[ \int_0^v (qt + \gamma v - \gamma)^b D_q f (ta + (1-t)b) d_q t \right. \\
 & \left. + \int_v^1 (qt + \gamma v - 1)^b D_q f (ta + (1-t)b) d_q t \right] \\
 & = \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma [vf(a) + (1-v)f(b)] - (1-\gamma)f(va + (1-v)b),
 \end{aligned} \tag{3.1}$$

where  $0 < q < 1$ .

*Proof* From Definition 2, it follows that

$${}^bD_q f (ta + (1-t)b) = \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)(b-a)t}.$$

By applying identical transformation, we obtain

$$\begin{aligned}
 (b-a) & \int_0^v (qt + \gamma v - \gamma)^b D_q f (ta + (1-t)b) d_q t \\
 & + (b-a) \int_v^1 (qt + \gamma v - 1)^b D_q f (ta + (1-t)b) d_q t \\
 & = (b-a) \int_0^1 (qt + \gamma v - 1)^b D_q f (ta + (1-t)b) d_q t \\
 & + (b-a) \int_0^v (1-\gamma)^b D_q f (ta + (1-t)b) d_q t.
 \end{aligned} \tag{3.2}$$

By the equality (2.1), we obtain

$$\begin{aligned}
 & \int_0^v {}^bD_q f (ta + (1-t)b) d_q t \\
 & = \int_0^v \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)(b-a)t} d_q t \\
 & = \frac{1}{b-a} \left[ \sum_{n=0}^{\infty} f(q^{n+1}va + (1-q^{n+1}v)b) - \sum_{n=0}^{\infty} f(q^n va + (1-q^n v)b) \right] \\
 & = \frac{1}{b-a} [f(b) - f(va + (1-v)b)]
 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
 & \int_0^1 {}^b D_q f(ta + (1-t)b) d_q t \tag{3.4} \\
 &= \int_0^1 \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)(b-a)t} d_q t \\
 &= \frac{1}{b-a} \left[ \sum_{n=0}^{\infty} f(q^{n+1}a + (1-q^{n+1})b) - \sum_{n=0}^{\infty} f(q^n a + (1-q^n)b) \right] \\
 &= \frac{1}{b-a} [f(b) - f(a)].
 \end{aligned}$$

It follows from (2.1) and Definition 4 that

$$\begin{aligned}
 & \int_0^1 t {}^b D_q f(ta + (1-t)b) d_q t \tag{3.5} \\
 &= \int_0^1 \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)(b-a)} d_q t \\
 &= \frac{1}{b-a} \left[ \sum_{n=0}^{\infty} q^n f(q^{n+1}a + (1-q^{n+1})b) - \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) \right] \\
 &= \frac{1}{b-a} \left[ \frac{1}{q} \sum_{n=0}^{\infty} q^{n+1} f(q^{n+1}a + (1-q^{n+1})b) - \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) \right] \\
 &= \frac{1}{b-a} \left[ \frac{1}{q} \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) - \frac{1}{q} f(a) - \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) \right] \\
 &= \frac{1}{b-a} \left[ \left( \frac{1}{q} - 1 \right) \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) - \frac{1}{q} f(a) \right] \\
 &= \frac{1}{b-a} \left( \frac{1-q}{q} \right) \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n)b) - \frac{1}{q(b-a)} f(a) \\
 &= \frac{1}{q(b-a)^2} \int_a^b f(x) {}^b d_q x - \frac{1}{q(b-a)} f(a).
 \end{aligned}$$

Using (3.3), (3.4), and (3.5) in (3.2), we obtain the required identity. □

*Remark 1* If we take  $q \rightarrow 1^-$  in the Lemma 1, then we have [25, Lemma 2.1].

