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Further results on Mittag-Leffler synchronization of fractional-order coupled neural networks

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Abstract

In this paper, we focus on the synchronization of fractional-order coupled neural networks (FCNNs). First, by taking information on activation functions into account, we construct a convex Lur'e–Postnikov Lyapunov function. Based on the convex Lyapunov function and a general convex quadratic function, we derive a novel Mittag-Leffler synchronization criterion for the FCNNs with symmetrical coupled matrix in the form of linear matrix inequalities (LMIs). Then we present a robust Mittag-Leffler synchronization criterion for the FCNNs with uncertain parameters. These two Mittag-Leffler synchronization criteria can be solved easily by LMI tools in Matlab. Moreover, we present a novel Lyapunov synchronization criterion for the FCNNs with unsymmetrical coupled matrix in the form of LMIs, which can be easily solved by YALMIP tools in Matlab. The feasibilities of the criteria obtained in this paper are shown by four numerical examples.

Keywords: Fractional-order coupled neural networks; Synchronization; LMIs

1 Introduction

The rapid development of modern network science and technology makes humans be aware of the importance and universality of the research on complex networks. Qualitative and quantitative analysis on various artificial and real complex network systems need to be conducted. As a typical complex network system, a neural network has a strong intelligence characteristic and learning functions, which plays an important role in artificial intelligence-related fields over the past few decades [1].

In the 1980s, Mandelbrot [2] found that fractional-order phenomena appeared in real life and engineering, which caused great repercussion both in engineering and academic circles. Different from classical calculus, fractional-order calculus has unique memorization and heredity. Thus fractional-order systems can more reasonably reflect the dynamical response process of the model [3]. In fact, since memory exists in the fractional-order derivatives and biological neurons, neurons in the human brain can be more accurately described and simulated by fractional-order neural networks. Nowadays, fractional-order neural networks play an important role in artificial intelligence-related fields, such as automatic control, intelligent robots, pattern recognition, and so on. An increasing number

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of scholars turn to the study on dynamical behaviors of fractional-order neural networks, and many excellent results are reported (see, e.g., [4–8] and references therein).

Synchronization of fractional-order neural networks, as a collective behavior, has received much attention and is extensively investigated [9–14]. In [9] the authors study the synchronization and robust synchronization issues for the FCNNs based on the general convex quadratic Lyapunov function. However, the synchronization criteria are in the form of nonlinear matrix inequalities (NLMIs), which brings certain difficulty in the solving process. In [10] the authors investigate the hybrid synchronization problem of two coupled complex networks with fractional-order dynamical nodes based on the general convex quadratic Lyapunov function by the fractional-order Lyapunov stability theorem. In [11] a Lyapunov function including fractional-order integral term is constructed to derive the synchronization stability conditions for Riemann–Liouville fractional-order delay-coupled complex neural networks. However, the relationship between node networks and coupling matrix is neglected.

Due to the complexity of fractional-order calculus theory, some existing synchronization criteria for fractional-order neural networks have not fully considered the information of FCNNs, such as activation functions and coupled matrix. Based on the fractional-order Lyapunov stability theory, Lyapunov functions play a key role in the synchronization discrimination of the fractional-order system and directly affect the conservatism of synchronization discrimination conditions. Constructing some Lyapunov functions including the information of activation function may be more reasonable and effective. Recently, the global Mittag-Leffler group consensus and group consensus in finite time for fractional-order multiagent systems are concerned in [15]. Under the fractional-order Filippov differential inclusion framework, by applying the Lur’e–Postnikov-type convex Lyapunov functional approach and Clarke’s nonsmooth analysis technique, some sufficient conditions are provided in terms of LMIs. However, the Lur’e–Postnikov Lyapunov function constructed in [15] requires that the activation function should be nondecreasing. In this paper, we remove this constraint. The activation function just needs to satisfy the general Lipschitz condition. Since the existing research methods have some limitations, further research is needed.

Motivated by the discussion above, in this paper, we study the synchronization of FCNNs. The results obtained enrich the theory of synchronization of FCNNs. The major contributions can be highlighted as follows:

- The Lur’e–Postnikov Lyapunov function is extended to a general case suitable for the activation functions satisfying the generalized Lipschitz condition;
- A novel Mittag-Leffler synchronization criterion is derived for the FCNNs with symmetrical coupled matrix in the form of LMIs. Then a robust Mittag-Leffler synchronization criterion is given for the FCNNs with uncertain parameters;
- A novel Lyapunov synchronization criterion is derived for the FCNNs with unsymmetrical coupled matrix in the form of LMIs, which can be easily solved by YALMIP tools in Matlab;
- The information of network node and coupling matrix is adequately considered in the synchronization criteria.

The paper is structured as follows. In Sect. 2, we present some definitions and main properties of Caputo fractional-order calculus. In Sect. 3, we present three novel synchro-

nization criteria for different types of FCNNs in the form of LMIs. Simulation examples are given in Sect. 4. Conclusions are given in Sect. 5.

2 Preliminaries

To study the synchronization of FCNNs, we first provide some definitions of Caputo fractional-order calculus and some useful lemmas.

Definition 1 ([16]) Let $\alpha \in R^+$. The operator $D^{-\alpha}$ defined on $L_1[0, b]$ by

$$D^{-\alpha}f(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \tag{1}$$

for $0 \leq t \leq b$ is called the Riemann–Liouville fractional-order integral of order α where Γ is the gamma function, $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t} dt$.

Definition 2 ([16]) Let $n - 1 \leq \alpha < n$, where $n \in N^+$. The operator D^α defined by

$$D^\alpha f(t) \triangleq \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds \tag{2}$$

is called the Caputo fractional-order differential of order α .

Lemma 1 ([17]) Let $V : \Omega \rightarrow R$ and $x : [0, \infty) \rightarrow \Omega$ be continuous differentiable functions, where $\Omega \subset R^n$. Suppose that $V(x(t))$ is convex over Ω and $V(0) = 0$. Then, for any time instant $t \geq 0$,

$$D^\alpha V(x(t)) \leq \left(\frac{\partial V(x(t))}{\partial x(t)} \right)^T D^\alpha x(t), \quad \forall \alpha \in (0, 1). \tag{3}$$

Lemma 1 allows us to construct some more useful convex Lyapunov functions. Based on Lemma 1, the following propositions obviously hold.

Proposition 1 ([18]) Let $x : [0, \infty) \rightarrow \Omega$ be a continuous differentiable function, where $\Omega \subset R^n$. Let $P \in R^{n \times n}$ be a positive definite matrix. Then, for any time instant $t \geq 0$,

$$D^\alpha [x^T(t)Px(t)] \leq 2x^T(t)PD^\alpha x(t), \quad \forall \alpha \in (0, 1). \tag{4}$$

Proposition 2 ([19]) Let $x : [0, \infty) \rightarrow R$ be a continuous differentiable function. Then, for any time instant $t \geq 0$,

$$D^\alpha |x(t)| \leq \text{sign}\{x(t)\}D^\alpha x(t), \quad \forall \alpha \in (0, 1), \tag{5}$$

almost everywhere.

Proposition 3 ([20]) Let $f : R \rightarrow R$ and $x : [0, \infty) \rightarrow R$ be continuous differentiable functions. Let f be a continuous nondecreasing function. Then, for any time instant $t \geq 0$,

$$D^\alpha \int_0^{x(t)} f(s) ds \leq f(x(t))D^\alpha x(t), \quad \forall \alpha \in (0, 1). \tag{6}$$

Moreover, we introduce Lemmas 2 and 3 for the discussion that follows.

Lemma 2 *Let α and β be real column vectors of dimensions of n_1 and n_2 , respectively. For real positive symmetric matrices $\Omega_1 \in R^{n_1 \times n_1}$ and $\Omega_2 \in R^{n_2 \times n_2}$, we have the following inequality for any matrix $S \in R^{n_1 \times n_2}$ satisfying $\begin{pmatrix} \Omega_1 & S \\ S^T & \Omega_2 \end{pmatrix} \geq 0$:*

$$2\alpha^T S \beta \leq \alpha^T \Omega_1 \alpha + \beta^T \Omega_2 \beta. \tag{7}$$

Proof The conclusion is obvious, and we omit the proof. □

This lemma is equivalent to the Moon inequality [21]. In this paper, we change its form for more convenient applications.

Lemma 3 ([22]) *Let $x = 0$ be an equilibrium point of system $D^\alpha x(t) = f(t, x)$, $0 \in D \subset R^n$. Let $V : [0, +\infty) \times D \rightarrow R$ be a continuously differentiable function that is locally Lipschitz in x and such that*

$$\begin{cases} \alpha_1 \|x(t)\|^a \leq V(t, x(t)) \leq \alpha_2 \|x(t)\|^{ab}, \\ D^\alpha V(t, x(t)) \leq -\alpha_3 \|x(t)\|^{ab}, \end{cases} \tag{8}$$

where $t \geq 0, x \in D, \alpha \in (0, 1)$, and $\alpha_1, \alpha_2, \alpha_3, a$, and b are arbitrary positive constants. Then $x = 0$ is Mittag-Leffler stable. Moreover, if $\alpha_3 = 0$, then $x = 0$ is Lyapunov stable.

3 Main results

In this paper, we consider the synchronization of the following FCNNs:

$$D^\alpha y_i(t) = -C y_i(t) + B g(y_i(t)) + J(t) + d \sum_{j=1}^N A_{ij} \Gamma y_j(t), \quad t \geq 0, \tag{9}$$

where $i = 1, 2, \dots, N, \alpha \in (0, 1), N$ is the number of nodes, $y_i(t) = (y_{i1}(t), y_{i2}(t), \dots, y_{in}(t))^T \in R^n$ is the state vector of node $i, g(y_i(t)) = [g_1(y_{i1}(t)), g_2(y_{i2}(t)), \dots, g_n(y_{in}(t))]^T$ with $g_i(0) = 0$ is the activation function of node $i, J(t)$ denotes the external input, $B = (B_{ij})_{n \times n}$ and $0 < C = \text{diag}\{C_1, C_2, \dots, C_n\}$ are real matrices, $0 < d \in R$ denotes the overall coupling strength, $A = (A_{ij})_{N \times N}$ is the coupled matrix of network, and $\Gamma = (\Gamma_{ij})_{n \times n}$ corresponds to the inner coupling matrix.

Throughout this paper, we make the following assumption on the FCNNs.

Assumption 1 ([23, 24]) The activation function g is continuous and bounded. For any $x_1, x_2 \in R, x_1 \neq x_2$, the activation functions g_j satisfy

$$k_j^- \leq \frac{g_j(x_1) - g_j(x_2)}{x_1 - x_2} \leq k_j^+, \quad j = 1, 2, \dots, n, \tag{10}$$

where k_j^+ and k_j^- are known constant values. For convenience, we denote $K^+ = \text{diag}\{k_j^+\}, K^- = \text{diag}\{k_j^-\}, k_j = \max\{|k_j^+|, |k_j^-|\}$, and $K = \text{diag}\{k_j\}$.

Remark 1 Assumption 1 for activation functions is introduced to investigate the Mittag-Leffler synchronization of the FCNNs to guarantee the existence and uniqueness of the equilibrium point. Most of the common activation functions satisfy the generalized Lipschitz condition, such as A linear excitation function, threshold or step excitation function, and so on.

