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On nontrivial solutions of nonlinear Schrödinger equations with sign-changing potential

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Abstract

In this paper, we consider the superlinear Schrödinger equation with bounded potential well. The potential here is allowed to be sign-changing. Without assuming the Ambrosetti–Rabinowitz-type condition, we prove the existence of a nontrivial solution and multiplicity results.

Keywords: Schrödinger equation; Superlinear; Potential well; Variational methods

1 Introduction and main results

This paper is concerned with the existence and multiplicity of nontrivial solutions for the superlinear Schrödinger equation of the form

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N), \quad N \geq 3. \end{cases} \tag{1.1}$$

With the aid of variational methods, problems of the form (1.1) have been extensively studied in the past decades. There are many works adopting various assumptions on V and f; see, for example, [1–13] and references therein.

Motivated by the above works, in this paper, we consider equation (1.1) with a sign-changing potential well. For the potential V, we assume:

(V) $V \in C(\mathbb{R}^N)$, $V(x) < V_\infty := \lim_{|x|\to\infty} V(x) < \infty$, $0 \notin \sigma(-\Delta + V)$, the spectrum of $-\Delta + V$.

Remark 1.1 Define the nondecreasing sequence of minimax values by

$$\lambda_n = \lim_{S\in\mathcal{S}_n} \sup_{u\in S\setminus\{\mathbf{0}\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x}{\int_{\mathbb{R}^N} u^2\,\mathrm{d}x}, \quad n \in \mathbb{N},$$

where $\mathcal{S}_n$ is a family of n-dimensional subspaces of $C_0^\infty(\mathbb{R}^N)$. We can see that $\sigma_{\mathrm{ess}}(-\Delta + V) \in (V_\infty, \infty)$ by (V), $\lambda_\infty := \lim_{k\to\infty} \lambda_n = \inf \sigma_{\mathrm{ess}}(-\Delta + V) < \infty$, and $\lambda_n \in \sigma_{pp}(-\Delta + V)$

whenever $\lambda_n < \lambda_\infty$, where $\sigma_{\mathrm{ess}}(-\Delta + V)$ denotes the essential spectrum of $-\Delta + V$, and $\sigma_{pp}(-\Delta + V)$ denotes the pure point spectrum of $-\Delta + V$ (see [14, 15] for details).

Besides (V), in [11, 12], it is also assumed that $\inf V > 0$, so that $\lambda_1 > 0$. Then the energy functional with respect to problem (1.1) has mountain pass geometry. In this work, we are interested in the case where the Schrödinger operator $-\Delta + V$ possesses a nontrivial negative space, which leads to more difficulty in verifying the compactness conditions. To the best of our knowledge, there are not many results in this case.

In this paper, we do not assume any compactness conditions on the potential function V. It is well known that the main difficulty in studying (1.1) in $\mathbb{R}^N$ is the lack of compactness. This difficulty can be avoided for (1.1) in bounded domains or if the potential function V possesses some compactness conditions. For example, if $\lim_{|x|\to\infty} V(x) = \infty$ or u is radially symmetric, we can get some compactness embedding, and then the Palias–Smale condition can be proved. We refer to [16] in this direction.

Denote $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$, $2^* := \frac{2N}{N-2}$, and $p' := \frac{p}{p-1}$, the conjugate exponent of p. We make the following assumptions on the nonlinearity f.

(f_1) $f \in C^1(\mathbb{R}^N \times \mathbb{R})$, and there exist constants $p \in (2, 2^*)$ and $c > 0$ such that

$$\bigl|f(x,t)\bigr| \le c\bigl(1 + |t|^{p-1}\bigr)$$

for $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

(f_2) $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^N$.

(f_3) $F(x,t)/t^2 \to \infty$ as $|t| \to \infty$ uniformly in $x \in \mathbb{R}^N$.

(f_4) $\lim_{|x|\to\infty} \sup_{|t|\le l} \frac{|f(x,t)|}{|t|} = 0$ for every $l > 0$.

(f_5) There exist $a, b > 0$ and $\alpha \in (0, \alpha_*)$ such that

$$0 < \left(2 + \frac{1}{a|t|^\alpha + b}\right)F(x,t) \le tf(x,t)$$

for $x \in \mathbb{R}^N$ and $t \ne 0$, where $\alpha_* = \min\{p', (2^* - 1)p' - 2^*\}$.