Before we present our main inequalities, we first give some calculated quantum integrals:

$$\begin{aligned}
 A_{1q}(\gamma, \nu) &= \int_0^\nu t |qt + \gamma\nu - \gamma| d_q t = \begin{cases} \frac{\nu^3 q}{[3]_q} - \frac{\nu^2 \gamma (1-\nu)}{[2]_q} + \frac{2\gamma^3 (1-\nu)^3}{([2]_q)([3]_q)}, & (\gamma + q)\nu > \gamma, \\ \frac{\nu^2 \gamma (1-\nu)}{[2]_q} - \frac{\nu^3 q}{[3]_q}, & (\gamma + q)\nu \leq \gamma, \end{cases} \\
 B_{1q}(\gamma, \nu) &= \int_0^\nu (1-t) |qt + \gamma\nu - \gamma| d_q t \\
 &= \int_0^\nu |qt + \gamma\nu - \gamma| d_q t - \int_0^\nu t |qt + \gamma\nu - \gamma| d_q t
 \end{aligned}$$

$$\begin{aligned}
 &= \begin{cases} \frac{v^2q+2\gamma^2(1-v)^2+v^2\gamma(1-v)}{[2]_q} - v\gamma(1-v) - \frac{v^3q}{[3]_q} - \frac{2\gamma^3(1-v)^3}{([2]_q)([3]_q)}, & (\gamma+q)v > \gamma, \\ v\gamma(1-v) - \frac{v^2q+v^2\gamma(1-v)}{[2]_q} + \frac{v^3q}{[3]_q}, & (\gamma+q)v \leq \gamma, \end{cases} \\
 A_{2q}(\gamma, v) &= \int_0^1 t|qt + \gamma v - 1| d_q t = \begin{cases} \frac{q}{[3]_q} - \frac{1-\gamma v}{[2]_q} + \frac{2(1-\gamma v)^3}{([2]_q)([3]_q)}, & \gamma v + q > 1, \\ \frac{1-\gamma v}{[2]_q} - \frac{q}{[3]_q}, & \gamma v + q \leq 1, \end{cases} \\
 B_{2q}(\gamma, v) &= \int_0^1 (1-t)|qt + \gamma v - 1| d_q t \\
 &= \int_0^1 |qt + \gamma v - 1| d_q t - \int_0^1 t|qt + \gamma v - 1| d_q t \\
 &= \begin{cases} \frac{q+2(1-\gamma v)^2+1-\gamma v}{[2]_q} - (1-\gamma v) - \frac{q}{[3]_q} - \frac{2(1-\gamma v)^3}{([2]_q)([3]_q)}, & \gamma v + q > 1, \\ 1 - \gamma v - \frac{q+1-\gamma v}{[2]_q} + \frac{q}{[3]_q}, & \gamma v + q \leq 1, \end{cases} \\
 A_{3q}(\gamma, v) &= \int_0^v t|qt + \gamma v - 1| d_q t = \begin{cases} \frac{qv^3}{[3]_q} - \frac{v^2(1-\gamma v)}{[2]_q} + \frac{2(1-\gamma v)^3}{([2]_q)([3]_q)}, & \gamma v + q > 1, \\ \frac{v^2(1-\gamma v)}{[2]_q} - \frac{qv^3}{[3]_q}, & \gamma v + q \leq 1, \end{cases} \\
 B_{3q}(\gamma, v) &= \int_0^v (1-t)|qt + \gamma v - 1| d_q t \\
 &= \int_0^v |qt + \gamma v - 1| d_q t - \int_0^v t|qt + \gamma v - 1| d_q t \\
 &= \begin{cases} \frac{v^2+2(1-\gamma v)^2+v^2(1-\gamma v)}{[2]_q} - v(1-\gamma v) - \frac{qv^3}{[3]_q} - \frac{2(1-\gamma v)^3}{([2]_q)([3]_q)}, & \gamma v + q > 1, \\ v(1-\gamma v) - \frac{qv^2+v^2(1-\gamma v)}{[2]_q} + \frac{qv^3}{[3]_q}, & \gamma v + q \leq 1, \end{cases} \\
 C_{1q}(\gamma, v) &= \int_0^1 |qt + \gamma v - 1| d_q t = \begin{cases} \frac{q+2(1-\gamma v)^2}{[2]_q} - (1-\gamma v), & \gamma v + q > 1, \\ (1-\gamma v) - \frac{q}{[2]_q}, & \gamma v + q \leq 1. \end{cases}
 \end{aligned}$$

**Theorem 4** *If  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  is a  $q^b$ -differentiable function on  $(a, b)$  such that  ${}^bD_q f$  is continuous and integrable on  $[a, b]$ , then we have the following inequality provided that  $|{}^bD_q f|$  is convex on  $[a, b]$ :*