Next, for easier understanding, we give the definition of synchronization.

Definition 3 ([9, 25]) The FCNNs (9) realize synchronization if

$$\lim_{t \rightarrow +\infty} \left\| y_i(t) - \frac{1}{N} \sum_{i=1}^N y_i(t) \right\| = 0, \quad i = 1, 2, \dots, N. \tag{11}$$

3.1 Mittag-Leffler synchronization analysis for FCNNs with symmetrical coupled matrix

In this section, we consider FCNNs (9) with symmetrical coupled matrix, that is, $A = (A_{ij})_{N \times N}$ is the coupled matrix of network satisfying $A_{ii} = -\sum_{j=1, j \neq i}^N A_{ij}$; $A_{ij} = A_{ji} > 0$ if $i \neq j$ and there is a connection between nodes i and j , otherwise, $A_{ij} = 0$, $i, j = 1, 2, \dots, N$.

Defining $\bar{y}(t) = \frac{1}{N} \sum_{i=1}^N y_i(t)$ and the error vector $e_i(t) = y_i(t) - \bar{y}(t)$, we have

$$D^\alpha e_i(t) = -C e_i(t) + B g(y_i(t)) - \frac{1}{N} \sum_{i=1}^N B g(y_i(t)) + d \sum_{j=1}^N A_{ij} \Gamma e_j(t), \quad i = 1, 2, \dots, N. \tag{12}$$

Thus the synchronization of FCNNs (9) with symmetrical coupled matrix is equivalent to the stability of system (12).

Notice that if we take $e(t) = [e_1^T(t), e_2^T(t), \dots, e_N^T(t)]^T$, then system (12) can be written as

$$D^\alpha e(t) = -[I_N \otimes C - d(A \otimes \Gamma)]e(t) + \left[\left(I_N - \frac{E_N}{N} \right) \otimes B \right] g(y(t)),$$

where $E_N = (E_{ij})_{N \times N}$ with all $E_{ij} = 1$.

For the synchronization of FCNNs (9) with symmetrical coupled matrix, we have the following result.

Theorem 1 *FCNNs (9) with symmetrical coupled matrix realize Mittag-Leffler synchronization if there exist positive definite matrices $P, \Theta_1, \Delta_1, \Gamma_2, \Psi_2 \in R^{n \times n}$, positive definite diagonal matrices D_i ($i = 1, 2, 3, 4$), $\Theta_2, \Delta_2, \Gamma_1, \Lambda_1, \Lambda_2, \Phi_1, \Phi_2, \Psi_1 \in R^{n \times n}$ and a matrix $\Omega \in R^{Nn \times Nn}$ such that the following linear matrix inequalities hold:*

$$\begin{pmatrix} \Theta_1 & PB \\ * & \Theta_2 \end{pmatrix} > 0,$$

$$\begin{pmatrix} \Delta_1 & [K^+ D_1 - K^- D_2 + K D_3 + K D_4] B \\ * & \Delta_2 \end{pmatrix} > 0,$$

$$\begin{pmatrix} \Gamma_1 & [D_1 - D_2 + D_3 - D_4] C \\ * & \Gamma_2 \end{pmatrix} > 0,$$

$$\begin{aligned}
 & \begin{pmatrix} \Lambda_1 & [-D_1 + D_2 - D_3 + D_4]B \\ * & \Lambda_2 \end{pmatrix} > 0, \\
 & \begin{pmatrix} \Phi_1 & [-D_1 + D_2 - D_3 + D_4]B \\ * & \Phi_2 \end{pmatrix} > 0, \\
 & \begin{pmatrix} \Psi_1 & d[-D_1 + D_2 - D_3 + D_4] \\ * & \Psi_2 \end{pmatrix} > 0, \\
 & \Pi < 0,
 \end{aligned} \tag{13}$$

where $\Pi = -I_N \otimes (PC + C^T P) + I_N \otimes \Theta_1 + I_N \otimes (K\Theta_2K) + dA \otimes (P\Gamma + \Gamma^T P) - I_N \otimes [(K^+D_1 - K^-D_2 + KD_3 + KD_4)C + C^T(K^+D_1 - K^-D_2 + KD_3 + KD_4)] + I_N \otimes \Delta_1 + I_N \otimes (K\Delta_2K) + dA \otimes [(K^+D_1 - K^-D_2 + KD_3 + KD_4)\Gamma + \Gamma^T(K^+D_1 - K^-D_2 + KD_3 + KD_4)] + I_N \otimes (K\Gamma_1K) + I_N \otimes \Gamma_2 + I_N \otimes (K\Lambda_1K) + I_N \otimes (K\Lambda_2K) + I_N \otimes (K\Phi_1K) + I_N \otimes (K\Phi_2K) + NI_N \otimes (K\Psi_1K) + \bar{A} \otimes (\Gamma^T\Psi_2\Gamma) + \Omega(E_N \otimes I_n) + (E_N \otimes I_n)\Omega^T$ and $\bar{A} = \text{diag}\{\sum_{i=1}^N A_{i1}^2, \sum_{i=1}^N A_{i2}^2, \dots, \sum_{i=1}^N A_{in}^2\}$.

Proof Construct the Lyapunov function

$$V(e(t)) = V_1(e(t)) + V_2(e(t)), \tag{14}$$

where

$$\begin{aligned}
 V_1(e(t)) &= \sum_{i=1}^N e_i^T(t) P e_i(t), \\
 V_2(e(t)) &= 2 \sum_{i=1}^N \sum_{j=1}^n \int_0^{e_{ij}(t)} d_{1j} [k_j^+ s - g_j(s)] ds \\
 &\quad + 2 \sum_{i=1}^N \sum_{j=1}^n \int_0^{e_{ij}(t)} d_{2j} [g_j(s) - k_j^- s] ds \\
 &\quad + 2 \sum_{i=1}^N \sum_{j=1}^n \int_0^{e_{ij}(t)} d_{3j} [k_j s - g_j(s)] ds \\
 &\quad + 2 \sum_{i=1}^N \sum_{j=1}^n \int_0^{e_{ij}(t)} d_{4j} [g_j(s) + k_j s] ds.
 \end{aligned} \tag{15}$$

To prove the synchronization of FCNNs (9) with symmetrical coupled matrix, the process involves three steps.

Step 1: We prove that $V(e(t))$ is positive.

Obviously, $V_1(e(t))$ is positive by the positive definiteness of P . Next, we prove that $V_2(e(t))$ is nonnegative.

From Assumption 1 we have

$$\frac{g_j(e_{ij}(t))}{e_{ij}(t)} \leq k_j^+, \quad j = 1, 2, \dots, n. \tag{16}$$

Then

$$[k_j^+ e_{ij}(t) - g_j(e_{ij}(t))]e_{ij}(t) \geq 0, \quad j = 1, 2, \dots, n, \tag{17}$$

which leads to $\int_0^{e_{ij}(t)} [k_j^+ s - g_j(s)] ds \geq 0$. Similarly, $\int_0^{e_{ij}(t)} [g_j(s) - k_j^- s] ds \geq 0$, $\int_0^{e_{ij}(t)} [k_j^- s - g_j(s)] ds \geq 0$, and $\int_0^{e_{ij}(t)} [g_j(s) + k_j s] ds \geq 0$.

Step 2: We prove that there exist positive constants α_1 and α_2 satisfying $\alpha_1 \|e(t)\|^2 \leq V(e(t)) \leq \alpha_2 \|e(t)\|^2$.

Obviously, we have $\lambda_{\min}\{P\} \|e(t)\|^2 \leq V(e(t)) \leq [\lambda_{\max}\{P\} + 8 \max_{1 \leq k \leq 4, 1 \leq j \leq n} \{d_{kj}\} \cdot \max_{1 \leq j \leq n} \{k_j\}] \|e(t)\|^2$.

Step 3: We prove that there exists a positive constant α_3 such that $D^\alpha V(e(t)) \leq -\alpha_3 \|e(t)\|^2$.

For $V_1(e(t))$, by Proposition 1 we have

$$\begin{aligned} D^\alpha V_1(e(t)) &\leq 2 \sum_{i=1}^N e_i^T(t) P D^\alpha e_i(t) \\ &= 2 \sum_{i=1}^N e_i^T(t) P \left[-C e_i(t) + B g(y_i(t)) - \frac{1}{N} \sum_{i=1}^N B g(y_i(t)) \right. \\ &\quad \left. + d \sum_{j=1}^N A_{ij} \Gamma e_j(t) \right]. \end{aligned} \tag{18}$$

Noticing that $\sum_{i=1}^N e_i^T(t) = 0$, we have

$$\begin{aligned} D^\alpha V_1(e(t)) &\leq 2 \sum_{i=1}^N e_i^T(t) P \left[-C e_i(t) + B g(y_i(t)) - B g(\bar{y}(t)) + d \sum_{j=1}^N A_{ij} \Gamma e_j(t) \right] \\ &= -2 \sum_{i=1}^N e_i^T(t) P C e_i(t) + 2 \sum_{i=1}^N e_i^T(t) P B (g(y_i(t)) - g(\bar{y}(t))) \\ &\quad + 2d \sum_{i=1}^N \sum_{j=1}^N e_i^T(t) P A_{ij} \Gamma e_j(t). \end{aligned} \tag{19}$$

For $2 \sum_{i=1}^N e_i^T(t) P B (g(y_i(t)) - g(\bar{y}(t)))$, by Lemma 2 there exist a positive definite matrix Θ_1 and a positive definite diagonal matrix Θ_2 such that

$$\begin{aligned} &2 \sum_{i=1}^N e_i^T(t) P B (g(y_i(t)) - g(\bar{y}(t))) \\ &\leq \sum_{i=1}^N e_i^T(t) \Theta_1 e_i(t) + \sum_{i=1}^N (g(y_i(t)) - g(\bar{y}(t)))^T \Theta_2 (g(y_i(t)) - g(\bar{y}(t))) \\ &\leq \sum_{i=1}^N e_i^T(t) \Theta_1 e_i(t) + \sum_{i=1}^N e_i^T(t) K \Theta_2 K e_i(t) \end{aligned} \tag{20}$$

and

$$\begin{pmatrix} \Theta_1 & PB \\ * & \Theta_2 \end{pmatrix} > 0.$$

So

$$\begin{aligned} D^\alpha V_1(e(t)) &\leq -2 \sum_{i=1}^N e_i^T(t)PCe_i(t) + \sum_{i=1}^N e_i^T(t)\Theta_1e_i(t) + \sum_{i=1}^N e_i^T(t)K\Theta_2Ke_i(t) \\ &\quad + 2d \sum_{i=1}^N \sum_{j=1}^N e_i^T(t)PA_{ij}\Gamma e_j(t) \\ &= e^T(t)[-I_N \otimes (PC + C^T P) + I_N \otimes \Theta_1 + I_N \otimes (K\Theta_2K) \\ &\quad + dA \otimes (P\Gamma + \Gamma^T P)]e(t). \end{aligned} \tag{21}$$

For $V_2(e(t))$, based on Proposition 3, we have

$$\begin{aligned} D^\alpha V_2(e(t)) &\leq 2 \sum_{i=1}^N [e_i^T(t)K^+ - g^T(e_i(t))]D_1 \left[-Ce_i(t) + Bg(y_i(t)) \right. \\ &\quad \left. - \frac{1}{N} \sum_{i=1}^N Bg(y_i(t)) + d \sum_{j=1}^N A_{ij}\Gamma e_j(t) \right] \\ &\quad + 2 \sum_{i=1}^N [g^T(e_i(t)) - e_i^T(t)K^-]D_2 \left[-Ce_i(t) + Bg(y_i(t)) \right. \\ &\quad \left. - \frac{1}{N} \sum_{i=1}^N Bg(y_i(t)) + d \sum_{j=1}^N A_{ij}\Gamma e_j(t) \right] \\ &\quad + 2 \sum_{i=1}^N [e_i^T(t)K - g^T(e_i(t))]D_3 \left[-Ce_i(t) + Bg(y_i(t)) \right. \\ &\quad \left. - \frac{1}{N} \sum_{i=1}^N Bg(y_i(t)) + d \sum_{j=1}^N A_{ij}\Gamma e_j(t) \right] \\ &\quad + 2 \sum_{i=1}^N [g^T(e_i(t)) + e_i^T(t)K]D_4 \left[-Ce_i(t) + Bg(y_i(t)) \right. \\ &\quad \left. - \frac{1}{N} \sum_{i=1}^N Bg(y_i(t)) + d \sum_{j=1}^N A_{ij}\Gamma e_j(t) \right], \end{aligned} \tag{22}$$

where $D_k = \text{diag}\{d_{k1}, d_{k2}, \dots, d_{kn}\}$, $k = 1, 2, 3, 4$.

