Aghazadeh et al. *Advances in Difference Equations* (2021) 2021:231
https://doi.org/10.1186/s13662-021-03388-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH | Open Access

Solving partial fractional differential equations by using the Laguerre wavelet-Adomian method

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Abstract

By using a nonlinear method, we try to solve partial fractional differential equations. In this way, we construct the Laguerre wavelets operational matrix of fractional integration. The method is proposed by utilizing Laguerre wavelets in conjunction with the Adomian decomposition method. We present the procedure of implementation and convergence analysis for the method. This method is tested on fractional Fisher's equation and the singular fractional Emden–Fowler equation. We compare the results produced by the present method with some well-known results.

MSC: Primary 35R11; 34G20; secondary 65T60; 65N22

Keywords: Adomian polynomials; Fractional Fisher equation; Laguerre wavelets; Operational matrix; Singular Emden–Fowler equation

1 Introduction

The fractional calculus has been extended extremely and investigated in distinct areas and applications by many research works (see, for example, [1–20]). In 1937, Fisher, Kolmogorov, Petrovsky, and Piscounov investigated independently the Fisher-KPP equation (or Fisher's equation; see [21, 22]). As you know, this equation is about population dynamics to describe the spatial spread of an advantageous allele and explores its traveling wave solutions. It has been used distinctly for obtaining approximate solutions of this equation (see, for example, [23–33]). Also, there are some chemical and biological applications for this famous equation and its fractional version (see, for example, [34–36]).

Many problems on the diffusion of heat and its equations in the mathematical physics and fluid dynamic are modeled by a form of the equations called Emden–Fowler equations:

$$u_{xx} + \frac{s}{x} u_x + a\phi(x,t)\psi(u) + \xi(x,t) = u_t, \quad \big(x,t \in [0,1], s > 0\big), \tag{1}$$

where $\phi(x,t)\psi(u)+\xi(x,t)$ denotes the heat source, u is the temperature, and time variable is t. Put $s=2$ and $\xi(x,t)=0$. Then relation (1) in one variable version reduces to

$$u_{xx}+\frac{2}{x}u_x+a\phi(x)\psi(u)=0 \quad \bigl(u(0)=u_0, u'(0)=0\bigr), \tag{2}$$

and for $\phi(x)=1$ and $\psi(u)=u^n$, we obtain the standard Lane–Emden equation [37, 38]. Based on the singularity point at $x=0$, many researchers have tried to solve these equations by using different numerical methods such as wavelets, Galerkin, or collocation [38–47].

By developing the Laguerre wavelets collocation method and using the Adomian decomposition technique, our aim is the investigation of the partial fractional differential equation

$${}^{C}D_t^{\alpha}u(x,t)+\frac{\partial^2 u(x,t)}{\partial x^2}+a(x)\frac{\partial u(x,t)}{\partial x}+F\bigl(u(x,t)\bigr)=0, \tag{3}$$

with boundary conditions $u(x,0)=g(x)$, $u(0,t)=y_1(t)$, $u(1,t)=y_2(t)$, where $0\le\alpha<1$, ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative, $g(x)$, $y_1(t)$, $y_2(t)$ are some functions, $F(u(x,t))$ is the nonlinear term, and $a(x)$ has singularity at the point $x=0$. One can find notions of fractional calculus such as the Riemann–Liouville integral and Caputo derivative in [48].

2 Laguerre wavelets

On the other hand, by using dilation and translation of a map (as the mother wavelet), we can construct wavelets. For example, we can consider the family of continuous wavelets

$$\psi_{a,b}(t)=|a|^{-1/2}\psi\left(\frac{t-b}{a}\right) \quad (a,b\in\mathbb{R}, a\neq 0),$$

where a and b are the dilation and translation parameters. If $a_0>1$, $b_0>0$, $a=a_0^{-k}$, $b=mb_0a_0^{-k}$ and k and m are positive integers, then it reduces to the discrete wavelets $\psi_{k,m}(t)=|a_0|^{k/2}\psi(a_0^k t-mb_0)$ which is a wavelet basis for $L^2(\mathbb{R})$ [15]. If $a_0=2$ and $b_0=1$, then $\{\psi_{k,m}(t)\}_{k,m\ge 0}$ is an orthonormal basis [15]. It is known that the Laguerre wavelets are defined on the interval $[0,1)$ as (see [15])

$$\psi_{n,m}(t)=\begin{cases}\frac{1}{m!}2^{\frac{k}{2}}L_m(2^k t-2n+1) & \frac{n-1}{2^{k-1}}\le t<\frac{n}{2^{k-1}},\\ 0 & \text{otherwise},\end{cases}$$

where $k\ge 1$, $n=1,2,3,\ldots,2^{k-1}$, t is the normalized time, $m=0,1,2,\ldots,M-1$, M is a fixed positive integer, $L_m(t)$ are the Laguerre polynomials of degree m which are orthogonal with respect to the weight function $\omega(t)=1$ on the interval $[0,\infty)$ and satisfy the recursive relation

$$L_0(t)=1, \qquad L_1(t)=1-t,$$
$$L_{m+1}(t)=\frac{(2m+1-t)L_m(t)-mL_{m-1}(t)}{m+1} \quad (m\ge 1).$$

Let $u(x) \in L_2(\mathbb{R})$ be a function defined over $[0,1)$. We say that u is expanded by Laguerre wavelets whenever

$$u(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x). \tag{4}$$

If the series in (4) is truncated, then it can be written by

$$u(x) \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = \mathbf{C}^T\Psi(x), \tag{5}$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C &= [c_{1,0},\dots,c_{2,0},\dots,c_{2^{k-1},M-1}]^T, \\ \Psi(t) &= [\psi_{1,0},\dots,\psi_{2,0},\dots,\psi_{2^{k-1},M-1}]^T. \end{aligned}$$

For simplicity, we rewrite (5) as

$$u(x) \cong \sum_{i=1}^{m'} c_i\psi_i = C^T\Psi(x), \tag{6}$$

where $c_i = c_{n,m}$, $\psi_i(t) = \psi_{n,m}(t)$ and $i = M(n-1)+m+1$. Hence, $C = [c_1, c_2, c_3, \dots, c_{m'}]^T$ and $\Psi(t) = [\psi_1, \psi_2, \psi_3, \dots, \psi_{m'}]^T$. Consider the collocation points $t_i = \frac{2i-1}{2^k M}$ for $i = 1, 2, \dots, 2^{k-1}M$. The Laguerre wavelet matrix $\Phi(x)_{m'\times m'}$ is defined by

$$\Phi_{m'\times m'} = \left[\Psi\left(\frac{1}{2m'}\right), \Psi\left(\frac{3}{2m'}\right), \dots, \Psi\left(\frac{2m'-1}{2m'}\right)\right],$$

where $m' = 2^{k-1}M$. If $M = 4$ and $k = 2$, then the Laguerre matrix is given by

$$\Phi_{8\times 8} = \begin{pmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{89}{32} & \frac{49}{32} & \frac{17}{32} & -\frac{7}{32} & 0 & 0 & 0 & 0 \\ \frac{533}{384} & \frac{709}{1152} & \frac{131}{1152} & -\frac{61}{384} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{89}{32} & \frac{49}{32} & \frac{17}{32} & -\frac{7}{32} \\ 0 & 0 & 0 & 0 & \frac{533}{384} & \frac{709}{1152} & \frac{131}{1152} & -\frac{61}{384} \end{pmatrix}.$$

Similarly, the function $u(x,t) \in L_2([0,1] \times [0,1])$ can be also approximated as

$$u(x,t) = \Psi^T(x)U\Psi(t), \tag{7}$$

in which U is an $m' \times m'$ matrix with $u_{ij} = \langle\psi_i(x), \langle u(x,t), \psi_j(t)\rangle\rangle$. We use the wavelet collocation method to determine the coefficients $u_{i,j}$.