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma [vf(a) + (1-v)f(b)] - (1-\gamma)f(va + (1-v)b) \right| \\
 &\leq (b-a) \left[ (A_{1q}(\gamma, v) + A_{2q}(\gamma, v) - A_{3q}(\gamma, v)) |{}^bD_q f(a)| \right. \\
 &\quad \left. + (B_{1q}(\gamma, v) + B_{2q}(\gamma, v) - B_{3q}(\gamma, v)) |{}^bD_q f(b)| \right],
 \end{aligned}$$

where  $0 < q < 1$ .

*Proof* On taking the modulus in Lemma 1 and applying the convexity of  $|{}^bD_q f|$ , we obtain

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma [vf(a) + (1-v)f(b)] - (1-\gamma)f(va + (1-v)b) \right| \\
 &\leq (b-a) \left[ \int_0^v |qt + \gamma v - \gamma| |{}^bD_q f(ta + (1-t)b)| d_q t \right. \\
 &\quad \left. + \int_v^1 |qt + \gamma v - 1| |{}^bD_q f(ta + (1-t)b)| d_q t \right]
 \end{aligned}$$

$$\begin{aligned}
 &= (b-a) \left[ \int_0^v |qt + \gamma v - \gamma|^b D_q f(ta + (1-t)b) d_q t \right. \\
 &\quad + \int_0^1 |qt + \gamma v - 1|^b D_q f(ta + (1-t)b) d_q t \\
 &\quad \left. - \int_0^v |qt + \gamma v - 1|^b D_q f(ta + (1-t)b) d_q t \right] \\
 &\leq (b-a) \left[ |{}^b D_q f(a)| \int_0^v t|qt + \gamma v - \gamma| d_q t + |{}^b D_q f(b)| \int_0^v (1-t)|qt + \gamma v - \gamma| d_q t \right. \\
 &\quad + |{}^b D_q f(a)| \int_0^1 t|qt + \gamma v - 1| d_q t + |{}^b D_q f(b)| \int_0^1 (1-t)|qt + \gamma v - 1| d_q t \\
 &\quad \left. - |{}^b D_q f(a)| \int_0^v t|qt + \gamma v - 1| d_q t - |{}^b D_q f(b)| \int_0^v (1-t)|qt + \gamma v - 1| d_q t \right] \\
 &= (b-a) \left[ (A_{1q}(\gamma, v) + A_{2q}(\gamma, v) - A_{3q}(\gamma, v)) |{}^b D_q f(a)| \right. \\
 &\quad \left. + (B_{1q}(\gamma, v) + B_{2q}(\gamma, v) - B_{3q}(\gamma, v)) |{}^b D_q f(b)| \right],
 \end{aligned}$$

which completes the proof. □

*Remark 2* Under the assumptions of Theorem 4 with  $q \rightarrow 1^-$ , we have [25, Corollary 2.3].

**Theorem 5** Suppose that  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  is a  $q^b$ -differentiable function on  $(a, b)$  and  ${}^b D_q f$  is continuous and integrable on  $[a, b]$ . If  $|{}^b D_q f|^{p_1}, p_1 > 1$ , is convex on  $[a, b]$ , then we have following inequality:

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma [vf(a) + (1-v)f(b)] - (1-\gamma)f(va + (1-v)b) \right| \\
 &\leq (b-a) (C_{1q}(\gamma, v))^{1-\frac{1}{p_1}} (A_{2q}(\gamma, v) |{}^b D_q f(a)|^{p_1} + B_{2q}(\gamma, v) |{}^b D_q f(b)|^{p_1})^{\frac{1}{p_1}} \\
 &\quad + (b-a) v^{1-\frac{1}{p_1}} \left( \frac{v^2}{[2]_q} |{}^b D_q f(a)|^{p_1} + \frac{v([2]_q - v)}{[2]_q} |{}^b D_q f(b)|^{p_1} \right)^{\frac{1}{p_1}},
 \end{aligned}$$

where  $0 < q < 1$ .