By simple calculation we get

$$\begin{aligned} D^\alpha V_2(e(t)) &\leq 2 \sum_{i=1}^N e_i^T(t)[K^+D_1 - K^-D_2 + KD_3 + KD_4] \left[-Ce_i(t) + Bg(y_i(t)) \right. \\ &\quad \left. - \frac{1}{N} \sum_{i=1}^N Bg(y_i(t)) + d \sum_{j=1}^N A_{ij}\Gamma e_j(t) \right] \end{aligned} \tag{23}$$

$$\begin{aligned}
 &+ 2 \sum_{i=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4] \left[-Ce_i(t) + Bg(y_i(t)) \right. \\
 &\left. - \frac{1}{N} \sum_{i=1}^N Bg(y_i(t)) + d \sum_{j=1}^N A_{ij}\Gamma e_j(t) \right].
 \end{aligned}$$

Similarly to (20), there exist a positive definite matrix Δ_1 and a positive definite diagonal matrix Δ_2 such that

$$\begin{aligned}
 &2 \sum_{i=1}^N e_i^T(t)[K^+D_1 - K^-D_2 + KD_3 + KD_4] \left[-Ce_i(t) + Bg(y_i(t)) \right. \\
 &\left. - \frac{1}{N} \sum_{i=1}^N Bg(y_i(t)) + d \sum_{j=1}^N A_{ij}\Gamma e_j(t) \right] \\
 &\leq -2 \sum_{i=1}^N e_i^T(t)[K^+D_1 - K^-D_2 + KD_3 + KD_4]Ce_i(t) \\
 &\quad + \sum_{i=1}^N e_i^T(t)\Delta_1 e_i(t) + \sum_{i=1}^N e_i^T(t)K\Delta_2 Ke_i(t) \tag{24} \\
 &\quad + 2d \sum_{i=1}^N \sum_{j=1}^N e_i^T(t)[K^+D_1 - K^-D_2 + KD_3 + KD_4]A_{ij}\Gamma e_j(t) \\
 &= e^T(t) \{ -I_N \otimes [(K^+D_1 - K^-D_2 + KD_3 + KD_4)C \\
 &\quad + C^T(K^+D_1 - K^-D_2 + KD_3 + KD_4)] + I_N \otimes \Delta_1 \\
 &\quad + I_N \otimes (K\Delta_2 K) + dA \otimes [(K^+D_1 - K^-D_2 + KD_3 + KD_4)\Gamma \\
 &\quad + \Gamma^T(K^+D_1 - K^-D_2 + KD_3 + KD_4)] \} e(t)
 \end{aligned}$$

and

$$\begin{pmatrix} \Delta_1 & [K^+D_1 - K^-D_2 + KD_3 + KD_4]B \\ * & \Delta_2 \end{pmatrix} > 0.$$

From (23) we have

$$\begin{aligned}
 &2 \sum_{i=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4] \left[-Ce_i(t) + Bg(y_i(t)) \right. \\
 &\left. - \frac{1}{N} \sum_{i=1}^N Bg(y_i(t)) + d \sum_{j=1}^N A_{ij}\Gamma e_j(t) \right] \\
 &= 2 \sum_{i=1}^N g^T(e_i(t))[D_1 - D_2 + D_3 - D_4]Ce_i(t) \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 &+ 2 \sum_{i=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(y_i(t)) - g(\bar{y}(t))] \\
 &+ \frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(\bar{y}(t)) - g(y_j(t))] \\
 &+ 2d \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]A_{ij}\Gamma e_j(t).
 \end{aligned}$$

For $2 \sum_{i=1}^N g^T(e_i(t))[D_1 - D_2 + D_3 - D_4]Ce_i(t)$, there exist a positive definite matrix Γ_1 and a positive definite diagonal matrix Γ_2 such that

$$\begin{aligned}
 &2 \sum_{i=1}^N g^T(e_i(t))[D_1 - D_2 + D_3 - D_4]Ce_i(t) \\
 &\leq \sum_{i=1}^N g^T(e_i(t))\Gamma_1 g(e_i(t)) + \sum_{i=1}^N e_i^T(t)\Gamma_2 e_i(t) \tag{26} \\
 &\leq \sum_{i=1}^N e_i^T(t)K\Gamma_1 K e_i(t) + \sum_{i=1}^N e_i^T(t)\Gamma_2 e_i(t) \\
 &= e^T(t)[I_N \otimes (K\Gamma_1 K)]e(t) + e^T(t)[I_N \otimes \Gamma_2]e(t)
 \end{aligned}$$

and

$$\begin{pmatrix} \Gamma_1 & [D_1 - D_2 + D_3 - D_4]C \\ * & \Gamma_2 \end{pmatrix} > 0.$$

For $2 \sum_{i=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(y_i(t)) - g(\bar{y}(t))]$, we have a positive definite matrix Λ_1 and a positive definite diagonal matrix Λ_2 such that

$$\begin{aligned}
 &2 \sum_{i=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(y_i(t)) - g(\bar{y}(t))] \\
 &\leq \sum_{i=1}^N g^T(e_i(t))\Lambda_1 g(e_i(t)) + \sum_{i=1}^N [g(y_i(t)) - g(\bar{y}(t))]^T \Lambda_2 [g(y_i(t)) - g(\bar{y}(t))] \tag{27} \\
 &\leq \sum_{i=1}^N e_i^T(t)K\Lambda_1 K e_i(t) + \sum_{i=1}^N e_i^T(t)K\Lambda_2 K e_i(t) \\
 &= e^T(t)[I_N \otimes (K\Lambda_1 K) + I_N \otimes (K\Lambda_2 K)]e(t)
 \end{aligned}$$

and

$$\begin{pmatrix} \Lambda_1 & [-D_1 + D_2 - D_3 + D_4]B \\ * & \Lambda_2 \end{pmatrix} > 0.$$

For $\frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(\bar{y}(t)) - g(y_j(t))]$, we have a positive definite matrix Φ_1 and a positive definite diagonal matrix Φ_2 such that

$$\begin{aligned} & \frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(\bar{y}(t)) - g(y_j(t))] \\ & \leq \sum_{i=1}^N g^T(e_i(t))\Phi_1 g(e_i(t)) + \sum_{i=1}^N [g(\bar{y}(t)) - g(y_i(t))]^T \Phi_2 [g(\bar{y}(t)) - g(y_i(t))] \\ & \leq \sum_{i=1}^N e_i^T(t)K\Phi_1 K e_i(t) + \sum_{i=1}^N e_i^T(t)K\Phi_2 K e_i(t) \\ & = e^T(t)[I_N \otimes (K\Phi_1 K) + I_N \otimes (K\Phi_2 K)]e(t) \end{aligned} \tag{28}$$

and

$$\begin{pmatrix} \Phi_1 & [-D_1 + D_2 - D_3 + D_4]B \\ * & \Phi_2 \end{pmatrix} > 0.$$

For $2d \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]A_{ij}\Gamma e_j(t)$, we have a positive definite matrix Ψ_1 and a positive definite diagonal matrix Ψ_2 such that

$$\begin{aligned} & 2d \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]A_{ij}\Gamma e_j(t) \\ & \leq \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))\Psi_1 g(e_i(t)) + \sum_{i=1}^N \sum_{j=1}^N [A_{ij}\Gamma e_j(t)]^T \Psi_2 [A_{ij}\Gamma e_j(t)] \\ & \leq N \sum_{i=1}^N e_i^T(t)K\Psi_1 K e_i(t) + \sum_{i=1}^N \sum_{j=1}^N [A_{ij}\Gamma e_j(t)]^T \Psi_2 A_{ij}\Gamma e_j(t) \\ & = e^T(t)[NI_N \otimes (K\Psi_1 K) + \bar{A} \otimes (\Gamma^T \Psi_2 \Gamma)]e(t) \end{aligned} \tag{29}$$

and

$$\begin{pmatrix} \Psi_1 & d[-D_1 + D_2 - D_3 + D_4] \\ * & \Psi_2 \end{pmatrix} > 0,$$

where $\bar{A} = \text{diag}\{\sum_{i=1}^N A_{i1}^2, \sum_{i=1}^N A_{i2}^2, \dots, \sum_{i=1}^N A_{iN}^2\}$.

Since $\sum_{i=1}^N e_i(t) = 0$, we get

$$2e^T(t)\Omega(E_N \otimes I_n)e(t) = 0, \tag{30}$$

where $\Omega \in R^{Nn \times Nn}$.

From (21), (24), and (26)–(30) we have

$$\begin{aligned}
 D^\alpha V(e(t)) &\leq e^T(t) \{ -I_N \otimes (PC + C^T P) + I_N \otimes \Theta_1 + I_N \otimes (K\Theta_2 K) \\
 &\quad + dA \otimes (P\Gamma + \Gamma^T P) - I_N \otimes [(K^+ D_1 - K^- D_2 + KD_3 + KD_4)C \\
 &\quad + C^T (K^+ D_1 - K^- D_2 + KD_3 + KD_4)] + I_N \otimes \Delta_1 \\
 &\quad + I_N \otimes (K\Delta_2 K) + dA \otimes [(K^+ D_1 - K^- D_2 + KD_3 + KD_4)\Gamma \\
 &\quad + \Gamma^T (K^+ D_1 - K^- D_2 + KD_3 + KD_4)] + I_N \otimes (K\Gamma_1 K) \\
 &\quad + I_N \otimes \Gamma_2 + I_N \otimes (K\Lambda_1 K) + I_N \otimes (K\Lambda_2 K) + I_N \otimes (K\Phi_1 K) \\
 &\quad + I_N \otimes (K\Phi_2 K) + NI_N \otimes (K\Psi_1 K) + \bar{A} \otimes (\Gamma^T \Psi_2 \Gamma) \\
 &\quad + \Omega(E_N \otimes I_n) + (E_N \otimes I_n)\Omega^T \} e(t) \\
 &= e^T(t) \Pi e(t).
 \end{aligned}
 \tag{31}$$

So there exists a positive constant α_3 such that $D^\alpha V(e(t)) \leq -\alpha_3 \|e(t)\|^2$ if $\Pi < 0$. Therefore by Lemma 3 FCNNs (9) with symmetrical coupled matrix realizes synchronization under condition (13).