3 Fractional integral of the Laguerre wavelets

Here, we review the Riemann–Liouville integral of the Laguerre wavelets.

Theorem 1 *The fractional integral of the Laguerre wavelets on* $[0,1]$ *is given by*

$$I^{\alpha}\psi_{n,m}(x)=\begin{cases}0, & x<\frac{n-1}{2^{k-1}},\\ \frac{2^{\frac{k}{2}}}{\Gamma(\alpha)}\sum_{r=0}^{m}\sum_{i=r}^{m}\sum_{j=0}^{r}T_{m,n,k}^{i-r,r}\frac{(-1)^j}{\alpha+j}C_r^j x^{r-j} & \\ \quad\times(x-\frac{n-1}{2^{k-1}})^{\alpha+j}, & \frac{n-1}{2^{k-1}}\le x\le\frac{n}{2^{k-1}},\\ \frac{2^{\frac{k}{2}}}{\Gamma(\alpha)}\sum_{r=0}^{m}\sum_{i=r}^{m}\sum_{j=0}^{r}T_{m,n,k}^{i-r,r}\frac{(-1)^j}{\alpha+j}C_r^j x^{r-j} & \\ \quad\times((x-\frac{n-1}{2^{k-1}})^{\alpha+j}-(x-\frac{n}{2^{k-1}})^{\alpha+j}), & x>\frac{n}{2^{k-1}},\end{cases}\tag{8}$$

where $T_{m,n,k}^{i-r,r}=(-1)^{2i-r}\frac{2^{rk}(2n-1)^{i-r}}{(m-i)!(i-r)!i!r!}$ *and* $C_r^j=\frac{r!}{j!(j-r)!}$.

Proof It is known that the Laguerre polynomials are given by

$$L_n(x)=\sum_{k=0}^{n}C_n^k\frac{(-1)^k}{k!}x^k,$$

where $C_n^k=\frac{n!}{k!(n-k)!}$. Hence, for Laguerre wavelets, we have

$$\begin{aligned}L_m\bigl(2^kx-2n+1\bigr)&=\sum_{i=0}^{m}C_m^i\frac{(-1)^i}{i!}\bigl(2^kx-2n+1\bigr)^i\\&=\sum_{i=0}^{m}C_m^i\frac{(-1)^i}{i!}2^{ki}\left(x-\frac{2n-1}{2^k}\right)^i\\&=\sum_{i=0}^{m}C_m^i\frac{(-1)^i}{i!}2^{ki}\sum_{r=0}^{i}\frac{i!}{r!(i-r)!}x^{i-r}\left(-\frac{2n-1}{2^k}\right)^r\\&=\sum_{i=0}^{m}\sum_{r=0}^{i}(-1)^{i+r}\frac{m!2^{k(i-r)}}{i!r!(m-i)!(i-r)!}(2n-1)^rx^{i-r},\end{aligned}\tag{9}$$

and so

$$L_m\bigl(2^kx-2n+1\bigr)=\sum_{r=0}^{m}\sum_{i=r}^{m}\frac{(-1)^{2i-r}2^{rk}(2n-1)^{i-r}m!}{i!r!(m-i)!(i-r)!}x^r.\tag{10}$$

If $(T')_{m,n,k}^{i-r,r}=m!(-1)^{2i-r}\frac{2^{rk}(2n-1)^{i-r}}{(m-i)!(i-r)!i!r!}$, then

$$L_m\bigl(2^kx-2n+1\bigr)=\sum_{r=0}^{m}\sum_{i=r}^{m}(T')_{m,n,k}^{i-r,r}x^r,\tag{11}$$

and so

$$\psi_{n,m}(x)=\begin{cases}\frac{1}{m!}2^{\frac{k}{2}}\sum_{r=0}^{m}\sum_{i=r}^{m}(T')_{m,n,k}^{i-r,r}x^r, & \frac{n-1}{2^{k-1}}\le x<\frac{n}{2^{k-1}},\\ 0, & \text{otherwise}.\end{cases}\tag{12}$$

On the other hand, by calculating the integrals, we get

$$I_1 = \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} t^r \, dt$$

and

$$I_2 = \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} t^r \, dt.$$

If $v = x - t$, then

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} t^r \, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_{0}^{x-(\frac{n-1}{2^{k-1}})} v^{\alpha-1} (x-v)^r \, dv \\ &= \frac{1}{\Gamma(\alpha)} \int_{0}^{x-(\frac{n-1}{2^{k-1}})} v^{\alpha-1} \sum_{j=0}^{r} C_r^j x^{r-j} (-v)^j \, dv \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{r} (-1)^j C_r^j x^{r-j} \int_{0}^{x-(\frac{n-1}{2^{k-1}})} v^{j+\alpha-1} \, dv \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} \left(x - \frac{n-1}{2^{k-1}} \right)^{j+\alpha}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} I_2 &= \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} t^r \, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} v^{\alpha-1} (x-v)^r \, dv \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j x^{r-j} \left(\left(x - \frac{n-1}{2^{k-1}} \right)^{j+\alpha} - \left(x - \frac{n}{2^{k-1}} \right)^{j+\alpha} \right). \end{aligned}$$

Now, we apply Riemann–Liouville fractional integration of order α with respect to x for the Laguerre wavelets. Thus, we obtain

$$I^\alpha \psi_{n,m}(x) = \begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} \psi_{n,m}(t) \, dt, & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} \psi_{n,m}(t) \, dt, & x > \frac{n}{2^{k-1}} \end{cases} \tag{13}$$

$$= \begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{2^{\frac{k}{2}}}{m!\Gamma(\alpha)} \sum_{r=0}^{m} \sum_{i=r}^{m} (T')_{m,n,k}^{i-r,r} \int_{\frac{n-1}{2^{k-1}}}^{x} (x-t)^{\alpha-1} t^r \, dt, & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{2^{\frac{k}{2}}}{m!\Gamma(\alpha)} \sum_{r=0}^{m} \sum_{i=r}^{m} (T')_{m,n,k}^{i-r,r} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (x-t)^{\alpha-1} t^r \, dt, & x > \frac{n}{2^{k-1}} \end{cases} \tag{14}$$

$$= \begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{2^{\frac{k}{2}}}{\Gamma(\alpha)} \sum_{r=0}^{m} \sum_{i=r}^{m} T_{m,n,k}^{i-r,r} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j & \\ \quad \times x^{r-j}(x - \frac{n-1}{2^{k-1}})^{j+\alpha}, & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ \frac{2^{\frac{k}{2}}}{\Gamma(\alpha)} \sum_{r=0}^{m} \sum_{i=r}^{m} T_{m,n,k}^{i-r,r} \sum_{j=0}^{r} \frac{(-1)^j}{j+\alpha} C_r^j & \\ \quad \times x^{r-j}((x - \frac{n-1}{2^{k-1}})^{j+\alpha} - (x - \frac{n}{2^{k-1}})^{j+\alpha}), & x > \frac{n}{2^{k-1}}. \end{cases} \tag{15}$$

