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Estimates and properties of certain q-Mellin transform on generalized q-calculus theory

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Abstract

This paper deals with the generalized q-theory of the q-Mellin transform and its certain properties in a set of q-generalized functions. Some related q-equivalence relations, q-quotients of sequences, q-convergence definitions, and q-delta sequences are represented. Along with that, a new q-convolution theorem of the estimated operator is obtained on the generalized context of q-Boehmians. On top of that, several results and q-Mellin spaces of q-Boehmians are discussed. Furthermore, certain continuous q-embeddings and an inversion formula are also discussed.

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1 Introduction and preliminaries

The quantum calculus or the q-calculus theory has been given a noticeable importance and popularity due to its wide application in various fields of mathematics, statistics, and physics [1]. The q-calculus theory has appeared as a connection between mathematics and physics. Recently, this topic has attracted the attention of several researchers, and a variety of results have been derived in various areas of research including number theory, hypergeometric functions, orthogonal polynomials, quantum theory, combinatorics, and electronics as well. The q-calculus begins with the definition of the q-analogue $d_q g$ of the differential

$$d_q g(t) = g(qt) - g(t)$$

of the function g, where q is a fixed real number such that $0 < q < 1$ (see [1–3]). Having said this, we immediately get the q-analogue of the derivative of g as

$$D_q g(t) := \frac{d_q g(t)}{d_q t} := \frac{g(t) - g(qt)}{(1-q)t} \quad \text{for } t \neq 0$$

and $D_q g(0) = \lim_{t \longrightarrow 0} D_q g(t) = g'(0)$ provided $g'(0)$ exists. Also, when g is differentiable, the q-derivative $D_q g$ tends to $g'(0)$ as q tends to 1. It also satisfies the q-analogue of the Leibniz rule

$$D_q\big(g_1(t)g(t)\big) = g(t)D_q g_1(t) + g_1(qt)D_q g(t).$$

The Jackson q-integrals from 0 to x and respectively from 0 to ∞ are defined by [1, 4]

$$\int_0^x g(t)\,d_qt = (1-q)t\sum_0^\infty g\big(tq^k\big)q^k, \tag{1}$$

$$\int_0^\infty g(t)\,d_qt = (1-q)t\sum_{-\infty}^\infty g\big(q^k\big)q^k, \tag{2}$$

when the sums converge absolutely. The Jackson q-integral on the generic interval $[a,b]$ is, therefore, given by [1, 5]

$$\int_a^b g(t)\,d_qt = \int_0^b g(t)\,d_qt - \int_0^a g(t)\,d_qt.$$

The q-integration by parts for two functions f and g is defined by

$$\int_0^b g_2(t)D_q g_1(t)\,d_qt = g_1(b)g_2(b) - g_1(a)g_2(a) - \int_a^b g_1(qt)D_q g_2(t)\,d_qt.$$

Arising from the notion of regular operators [6], the notion of a Boehmian was firstly introduced by Mikusinski and Mikusinski [7] to generalize distributions and regular operators [8]. Boehmians are equivalence classes of quotients of sequences constructed from an integral domain when the operations are interpreted as addition and convolution, see, e.g., [9–20]. In terms of the q-calculus concept, we introduce the concept of q-Boehmians to popularize the concept of q-calculus theory as follows:

For a complex linear space V and a subspace $(W, *^q)$ of V, let $\overset{q}{\bullet} : V \times W \to V$ be a binary operation such that the undermentioned axioms (1)–(5) hold:

(1) $(g_1 + g_2)\overset{q}{\bullet}\psi = g_1\overset{q}{\bullet}\psi + g_2\overset{q}{\bullet}\psi, \forall g_1, g_2 \in V$ and $\psi \in W$.
(2) $(\alpha g)\overset{q}{\bullet}\psi = \alpha(g\overset{q}{\bullet}\psi), \forall\alpha \in \mathbb{C}, \forall g \in V$ and $\psi \in W$.
(3) $g\overset{q}{\bullet}(\psi_1\overset{q}{\bullet}\psi_2) = (g\overset{q}{\bullet}\psi_1)\overset{q}{\bullet}\psi_2, \forall g \in V$ and $\psi_1, \psi_2 \in W$.
(4)

$$\begin{aligned} &\text{If } g_n \to g \text{ in } V \text{ as } n \to \infty \text{ and } \psi \in W, \\ &\quad \text{then } g_n\overset{q}{\bullet}\psi \to g\overset{q}{\bullet}\psi \text{ as } n \to \infty \text{ in } V. \end{aligned} \tag{3}$$

(5) A collection Δ_q of sequences from W such that, for all $(\varepsilon_n), (\phi_n) \in \Delta_q$ and $(g_n) \in W$, we have $\varepsilon_n\overset{q}{\bullet}\phi_n \in \Delta_q$ and

$$\text{if } g_n \to g \text{ in } V \text{ as } n \to \infty, \text{ then } g_n\overset{q}{\bullet}\varepsilon_n \to g \text{ as } n \to \infty.$$

Once the preceding axioms are applied, the name of a q-Boehmian is set to mean the equivalence class $\frac{g_n}{\delta_n}$ that arises from the equivalence relation

$$g_n\overset{q}{\bullet}\varepsilon_m = g_m\overset{q}{\bullet}\varepsilon_n, \quad \forall m, n \in \mathbb{N}, \tag{4}$$

where $(g_n) \in V$ and $(\varepsilon_n) \in \Delta_q$. The collection of all q-Boehmians is denoted by $\mathbb{B}_q$ which is the so-called Boehmian space. The classical linear space V is identified as a subset of the space $\mathbb{B}_q$ which can be recognized from the relation

$$g \longrightarrow \frac{g \overset{q}{\bullet} \varepsilon_n}{\varepsilon_n}, \tag{5}$$

where $(\varepsilon_n) \in \Delta_q$ is arbitrary. Two q-Boehmians $\frac{g_n}{\varepsilon_n}$ and $\frac{\varphi_n}{\epsilon_n}$ are said to be equal in $\mathbb{B}_q$ if $g_n \overset{q}{\bullet} \epsilon_m = \varphi_m \overset{q}{\bullet} \varepsilon_n, \forall m, n \in \mathbb{N}$. Addition in the space $\mathbb{B}_q$ is defined as

$$\frac{g_n}{\varepsilon_n} + \frac{\varphi_n}{\epsilon_n} = \frac{g_n \overset{q}{\bullet} \epsilon_n + \varphi_n \overset{q}{\bullet} \varepsilon_n}{\varepsilon_n \overset{q}{\bullet} \epsilon_n}. \tag{6}$$

The scalar multiplication in the space $\mathbb{B}_q$ is defined as

$$\alpha \frac{g_n}{\varepsilon_n} = \frac{\alpha g_n}{\varepsilon_n}, \quad \alpha \in \mathbb{C}.$$

The q-convergence of type δ, $\beta_n \xrightarrow{\delta} \beta$, is defined in the space $\mathbb{B}_q$ when for $(\psi_n) \in \Delta_q$ and each $k \in \mathbb{N}$ such that

$$\beta_n \overset{q}{\bullet} \varepsilon_k \in V, \quad \forall k, n \in \mathbb{N}, \beta \overset{q}{\bullet} \varepsilon_k \in V, \tag{7}$$

we have $\beta_n \overset{q}{\bullet} \varepsilon_k \to \beta \overset{q}{\bullet} \varepsilon_k$ as $n \to \infty$ in V. The q-convergence $\beta_n \xrightarrow{\Delta_q} \beta$ of type Δ_q is defined when for some $(\varepsilon_n) \in \Delta_q$ we have

$$(\beta_n - \beta) \overset{q}{\bullet} \varepsilon_n \in V, \quad \forall n \in \mathbb{N} \quad \text{and} \quad (\beta_n - \beta) \overset{q}{\bullet} \varepsilon_n \to 0 \quad \text{as } n \to \infty \text{ in } V. \tag{8}$$

The space of q-Boehmians emerging from the q-convergence assigns a complete quasi-normed space.