The proof is completed. □

3.2 Mittag-Leffler synchronization analysis for FCNNs with uncertain parameters

FCNNs may contain uncertain parameters due to the existence of environmental noises or model errors in many circumstances. In this section, we consider the following FCNNs with uncertain parameters:

$$\begin{aligned}
 D^\alpha y_i(t) &= -(C + \Delta C(t))y_i(t) + (B + \Delta B(t))g(y_i(t)) + J(t) \\
 &\quad + d \sum_{j=1}^N A_{ij} \Gamma y_j(t), \quad t \geq 0,
 \end{aligned}
 \tag{32}$$

where $A = (A_{ij})_{N \times N}$ is the coupled matrix of network that satisfies $A_{ij} = A_{ji} > 0$ if $i \neq j$ and there is a connection between node i and node j , otherwise, $A_{ij} = 0$, and $A_{ii} = -\sum_{j=1, j \neq i}^N A_{ij}$, $i = 1, 2, \dots, N$.

For the uncertain parameters $\Delta C(t)$ and $\Delta B(t)$ in (32), we make the following assumption.

Assumption 2 The parametric uncertainties $\Delta C(t)$ and $\Delta B(t)$ are of the form

$$[\Delta C(t), \Delta B(t)] = MF(t)[H_C, H_B],
 \tag{33}$$

where M, H_C , and $H_B \in R^{n \times n}$ are known real constant matrices, and the uncertain matrix $F(t)$ is unknown real time-varying matrix satisfying $F^T(t)F(t) \leq I_n$.

To study the synchronization of (32), we need the following lemma.

Lemma 4 ([26]) *For given matrices Y, D , and E of proper dimensions, assume that Y satisfies $Y^T = Y$. Then*

$$Y + DFE + E^T F^T D^T < 0 \tag{34}$$

for any F satisfying $F^T F \leq I$ if and only if there is a real number $\varepsilon > 0$ such that

$$Y + \varepsilon DD^T + \varepsilon^{-1} E^T E < 0. \tag{35}$$

For the synchronization of FCNNs (32), we have the following result.

Theorem 2 *FCNNs (32) realize Mittag-Leffler synchronization if there exist positive definite matrices $P, \Theta_1, \Delta_1, \Gamma_2, \Psi_2 \in R^{n \times n}$, positive definite diagonal matrices D_i ($i = 1, 2, 3, 4$), $\Theta_2, \Delta_2, \Gamma_1, \Lambda_1, \Lambda_2, \Phi_1, \Phi_2, \Psi_1 \in R^{n \times n}$, a matrix $\Omega \in R^{Nn \times Nn}$, and positive real numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7$ such that the following linear matrix inequalities hold:*

$$\begin{aligned} & \begin{pmatrix} \Psi_1 & d[-D_1 + D_2 - D_3 + D_4] \\ * & \Psi_2 \end{pmatrix} > 0, & \begin{pmatrix} \Theta_1 & PB & PM \\ * & \Theta_2 - \varepsilon_1 H_B^T H_B & O \\ * & * & \varepsilon_1 I_n \end{pmatrix} > 0, \\ & \begin{pmatrix} \Delta_1 & [K^+ D_1 - K^- D_2 + K D_3 + K D_4] B & [K^+ D_1 - K^- D_2 + K D_3 + K D_4] M \\ * & \Delta_2 - \varepsilon_2 H_B^T H_B & O \\ * & * & \varepsilon_2 I_n \end{pmatrix} > 0, \\ & \begin{pmatrix} \Gamma_1 & [D_1 - D_2 + D_3 - D_4] C & [D_1 - D_2 + D_3 - D_4] M \\ * & \Gamma_2 - \varepsilon_3 H_C^T H_C & O \\ * & * & \varepsilon_3 I_n \end{pmatrix} > 0, \\ & \begin{pmatrix} \Lambda_1 & [-D_1 + D_2 - D_3 + D_4] B & [-D_1 + D_2 - D_3 + D_4] M \\ * & \Lambda_2 - \varepsilon_4 H_B^T H_B & O \\ * & * & \varepsilon_4 I_n \end{pmatrix} > 0, \\ & \begin{pmatrix} \Phi_1 & [-D_1 + D_2 - D_3 + D_4] B & [-D_1 + D_2 - D_3 + D_4] M \\ * & \Phi_2 - \varepsilon_5 H_B^T H_B & O \\ * & * & \varepsilon_5 I_n \end{pmatrix} > 0, \\ & \begin{pmatrix} \Pi + \varepsilon_6 \Pi_1 + \varepsilon_7 \Pi_2 & I_N \otimes (PM) & I_N \otimes [(K^+ D_1 - K^- D_2 + K D_3 + K D_4) M] \\ * & -\varepsilon_6 I_{Nn} & O \\ * & * & -\varepsilon_7 I_{Nn} \end{pmatrix} < 0, \end{aligned} \tag{36}$$

where $\Pi_1 = \Pi_2 = [I_N \otimes H_C^T][I_N \otimes H_C]$, and Π is the same as in Theorem 1.

Proof By Theorem 1 FCNNs (32) realize Mittag-Leffler synchronization if the following matrix inequalities hold:

$$\begin{pmatrix} \Theta_1 & P(B + MF(t)H_B) \\ * & \Theta_2 \end{pmatrix} > 0,$$

$$\begin{aligned}
 & \begin{pmatrix} \Delta_1 & [K^+D_1 - K^-D_2 + KD_3 + KD_4](B + MF(t)H_B) \\ * & \Delta_2 \end{pmatrix} > 0, \\
 & \begin{pmatrix} \Gamma_1 & [D_1 - D_2 + D_3 - D_4](C + MF(t)H_C) \\ * & \Gamma_2 \end{pmatrix} > 0, \\
 & \begin{pmatrix} \Lambda_1 & [-D_1 + D_2 - D_3 + D_4](B + MF(t)H_B) \\ * & \Lambda_2 \end{pmatrix} > 0, \\
 & \begin{pmatrix} \Phi_1 & [-D_1 + D_2 - D_3 + D_4](B + MF(t)H_B) \\ * & \Phi_2 \end{pmatrix} > 0, \\
 & \begin{pmatrix} \Psi_1 & d[-D_1 + D_2 - D_3 + D_4] \\ * & \Psi_2 \end{pmatrix} > 0, \\
 \Pi^* &= -I_N \otimes [P(C + MF(t)H_C) + (C + MF(t)H_C)^T P] + I_N \otimes \Theta_1 \\
 &+ I_N \otimes (K\Theta_2K) + dA \otimes (P\Gamma + \Gamma^T P) \\
 &- I_N \otimes [(K^+D_1 - K^-D_2 + KD_3 + KD_4)(C + MF(t)H_C) \\
 &+ (C + MF(t)H_C)^T (K^+D_1 - K^-D_2 + KD_3 + KD_4)] \\
 &+ I_N \otimes \Delta_1 + I_N \otimes (K\Delta_2K) + dA \otimes [(K^+D_1 - K^-D_2 + KD_3 + KD_4)\Gamma \\
 &+ \Gamma^T (K^+D_1 - K^-D_2 + KD_3 + KD_4)] + I_N \otimes (K\Gamma_1K) \\
 &+ I_N \otimes \Gamma_2 + I_N \otimes (K\Lambda_1K) + I_N \otimes (K\Lambda_2K) + I_N \otimes (K\Phi_1K) \\
 &+ I_N \otimes (K\Phi_2K) + NI_N \otimes (K\Psi_1K) + \bar{A} \otimes (\Gamma^T \Psi_2 \Gamma) \\
 &+ \Omega(E_N \otimes I_n) + (E_N \otimes I_n)\Omega^T < 0.
 \end{aligned} \tag{37}$$

Let $e_1 = [I_n, O]$ and $e_2 = [O, I_n]$. Then

$$\begin{aligned}
 & \begin{pmatrix} \Theta_1 & P(B + MF(t)H_B) \\ * & \Theta_2 \end{pmatrix} \\
 &= \begin{pmatrix} \Theta_1 & PB + PMF(t)H_B \\ * & \Theta_2 \end{pmatrix} \\
 &= \begin{pmatrix} \Theta_1 & PB \\ * & \Theta_2 \end{pmatrix} + e_1^T PMF(t)H_B e_2 + e_2^T H_B^T F^T(t)M^T P e_1.
 \end{aligned} \tag{38}$$

Lemma 4 implies that $\begin{pmatrix} \Theta_1 & PB \\ * & \Theta_2 \end{pmatrix} + e_1^T PMF(t)H_B e_2 + e_2^T H_B^T F^T(t)M^T P e_1 > 0$ if and only if there exists $\varepsilon_\Theta > 0$ such that

$$\begin{pmatrix} \Theta_1 & PB \\ * & \Theta_2 \end{pmatrix} - \varepsilon_\Theta^{-1} e_1^T P M M^T P e_1 - \varepsilon_\Theta e_2^T H_B^T F^T H_B e_2 > 0. \tag{39}$$

By the Schur complement theorem, $\begin{pmatrix} \Theta_1 & PB \\ * & \Theta_2 \end{pmatrix} - \varepsilon_{\Theta}^{-1} e_1^T P M M^T P e_1 - \varepsilon_{\Theta} e_2^T H_B^T H_B e_2 > 0$ if and only if

$$\begin{pmatrix} \begin{pmatrix} \Theta_1 & PB \\ * & \Theta_2 \end{pmatrix} - \varepsilon_{\Theta} e_2^T H_B^T H_B e_2 & e_1^T P M \\ * & \varepsilon_{\Theta} I \end{pmatrix} = \begin{pmatrix} \Theta_1 & PB & P M \\ * & \Theta_2 - \varepsilon_{\Theta} H_B^T H_B & O \\ * & * & \varepsilon_{\Theta} I_n \end{pmatrix} > 0. \tag{40}$$

So $\begin{pmatrix} \Theta_1 & P(B+MF(t)H_B) \\ * & \Theta_2 \end{pmatrix} > 0$ is equivalent to

$$\begin{pmatrix} \Theta_1 & PB & P M \\ * & \Theta_2 - \varepsilon_{\Theta} H_B^T H_B & O \\ * & * & \varepsilon_{\Theta} I_n \end{pmatrix} > 0 \tag{41}$$

for some $\varepsilon_{\Theta} > 0$.