This completes the proof. □

For instance, for $k = 2$, $M = 4$, $x = 0.6$, $\alpha = 0.9$, we obtain

$$I^{0.9}\Psi_{8\times1}(0.6) = \begin{pmatrix} 1.0513 \\ 1.02266 \\ 0.585489 \\ 0.248884 \\ 0.261795 \\ 0.468475 \\ 0.37927 \\ 0.192481 \end{pmatrix},$$

where $\Psi_{8\times1} = (\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{1,3}(x), \psi_{2,0}(x), \psi_{2,1}(x), \psi_{12,2}(x), \psi_{2,3}(x))^T$. Now, by using the collocation points in (8), we can calculate the integration matrix $P^{\alpha}_{m'\times m'} = I^{\alpha}\psi_{n,m}(x)$ as

$$P^{\alpha}_{2^{k-1}M\times2^{k-1}M} = \begin{pmatrix} I^{\alpha}\psi_{1,0}(x(1)) & I^{\alpha}\psi_{1,0}(x(2)) & \dots & I^{\alpha}\psi_{1,0}(x(2^{k-1}M)) \\ I^{\alpha}\psi_{1,1}(x(1)) & I^{\alpha}\psi_{1,1}(x(2)) & \dots & I^{\alpha}\psi_{1,1}(x(2^{k-1}M)) \\ \vdots & \vdots & \ddots & \vdots \\ I^{\alpha}\psi_{2^{k-1},M}(x(1)) & I^{\alpha}\psi_{2^{k-1},M}(x(2)) & \dots & I^{\alpha}\psi_{2^{k-1},M}(x(2^{k-1}M)) \end{pmatrix}.$$

For $k = 2$, $M = 4$, and $\alpha = 0.9$, we get

$$P^{0.9}_{8\times8} = \begin{pmatrix} 0.17149 & 0.46095 & 0.73000 & 0.98819 & 1.0675 & 1.02329 & 0.99504 & 0.97397 \\ 0.32042 & 0.73996 & 0.97974 & 1.06621 & 1.03333 & 1.00222 & 0.97962 & 0.96181 \\ 0.26724 & 0.55727 & 0.66061 & 0.63869 & 0.58976 & 0.57631 & 0.56528 & 0.55615 \\ 0.13879 & 0.26809 & 0.29611 & 0.27235 & 0.25042 & 0.24540 & 0.24105 & 0.23737 \\ 0 & 0 & 0 & 0 & 0.17149 & 0.46095 & 0.73000 & 0.98819 \\ 0 & 0 & 0 & 0 & 0.32042 & 0.73996 & 0.97974 & 1.06621 \\ 0 & 0 & 0 & 0 & 0.26724 & 0.55727 & 0.66061 & 0.63869 \\ 0 & 0 & 0 & 0 & 0.13879 & 0.26809 & 0.29611 & 0.27235 \end{pmatrix}.$$

Suppose that $\eta > 0$ and $g : [0, \eta] \to R$ is a continuous function. Put

$$g(x)I^{\alpha}\psi_{n,m}(\eta) = V^{\alpha,\eta,g(x)}. \tag{16}$$

By using the collocation points $x_i = \frac{2i-1}{2^k M}$ for $i = 1, 2, \ldots, 2^{k-1}M$ in (8), we get

$$V^{\alpha,\eta,g(x)}_{2^{k-1}M\times 2^{k-1}M} = \begin{pmatrix} g(x_1)I^\alpha\psi_{1,0}(\eta) & g(x_2)I^\alpha\psi_{1,0}(\eta) & \ldots & g(x_{2^{k-1}M})I^\alpha\psi_{1,0}(\eta) \\ g(x_1)I^\alpha\psi_{1,1}(\eta) & g(x_2)I^\alpha\psi_{1,1}(\eta) & \ldots & g(x_{2^{k-1}M})I^\alpha\psi_{1,1}(\eta) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_1)I^\alpha\psi_{2^{k-1},M-1}(\eta) & g(x_2)I^\alpha\psi_{2^{k-1},M-1}(\eta) & \ldots & g(x_{2^{k-1}M})I^\alpha\psi_{2^{k-1},M-1}(\eta) \end{pmatrix}.$$

For $\eta = 1$, $g(x) = x$, $\alpha = 0.9$, $k = 2$, and $M = 4$, we obtain

$$V^{0.9,1,x}_{8\times 8} = \begin{pmatrix} 0.0603 & 0.1810 & 0.3016 & 0.4222 & 0.5429 & 0.6635 & 0.7842 & 0.9048 \\ 0.0596 & 0.1789 & 0.2982 & 0.4174 & 0.5367 & 0.6560 & 0.7752 & 0.8945 \\ 0.0345 & 0.1035 & 0.1725 & 0.2416 & 0.3106 & 0.3796 & 0.4486 & 0.5176 \\ 0.0147 & 0.0442 & 0.0737 & 0.1031 & 0.1326 & 0.1621 & 0.1915 & 0.2210 \\ 0.0696 & 0.2089 & 0.3482 & 0.4875 & 0.6268 & 0.7661 & 0.9054 & 1.0447 \\ 0.0660 & 0.1979 & 0.3299 & 0.4619 & 0.5938 & 0.7258 & 0.8578 & 0.9897 \\ 0.0372 & 0.1116 & 0.1860 & 0.2604 & 0.3348 & 0.4091 & 0.4835 & 0.5579 \\ 0.0157 & 0.0472 & 0.0787 & 0.1102 & 0.1417 & 0.1732 & 0.2046 & 0.2361 \end{pmatrix}.$$

4 Method of solution

Now, we review the method for the partial fractional differential equation. The Adomian polynomials are used to convert the nonlinear terms of the partial differential equation into a set of polynomials. No linearization process is required for the suggested method. We describe the procedure of implementation in more detail, which enables the readers to understand the method more efficiently. Consider the partial fractional differential equation

$${}^C D_t^\alpha u(x,t) + \frac{\partial^2 u(x,t)}{\partial x^2} + a(x)\frac{\partial u(x,t)}{\partial x} + F\big(u(x,t)\big) = 0, \quad 0 < \alpha \le 1, \tag{17}$$

with the boundary conditions

$$u(x,0) = g(x), \qquad u(0,t) = y_1(t), \qquad u(1,t) = y_2(t),$$

where $a(x)$ has singularity at the point $x = 0$ and $F(u(x,t))$ is the nonlinear term of the problem. By applying the Adomian decomposition method, we can express the solution of (17) as

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{18}$$

We approximate the solution of (18) by using the truncated Adomian series as follows:

$$u(x,t) \approx \sum_{i=0}^{N} u_i(x,t) \quad (N \in \mathbb{N}). \tag{19}$$

Moreover, the nonlinear term $F(u(x,t))$ in (17) is decomposed in terms of Adomian polynomials as

$$F\big(u(x,t)\big) \approx \sum_{i=0}^{N-1} A_i\big(u_0(x,t), u_1(x,t), \dots, u_i(x,t)\big), \tag{20}$$

where $A_i = \frac{1}{i!}\frac{d^i}{dp^i}[F(\sum_{j=0}^{i} p^j u_j(x,t)]_{p=0}$, $i = 0,1,2,\dots$, are the Adomian polynomials. By applying (19) and (20) in (17), we obtain

$$ {}^{C}D_t^{\alpha} \sum_{i=0}^{N} u_i(x,t) + \frac{\partial^2}{\partial x^2}\sum_{i=0}^{N} u_i(x,t) + a(x)\frac{\partial}{\partial x}\sum_{i=0}^{N} u_i(x,t) + \sum_{i=0}^{N-1} A_i = 0, \tag{21}$$

where $0 \le \alpha < 1$. Problem (17) can be decomposed into $N+1$ subproblems by the principle of superposition as follows:

$$\begin{aligned} &{}^{C}D_t^{\alpha} u_0(x,t) + \frac{\partial^2}{\partial x^2}u_0(x,t) + a(x)\frac{\partial}{\partial x}u_0(x,t) = 0, \\ &u_0(x,0) = g(x), \qquad u_0(0,t) = y_1(t), \qquad u_0(1,t) = y_2(t) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &{}^{C}D_t^{\alpha} u_i(x,t) + \frac{\partial^2}{\partial x^2}u_i(x,t) + a(x)\frac{\partial}{\partial x}u_i(x,t) = -A_{i-1}, \\ &u_i(x,0) = 0, \qquad u_i(0,t) = 0, \qquad u_i(1,t) = 0, \end{aligned} \tag{23}$$

where $0 \le \alpha < 1$ and $i = 1,2,\dots,N$. By using the Laguerre wavelet method on (22), we approximate it as

$$\frac{\partial^2}{\partial x^2}u_0(x,t) \approx \sum_{i=1}^{m'}\sum_{j=1}^{m'} c_{i,j}^0 \psi_i(x)\psi_j(t) = \Psi^T(x)C^0\Psi(t). \tag{24}$$