*Proof* Taking the modulus in Lemma 1 and applying the well-known power-mean inequality for quantum integrals, we have

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma [vf(a) + (1-v)f(b)] - (1-\gamma)f(va + (1-v)b) \right| \\
 &= (b-a) \left| \int_0^v (qt + \gamma v - \gamma)^b D_q f(ta + (1-t)b) d_q t \right. \\
 &\quad \left. + \int_0^1 (qt + \gamma v - 1)^b D_q f(ta + (1-t)b) d_q t \right| \\
 &= (b-a) \left| \int_0^1 (qt - \gamma v - 1)^b D_q f(ta + (1-t)b) d_q t \right. \\
 &\quad \left. + \int_0^v (1-\gamma)^b D_q f(ta + (1-t)b) d_q t \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq (b-a) \int_0^1 |qt - \gamma v - 1|^b |{}^bD_q f(ta + (1-t)b)| d_q t \\
 &\quad + (b-a)(1-\gamma) \int_0^v |{}^bD_q f(ta + (1-t)b)| d_q t \\
 &\leq (b-a) \left( \int_0^1 |qt + \gamma v - 1| d_q t \right)^{1-\frac{1}{p_1}} \\
 &\quad \times \left( \int_0^1 |qt + \gamma v - 1|^b |{}^bD_q f(ta + (1-t)b)|^{p_1} d_q t \right)^{\frac{1}{p_1}} \\
 &\quad + (b-a)(1-\gamma) \left( \int_0^v 1 d_q t \right)^{1-\frac{1}{p_1}} \left( \int_0^v |{}^bD_q f(ta + (1-t)b)|^{p_1} d_q t \right)^{\frac{1}{p_1}}.
 \end{aligned}$$

By the convexity of  $|{}^bD_q f|^{p_1}$ , we have

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma [vf(a) + (1-v)f(b)] - (1-\gamma)f(va + (1-v)b) \right| \\
 &\leq (b-a) \left( \int_0^1 |qt + \gamma v - 1| d_q t \right)^{1-\frac{1}{p_1}} \\
 &\quad \times \left( |{}^bD_q f(a)|^{p_1} \int_0^1 t |qt + \gamma v - 1| d_q t \right. \\
 &\quad \left. + |{}^bD_q f(b)|^{p_1} \int_0^1 (1-t) |qt + \gamma v - 1| d_q t \right)^{\frac{1}{p_1}} \\
 &\quad + (b-a)(1-\gamma) \left( \int_0^v 1 d_q t \right)^{1-\frac{1}{p_1}} \\
 &\quad \times \left( \int_0^v |{}^bD_q f(a)|^{p_1} \int_0^v t d_q t + |{}^bD_q f(b)|^{p_1} \int_0^v (1-t) d_q t \right)^{\frac{1}{p_1}} \\
 &= (b-a) (C_{1q}(\gamma, v))^{1-\frac{1}{p_1}} (A_{2q}(\gamma, v) |{}^bD_q f(a)|^{p_1} + B_{2q}(\gamma, v) |{}^bD_q f(b)|^{p_1})^{\frac{1}{p_1}} \\
 &\quad + (b-a)v^{1-\frac{1}{p_1}} \left( \frac{v^2}{[2]_q} |{}^bD_q f(a)|^{p_1} + \frac{v([2]_q - v)}{[2]_q} |{}^bD_q f(b)|^{p_1} \right)^{\frac{1}{p_1}},
 \end{aligned}$$

which completes the proof. □

*Remark 3* If we take  $q \rightarrow 1^-$  in Theorem 5, then we have [25, Theorem 2.2].

**Theorem 6** Let  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a  $q^b$ -differentiable function on  $(a, b)$  and  ${}^bD_q f$  a continuous and integrable function on  $[a, b]$ . If  $|{}^bD_q f|^{p_1}$  is convex on  $[a, b]$  for some  $p_1 > 1$  and  $\frac{1}{r_1} + \frac{1}{p_1} = 1$ , then we have