Then we have the following two results.

Theorem 1.2 *Under assumptions* (V) *and* (f_1)–(f_5), *problem* (1.1) *possesses at least one nontrivial solution.*

Theorem 1.3 *Under assumptions of Theorem* 1.2, *if* $f(x,t)$ *is odd in* t, *then problem* (1.1) *possesses infinitely many solutions.*

Remark 1.4 To produce critical points of the variational functional of (1.1), we will eventually encounter the compactness problem. For this issue, we introduced assumption (f_4). It is easy to see that if $a : \mathbb{R}^N \to \mathbb{R}$ is continuous, $\lim_{|x|\to\infty} a(x) = 0$, and $p \in (2, 2^*)$, then

$$f(x,t) = a(x)|t|^{p-2}t$$

satisfies (f_1)–(f_5).

Remark 1.5 Most papers concerned with the superlinear Schrödinger equations involve the following classical condition of Ambrosetti and Rabinowitz:

(AR) There exists $\mu > 2$ such that $0 < \mu F(x,t) \le tf(x,t)$ for all $x \in \mathbb{R}^N$ and $t \neq 0$.
Condition (AR) plays a crucial role in proving the boundedness of Palias–Smale or Cerami sequences. Instead, we introduce a new condition (f_5), and we will illustrate a general technique to establish the boundedness of Cerami sequences. It is well known that many superlinear nonlinearities such as

$$f(x,t) = t\ln\bigl(1+|t|\bigr)$$

do not satisfy condition (AR). Note that $\frac{1}{a|t|^\alpha+b} \to 0$ as $|t| \to \infty$, which indicates that (f_5) is somewhat weaker than (AR). Note also that $(2^*-1)p' - 2^* > 0$ whenever $p < 2^*$. So the parameter $\alpha \in (0,\alpha_*)$ is available. It is also worth pointing out that (f_5) is not a superlinear condition. Indeed, there are asymptotically linear functions satisfying (f_5).

2 Preliminaries

We denote by $E := H^1(\mathbb{R}^N)$ the usual Sobolev space. Define the functional $\Phi : E \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N}\bigl(|\nabla u|^2 + V(x)u^2\bigr)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x.$$

Our assumptions on V and f stated above imply that the Schrödnger operator $-\Delta + V$ is selfadjoint and semibounded in $L^2(\mathbb{R}^N)$ and $\Phi \in C^1(E,\mathbb{R})$. A direct computation gives that, for all $u, v \in E$,

$$\bigl\langle \Phi'(u), v\bigr\rangle = \int_{\mathbb{R}^N}\bigl(\nabla u\nabla v + V(x)uv\bigr)\,\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x.$$

It is well known that the critical points of Φ are solutions of problem (1.1).

By (V) 0 is not an eigenvalue of $-\Delta + V$. If $\lambda_1 > 0$,we easily see that Φ has the mountain pass geometry. This case is simple, and we omit it here. In view of Remark 1.1, we arrange the eigenvalues (counted with multiplicity) of $-\Delta + V$ as

$$-\infty < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_\ell < 0 < \lambda_{\ell+1} \le \cdots < \lambda_\infty \tag{2.1}$$

and denote by e_j the corresponding eigenfunction of λ_j. Let $E^- = \operatorname{span}\{e_1,\dots,e_\ell\}$ and $E^+ = (E^-)^\perp$. From (V) we deduce that $E = E^- \oplus E^+$, where E^- and E^+ are the negative and positive eigenspaces of the operator $-\Delta + V$, and that $\dim E^- < \infty$. For $u, v \in E$, define

$$(u,v) = \int_{\mathbb{R}^N}\bigl(\nabla u^+\nabla v^+ + V(x)u^+v^+\bigr)\,\mathrm{d}x - \int_{\mathbb{R}^N}\bigl(\nabla u^-\nabla v^- + V(x)u^-v^-\bigr)\,\mathrm{d}x, \tag{2.2}$$

where $u = u^- + u^+$ with $u^- \in E^-$ and $u^+ \in E^+$. Then $(\cdot,\cdot)$ is an inner product on E. Therefore E is a Hilbert space with the norm $\|\cdot\| := \sqrt{(\cdot,\cdot)}$. We easily see that

$$\Phi(u) = \frac{1}{2}\bigl\|u^+\bigr\|^2 - \frac{1}{2}\bigl\|u^-\bigr\|^2 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \tag{2.3}$$

and

$$\bigl\langle \Phi'(u), v\bigr\rangle = \bigl(u^+,v^+\bigr) - \bigl(u^-,v^-\bigr) - \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x.$$

For any $s \in [2, 2^*]$, the imbedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is continuous. Consequently, there exists a constant $\tau_s > 0$ such that

$$|u|_s \le \tau_s \|u\|, \quad \forall u \in E, \tag{2.4}$$

where $|\cdot|_s$ denotes the L^s norm.