Now, apply I_x^2 on (24) to obtain

$$u_0(x,t) \approx \big(P_x^2\big)^T C^0\Psi(t) + p(t)x + q(t), \tag{25}$$

where $p(t)$ and $q(t)$ are some mappings of t, and we use the boundary conditions and (13) and (16) to get

$$\begin{aligned} &x = 0: \quad q(t) = y_1(t), \\ &x = 1: \quad p(t) = -\big(\big(P_x^2(1)\big)^T C^0\Psi(t)\big) + y_2(t) - y_1(t). \end{aligned} \tag{26}$$

We can write (25) as

$$\begin{aligned} u_0(x,t) \approx{} & \big(P_x^2\big)^T C^0\Psi(t) - x\big(\big(P_x^2(1)\big)^T C^0\Psi(t)\big) \\ & + x\big(y_2(t) - y_1(t)\big) + y_1(t), \end{aligned} \tag{27}$$

and so

$$\frac{\partial u_0(x,t)}{\partial x} \approx \left(P_x^1\right)^T C^0 \Psi(t) - \left(P_x^2(1)\right)^T C^0 \Psi(t) + \left(y_2(t) - y_1(t)\right). \tag{28}$$

By substituting (28), (24) in (22), we obtain

$$\frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} \approx -\Psi(x)^T C^0 \Psi(t) - a(x)\left(\left(P_x^1\right)^T C^0 \Psi(t) - \left(P_x^2(1)\right)^T C^0 \Psi(t) + y_2(t) - y_1(t)\right), \tag{29}$$

and by integrating, we get

$$u_0(x,t) \approx -\Psi^T(x) C^0 P_t^\alpha - a(x)\left(\left(P_x^1\right)^T C^0 P_t^\alpha - \left(P_x^2(1)\right)^T C^0 P_t^\alpha + I_t^\alpha\left(y_2(t) - Y_1(t)\right)\right) + g(x). \tag{30}$$

Put $K(x,t) = g(x) - x(y_2(t) - y_1(t)) - I_t^\alpha(a(x)(y_2(t) - y_1(t)))$. From (30), (27), we have

$$\begin{aligned}&\left(P_x^2\right)^T C^0 \Psi(t) - x\left(\left(P_x^2(1)\right)^T C^0 \Psi(t)\right)\\ &\quad\approx \Psi^T(x) C^0 P_t^\alpha\\ &\qquad + a(x)\left(\left(P_x^1\right)^T C^0 P_t^\alpha - \left(P_x^2(1)\right)^T C^0 P_t^\alpha\right) + K(x,t).\end{aligned} \tag{31}$$

By using the collocation points and replacing $\approx$ with $=$, we obtain the matrix version of (31) in a discrete form as follows:

$$\begin{aligned}&\left(P_x^2\right)^T C^0 \Psi - V^{2,1,x} C^0 \Psi - \Psi^T C^0 P_t^\alpha\\ &\quad - a(x)\left(\left(P^1\right)^T C^0 P_t^\alpha - \left(P_x^2(1)\right)^T C^0 P_t^\alpha\right) - K = 0,\end{aligned} \tag{32}$$

where Ψ is the $2^{k-1}M \times 2^{k-1}M$ Laguerre wavelets matrix, $V^{2,1,x} = xP_x^2(1)$ is the $2^{k-1}M \times 2^{k-1}M$ fractional matrix, and $P_x^2 = I_x^2 \Psi^T$, $P_t^\alpha = I_t^\alpha \Psi$ are $2^{k-1}M \times 2^{k-1}M$ matrices of fractional integration of the Laguerre wavelets. Now, put $L := (\Psi^T + A((P^1)^T - (V^{2,1})^T)^{-1}$, where

$$A = \begin{pmatrix} a(x(1)) & 0 & \dots & 0\\ 0 & a(x(2)) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a(x(2^{k-1}M)) \end{pmatrix}.$$

Thus, relation (32) can be written as

$$L\left(P_x^2 - V^{2,1,x}\right) C^0 - C^0 P_t^\alpha \Psi^{-1} = LK. \tag{33}$$

If we solve (33) for C^0 and substitute in (30) or (27), we obtain the solution u_0 at the collocation points. Similarly, we apply the Laguerre wavelet method on (23) by approximating higher order derivative by Laguerre wavelet series as follows:

$$\frac{\partial^2}{\partial x^2}u_i(x,t) \approx \sum_{l=1}^{m'}\sum_{j=1}^{m'} c_{l,j}^i \psi_l(x)\psi_j(t) = \Psi^T(x)C^i\Psi(t). \tag{34}$$

Now, by integrating I_x^2 on (34), we get

$$u_i(x,t) \approx \left(P_x^2\right)^T C^i\Psi(t) - x\left(P_x^2(1)\right)^T C^i\Psi(t), \tag{35}$$

and so

$$\frac{\partial u_i(x,t)}{\partial x} \approx \left(P_x^1\right)^T C^i\Psi(t) - \left(P_x^2(1)\right)^T C^i\Psi(t). \tag{36}$$

By substituting (36), (34) in (23), we obtain

$$\begin{aligned}\frac{\partial^\alpha u_i(x,t)}{\partial t^\alpha} &\approx -\Psi(x)^T C^i\Psi(t) \\ &\quad - a(x)\left(\left(P_x^1\right)^T C^i\Psi(t) - \left(P_x^2(1)\right)^T C^i\Psi(t)\right) - A_{i-1}.\end{aligned} \tag{37}$$

By applying fractional integral operator I_t^α to (37) and using the initial condition, we get

$$\begin{aligned}u_i(x,t) &\approx -\Psi^T(x)C^iP_t^\alpha - a(x)(\left(P_x^1\right)^T C^iP_t^\alpha \\ &\quad - \left(P_x^2(1)\right)^T C^iP_t^\alpha - I_t^\alpha A_{i-1}.\end{aligned} \tag{38}$$

From (38) and (35), we have

$$\begin{aligned}&\left(P_x^2\right)^T C^i\Psi(t) - x\left(\left(P_x^2(1)\right)^T C^i\Psi(t)\right) \\ &\quad\approx -\Psi^T(x)C^iP_t^\alpha - a(x)\left(\left(P_x^1\right)^T C^iP_t^\alpha - \left(P_x^2(1)\right)^T C^iP_t^\alpha\right) - I_t^\alpha A_{i-1}.\end{aligned} \tag{39}$$