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma [vf(a) + (1-v)f(b)] - (1-\gamma)f(va + (1-v)b) \right| \tag{3.6} \\
 &\leq (b-a) (u_{1q}(\gamma, v))^{\frac{1}{r_1}} \left( \frac{|{}^bD_q f(a)|^{p_1} + q |{}^bD_q f(b)|^{p_1}}{[2]_q} \right)^{\frac{1}{p_1}} \\
 &\quad + (b-a)v^{\frac{1}{r_1}} \left( \frac{v^2}{[2]_q} |{}^bD_q f(a)|^{p_1} + \frac{v([2]_q - v)}{[2]_q} |{}^bD_q f(b)|^{p_1} \right)^{\frac{1}{p_1}},
 \end{aligned}$$



where  $0 < q < 1$  and

$$u_{1q}(\gamma, \nu) = \int_0^1 |qt + \gamma\nu - 1|^{r_1} d_q t.$$

*Proof* Taking the modulus in the Lemma 1 and applying well-known Hölder’s inequality for quantum integrals, we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma [ \nu f(a) + (1-\nu)f(b) ] - (1-\gamma)f(\nu a + (1-\nu)b) \right| \\ &= (b-a) \left| \int_0^\nu (qt + \gamma\nu - \gamma)^b D_q f(ta + (1-t)b) d_q t \right. \\ & \quad \left. + \int_\nu^1 (qt + \gamma\nu - 1)^b D_q f(ta + (1-t)b) d_q t \right| \\ &= (b-a) \left| \int_0^1 (qt - \gamma\nu - 1)^b D_q f(ta + (1-t)b) d_q t \right. \\ & \quad \left. + \int_0^\nu (1-\gamma)^b D_q f(ta + (1-t)b) d_q t \right| \\ &\leq (b-a) \int_0^1 |qt - \gamma\nu - 1|^b |D_q f(ta + (1-t)b)| d_q t \\ & \quad + (b-a)(1-\gamma) \int_0^\nu |D_q f(ta + (1-t)b)| d_q t \\ &\leq (b-a) \left( \int_0^1 |qt + \gamma\nu - 1|^{r_1} d_q t \right)^{\frac{1}{r_1}} \left( \int_0^1 |D_q f(ta + (1-t)b)|^{p_1} d_q t \right)^{\frac{1}{p_1}} \\ & \quad + (b-a)(1-\gamma) \left( \int_0^\nu 1^{r_1} d_q t \right)^{\frac{1}{r_1}} \left( \int_0^\nu |D_q f(ta + (1-t)b)|^{p_1} d_q t \right)^{\frac{1}{p_1}}. \end{aligned}$$

Using the fact that  $|^b D_q f|^{p_1}$  is convex, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma [ \nu f(a) + (1-\nu)f(b) ] - (1-\gamma)f(\nu a + (1-\nu)b) \right| \\ &\leq (b-a) \left( \int_0^1 |qt + \gamma\nu - 1|^{r_1} d_q t \right)^{\frac{1}{r_1}} \\ & \quad \times \left( |^b D_q f(a)|^{p_1} \int_0^1 t d_q t + |^b D_q f(b)|^{p_1} \int_0^1 (1-t) d_q t \right)^{\frac{1}{p_1}} \\ & \quad + (b-a)(1-\gamma) \left( \int_0^\nu 1^{r_1} d_q t \right)^{\frac{1}{r_1}} \\ & \quad \times \left( |^b D_q f(a)|^{p_1} \int_0^\nu t d_q t + |^b D_q f(b)|^{p_1} \int_0^\nu (1-t) d_q t \right)^{\frac{1}{p_1}} \\ &= (b-a) (u_{1q}(\gamma, \nu))^{\frac{1}{r_1}} \left( \frac{|^b D_q f(a)|^{p_1} + q|^b D_q f(b)|^{p_1}}{[2]_q} \right)^{\frac{1}{p_1}} \\ & \quad + (b-a) \nu^{\frac{1}{r_1}} \left( \frac{\nu^2}{[2]_q} |^b D_q f(a)|^{p_1} + \frac{\nu([2]_q - \nu)}{[2]_q} |^b D_q f(b)|^{p_1} \right)^{\frac{1}{p_1}}, \end{aligned}$$

which completes the proof. □

*Remark 4* On taking limit as  $q \rightarrow 1^-$  in Theorem 6, we have [25, Theorem 2.8].

### 4 Special cases

In this section, we discuss special cases of our main results and the relationship between the results presented herein and comparable results in the literature.