Similarly, $\begin{pmatrix} \Delta_1 & [K^+ D_1 - K^- D_2 + K D_3 + K D_4] (B+MF(t)H_B) \\ * & \Delta_2 \end{pmatrix} > 0$ if and only if

$$\begin{pmatrix} \Delta_1 & [K^+ D_1 - K^- D_2 + K D_3 + K D_4] B & [K^+ D_1 - K^- D_2 + K D_3 + K D_4] M \\ * & \Delta_2 - \varepsilon_{\Delta} H_B^T H_B & O \\ * & * & \varepsilon_{\Delta} I_n \end{pmatrix} > 0 \tag{42}$$

for some $\varepsilon_{\Delta} > 0$;

$\begin{pmatrix} \Gamma_1 & [D_1 - D_2 + D_3 - D_4] (C+MF(t)H_C) \\ * & \Gamma_2 \end{pmatrix} > 0$ if and only if

$$\begin{pmatrix} \Gamma_1 & [D_1 - D_2 + D_3 - D_4] C & [D_1 - D_2 + D_3 - D_4] M \\ * & \Gamma_2 - \varepsilon_3 H_C^T H_C & O \\ * & * & \varepsilon_3 I_n \end{pmatrix} > 0 \tag{43}$$

for some $\varepsilon_3 > 0$;

$\begin{pmatrix} \Lambda_1 & [-D_1 + D_2 - D_3 + D_4] (B+MF(t)H_B) \\ * & \Lambda_2 \end{pmatrix} > 0$ if and only if

$$\begin{pmatrix} \Lambda_1 & [-D_1 + D_2 - D_3 + D_4] B & [-D_1 + D_2 - D_3 + D_4] M \\ * & \Lambda_2 - \varepsilon_4 H_B^T H_B & O \\ * & * & \varepsilon_4 I_n \end{pmatrix} > 0 \tag{44}$$

for some $\varepsilon_4 > 0$; and

$\begin{pmatrix} \Phi_1 & [-D_1 + D_2 - D_3 + D_4] (B+MF(t)H_B) \\ * & \Phi_2 \end{pmatrix} > 0$ if and only if

$$\begin{pmatrix} \Phi_1 & [-D_1 + D_2 - D_3 + D_4] B & [-D_1 + D_2 - D_3 + D_4] M \\ * & \Phi_2 - \varepsilon_5 H_B^T H_B & O \\ * & * & \varepsilon_5 I_n \end{pmatrix} > 0 \tag{45}$$

for some $\varepsilon_5 > 0$.

Moreover, for Π^* , we have

$$\begin{aligned}
 \Pi^* &= \Pi - I_N \otimes [PMF(t)H_C + (MF(t)H_C)^T P] \\
 &\quad - I_N \otimes [(K^+D_1 - K^-D_2 + KD_3 + KD_4)MF(t)H_C \\
 &\quad + (MF(t)H_C)^T (K^+D_1 - K^-D_2 + KD_3 + KD_4)] \\
 &= \Pi - [I_N \otimes (PM)][I_N \otimes F(t)][I_N \otimes H_C] \\
 &\quad - [I_N \otimes H_C^T][I_N \otimes F^T(t)][I_N \otimes (PM)^T] \\
 &\quad - [I_N \otimes ((K^+D_1 - K^-D_2 + KD_3 + KD_4)M)][I_N \otimes F(t)][I_N \otimes H_C] \\
 &\quad - [I_N \otimes H_C^T][I_N \otimes F^T(t)][I_N \otimes ((K^+D_1 - K^-D_2 + KD_3 + KD_4)M)^T],
 \end{aligned} \tag{46}$$

where $[I_N \otimes F^T(t)][I_N \otimes F(t)] \leq I_{Nn}$.

Thus by the Schur complement theorem, $\Pi^* < 0$ is equivalent to

$$\begin{pmatrix}
 \Pi + \varepsilon_6 \Pi_1 + \varepsilon_7 \Pi_2 & I_N \otimes (PM) & I_N \otimes [(K^+D_1 - K^-D_2 + KD_3 + KD_4)M] \\
 * & -\varepsilon_6 I_{Nn} & O \\
 * & * & -\varepsilon_7 I_{Nn}
 \end{pmatrix} < 0 \tag{47}$$

for some $\varepsilon_6 > 0$ and $\varepsilon_7 > 0$, where $\Pi_1 = \Pi_2 = [I_N \otimes H_C^T][I_N \otimes H_C]$.

By Lemma 3 FCNNs (32) realize synchronization under condition (36).

The proof is completed. □

3.3 Synchronization analysis for FCNNs with unsymmetrical coupled matrix

In this section, we consider FCNNs (9) with unsymmetrical coupled matrix, that is, $A = (A_{ij})_{N \times N}$ is an unsymmetrical coupled matrix representing the coupling strength and topological structure of the networks.

Let $\bar{y}(t) = \frac{1}{N} \sum_{i=1}^N y_i(t)$ and define the error vector $e_i(t) = y_i(t) - \bar{y}(t)$. From FCNNs (9) we have

$$\begin{aligned}
 D^\alpha e_i(t) &= -C e_i(t) + B g(y_i(t)) - \frac{1}{N} \sum_{i=1}^N B g(y_i(t)) \\
 &\quad + d \sum_{j=1}^N \left(A_{ij} - \frac{1}{N} \sum_{i=1}^N A_{ij} \right) \Gamma y_j(t), \quad i = 1, 2, \dots, N.
 \end{aligned} \tag{48}$$

The synchronization of FCNNs (9) with unsymmetrical coupled matrix is equivalent to the stability of system (48).

Theorem 3 *FCNNs (9) with unsymmetrical coupled matrix realize synchronization in the sense of Lyapunov if there exist positive definite matrices $P \in R^{n \times n}$, positive definite diagonal matrices D_i ($i = 1, 2, 3, 4$), positive semidefinite matrices $\Theta_i, \Delta_i, \Gamma_i, \Lambda_i, \Phi_i, \Psi_i \in R^{n \times n}$, $i =$*

1, 2, and matrices $\Omega_1 \in R^{Nn \times Nn}$ such that the following linear matrix inequalities hold:

$$\begin{aligned} \Pi \leq 0, \quad & \begin{pmatrix} \Theta_1 & PB \\ * & \Theta_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} \Delta_1 & [K^+D_1 - K^-D_2 + KD_3 + KD_4]B \\ * & \Delta_2 \end{pmatrix} \geq 0, \\ & \begin{pmatrix} \Gamma_1 & [D_1 - D_2 + D_3 - D_4]C \\ * & \Gamma_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} \Lambda_1 & [-D_1 + D_2 - D_3 + D_4]B \\ * & \Lambda_2 \end{pmatrix} \geq 0, \quad (49) \\ & \begin{pmatrix} \Phi_1 & [-D_1 + D_2 - D_3 + D_4]B \\ * & \Phi_2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} \Psi_1 & -D_1 + D_2 - D_3 + D_4 \\ * & \Psi_2 \end{pmatrix} \geq 0, \end{aligned}$$

where $\Pi = (I_{Nn} - \frac{E_N \otimes I_n}{N})[-I_N \otimes (PC + C^T P) + I_N \otimes \Theta_1 + I_N \otimes (K\Theta_2 K) - I_N \otimes ((K^+D_1 - K^-D_2 + KD_3 + KD_4)C + C^T(K^+D_1 - K^-D_2 + KD_3 + KD_4)) + I_N \otimes \Delta_1 + I_N \otimes (K\Delta_2 K) + I_N \otimes (K\Gamma_1 K) + I_N \otimes \Gamma_2 + I_N \otimes (K\Lambda_1 K) + I_N \otimes (K\Lambda_2 K) + I_N \otimes (K\Phi_1 K) + I_N \otimes (K\Phi_2 K) + NI_N \otimes (K\Psi_1 K)](I_{Nn} - \frac{E_N \otimes I_n}{N}) + \text{sym}\{\Omega_1(E_N \otimes I_n)\}(I_{Nn} - \frac{E_N \otimes I_n}{N}) + \text{sym}\{d(I_{Nn} - \frac{E_N \otimes I_n}{N})[\underline{A} \otimes ((P + K^+D_1 - K^-D_2 + KD_3 + KD_4)\Gamma)] + d^2 \underline{A}^* \otimes (\Gamma^T \Psi_2 \Gamma), \underline{A} = (A_{ij} - A_{\bar{j}})_{N \times N}, A_{\bar{j}} = \frac{1}{N} \sum_{i=1}^N A_{ij}$, and $\underline{A}^* = \text{diag}\{\sum_{i=1}^N (A_{i1} - A_{\bar{1}})^2, \sum_{i=1}^N (A_{i2} - A_{\bar{2}})^2, \dots, \sum_{i=1}^N (A_{iN} - A_{\bar{N}})^2\}$.

Proof Construct the Lyapunov function

$$V(e(t)) = V_1(e(t)) + V_2(e(t)), \tag{50}$$

where

$$\begin{aligned} V_1(e(t)) &= \sum_{i=1}^N e_i^T(t) P e_i(t), \\ V_2(e(t)) &= 2 \sum_{i=1}^N \sum_{j=1}^n \int_0^{e_{ij}(t)} d_{1j} [k_j^+ s - g_j(s)] ds \\ &\quad + 2 \sum_{i=1}^N \sum_{j=1}^n \int_0^{e_{ij}(t)} d_{2j} [g_j(s) - k_j^- s] ds \\ &\quad + 2 \sum_{i=1}^N \sum_{j=1}^n \int_0^{e_{ij}(t)} d_{3j} [k_j s - g_j(s)] ds \\ &\quad + 2 \sum_{i=1}^N \sum_{j=1}^n \int_0^{e_{ij}(t)} d_{4j} [g_j(s) + k_j s] ds. \end{aligned} \tag{51}$$

Proposition 1 implies

$$\begin{aligned} D^\alpha V_1(e(t)) &\leq 2 \sum_{i=1}^N e_i^T(t) P D^\alpha e_i(t) \\ &= 2 \sum_{i=1}^N e_i^T(t) P \left[-C e_i(t) + B g(y_i(t)) - B g(\bar{y}(t)) \right. \\ &\quad \left. + B g(\bar{y}(t)) - \frac{1}{N} \sum_{i=1}^N B g(y_i(t)) + d \sum_{j=1}^N (A_{ij} - A_{\bar{j}}) \Gamma y_j(t) \right], \end{aligned} \tag{52}$$

where $A_{\bar{j}} = \frac{1}{N} \sum_{i=1}^N A_{ij}$.