In recent work, several remarkable integral transforms were given different q-analogues in a q-calculus context [4, 21–24]. In the sequence of such q-integral transforms, we recall the q-Laplace integral transform [25–29], the q-Sumudu integral transform [2, 30–32], the q-Weyl fractional integral transform [33], the q-wavelet integral transform [34], the q-Mellin type integral transform [35], the Mangontarum integral transform [36, 37], the $E_{2;1}$ integral transform [38, 39], the natural integral transform [3], and many others, to mention but a few. In this paper, we discuss the generalized q-theory of the q-Mellin transform and obtain several results.

Let g be a function defined on $\mathbb{R}_{q,+}, \mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$, then the q-Mellin transform was defined by [40], p. 521 as

$$M_q\bigl(g(t)\bigr)(\zeta) = \int_0^\infty t^{\zeta-1} g(t)\, d_q t, \tag{9}$$

provided the q-integral converges. The integral (9) is analytic on the fundamental strip $\langle \alpha_{q,g}; \beta_{q,g} \rangle$ and periodic with period $2i\pi \log(q)$. The inversion formula for the q-analogue

(9) is given by

$$g(t)=\frac{\log(q)}{2i\pi(1-q)}\int_{c-\frac{i\pi}{\log(q)}}^{c+\frac{i\pi}{\log(q)}} M_q(g)(\zeta)t^{-\zeta}\,d\zeta,\quad t\in\mathbb{R}_{q,+},$$

where $\alpha_{q,g}<c<\beta_{q,g}$. The asymptotic properties as well as the asymptotic singularities of the q-Mellin transform into asymptotic expansions of the original function for $x\to 0$ and $x\to\infty$ are given in [40]. Additionally, the asymptotic behavior at 0 or ∞ is studied using the q-Mellin transform.

Definition 1 The function g is said to be q-integrable on an interval $[0,\infty[$ provided the infinite series

$$\sum_{n\in\mathbb{Z}} q^n g(q^n)$$

converges absolutely. The space of all q-integrable functions on $[0,\infty[$ is denoted by $L_q^1(\mathbb{R}_{q,+})$. In a better recognition, the space $L_q^1(\mathbb{R}_{q,+})$ is defined to be the space of all q-integrable functions g on $\mathbb{R}_{q,+}$ such that

$$L_q^1 g(t)=\frac{1}{1-q}\int_0^\infty \bigl|g(t)\bigr|\,d_q t<\infty. \tag{10}$$

We denote by $\mathbb{D}_q$ the q-space of test functions of compact supports on $\mathbb{R}_{q,+}$, i.e., $\mathbb{D}_q$ is the q-space of all smooth functions $\kappa\in C^\infty(\mathbb{R}_{q,+})$ such that

$$\mathbb{D}_q=\Bigl\{\kappa\in C^\infty(\mathbb{R}_{q,+}):\sup_{0<t<\infty}\bigl|D_q\kappa(t)\bigr|<\infty\Bigr\}. \tag{11}$$

However, this theory is new and might be developing a new area of research. It investigates a generalization to the q-theory of calculus [40] and hence all results can be popularized. Every element in the space $L_q^1(\mathbb{R}_{q,+})$ is identified as a member in the generalized theory. To this aim, we spread our results into five sections. In Sect. 1, we recall some definitions and preliminaries from the q-calculus theory. In Sect. 2, we derive q-delta sequences, q-convolution theorems and establish a space of q-Boehmians. In Sect. 3, we establish a space of q-ultraBoehmians. In Sect. 4, we generalize definitions and obtain several properties of the q-Mellin transform. In Sect. 5 we include several results.