*Remark 5* From Lemma 1, we have following observations:

(i) If we take  $\nu = 0$ , then we obtain following new identity:

$$\begin{aligned} & (b - a) \int_0^1 (qt - 1)^b D_q f (ta + (1 - t)b) d_q t \\ &= \frac{1}{b - a} \int_a^b f(x)^b d_q x - f(b). \end{aligned} \tag{4.1}$$

(ii) By setting  $\nu = 1$  in Lemma 1, we obtain following new identity:

$$\begin{aligned} & (b - a) \int_0^1 qt^b D_q f (ta + (1 - t)b) d_q t \\ &= \frac{1}{b - a} \int_a^b f(x)^b d_q x - f(a). \end{aligned} \tag{4.2}$$

(iii) Set  $\nu = \frac{1}{[2]_q}$  to obtain the following new identity:

$$\begin{aligned} & (b - a) \left[ \int_0^{\frac{1}{[2]_q}} \left( qt - \frac{\gamma q}{[2]_q} \right)^b D_q f (ta + (1 - t)b) d_q t \right. \\ & \quad \left. + \int_{\frac{1}{[2]_q}}^1 \left( qt + \frac{\gamma}{[2]_q} - 1 \right)^b D_q f (ta + (1 - t)b) d_q t \right] \\ &= \frac{1}{b - a} \int_a^b f(x)^b d_q x - \gamma \frac{f(a) + qf(b)}{[2]_q} - (1 - \gamma) f \left( \frac{a + qb}{[2]_q} \right). \end{aligned} \tag{4.3}$$

*Remark 6* In the following, we give the different variants of Lemma 1.

(i) If we set  $\gamma = 0$ , in Lemma 1, we have following new identity:

$$\begin{aligned} & (b - a) \left[ \int_0^\nu qt^b D_q f (ta + (1 - t)b) d_q t + \int_\nu^1 (qt - 1)^b D_q f (ta + (1 - t)b) d_q t \right] \\ &= \frac{1}{b - a} \int_a^b f(x)^b d_q x - f(\nu a + (1 - \nu)b). \end{aligned} \tag{4.4}$$

Specifically, for  $\nu = \frac{1}{[2]_q}$ , we have following midpoint-like identity given in [17, Lemma 2]:

$$\begin{aligned} & (b - a) \left[ \int_0^{\frac{1}{[2]_q}} qt^b D_q f (ta + (1 - t)b) d_q t + \int_{\frac{1}{[2]_q}}^1 (qt - 1)^b D_q f (ta + (1 - t)b) d_q t \right] \\ &= \frac{1}{b - a} \int_a^b f(x)^b d_q x - f \left( \frac{a + qb}{[2]_q} \right). \end{aligned}$$

(ii) Taking  $\gamma = \frac{1}{3}$  in Lemma 1, we have following new identity:

$$\begin{aligned}
 (b-a) & \left[ \int_0^v \left( qt + \frac{1}{3}v - \frac{1}{3} \right)^b D_q f (ta + (1-t)b) d_q t \right. \\
 & \left. + \int_v^1 \left( qt + \frac{1}{3}v - 1 \right)^b D_q f (ta + (1-t)b) d_q t \right] \\
 & = \frac{1}{b-a} \int_a^b f(x)^b d_q x - \frac{1}{3} [vf(a) + (1-v)f(b) + 2f(va + (1-v)b)].
 \end{aligned} \tag{4.5}$$

Specifically, for  $v = \frac{1}{[2]_q}$ , we have Simpson-like integral identity given as follows:

$$\begin{aligned}
 (b-a) & \left[ \int_0^{\frac{1}{[2]_q}} \left( qt - \frac{q}{3([2]_q)} \right)^b D_q f (ta + (1-t)b) d_q t \right. \\
 & \left. + \int_{\frac{1}{[2]_q}}^1 \left( qt + \frac{1}{3([2]_q)} - 1 \right)^b D_q f (ta + (1-t)b) d_q t \right] \\
 & = \frac{1}{b-a} \int_a^b f(x)^b d_q x - \frac{1}{3} \left[ \frac{f(a) + qf(b)}{[2]_q} + 2f\left(\frac{a+qb}{[2]_q}\right) \right].
 \end{aligned} \tag{4.6}$$