By using the collocation points and replacing $\approx$ with $=$, we obtain the matrix form of (39) as follows:

$$\begin{aligned}&\left(P_x^2\right)^T C^i\Psi - V^{2,1,x}C^i\Psi - \Psi^T C^0P_t^\alpha \\ &\quad - a(x)\left(\left(P^1\right)^T C^0P_t^\alpha - \left(P_x^2(1)\right)^T C^0P_t^\alpha\right) = -I_t^\alpha A_{i-1},\end{aligned} \tag{40}$$

where Ψ is the Laguerre wavelets matrix, $V^{2,1,x} = xP_x^2(1)$ and $P_x^2 = I_x^2\Psi^T$ and $P_t^\alpha = I_t^\alpha\Psi$ are $2^{k-1}M \times 2^{k-1}M$ matrices of fractional integration of the Laguerre wavelets. Put $L := (\Psi^T + A((P^1)^T - (V^{2,1})^T)^{-1}$, where

$$A = \begin{pmatrix} a(x(1)) & 0 & \dots & 0 \\ 0 & a(x(2)) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a(x(2^{k-1}M)) \end{pmatrix}.$$

Relation (32) can be written as

$$L\big(P_x^2 - V^{2,1,x}\big)C^i - C^i P_t^\alpha \Psi^{-1} = L I_t^\alpha A_{i-1}, \tag{41}$$

which is the Sylvester equation. Fix $i = 1$ and use the obtained approximation $u_0(x,t)$ in the calculation of Adomian's polynomials A_0. By solving (41) for C^1 and replacing in Eq. (35), we obtain an approximate solution of $u_1(x,t)$. This process is repeated by using the approximate solutions $u_i(x,t)$, $i = 0,1,\dots,k$, in the calculation of Adomian's polynomials A_k and use it in Eq. (41) to get C^i, which is used in Eq. (35) to obtain an approximate solution $u_i(x,t)$. In this way, we obtain a sequence of approximations $\{u_i(x,t)\}$, $i = 0,1,\dots,N$, where N is an arbitrary natural number. Thus the approximate solution of (17) is obtained as $\sum_{i=0}^{N} u_i(x,t)$.

5 Error analysis

Here, we are going to review the error analysis of the method by expansion of a function in terms of Laguerre wavelets.

Theorem 2 *Assume that $u_{m,m'}(x,t)$ is the Laguerre wavelets expansion of a smooth function $u(x,t) \in \Omega$. There are real numbers C_1, C_2, and C_3 such that*

$$\begin{aligned}&\big\|u(x,t) - u_{m,m'}(x,t)\big\|_2 \\ &\quad \le + \frac{C_1}{M!2^{(k+1)M-1}} + \frac{C_2}{M'!2^{(k'+1)M'-1}} + \frac{C_3}{M!2^{(k+1)M-1}M'!2^{(k'+1)M'-1}}.\end{aligned}$$

Proof Consider

$$V_{m,m'} = span\big\{\psi_{n,m_1}(x)\psi_{n',m_2}(t)\big\},$$

where $n = 1,2,\dots,2^{k-1}$, $n' = 1,2,\dots,2^{k'-1}$, $m_1 = 0,1,\dots,M-1$, $m_2 = 0,1,\dots,M'-1$, and $m = 2^{k-1}M$, $m' = 2^{k'-1}M'$. Let $u_{m,m'}(x,t)$ be the best approximation of $u(x,t)$. In this case, we have $\|u(x,t) - u_{m,m'}(x,t)\|_2 \le \|u(x,t) - v_{m,m'}(x,t)\|_2$ for all $v_{m,m'}(x,t) \in V_{m,m'}$. One can check that the last inequality holds whenever $v_{m,m'}(x,t)$ is an interpolating polynomial for $u(x,t)$. Let $P_{m,m'}(x,t)$ be the interpolating polynomial of $u(x,t)$ on Ω and $p_{m,m'}(x,t)$ is the interpolating polynomial of $u(x,t)$ at points (x_i,t_j), where x_i, $i = 0,1,\dots,M-1$, are the roots of the M-degree shifted Chebyshev polynomial in $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$ and t_j, $j = 0,1,\dots,M'-1$, are the roots of the M'-degree shifted Chebyshev polynomial in $[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}})$. In this case,

$$\begin{aligned}u(x,t) - p_{m,m'}(x,t) &= \frac{\partial^M u(\xi,t)}{\partial x^M M!}\prod_{i=0}^{M-1}(x-x_i) + \frac{\partial^{M'} u(x,\zeta)}{\partial x^{M'} M'!}\prod_{j=0}^{M'-1}(t-t_j) \\ &\quad + \frac{\partial^{M+M'} u(\xi',\zeta')}{\partial x^M\,\partial t^{M'} M'!M!}\prod_{i=0}^{M-1}(x-x_i)\prod_{j=0}^{M'-1}(t-t_j),\end{aligned}$$

where $\xi,\xi' \in I_{k,n} = [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$ and $\zeta,\zeta' \in I_{n',k'} = [\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}})$ (see [49]). Let $\Delta = I_{n,k} \times I_{n',k'}$, we get

$$\begin{aligned}
\left|u(x,t) - p_{m,m'}(x,t)\right| \leq{}& max_{(x,t)\in\Delta}\left|\frac{\partial^M u(x,t)}{\partial x^M}\right|\frac{\prod_{i=0}^{M-1}|(x-x_i)|}{M!} \\
&+ max_{(x,t)\in\Delta}\left|\frac{\partial^{M'} u(x,t)}{\partial t^{M'}}\right|\frac{\prod_{j=0}^{M'-1}|(t-t_j)|}{M'!} \\
&+ max_{(x,t)\in\Delta}\left|\frac{\partial^{M+M'} u(x,t)}{\partial x^M\,\partial t^{M'}}\right|\frac{\prod_{i=0}^{M-1}|(x-x_i)|\prod_{j=0}^{M'-1}|(t-t_j)|}{M!M'!}.
\end{aligned}$$

By using Theorem 2.2.3 in [50] for error of Chebyshev interpolation nodes, we obtain

$$\begin{aligned}
\left|u(x,t) - p_{m,m'}(x,t)\right| \leq{}& max_{(x,t)\in\Delta}\left|\frac{\partial^M u(x,t)}{\partial x^M}\right|\frac{1}{M!2^{M(k+1)-1}} \\
&+ max_{(x,t)\in\Delta}\left|\frac{\partial^{M'} u(x,t)}{\partial t^{M'}}\right|\frac{1}{M'!2^{M'(k'+1)-1}} \\
&+ max_{(x,t)\in\Delta}\left|\frac{\partial^{M+M'} u(x,t)}{\partial x^M\,\partial t^{M'}}\right|\frac{1}{M!M'!2^{M(k+1)-1}2^{M'(k'+1)-1}}.
\end{aligned}$$