Because $\sum_{i=1}^N e_i^T(t) = 0$, we get

$$\begin{aligned}
 D^\alpha V_1(e(t)) &\leq 2 \sum_{i=1}^N e_i^T(t) P \left[-Ce_i(t) + Bg(y_i(t)) - Bg(\bar{y}(t)) \right. \\
 &\quad \left. + d \sum_{j=1}^N (A_{ij} - A_{j\bar{i}}) \Gamma y_j(t) \right] \\
 &= -2 \sum_{i=1}^N e_i^T(t) PCe_i(t) + 2 \sum_{i=1}^N e_i^T(t) PB(g(y_i(t)) - g(\bar{y}(t))) \\
 &\quad + 2d \sum_{i=1}^N \sum_{j=1}^N e_i^T(t) P(A_{ij} - A_{j\bar{i}}) \Gamma y_j(t).
 \end{aligned} \tag{53}$$

By Lemma 2 there exist a positive definite matrix Θ_1 and a positive semidefinite matrix Θ_2 such that

$$2 \sum_{i=1}^N e_i^T(t) PB(g(y_i(t)) - g(\bar{y}(t))) \leq \sum_{i=1}^N e_i^T(t) \Theta_1 e_i(t) + \sum_{i=1}^N e_i^T(t) K \Theta_2 K e_i(t) \tag{54}$$

and

$$\begin{pmatrix} \Theta_1 & PB \\ * & \Theta_2 \end{pmatrix} \geq 0.$$

So

$$\begin{aligned}
 D^\alpha V_1(e(t)) &\leq -2 \sum_{i=1}^N e_i^T(t) PCe_i(t) + \sum_{i=1}^N e_i^T(t) \Theta_1 e_i(t) + \sum_{i=1}^N e_i^T(t) K \Theta_2 K e_i(t) \\
 &\quad + 2d \sum_{i=1}^N \sum_{j=1}^N e_i^T(t) P(A_{ij} - A_{j\bar{i}}) \Gamma y_j(t) \\
 &= e^T(t) \left[-I_N \otimes (PC + C^T P) + I_N \otimes \Theta_1 + I_N \otimes (K \Theta_2 K) \right] e(t) \\
 &\quad + 2d e^T(t) \underline{A} \otimes (P \Gamma) y(t),
 \end{aligned} \tag{55}$$

where $\underline{A} = (A_{ij} - A_{j\bar{i}})_{N \times N}$.

Noting that $e(t) = (I_{Nn} - \frac{E_N \otimes I_n}{N}) y(t)$, we have

$$\begin{aligned}
 D^\alpha V_1(e(t)) &\leq y^T(t) \left\{ \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \left[-I_N \otimes (PC + C^T P) \right. \right. \\
 &\quad \left. \left. + I_N \otimes \Theta_1 + I_N \otimes (K \Theta_2 K) \right] \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \right. \\
 &\quad \left. + 2d \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \left[\underline{A} \otimes (P \Gamma) \right] \right\} y(t).
 \end{aligned} \tag{56}$$

For $V_2(e(t))$, by Proposition 3 we have

$$\begin{aligned}
 D^\alpha V_2(e(t)) \leq & 2 \sum_{i=1}^N e_i^T(t) [K^+ D_1 - K^- D_2 + K D_3 + K D_4] \left[-C e_i(t) \right. \\
 & \left. + B g(y_i(t)) - \frac{1}{N} \sum_{i=1}^N B g(y_i(t)) + d \sum_{j=1}^N (A_{ij} - A_j) \Gamma y_j(t) \right] \\
 & + 2 \sum_{i=1}^N g^T(e_i(t)) [-D_1 + D_2 - D_3 + D_4] \left[-C e_i(t) + B g(y_i(t)) \right. \\
 & \left. - \frac{1}{N} \sum_{i=1}^N B g(y_i(t)) + d \sum_{j=1}^N (A_{ij} - A_j) \Gamma y_j(t) \right], \tag{57}
 \end{aligned}$$

where $D_k = \text{diag}\{d_{k1}, d_{k2}, \dots, d_{kn}\}$, $k = 1, 2, 3, 4$.

Similarly, there are a positive definite matrix Δ_1 and a positive semi-definite matrix Δ_2 such that

$$\begin{aligned}
 & 2 \sum_{i=1}^N e_i^T(t) [K^+ D_1 - K^- D_2 + K D_3 + K D_4] \left[-C e_i(t) + B g(y_i(t)) \right. \\
 & \left. - \frac{1}{N} \sum_{i=1}^N B g(y_i(t)) + d \sum_{j=1}^N (A_{ij} - A_j) \Gamma y_j(t) \right] \\
 & \leq -2 \sum_{i=1}^N e_i^T(t) [K^+ D_1 - K^- D_2 + K D_3 + K D_4] C e_i(t) \\
 & \quad + \sum_{i=1}^N e_i^T(t) \Delta_1 e_i(t) + \sum_{i=1}^N e_i^T(t) K \Delta_2 K e_i(t) \\
 & \quad + 2d \sum_{i=1}^N \sum_{j=1}^N e_i^T(t) [K^+ D_1 - K^- D_2 + K D_3 + K D_4] (A_{ij} - A_j) \Gamma y_j(t) \\
 & = y^T(t) \left\{ \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) [-I_N \otimes ((K^+ D_1 - K^- D_2 + K D_3 + K D_4) C \right. \\
 & \quad + C^T (K^+ D_1 - K^- D_2 + K D_3 + K D_4)) \\
 & \quad + I_N \otimes \Delta_1 + I_N \otimes (K \Delta_2 K)] \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \right. \\
 & \quad \left. + 2d \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) [A \otimes ((K^+ D_1 - K^- D_2 + K D_3 + K D_4) \Gamma)] \right\} y(t) \tag{58}
 \end{aligned}$$

and

$$\begin{pmatrix} \Delta_1 & [K^+ D_1 - K^- D_2 + K D_3 + K D_4] B \\ * & \Delta_2 \end{pmatrix} \geq 0.$$

Simple calculation yields

$$\begin{aligned}
 & 2 \sum_{i=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4] \left[-Ce_i(t) + Bg(y_i(t)) \right. \\
 & \quad \left. - \frac{1}{N} \sum_{i=1}^N Bg(y_i(t)) + d \sum_{j=1}^N (A_{ij} - A_j)\Gamma y_j(t) \right] \\
 & = 2 \sum_{i=1}^N g^T(e_i(t))[D_1 - D_2 + D_3 - D_4]Ce_i(t) \\
 & \quad + 2 \sum_{i=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(y_i(t)) - g(\bar{y}(t))] \\
 & \quad + \frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(\bar{y}(t)) - g(y_j(t))] \\
 & \quad + 2d \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4](A_{ij} - A_j)\Gamma y_j(t).
 \end{aligned} \tag{59}$$

There exist a positive definite matrix Γ_1 and a positive semidefinite matrix Γ_2 such that

$$\begin{aligned}
 & 2 \sum_{i=1}^N g^T(e_i(t))[D_1 - D_2 + D_3 - D_4]Ce_i(t) \\
 & \leq \sum_{i=1}^N e_i^T(t)K\Gamma_1Ke_i(t) + \sum_{i=1}^N e_i^T(t)\Gamma_2e_i(t) \\
 & = y^T(t) \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) [I_N \otimes (K\Gamma_1K) + I_N \otimes \Gamma_2] \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) y(t)
 \end{aligned} \tag{60}$$

and

$$\begin{pmatrix} \Gamma_1 & [D_1 - D_2 + D_3 - D_4]C \\ * & \Gamma_2 \end{pmatrix} \geq 0.$$

Similarly, we have a positive definite matrix Λ_1 and a positive semidefinite matrix Λ_2 such that

$$\begin{aligned}
 & 2 \sum_{i=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(y_i(t)) - g(\bar{y}(t))] \\
 & \leq \sum_{i=1}^N e_i^T(t)K\Lambda_1Ke_i(t) + \sum_{i=1}^N e_i^T(t)K\Lambda_2Ke_i(t) \\
 & = y^T(t) \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) [I_N \otimes (K\Lambda_1K) \\
 & \quad + I_N \otimes (K\Lambda_2K)] \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) y(t)
 \end{aligned} \tag{61}$$

and

$$\begin{pmatrix} \Lambda_1 & [-D_1 + D_2 - D_3 + D_4]B \\ * & \Lambda_2 \end{pmatrix} \geq 0.$$

By Lemma 2 we have a positive definite matrix Φ_1 and a positive semidefinite matrix Φ_2 satisfying the matrix inequalities

$$\begin{aligned} & \frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4]B[g(\bar{y}(t)) - g(y_j(t))] \\ & \leq \sum_{i=1}^N e_i^T(t)K\Phi_1Ke_i(t) + \sum_{i=1}^N e_i^T(t)K\Phi_2Ke_i(t) \tag{62} \\ & = y^T(t) \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) [I_N \otimes (K\Phi_1K) + I_N \otimes (K\Phi_2K)] \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) y(t) \end{aligned}$$

and

$$\begin{pmatrix} \Phi_1 & [-D_1 + D_2 - D_3 + D_4]B \\ * & \Phi_2 \end{pmatrix} \geq 0,$$

where Φ_1 and Φ_2 are diagonal matrices.

For $2d \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4](A_{ij} - A_j)\Gamma y_j(t)$, by Lemma 2 there exist a positive definite matrix Ψ_1 and a positive semidefinite matrix Ψ_2 such that

$$\begin{aligned} & 2d \sum_{i=1}^N \sum_{j=1}^N g^T(e_i(t))[-D_1 + D_2 - D_3 + D_4](A_{ij} - A_j)\Gamma y_j(t) \\ & \leq N \sum_{i=1}^N e_i^T(t)K\Psi_1Ke_i(t) + d^2 \sum_{i=1}^N \sum_{j=1}^N [(A_{ij} - A_j)\Gamma y_j(t)]^T \Psi_2 (A_{ij} - A_j)\Gamma y_j(t) \tag{63} \\ & = y^T(t) \left\{ \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) [NI_N \otimes (K\Psi_1K)] \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \right. \\ & \quad \left. + d^2 \underline{A}^* \otimes (\Gamma^T \Psi_2 \Gamma) \right\} y(t) \end{aligned}$$

and

$$\begin{pmatrix} \Psi_1 & -D_1 + D_2 - D_3 + D_4 \\ * & \Psi_2 \end{pmatrix} \geq 0,$$

where $\underline{A}^* = \text{diag}\{\sum_{i=1}^N (A_{i1} - A_1)^2, \sum_{i=1}^N (A_{i2} - A_2)^2, \dots, \sum_{i=1}^N (A_{iN} - A_N)^2\}$.