(iii) If we set  $\gamma = \frac{1}{2}$  in Lemma 1, then we have following new identity:

$$\begin{aligned}
 (b-a) & \left[ \int_0^v \left( qt + \frac{1}{2}v - \frac{1}{2} \right)^b D_q f (ta + (1-t)b) d_q t \right. \\
 & \left. + \int_v^1 \left( qt + \frac{1}{2}v - 1 \right)^b D_q f (ta + (1-t)b) d_q t \right] \\
 & = \frac{1}{b-a} \int_a^b f(x)^b d_q x - \frac{1}{2} [vf(a) + (1-v)f(b) + f(va + (1-v)b)].
 \end{aligned} \tag{4.7}$$

Specifically, for  $v = \frac{1}{[2]_q}$ , we get following averaged midpoint–trapezoid-type integral identity:

$$\begin{aligned}
 (b-a) & \left[ \int_0^{\frac{1}{[2]_q}} \left( qt - \frac{q}{2([2]_q)} \right)^b D_q f (ta + (1-t)b) d_q t \right. \\
 & \left. + \int_{\frac{1}{[2]_q}}^1 \left( qt + \frac{1}{2([2]_q)} - 1 \right)^b D_q f (ta + (1-t)b) d_q t \right] \\
 & = \frac{1}{b-a} \int_a^b f(x)^b d_q x - \frac{1}{2} \left[ \frac{f(a) + qf(b)}{[2]_q} + f\left(\frac{a+qb}{[2]_q}\right) \right].
 \end{aligned} \tag{4.8}$$

(iv) By setting  $\gamma = 1$  in Lemma 1, we have following new identity

$$\begin{aligned}
 (b-a) & \left[ \int_0^v (qt + v - 1)^b D_q f (ta + (1-t)b) d_q t \right. \\
 & \left. + \int_v^1 (qt + v - 1)^b D_q f (ta + (1-t)b) d_q t \right] \\
 & = \frac{1}{b-a} \int_a^b f(x)^b d_q x - [vf(a) + (1-v)f(b)].
 \end{aligned} \tag{4.9}$$

Specifically, for  $\nu = \frac{1}{[2]_q}$ , we have following trapezoid-like integral identity given in [17, Lemma 1]:

$$\begin{aligned} & (b-a) \left[ \int_0^{\frac{1}{[2]_q}} \left( qt + \frac{1}{[2]_q} - 1 \right)^b D_q f(ta + (1-t)b) d_q t \right. \\ & \quad \left. + \int_{\frac{1}{[2]_q}}^1 \left( qt + \frac{1}{[2]_q} - 1 \right)^b D_q f(ta + (1-t)b) d_q t \right] \\ & = \frac{1}{b-a} \int_a^b f(x)^b d_q x - \frac{f(a) + qf(b)}{[2]_q}. \end{aligned}$$

**Corollary 2** Under the assumptions of Theorem 4 with  $\nu = \frac{1}{[2]_q}$ , we have following new inequality:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \gamma \left[ \frac{f(a) + qf(b)}{[2]_q} \right] - (1-\gamma) f\left(\frac{a+qb}{[2]_q}\right) \right| \tag{4.10} \\ & \leq (b-a) \left[ \left( A_{1q}\left(\gamma, \frac{1}{[2]_q}\right) + A_{2q}\left(\gamma, \frac{1}{[2]_q}\right) - A_{3q}\left(\gamma, \frac{1}{[2]_q}\right) \right) |{}^b D_q f(a)| \right. \\ & \quad \left. + \left( B_{1q}\left(\gamma, \frac{1}{[2]_q}\right) + B_{2q}\left(\gamma, \frac{1}{[2]_q}\right) - B_{3q}\left(\gamma, \frac{1}{[2]_q}\right) \right) |{}^b D_q f(b)| \right]. \end{aligned}$$

*Remark 7* We now deduce some more inequalities from Corollary 2.