We next recall some abstract critical point theorems, which will be used in the proofs of our main results.

Definition 2.1 Let E be a Banach space, and let $\Phi \in C^1(E, \mathbb{R})$. Given $c \in \mathbb{R}$, a sequence $\{u_n\} \subset E$ is called a *Cerami sequence* of Φ at level c (shortly, a $(C)_c$ sequence) if

$$\Phi(u_n) \to c, \quad \bigl(1 + \|u_n\|\bigr)\bigl\|\Phi'(u_n)\bigr\| \to 0. \tag{2.5}$$

We say that Φ satisfies the *Cerami condition at level* c (shortly, condition $(C)_c$) if every $(C)_c$ sequence of Φ contains a convergent subsequence. If Φ satisfies condition $(C)_c$ for every $c \in \mathbb{R}$, then we say that Φ satisfies the *Cerami condition* (shortly, condition (C)).

Obviously, condition (C) is weaker than the Palais–Smale condition. However, as was shown in [17], the deformation theorem is still valid under the Cerami condition. Thus we have the following theorems.

Theorem 2.2 (Linking theorem [18]) *Let* $E = E^- \oplus E^+$ *be a Banach space with* $\dim E^- < \infty$. *Let* $R > r > 0$, *and let* $\phi \in E^+ \backslash \{0\}$. *Define*

$$M := \bigl\{u = u^- + \lambda\phi \mid u^- \in E^-, \|u\| \le R, \lambda \ge 0\bigr\}, \qquad N := \bigl\{u \in E^+ \mid \|u\| = r\bigr\}.$$

If $\Phi \in C^1(E, \mathbb{R})$ *satisfies condition* (C) *and*

$$\inf_N \Phi > \max_{\partial M} \Phi,$$

then Φ *has a nontrivial critical point*.

For the proofs of Theorems 1.2–1.3, we will use the following fountain theorem, which is a generalization of the classical fountain theorem of Bartsch [19] (see also [10]). For $k \in \mathbb{N}$, let

$$Y_k = \operatorname{span}\{e_1, \dots e_k\}, \qquad Z_k = Y_k^\perp. \tag{2.6}$$

Theorem 2.3 (Fountain theorem [20]) *Suppose that the functional* $\Phi \in C^1(E, \mathbb{R})$ *is even and satisfies condition* (C). *Suppose that for every* $k \ge k_0$ *for some constant* $k_0 > 0$, *there exist* $\rho_k > r_k > 0$ *such that*

(A_1) $b_k = \inf_{u \in Z_k, \|u\| = r_k} \Phi(u) \to \infty$ *as* $k \to \infty$, *and*

(A_2) $a_k = \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \le 0$.

Then Φ *has a sequence of critical points* $\{u_k\}$ *such that* $\Phi(u_k) \to \infty$.

3 Proof of main results

Lemma 3.1 *Suppose that* (V), (f_1), *and* (f_2) *are satisfied. Then there exists* $r > 0$ *such that* $\inf \Phi(\partial B_r(\mathbf{0}) \cap E^+) > 0$.

Proof It follows from (f_1) and (f_2) that, for given $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\left|F(x,t)\right| \le \varepsilon |t|^2 + C_\varepsilon |t|^p \tag{3.1}$$

and

$$\left|f(x,t)\right| \le \varepsilon |t| + C_\varepsilon |t|^{p-1}. \tag{3.2}$$

For $u \in E^+$, we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\ &\ge \frac{1}{2}\|u\|^2 - \varepsilon |u|_2^2 - C_\varepsilon |u|_p^p \\ &\ge \left(\frac{1}{2} - \varepsilon\tau_2\right)\|u\|^2 - \tau_p C_\varepsilon \|u\|^p,\end{aligned}$$

where τ_2 and τ_p are constants in (2.4). Let $\varepsilon = \frac{1}{4\tau_2}$. Since $p > 2$, we can fix some r small enough such that

$$\inf_{u \in E^+, \|u\| = r} \Phi(u) > 0.$$

The proof is completed. □

Lemma 3.2 *Suppose that* (V) *and* (f_1)–(f_3) *are satisfied. Then, for any nontrivial finite-dimensional subspace* W *of* E^+, *there exists* $R > r$ *such that* $\Phi \le 0$ *in* $(E^- \oplus W)\backslash B_R(0)$, *where* $r > 0$ *is the constant given by Lemma* 3.1.