Combining (58) with (60)–(63) gives

$$\begin{aligned}
 D^\alpha V_2(e(t)) \leq & y^T(t) \left\{ \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \left[-I_N \otimes ((K^+ D_1 - K^- D_2 + K D_3 + K D_4) C \right. \right. \\
 & + C^T (K^+ D_1 - K^- D_2 + K D_3 + K D_4)) + I_N \otimes \Delta_1 \\
 & + I_N \otimes (K \Delta_2 K) + I_N \otimes (K \Gamma_1 K) + I_N \otimes \Gamma_2 + I_N \otimes (K \Lambda_1 K) \\
 & + I_N \otimes (K \Lambda_2 K) + I_N \otimes (K \Phi_1 K) + I_N \otimes (K \Phi_2 K) \\
 & + N I_N \otimes (K \Psi_1 K) \left. \right] \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \\
 & + 2d \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \left[\underline{A} \otimes ((K^+ D_1 - K^- D_2 + K D_3 + K D_4) \Gamma) \right] \\
 & \left. + [\underline{A}^* \otimes (\Gamma^T \Psi_2 \Gamma)] \right\} y(t).
 \end{aligned} \tag{64}$$

Since $\sum_{i=1}^N e_i^T(t) = 0$, we have $2y^T(t) \Omega_1 (E_N \otimes I_n) (I_{Nn} - \frac{E_N \otimes I_n}{N}) y(t) = 0$ for any matrix $\Omega_1 \in \mathbb{R}^{Nn \times Nn}$. Then

$$\begin{aligned}
 D^\alpha V(e(t)) \leq & y^T(t) \left\{ \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \left[-I_N \otimes (PC + C^T P) + I_N \otimes \Theta_1 \right. \right. \\
 & + I_N \otimes (K \Theta_2 K) - I_N \otimes ((K^+ D_1 - K^- D_2 + K D_3 + K D_4) C \\
 & + C^T (K^+ D_1 - K^- D_2 + K D_3 + K D_4)) + I_N \otimes \Delta_1 \\
 & + I_N \otimes (K \Delta_2 K) + I_N \otimes (K \Gamma_1 K) + I_N \otimes \Gamma_2 + I_N \otimes (K \Lambda_1 K) \\
 & + I_N \otimes (K \Lambda_2 K) + I_N \otimes (K \Phi_1 K) + I_N \otimes (K \Phi_2 K) \\
 & + N I_N \otimes (K \Psi_1 K) \left. \right] \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \\
 & + 2\Omega_1 (E_N \otimes I_n) \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \\
 & + 2d \left(I_{Nn} - \frac{E_N \otimes I_n}{N} \right) \left[\underline{A} \otimes ((P + K^+ D_1 - K^- D_2 + K D_3 + K D_4) \Gamma) \right] \\
 & \left. + d^2 \underline{A}^* \otimes (\Gamma^T \Psi_2 \Gamma) \right\} y(t) \\
 = & y^T(t) \Pi y(t).
 \end{aligned} \tag{65}$$

So $D^\alpha V(e(t)) \leq 0$ if $y^T(t) \Pi y(t) \leq 0$. Therefore by Lemma 3 FCNNs (9) with unsymmetrical coupled matrix realize synchronization in the sense of Lyapunov under condition (49). The proof is completed. \square

The Lyapunov synchronization of the FCNNs with unsymmetrical coupled matrix in Theorem 3 is weaker than the Mittag-Leffler synchronization.

Remark 2 Each sufficient synchronization condition proposed in Theorems 1–3 includes several LMIs. The forms of sufficient synchronization conditions seem to be complicated, but they can be easily solved by Matlab. The sufficient conditions in Theorems 1–3 require

$(0.5N^2 + 2.5)n^2 + (14.5 + 0.5N)n$, $(0.5N^2 + 2.5)n^2 + (14.5 + 0.5N)n + 7$ and $(0.5N^2 + 2.5)n^2 + (14.5 + 0.5N)n$ decision variables, respectively, where N stands for the number of nodes, and n is the dimension of the state vector for each node.

4 Numerical examples

In this section, we provide four numerical examples to confirm the correctness of the obtained synchronization criteria.

Example 1 Consider FCNNs (9) consisting of 5 identical 2-D fractional-order neural networks, where $\alpha = 0.97, f_j(\xi) = \frac{|\xi+1|-|\xi-1|}{4}$ ($j = 1, 2$), $d = 0.7$,

$$\begin{aligned}
 C &= \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}, & B &= \begin{pmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{pmatrix}, & \Gamma &= \begin{pmatrix} 0.5 & 0.6 \\ 0.7 & 0.3 \end{pmatrix}, \\
 J(t) &= \begin{pmatrix} 0.1 \\ 0.05 \end{pmatrix}, & A &= \begin{pmatrix} -0.4 & 0.2 & 0.1 & 0 & 0.1 \\ 0.2 & -0.5 & 0 & 0.3 & 0 \\ 0.1 & 0 & -0.5 & 0.3 & 0.1 \\ 0 & 0.3 & 0.3 & -0.6 & 0 \\ 0.1 & 0 & 0.1 & 0 & -0.2 \end{pmatrix}.
 \end{aligned} \tag{66}$$

It is clear that f_j satisfy Assumption 1 with $k_j^- = -0.5$ and $k_j^+ = 0.5$. Solving LMIs in Theorem 1 by using LMI tools in Matlab, we get $t_{\min} = -0.3441$. So the FCNNs in this example realize synchronization.

Take $y_{11}(0) = 0.2, y_{12}(0) = -0.1, y_{21}(0) = 0.15, y_{22}(0) = -0.3, y_{31}(0) = -0.26, y_{32}(0) = 0.16, y_{41}(0) = 0.05, y_{42}(0) = -0.07, y_{51}(0) = 0.1,$ and $y_{52}(0) = -0.19$. The trajectories of the state and error systems are shown in Figs. 1 and 2, respectively. Figure 2 shows that FCNNs (9) with coefficient (66) achieve synchronization.

Remark 3 This example was studied in [9]. The fractional-order neural network is synchronized under a pinning controller $U(t) = Ke(t)$ with $K = \text{diag}\{0.8I, 1.6I, O, O, O\}$. However, the synchronization criteria are in the form of NLMIs, which brings certain difficulty in the solving process. Differently from the method used in [9], Lemma 2 is adopted to get a synchronization criterion in the form of LMIs. Moreover, a novel convex Lyapunov function V_2 is constructed to take into account the information of the activation functions.

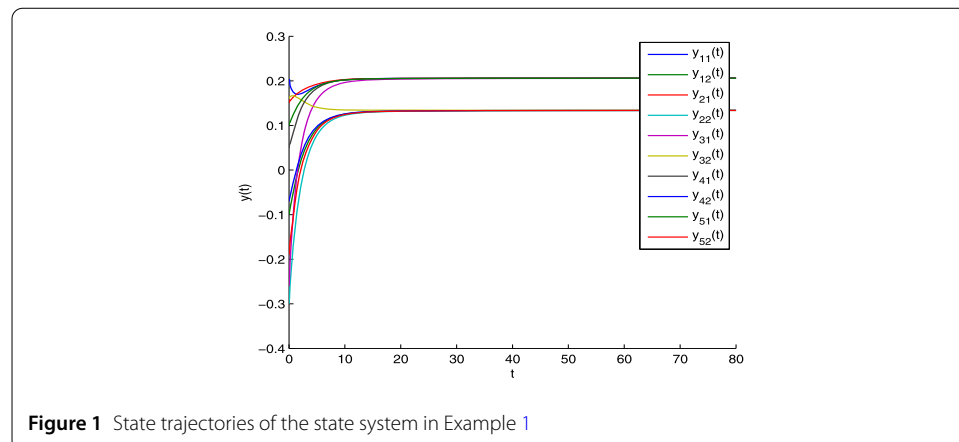
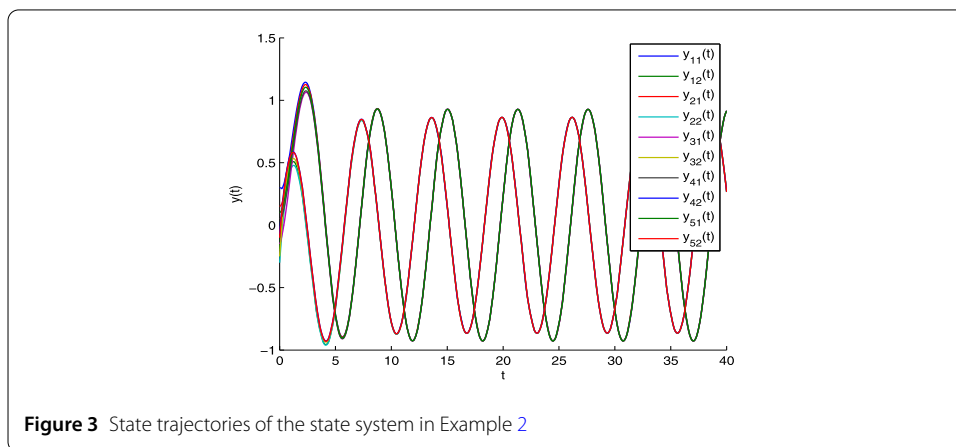
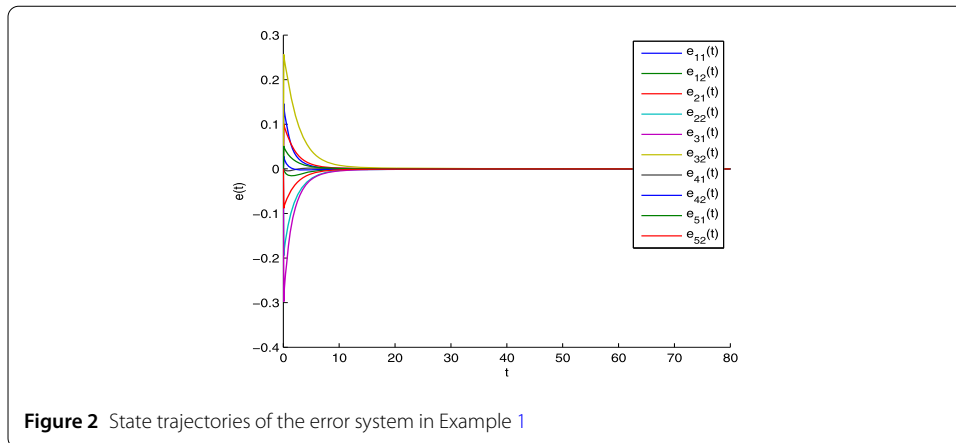


Figure 1 State trajectories of the state system in Example 1

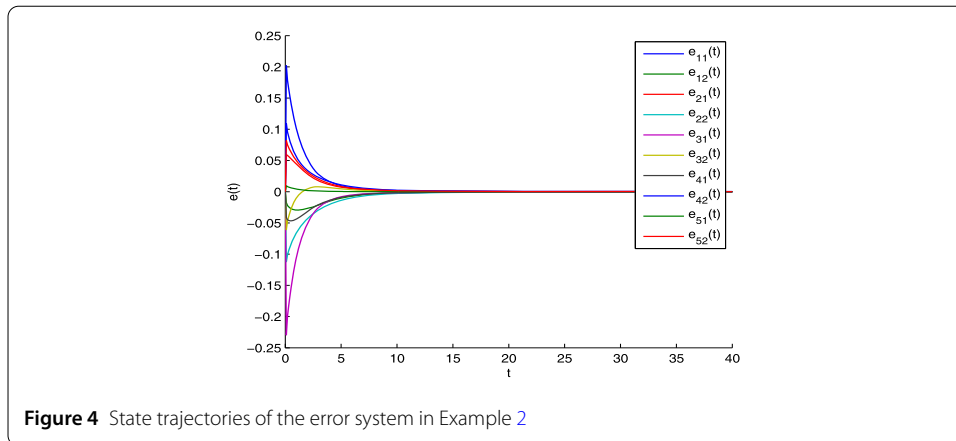


Example 2 Consider FCNNs (9) with uncertain parameters consisting of 5 identical 2-D fractional-order neural networks, where $\alpha = 0.97, f_j(\xi) = \frac{|\xi+1|-|\xi-1|}{4}$ ($j = 1, 2$), $d = 0.7$,

$$\begin{aligned}
 C &= \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}, & B &= \begin{pmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{pmatrix}, & \Gamma &= \begin{pmatrix} 0.5 & 0.6 \\ 0.7 & 0.3 \end{pmatrix}, \\
 J(t) &= \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, & A &= \begin{pmatrix} -0.4 & 0.2 & 0.1 & 0 & 0.1 \\ 0.2 & -0.5 & 0 & 0.3 & 0 \\ 0.1 & 0 & -0.5 & 0.3 & 0.1 \\ 0 & 0.3 & 0.3 & -0.6 & 0 \\ 0.1 & 0 & 0.1 & 0 & -0.2 \end{pmatrix}, & & (67) \\
 M &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, & H_C &= \begin{pmatrix} 0.05 & 0 \\ 0 & -0.05 \end{pmatrix}, & H_B &= \begin{pmatrix} -0.03 & 0 \\ 0 & 0.03 \end{pmatrix}.
 \end{aligned}$$

It is clear that f_j satisfy Assumption 1 with $k_j^- = -0.5$ and $k_j^+ = 0.5$. Solving LMIs in Theorem 2 by LMI tools in Matlab, we get $t_{\min} = -0.2642$. So FCNNs with uncertain parameters realize synchronization.