- (i) By putting  $\gamma = 0$  in Corollary 2, we obtain following midpoint-type inequality [17, Theorem 5]:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - f\left(\frac{a+qb}{[2]_q}\right) \right| \\ & \leq (b-a) \left[ |{}^b D_q f(a)| \frac{3q}{([2]_q)^3 ([3]_q)} + |{}^b D_q f(b)| \frac{-q + 2q^2 + q^3}{([2]_q)^3 ([3]_q)} \right]. \end{aligned}$$

- (ii) If we take  $\gamma = \frac{1}{3}$  in Corollary 2, then we get following Simpson’s-type inequality:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \frac{1}{3} \left[ \frac{f(a) + qf(b)}{[2]_q} + 2f\left(\frac{a+qb}{[2]_q}\right) \right] \right| \tag{4.11} \\ & \leq (b-a) \left[ \left( A_{1q}\left(\frac{1}{3}, \frac{1}{[2]_q}\right) + A_{2q}\left(\frac{1}{3}, \frac{1}{[2]_q}\right) - A_{3q}\left(\frac{1}{3}, \frac{1}{[2]_q}\right) \right) |{}^b D_q f(a)| \right. \\ & \quad \left. + \left( B_{1q}\left(\frac{1}{3}, \frac{1}{[2]_q}\right) + B_{2q}\left(\frac{1}{3}, \frac{1}{[2]_q}\right) - B_{3q}\left(\frac{1}{3}, \frac{1}{[2]_q}\right) \right) |{}^b D_q f(b)| \right]. \end{aligned}$$

Specifically for  $q \rightarrow 1^-$ , we have following inequality given by Alomori et al. [9, Corollary 1]:

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{5(b-a)}{72} [|f'(b)| + |f'(a)|].$$

(iii) Take  $\gamma = \frac{1}{2}$  in Corollary 2 to obtain the following averaged midpoint–trapezoid-type inequality:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \frac{1}{2} \left[ \frac{f(a) + qf(b)}{[2]_q} + f\left(\frac{a+qb}{[2]_q}\right) \right] \right| \\ & \leq (b-a) \left[ \left( A_{1q}\left(\frac{1}{2}, \frac{1}{[2]_q}\right) + A_{2q}\left(\frac{1}{2}, \frac{1}{[2]_q}\right) - A_{3q}\left(\frac{1}{2}, \frac{1}{[2]_q}\right) \right) |{}^b D_q f(a)| \right. \\ & \quad \left. + \left( B_{1q}\left(\frac{1}{2}, \frac{1}{[2]_q}\right) + B_{2q}\left(\frac{1}{2}, \frac{1}{[2]_q}\right) - B_{3q}\left(\frac{1}{2}, \frac{1}{[2]_q}\right) \right) |{}^b D_q f(b)| \right]. \end{aligned} \tag{4.12}$$

Specifically, for  $q \rightarrow 1^-$ , we obtain the following inequality in [44, Corollary 3.4]:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{b-a}{16} [|f'(b)| + |f'(a)|]. \end{aligned}$$

(iv) If we take  $\gamma = 1$  in Corollary 2, then we obtain following trapezoid-type integral inequality:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \frac{f(a) + qf(b)}{[2]_q} \right| \\ & \leq (b-a) \left[ |{}^b D_q f(a)| \frac{q^2(1+4q+q^2)}{([2]_q)^4([3]_q)} + |{}^b D_q f(b)| \frac{q^2(1+3q^2+2q^3)}{([2]_q)^4([3]_q)} \right], \end{aligned}$$

which is given by Budak in [17, Theorem 3].

*Remark 8* Under the assumptions of Theorems 5 and 6 with  $\nu = \frac{1}{[2]_q}$ , we may choose  $\gamma = 0$ ,  $\gamma = \frac{1}{3}$ ,  $\gamma = \frac{1}{2}$ , and  $\gamma = 1$ , to obtain the midpoint-like integral inequality, the Simpson-like integral inequality, the averaged midpoint–trapezoid-like integral inequality and the trapezoid-like integral inequality, respectively.

### 5 Conclusions

In this paper, some new bounds of Hermite–Hadamard inequalities for convex functions by applying the notions of the newly defined  $q^b$ -integral and  $q^b$ -derivative are obtained. It is shown that the results proved herein are generalizations of the existing comparable results in the literature, like the results proved in [9, 17, 25, 44]. It is an interesting and new problem that the upcoming researchers can offer similar inequalities for different kinds of convexity in their future work.

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#### Availability of data and materials

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

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