Since the interval $[0,1)$ is divided into 2^{k-1} (or 2^{k-1}) subintervals $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$ (or $[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}})$), the function $u(x,t)$ is approximated on them by using the Laguerre wavelets method as a polynomial of Mth (or Mth) degree at most with the least-square property, we get

$$\begin{aligned}
&\left\|u(x,t) - u_{m,m'}(x,t)\right\|_2^2 \qquad (42)\\
&= \int_0^1\int_0^1\left[u(x,t) - u_{m,m'}(x,t)\right]^2 dx\,dt \\
&\leq \int_0^1\int_0^1\left[u(x,t) - P_{m,m'}(x,t)\right]^2 dx\,dt \\
&\leq \sum_{n'=1}^{2^{k'-1}}\sum_{n=1}^{2^{k-1}}\int_{I_{n',k'}}\int_{I_{n,k}}\left[u(x,t) - p_{m,m'}(x,t)\right]^2 dx\,dt \\
&\leq \sum_{n'=1}^{2^{k'-1}}\sum_{n=1}^{2^{k-1}}\int_{I_{n',k'}}\int_{I_{n,k}}\Bigg[max_{(x,t)\in\Delta}\left|\frac{\partial^M u(x,t)}{\partial x^M}\right|\frac{1}{M!2^{M(k+1)-1}} \\
&\quad + max_{(x,t)\in\Delta}\left|\frac{\partial^{M'} u(x,t)}{\partial t^{M'}}\right|\frac{1}{M'!2^{M'(k'+1)-1}} \\
&\quad + max_{(x,t)\in\Delta}\left|\frac{\partial^{M+M'} u(\xi',\zeta')}{\partial x^M\,\partial t^{M'}}\right|\frac{1}{M!M'!2^{M(k+1)-1}2^{M'(k'+1)-1}}\Bigg]^2 dx\,dt \\
&\leq \int_{I_{n',k'}}\int_{I_{n,k}}\Bigg[max_{(x,t)\in\Omega}\left|\frac{\partial^M u(x,t)}{\partial x^M}\right|\frac{1}{M!2^{M(k+1)-1}} \\
&\quad + max_{(x,t)\in\Omega}\left|\frac{\partial^{M'} u(x,t)}{\partial t^{M'}}\right|\frac{1}{M'!2^{M'(k'+1)-1}}
\end{aligned}$$

$$+\, max_{(x,t)\in\Omega}\left|\frac{\partial^{M+M'}u(x,t)}{\partial x^M\,\partial t^{M'}}\right|\frac{1}{M!M'!2^{M(k+1)-1}2^{M'(k'+1)-1}}\Bigg]^2\,dx\,dt.$$

Now, choose real numbers C_1, C_2, and C_3 such that

$$max_{(x,t)\in\Omega}\left|\frac{\partial^{M}u(x,t)}{\partial x^M}\right|\le C_1, \tag{43}$$

$$max_{(x,t)\in\Omega}\left|\frac{\partial^{M'}u(x,t)}{\partial t^{M'}}\right|\le C_2, \tag{44}$$

$$max_{(x,t)\in\Omega}\left|\frac{\partial^{M+M'}u(x,t)}{\partial x^M\,\partial t^{M'}}\right|\le C_3. \tag{45}$$

By replacing (43), (44), and (45) in (42), we obtain

$$\begin{aligned}&\left\|u(x,t)-u_{m,m'}(x,t)\right\|_2\\&\quad\le\frac{C_1}{M!2^{(k+1)M-1}}+\frac{C_2}{M'!2^{(k'+1)M'-1}}+\frac{C_3}{M!2^{(k+1)M-1}M'!2^{(k'+1)M'-1}}.\end{aligned} \tag{46}$$

Relation (46) ensures the convergence of Laguerre wavelet approximation $u_{m,m'}(x,t)$ for components of the Adomian series $u_i(x)$ at higher level of k and M, that is, when k and M approach infinity. According to the convergence of the Adomian method [51], $\sum_{i=0}^{N}u_i(x,t)$ converges to $u(x,t)$ when $N\to\infty$. According to this analysis, we conclude that the present method converges to the exact solution of (42) whenever N and k, M approach infinity. This completes the proof. □

For the special case $M=M'$ and $k=k'$, we have

$$\left\|u(x,t)-u_{m,m'}(x,t)\right\|_2\le\frac{C'}{M!2^{(k+1)M-1}}+\frac{C'_1}{(M!)^2 2^{2M(k+1)-2}},$$

where $C'=C_1+C_2$, $C'_1=C_3$, and $u_{m,m'}(x,t)$ is the best approximation of $u(x,t)$.

6 Numerical examples

Now, using the method, we provide some illustrative examples. In the examples, exact solutions are available and a comparison is made between the approximate Laguerre technique and the exact solutions to show the efficiency of the method.

Example 1 Consider the fractional Fisher equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^2 u(x,t)}{\partial x^2}+6u(x,t)\big(1-u(x,t)\big),\quad 0\le x,t\le 1, 0<\alpha\le 1, \tag{47}$$

with boundary conditions

$$u(x,0)=\frac{1}{(1+e^x)^2},\qquad u(0,t)=\frac{1}{(1+e^{-5t})^2},\qquad u(1,t)=\frac{1}{(1+e^{1-5t})^2}.$$

For $\alpha=1$, the exact solution of (47) is $u(x,t)=\frac{1}{(1+e^{x-5t})^2}$. By solving (47) for $k=3$ and $M=5$ by the Laguerre wavelet Adomian method (LWAM), the approximate solution obtained

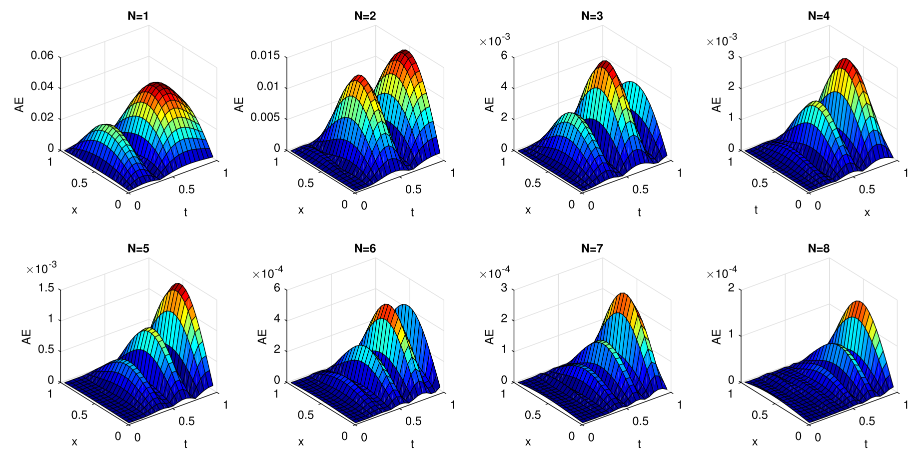


Figure 1 Absolute errors(AE) for different values of N, $\alpha = 1$, $k = 3$, $M = 5$ in Example 1

Table 1 Absolute errors for $N = 8$, $k = 3$, $M = 5$, various values of α when it goes to $\alpha = 1$, and comparison of the absolute error with HPM [33] and MVIM [36] in Example 1

$x = t$	E_{LWAM}					*HPM*	*MVIM*
	$\alpha = 0.35$	$\alpha = 0.55$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$	$\alpha = 1$	$\alpha = 1$
0	2.5654e–07	7.7371e–07	1.3304e–06	2.4878e–06	2.0852e–08	0	0
0.1	3.0943e–03	2.5784e–03	1.8021e–03	4.2893e–04	1.0496e–06	6.4221e–04	1.0323e–04
0.2	5.0069e–05	1.4518e–04	5.8107e–04	1.5436e–04	1.2781e–06	9.8905e–03	1.9372e–03
0.3	6.0134e–03	4.7295e–03	2.9859e–03	5.8153e–04	2.6500e–06	4.7274e–02	1.3430e–02
0.4	9.8473e–03	7.7664e–03	4.9873e–03	1.1538e–03	9.4140e–06	1.3911e–01	4.2501e–02
0.5	9.3915e–03	7.2676e–03	4.6473e–03	1.1361e–03	2.0053e–05	3.1320e–01	9.4534e–02
0.6	6.2457e–03	4.6540e–03	2.8907e–03	7.0416e–04	1.9822e–05	5.9479e–01	1.7111e–01
0.7	2.8244e–03	1.9277e–03	1.1064e–03	2.1850e–04	4.4881e–05	$1.0034e + 00$	2.7047e–01
0.8	5.3751e–04	1.9959e–04	1.2624e–04	2.4006e–05	2.4770e–05	$1.5504e + 00$	3.8837e–01
0.9	3.1315e–04	3.5580e–04	2.9325e–04	6.4162e–05	2.3675e–05	$2.2365e + 00$	5.1885e–01
1	1.6365e–07	3.8611e–08	2.5677e–08	6.8794e–07	6.9133e–07	$3.0511e + 00$	6.5484e–01

by this method for $N = 8$ is $u_{LWAM} = \sum_{i=0}^{8} u_i(x,t)$. We plotted in Fig. 1 the absolute errors for various values of $N = 1, 2, \ldots, 8$. As can be seen, by increasing the values of N absolute errors are decreasing. Table 1 shows the comparison of absolute errors for different values of α and the methods introduced in [33, 36]. Table 2 shows the comparison of absolute errors for different values of M. Also, it says that by increasing of M absolute errors are decreasing.