Take $y_{11}(0) = 0.3, y_{12}(0) = -0.2, y_{21}(0) = 0.15, y_{22}(0) = -0.3, y_{31}(0) = -0.15, y_{32}(0) = -0.25, y_{41}(0) = 0.05, y_{42}(0) = -0.07, y_{51}(0) = 0.1$, and $y_{52}(0) = -0.1$. The trajectories of the state



system are shown in Fig. 3. The trajectories of the error system are shown in Fig. 4. From Fig. 4 we can see that FCNNs with coefficient (67) are synchronized.

Example 3 Consider the FCNNs (9) with unsymmetrical coupled matrix consisting of 2 identical 2-D fractional-order neural networks, where $\alpha = 0.97, f_j(\xi) = \frac{|\xi+1|-|\xi-1|}{4}$ ($j = 1, 2$), $d = 0.5$,

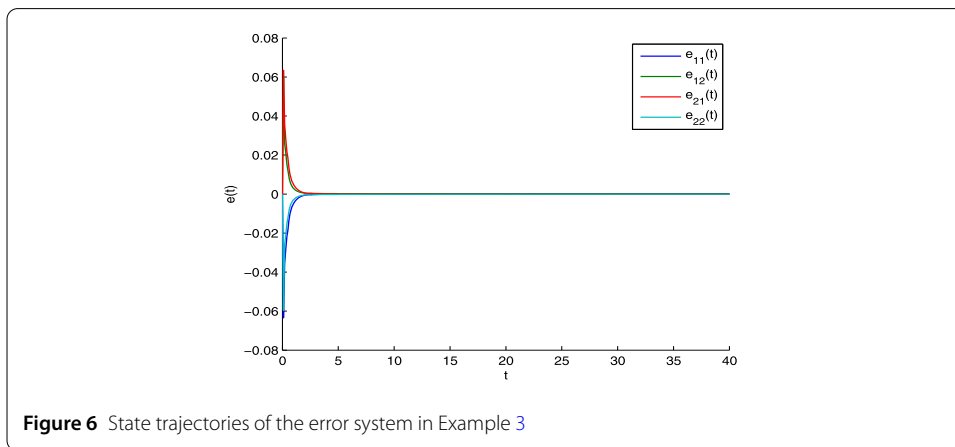
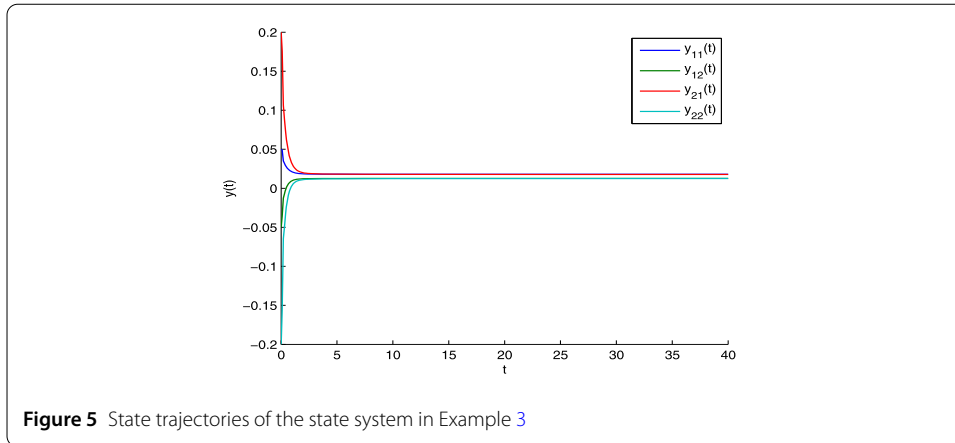
$$\begin{aligned}
 C &= \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, & B &= \begin{pmatrix} 0.5 & -0.1 \\ -0.2 & 0.4 \end{pmatrix}, & \Gamma &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
 A &= \begin{pmatrix} -0.1 & 0.1 \\ 0.3 & -0.4 \end{pmatrix}, & J(t) &= \begin{pmatrix} 0.05 \\ 0.05 \end{pmatrix}.
 \end{aligned} \tag{68}$$

It is clear that f_j satisfy Assumption 1 with $k_j^- = -0.5$ and $k_j^+ = 0.5$. Solving LMIs in Theorem 3 by YALMIP tools in Matlab, we get

$$\Pi = 1.0e-06 * \begin{pmatrix} -0.0793 & 0.0020 & 0.0803 & -0.0020 \\ 0.0020 & -0.1166 & -0.0020 & 0.1177 \\ 0.0803 & -0.0020 & -0.0812 & 0.0020 \\ -0.0020 & 0.1177 & 0.0020 & -0.1188 \end{pmatrix} \leq 0.$$

Since Π is negative semidefinite, FCNNs with uncertain parameters in this example realize synchronization in the sense of Lyapunov.

Take $y_{11}(0) = 0.05, y_{12}(0) = -0.05, y_{21}(0) = 0.2$, and $y_{22}(0) = -0.2$. The trajectories of the state system are shown in Fig. 5. The trajectories of the error system are shown in Fig. 6. Figure 6 shows that FCNNs with coefficient (68) are synchronized.



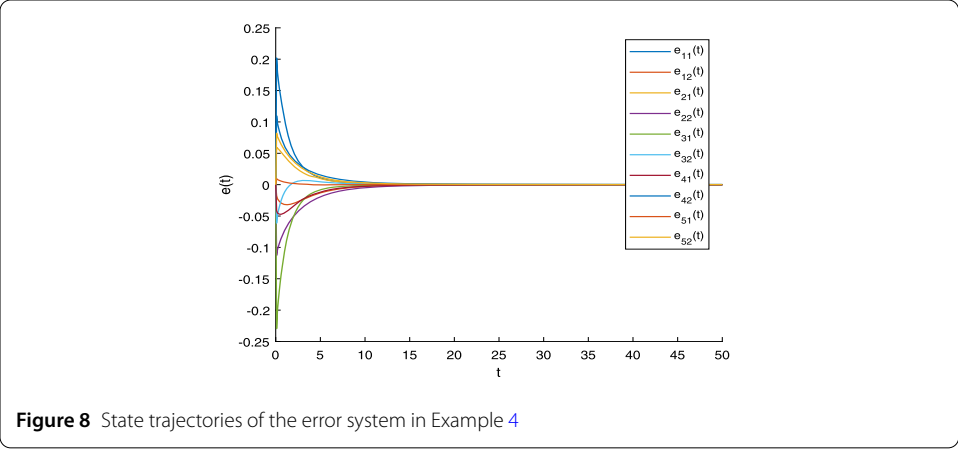
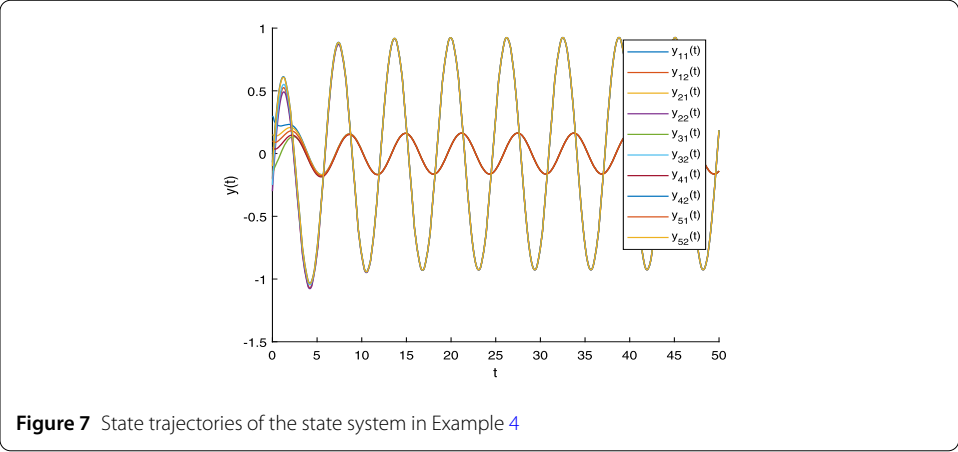
Example 4 Consider FCNNs (9) consisting of 5 identical 2-D fractional-order neural networks. Let $\alpha = 0.97, f_j(\xi) = \frac{|\xi+1|-|\xi-1|}{4}$ ($j = 1, 2$), $d = 0.7$,

$$\begin{aligned}
 C &= \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}, & B &= \begin{pmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{pmatrix}, & \Gamma &= \begin{pmatrix} 0.5 & 0.6 \\ 0.7 & 0.3 \end{pmatrix}, \\
 J(t) &= \begin{pmatrix} 0.1 \sin(t) \\ \cos(t) \end{pmatrix}, & A &= \begin{pmatrix} -0.4 & 0.2 & 0.1 & 0 & 0.1 \\ 0.2 & -0.5 & 0 & 0.3 & 0 \\ 0.1 & 0 & -0.5 & 0.3 & 0.1 \\ 0 & 0.3 & 0.3 & -0.6 & 0 \\ 0.1 & 0 & 0.1 & 0 & -0.2 \end{pmatrix}.
 \end{aligned} \tag{69}$$

It is clear that f_j satisfy Assumption 1 with $k_j^- = -0.5$ and $k_j^+ = 0.5$.

Since all conditions in Theorem 1 are satisfied, FCNNs in this example can realize synchronization.

The trajectories of the state system are shown in Fig. 7. From Fig. 7 we see that FCNN system is unstable. The trajectories of the error system are shown in Fig. 8. Figure 8 shows that FCNNs with coefficient (69) realize synchronization.



5 Conclusions

In this paper, we study the synchronization of FCNNs. We construct some novel convex Lyapunov functions containing the activation function information, based on which, we present several novel Mittag-Leffler synchronization criteria for FCNNs with and without uncertain parameters. Then we establish a novel synchronization criterion in the sense of Lyapunov for FCNNs with unsymmetrical coupled matrix. The benefits of the synchronization criteria obtained in this paper are illustrated by four numerical examples. The global synchronization for stochastic dynamic networks [27, 28] and the synchronization in fixed or finite time for fractional-order network [29, 30] have attracted considerable attention in the past few decades. In the future, we will study the global synchronization of fractional-order stochastic neural networks basing on the event-triggered strategy.

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Availability of data and materials

All the data and materials are available in this paper.

Competing interests

The authors declare that they have no competing interests.

Consent for publication

This manuscript is approved by all authors for publication. I declare on behalf of my coauthors that the work described was original research that has not been published previously.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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