Proof This lemma is a corollary of [13, Lemma 2.5]. We omit the proof. □

Lemma 3.3 *Suppose that* (V), (f_1)–(f_3), *and* (f_5) *are satisfied and* $c \in \mathbb{R}$. *Then any* $(C)_c$ *sequence of* Φ *is bounded.*

Proof It follows from (f_5) that, for all $t \neq 0$ and $x \in \mathbb{R}^N$,

$$tf(x,t) - 2F(x,t) \ge \frac{1}{2a|t|^\alpha + 2b + 1} tf(x,t) > 0.$$

Let $\{u_n\}$ be a $(C)_c$ sequence of Φ, that is, a sequence satisfying (2.5). Set $\Pi_n := \{x \in \mathbb{R}^N \,|\, |u_n(x)| < 1\}$ and $\Pi_n^c := \mathbb{R}^N \backslash \Pi_n$. Then there are constants $c_1, c_2 > 0$ such that

$$2a|u_n|^\alpha + 2b + 1 \le 1/c_1, \quad \forall x \in \Pi_n,$$

and

$$2a|u_n|^\alpha + 2b + 1 \le |u_n|^\alpha / c_2, \quad \forall x \in \Pi_n^c.$$

For n sufficient large, it follows that

$$\begin{aligned}
D &\ge 2\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle \\
&= \int_{\mathbb{R}^N} \bigl(u_n f(x,u_n) - 2F(x,u_n)\bigr)\,\mathrm{d}x \\
&\ge \int_{\mathbb{R}^N} \frac{u_n f(x,u_n)}{2a|u_n|^\alpha + 2b + 1}\,\mathrm{d}x \\
&\ge c_1 \int_{\Pi_n} u_n f(x,u_n)\,\mathrm{d}x + c_2 \int_{\Pi_n^c} |u_n|^{-\alpha} u_n f(x,u_n)\,\mathrm{d}x
\end{aligned} \tag{3.3}$$

for some constant $D > 0$.

Note that $\alpha < (2^* - 1)p' - 2^*$ by (f_5). We have

$$\frac{1}{p'} < \frac{2^*}{2^* - 1}\frac{1}{p'} < \frac{2^*}{2^* + \alpha} \quad \text{and} \quad \frac{2}{2+\alpha} < \frac{2^*}{2^* + \alpha}.$$

Then we can choose a constant $r \in (0,1)$ such that

$$\max\left\{ \frac{2^*}{2^* - 1}\frac{1}{p'}, \frac{2}{2+\alpha} \right\} < r < \frac{2^*}{2^* + \alpha}. \tag{3.4}$$

Let $s := r/(1-r) > 0$. Then $\frac{1}{r} + \frac{1}{-s} = 1$. By (3.3) and the inverse Hölder inequality we have

$$\begin{aligned}
D &\ge c_1 \int_{\Pi_n} u_n f(x,u_n)\,\mathrm{d}x + c_2 \left(\int_{\Pi_n^c} \bigl(u_n f(x,u_n)\bigr)^r \,\mathrm{d}x \right)^{1/r} \left(\int_{\Pi_n^c} |u_n|^{\alpha s}\,\mathrm{d}x \right)^{1/(-s)} \\
&\ge c_1 \int_{\Pi_n} u_n f(x,u_n)\,\mathrm{d}x + c_2 \frac{(\int_{\Pi_n^c} (u_n f(x,u_n))^r \,\mathrm{d}x)^{1/r}}{|u_n|_{\alpha s}^\alpha}.
\end{aligned} \tag{3.5}$$