Example 2 Consider the fractional Fisher equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\bigl(1 - u(x,t)^6\bigr), \quad 0 \le x, t \le 1, 0 < \alpha \le 1, \tag{48}$$

with boundary conditions

$$u(x,0) = \sqrt[3]{\frac{1}{e^{\frac{3x}{2}} + 1}}, \qquad u(0,t) = \sqrt[3]{\frac{1}{e^{\frac{-15t}{4}} + 1}}, \qquad u(1,t) = \sqrt[3]{\frac{1}{e^{\frac{-15t+6}{4}} + 1}}.$$

Table 2 Absolute errors for $N = 8$, different values of M, $\alpha = 1$ in Example 1

$x = t$	E_{LWAM}			
	$k = 2, M = 3$	$k = 2, M = 4$	$k = 2, M = 5$	$k = 2, M = 6$
0	4.6930e–05	1.0559e–05	1.8124e–07	5.9985e–08
0.1	1.5318e–04	2.3859e–04	3.2965e–05	1.5931e–05
0.2	5.2275e–04	3.9222e–05	3.0829e–05	1.5770e–05
0.3	9.9374e–04	2.2407e–04	3.8803e–05	1.2037e–05
0.4	1.4868e–03	4.0119e–04	9.3107e–05	1.9967e–05
0.5	1.9331e–03	6.9119e–04	2.1975e–04	2.0484e–05
0.6	1.4577e–03	6.5306e–04	1.4629e–04	1.0167e–05
0.7	4.2596e–04	1.7905e–05	6.3265e–05	5.0338e–05
0.8	1.9582e–04	1.5374e–05	3.1038e–05	3.0724e–05
0.9	2.4997e–04	8.9176e–05	7.1429e–06	1.7008e–06
1	1.7706e–03	4.4301e–04	5.0532e–06	7.7092e–07

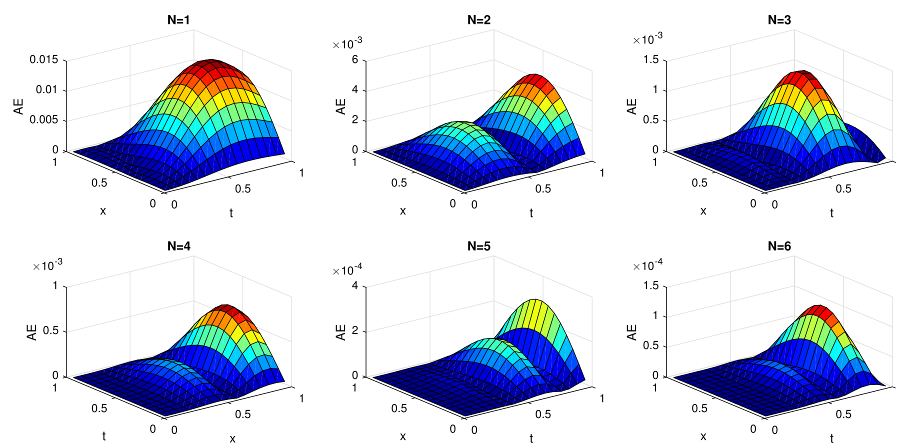


Figure 2 Absolute errors(AE) for different values of N, $\alpha = 1$, $k = 2$, $M = 8$ in Example 2

For $\alpha = 1$, the exact solution of (48) is $u(x,t) = \sqrt[3]{\frac{1}{e^{\frac{-15t+6x}{4}}+1}}$. We solve (48) for $k = 3$ and $M = 5$ by the LWAM. The approximate solution for $N = 6$ is $u_{LWAM} = \sum_{i=0}^{6} u_i(x,t)$. We plotted in Fig. 2 the absolute errors for various values of $N = 1, 2, \dots, 6$. One can check that by increasing the values of N absolute errors are decreasing. Table 3 shows the comparison of absolute errors for different values of α and the method introduced in [33, 36]. Table 4 shows the comparison of absolute errors for different values of k and M. Also it shows that by increasing of k and M absolute errors are decreasing.

Example 3 Consider the following singular fractional time-dependent Emden–Fowler equation (see [38]):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{2}{x}\frac{\partial u(x,t)}{\partial x} - 2t(x^2 - 3t)e^{u(x,t)} - 4t^4 x^2 e^{2u(x,t)} \quad (0 < \alpha \le 1), \qquad (49)$$

with boundary conditions

$$u(x,0) = -ln(3), \qquad u(0,t) = -ln(3), \qquad u(1,t) = -ln(3 + t^2) \quad (0 < x, t \le 1).$$

Table 3 Absolute errors for $N = 6$, $k = 3$, $M = 5$, when α goes to 1, and comparison with HPM [33] and MVIM [36] in Example 2

$x = t$	E_{LWAM}					*MVIM*	*HPM*
	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$	$\alpha = 1$	$\alpha = 1$
0	3.0971e–07	7.8336e–07	1.9749e–07	2.1290e–07	4.9380e–08	0	0
0.1	2.6629e–03	2.1378e–03	1.4414e–03	5.4346e–04	1.7958e–08	2.1033e–02	3.3842e–02
0.2	5.1413e–03	4.0180e–03	2.6911e–03	9.7438e–04	9.3708e–08	4.1746e–02	8.2808e–02
0.3	6.4672e–03	4.9687e–03	3.2217e–03	1.1902e–03	1.7236e–06	5.7486e–02	1.5791e–01
0.4	6.2466e–03	4.6376e–03	2.9121e–03	1.0360e–03	1.2635e–05	6.3948e–02	2.7260e–01
0.5	4.6054e–03	3.2695e–03	1.9703e–03	6.5305e–04	2.7198e–05	5.8513e–02	4.4153e–01
0.6	2.3633e–03	1.4515e–03	7.7456e–04	1.9034e–04	1.4013e–05	4.1063e–02	6.7895e–01
0.7	2.5723e–04	1.6945e–04	3.7950e–04	1.8025e–04	3.2969e–05	1.3997e–02	9.9730e–01
0.8	1.0251e–03	1.0588e–03	8.4048e–04	3.3244e–04	5.6001e–05	1.8477e–02	1.4059e+00
0.9	1.1122e–03	1.0186e–03	7.6367e–04	2.8646e–04	2.4918e–05	5.1444e–02	1.9104e+00
1	1.9387e–06	2.2032e–07	2.3431e–07	3.0510e–07	4.7726e–08	8.0369e–02	2.5119e+00

Table 4 Absolute errors for $N = 6$, different values of k, M, $\alpha = 1$ in Example 2