By (f_1) and (f_2) we have

$$\begin{aligned}
&|f(x,u)|^{p'r} \le \bigl(c_3 |u|^{(p-1)(p'-1)} |f(x,u)|\bigr)^r = c_4 \bigl(uf(x,u)\bigr)^r, \quad \forall |u| \ge 1, \\
&|f(x,u)|^2 \le c_5 |u| |f(x,u)| = c_5 u f(x,u), \quad \forall |u| < 1.
\end{aligned}$$

Therefore by (3.5) we have

$$\left(\int_{\Pi_n^c} |f(x,u_n)|^{p'r}\,\mathrm{d}x \right)^{1/p'r} \le c_6 |u_n|_{\alpha s}^{\alpha/p'}, \tag{3.6}$$

$$\left(\int_{\Pi_n} |f(x,u_n)|^2\,\mathrm{d}x \right)^{1/2} \le c_7. \tag{3.7}$$

In view of (3.4), we easily check that $p'r > 1$ and $(p'r)', \alpha s \in [2, 2^*]$, where $(p'r)' := p'r/(p'r-1)$. Consequently, it follows from (3.6) and (3.7), Hölder's inequality, and

Sobolev's inequality that, for n large enough and some constants $c_8, c_9 > 0$,

$$\begin{aligned}\|u_n^+\|^2 &= \langle \Phi'(u_n),\ u_n^+\rangle + \int_{\mathbb{R}^N} f(x,u_n)u_n^+\,\mathrm{d}x \\ &\leq \|u_n^+\| + \left(\int_{\Pi_n} |f(x,u_n)|^2\,\mathrm{d}x\right)^{1/2} |u_n^+|_2 + \left(\int_{\Pi_n^c} |f(x,u_n)|^{p'r}\,\mathrm{d}x\right)^{1/p'r} |u_n^+|_{(p'r)'} \\ &\leq \|u_n^+\| + c_7|u_n^+|_2 + c_6|u_n|_{\alpha s}^{\alpha/p'} |u_n^+|_{(p'r)'} \leq c_8\|u_n^+\| + c_9\|u_n^+\|\|u_n\|^{\alpha/p'}.\end{aligned}$$

Therefore we obtain

$$\|u_n^+\| \leq c_8 + c_9\|u_n\|^{\alpha/p'}$$

and, similarly,

$$\|u_n^-\| \leq c_8 + c_9\|u_n\|^{\alpha/p'}.$$

Note that $\alpha < p'$. Then we easily verify that $\|u_n\|^2 = \|u_n^-\|^2 + \|u_n^+\|^2$ is bounded. □

Lemma 3.4 *Suppose that* (V) *and* (f_1)–(f_4) *are satisfied. Then any bounded* $(C)_c$ *sequence of* Φ *contains a convergent subsequence.*

Proof Suppose $\{u_n\}$ is a bounded $(\mathrm{C})_c$ sequence of Φ. Then, passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in E. Since $\dim E^- < \infty$, we have $u_n^+ \rightharpoonup u^+$ in E^+, $u_n^- \to u^-$ in E^-, and $u_n^+ \to u^+$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $s \in [2, 2^*)$. To establish the strong convergence, it suffices to prove that

$$\|u_n^+\| \to \|u^+\|. \tag{3.8}$$

Since

$$\langle \Phi'(u_n), u_n^+ - u^+\rangle = (u_n^+, u_n^+ - u^+) - \int_{\mathbb{R}^N} f(x,u_n)(u_n^+ - u^+)\,\mathrm{d}x \to 0,$$

we have

$$\begin{aligned}0 &\leq \limsup_{n\to\infty}\left(\|u_n^+\|^2 - \|u^+\|^2\right) \\ &= \limsup_{n\to\infty}(u_n^+, u_n^+ - u^+) = \limsup_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)(u_n^+ - u^+)\,\mathrm{d}x.\end{aligned} \tag{3.9}$$

Next, let $\varepsilon > 0$. For $l \geq 1$, from (f_1) and Hölder's inequality it follows that

$$\begin{aligned}\int_{|u_n|\geq l} f(x,u_n)(u_n^+ - u^+)\,\mathrm{d}x &\leq 2c\int_{|u_n|\geq l} |u_n|^{p-1}|u_n^+ - u^+|\,\mathrm{d}x \\ &\leq 2cl^{p-2^*}\int_{|u_n|\geq l} |u_n|^{2^*-1}|u_n^+ - u^+|\,\mathrm{d}x \\ &\leq 2cl^{p-2^*}|u_n|_{2^*}^{2^*-1}|u_n^+ - u^+|_{2^*}.\end{aligned}$$