$x = t$	E_{LWAM}		
	$k = 2, M = 3$	$k = 3, M = 4$	$k = 4, M = 5$
0	2.4316e–05	6.9503e–10	7.8160e–13
0.1	9.6999e–06	6.7663e–08	3.8961e–09
0.2	1.3778e–04	8.0884e–07	8.2865e–08
0.3	2.9538e–05	1.2498e–05	1.7223e–06
0.4	2.0527e–04	1.2532e–05	1.1628e–05
0.5	1.4936e–05	2.7802e–05	2.5275e–05
0.6	1.9831e–04	1.3978e–05	1.1878e–05
0.7	1.3270e–04	3.3504e–05	3.2985e–05
0.8	2.2554e–04	5.4850e–05	5.1029e–05
0.9	1.1613e–04	2.4786e–05	2.4947e–06
1	1.3348e–06	1.5250e–07	2.9396e–10

For $\alpha = 1$, the exact solutions of (49) is $u(x,t) = -ln(3 + (xt)^2)$. We solve (49) for $k = 2$ and $M = 6$ by the LWAM. The approximate solution obtained by this method for $N = 6$ is $u_{LWAM} = \sum_{i=0}^{6} u_i(x,t)$. We plotted in Fig. 3 the absolute errors for various values of $N = 1, 2, \ldots, 6$. You can see that by increasing the values of N absolute errors are decreasing. Table 5 shows the comparison of absolute errors for different values of α. For the case $\alpha = 1$, with the method introduced in [38]. In Figs. 4, 5 and 6, we plotted the Laguerre wavelet Adomian approximate solution, the exact solution, and the absolute error for $k = 2$, $M = 6$, $\alpha = 1$, and $N = 6$.

7 Conclusion

By using the Laguerre wavelets and the Adomian decomposition method, we tried to provide appropriate numerical solutions for some partial fractional differential equations. We compared our results with some other methods. Also, we gave some illustrative examples which showed that the method is an effective tool to solve the time-fractional order Fisher equations and the singular nonlinear Emden–Fowler equation. We summarize the advantages of the present methods as follows.

1) Instead of operational derivative, we used the operational integral matrix with initial conditions taken into automatically, so we did not need to choose the base to satisfy them.

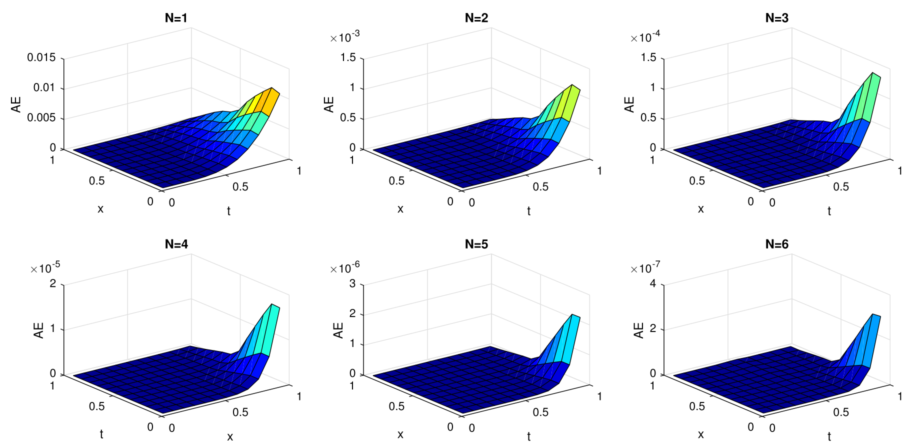
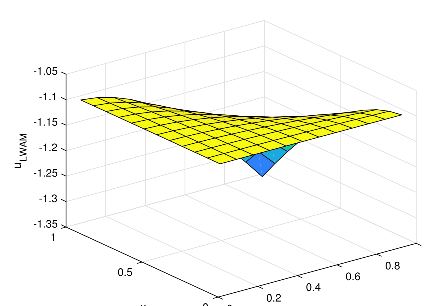


Figure 3 Absolute errors(AE) for different values of N, $\alpha = 1$, $k = 2$, $M = 6$ in Example 3

Table 5 Absolute errors for $N = 6$, $k = 2$, $M = 6$, $t = 0.5$, different values of α going to $\alpha = 1$, and comparison with MHPM [38] in Example 3

x	E_{LWAM}					MHPM
	$\alpha = 0.35$	$\alpha = 0.55$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$	$\alpha = 1$
0	4.6662e–03	3.4950e–03	2.1218e–03	4.6913e–04	3.8159e–10	7.6700e–05
0.1	5.8458e–03	4.3789e–03	2.6579e–03	5.8746e–04	4.9973e–10	7.5500e–05
0.2	5.7294e–03	4.2927e–03	2.6040e–03	5.7483e–04	5.7220e–10	7.2100e–05
0.3	5.6445e–03	4.2305e–03	2.5638e–03	5.6477e–04	6.9755e–10	6.6600e–05
0.4	5.4712e–03	4.1021e–03	2.4826e–03	5.4528e–04	8.5924e–10	5.9100e–05
0.5	5.2732e–03	3.9547e–03	2.3895e–03	5.2296e–04	1.0573e–09	5.0200e–05
0.6	5.5037e–03	4.1274e–03	2.4911e–03	5.4372e–04	1.3137e–09	4.0100e–05
0.7	5.2611e–03	3.9447e–03	2.3779e–03	5.1743e–04	1.4914e–09	2.9500e–05
0.8	4.3288e–03	3.2448e–03	1.9531e–03	4.2336e–04	1.4557e–09	1.8900e–05
0.9	2.6220e–03	1.9646e–03	1.1806e–03	2.5472e–04	1.0271e–09	8.9000e–06
1	5.7437e–06	4.1966e–06	2.4787e–06	4.6029e–07	8.8132e–12	2.2200e–16

Figure 4 Approximate solution for $k = 2$, $M = 6$, $\alpha = 1$, $N = 6$ in Example 3



2) Instead of approximating the integral operation by the use of black-pulse functions or any approximation, the fractional integral operation has been directly obtained to get a better approximation.

3) With respect to the wavelet bases used and transforming the nonlinear problem into the algebraic equations, we obtained good results by performing few calculations and resolution.

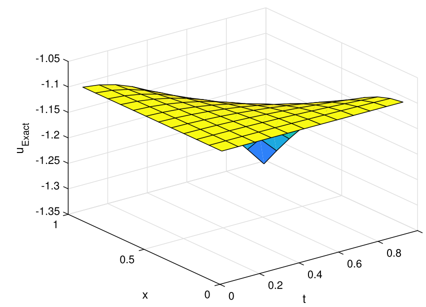


Figure 5 Exact solution for $k = 2$, $M = 6$, $\alpha = 1$, $N = 6$ in Example 3

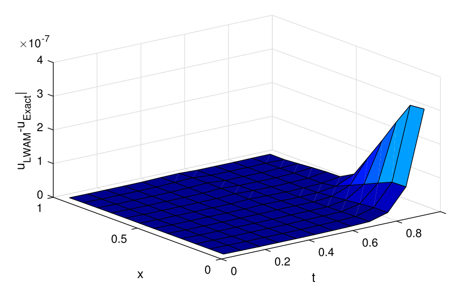


Figure 6 Absolute error for $k = 2$, $M = 6$, $\alpha = 1$, $N = 6$ in Example 3

4) Operational Laguerre wavelet matrix is sparse, so solving a system of algebraic equations obtained by using LWAM is simple and fast.

Acknowledgements
The authors were supported by Azarbaijan Shahid Madani University. Also, the authors express their gratitude to dear unknown referees for their helpful suggestions which improved the final version of this paper.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Publisher's Note

Received: 11 March 2021 Accepted: 20 April 2021 Published online: 01 May 2021

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