Since $p < 2^*$, we may fix l large enough such that

$$\int_{|u_n|\geq l} f(x,u_n)\big(u_n^+ - u^+\big)\,\mathrm{d}x \leq \frac{\varepsilon}{3} \tag{3.10}$$

for all n. Moreover, by (f_4) there exists $L > 0$ such that

$$\int_{\substack{|x|\geq L\\ |u_n|\leq l}} f(x,u_n)\big(u_n^+ - u^+\big)\,\mathrm{d}x \leq |u_n|_2\big|u_n^+ - u^+\big|_2 \sup_{|t|\leq l, |x|\geq L} \frac{|f(x,t)|}{|t|} \leq \frac{\varepsilon}{3} \tag{3.11}$$

for all n. Finally, since $u_n^+ \to u^+$ in $L^s(B_L(\mathbf{0}))$ for $s \in [2, 2^*)$, from (3.2) it follows that

$$\begin{aligned}\int_{\substack{|x|\leq L\\ |u_n|\leq l}} f(x,u_n)\big(u_n^+ - u^+\big)\,\mathrm{d}x &\leq \int_{\substack{|x|\leq L\\ |u_n|\leq l}} |u_n|\big|u_n^+ - u^+\big|\,\mathrm{d}x + C_1\int_{\substack{|x|\leq L\\ |u_n|\leq l}} |u_n|^{p-1}\big|u_n^+ - u^+\big|\,\mathrm{d}x\\ &\leq |u_n|_2\big|u_n^+ - u^+\big|_{L^2(B_L(\mathbf{0}))} + C_1|u_n|_p^{p-1}\big|u_n^+ - u^+\big|_{L^p(B_L(\mathbf{0}))}\\ &\leq \frac{\varepsilon}{3}\end{aligned} \tag{3.12}$$

for n large enough. Combining (3.10)–(3.12), we conclude that

$$\int_{\mathbb{R}^N} f(x,u_n)\big(u_n^+ - u^+\big)\,\mathrm{d}x \leq \varepsilon$$

for n large enough. Since ε is arbitrary, this, together with (3.9), implies (3.8). The lemma is proved. □

Proof of Theorem 1.2 For $u \in E^-$, since $F(x,t) \geq 0$ by (f_5), we obtain that

$$\Phi(u) = -\frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \leq 0.$$

This, together with Lemmas 3.1 and 3.2, implies that there exist $R > r > 0$ such that

$$\inf_N \Phi > 0 \geq \max_{\partial M} \Phi.$$

In view of Lemmas 3.3 and 3.4, Φ satisfies condition (C). By Theorem 2.2 we have that Φ possesses at least one nontrivial critical point, which is the nontrivial solution of problem (1.1). □

Proof of Theorem 1.3 Since f is odd, Φ is an even functional. By Lemmas 3.3 and 3.4 we know that Φ satisfies condition (C). To apply Theorem 2.3, it suffices to verify (A_1) and (A_2).

Define Y_k and Z_k as in (2.6). Recall that $\lambda_\ell < 0 < \lambda_{\ell+1}$. If $k > \ell$, then we have $Z_k \subset E^+$. Define $\beta_k := \sup_{u\in Z_k, \|u\|=1} |u|_p$. Therefore by (2.4) and (3.1) with $\varepsilon = 1/4\tau_2^2$ we have

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4\tau_2^2}|u|_2^2 - C|u|_p^p\\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4\tau_2^2}|u|_2^2 - C\beta_k^p\|u\|^p \geq \frac{1}{4}\|u\|^2 - C\beta_k^p\|u\|^p.\end{aligned}$$

Let $r_k = (2pC\beta_k^p)^{1/(2-p)}$. Then for $u \in Z_k$ with $\|u\| = r_k$, we have

$$\Phi(u) \geq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)\left(2pC\beta_k^p\right)^{1/(2-p)}.$$

Since $\beta_k \to 0$ as $k \to \infty$ by [10, Lemma 3.8] and $p > 2$, it follows that

$$b_k = \inf_{u \in Z_k, \|u\| = r_k} \Phi(u) \to \infty.$$

Hence (A_1) is satisfied. Finally, by Lemma 3.2 with $W = \bigoplus_{j=0}^{k} \mathbb{R}e_j$ we easily see that (A_2) holds. □

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