2 The space $\mathbb{B}$

In this section, we strive to establish the space $\mathbb{B}$ of q-Boehmians. Henceforth, we denote by Δ_q the set of all sequences from $\mathbb{D}_q$ such that the undermentioned identities $\Delta_q^1-\Delta_q^3$ hold, where

$$\begin{aligned}
&\Delta_q^1:\int_0^\infty \bigl|\varepsilon_n(t)\bigr|\,d_q t=1,\quad \forall n\in\mathbb{N},\\
&\Delta_q^2:\bigl|\varepsilon_n(t)\bigr|<M,\quad M>0, M\in\mathbb{R}_+,\\
&\Delta_q^3:\operatorname{supp}(\varepsilon_n)\subseteq(0,b_n),\quad b_n\to 0 \text{ as } n\to\infty, 0<b_n,\forall n\in\mathbb{N}.
\end{aligned} \tag{12}$$

On the other hand, we denote by $\overset{q}{\bullet}$ the Mellin type q-convolution product defined on $L_q^1(\mathbb{R}_{q,+})$ by

$$(g_1 \overset{q}{\bullet} g_2)(x) = \int_0^\infty t^{-1} g_1\big(t^{-1}x\big) g_2(t)\, d_q t, \tag{13}$$

provided the integral part exists for every $x > 0$. It is clear from the context that $g_1 \overset{q}{\bullet} g_2 \in L_q^1(\mathbb{R}_{q,+})$ for all g_1 and g_2 in $L_q^1(\mathbb{R}_{q,+})$. On that account, the q-convolution theorem of the q-Mellin transform of the product $g_1 \overset{q}{\bullet} g_2$ can be easily established as follows.

Theorem 2 *Let $L_q^1(\mathbb{R}_{q,+})$ be the space of all q-integrable functions on $\mathbb{R}_{q,+}$. Then the q-convolution theorem of the transform M_q is given by*

$$M_q(g_1 \overset{q}{\bullet} g_2)(\zeta) = M_q g_1(\zeta) M_q g_2(\zeta) \quad \textit{for } g_1 \textit{ and } g_2 \textit{ in } L_q^1(\mathbb{R}_{q,+}).$$

Proof By applying the definition of the M_q transform to the product $g_1 \overset{q}{\bullet} g_2$, we get

$$\begin{aligned} M_q(g_1 \overset{q}{\bullet} g_2)(\zeta) &= \int_0^\infty (g_1 \overset{q}{\bullet} g_2)(x) x^{\zeta-1}\, d_q x \\ &= \int_0^\infty \left(\int_0^\infty g_1(t) g_2\big(t^{-1}x\big) x^{-1}\, d_q t \right) x^{\zeta-1}\, d_q x. \end{aligned}$$

Therefore, employing the substitution $z = t^{-1}x$ and, hence, $d_q z = t^{-1}\, d_q x$, in collaboration with simple computations, reveals

$$M_q(g_1 \overset{q}{\bullet} g_2)(\zeta) = M_q(g_1)(\zeta) M_q(g_2)(\zeta).$$

Hence, the proof of this theorem is completed. □

The following is an imperative result for initiating the q-delta sequence concept.

Lemma 3 *Let (ε_n) and (ϵ_n) be sequences in Δ_q. Then $(\varepsilon_n \overset{q}{\bullet} \epsilon_n)$ is a sequence in Δ_q.*

Proof To establish this lemma, we examine the performance of the sequence $(\varepsilon_n \overset{q}{\bullet} \epsilon_n)$. To inspect the correctness of the property Δ_q^1, we use the integral equation (3) to get

$$\int_0^\infty (\varepsilon_n \overset{q}{\bullet} \epsilon_n)(x)\, d_q x = \int_0^\infty t^{-1} \epsilon_n(t) \left(\int_0^\infty \varepsilon_n\big(t^{-1}x\big)\, d_q x \right) d_q t. \tag{14}$$

Therefore, by using the change of variables $t^{-1}x = y$ and, hence, $d_q x = t\, d_q y$, (14) we indicate

$$\int_0^\infty (\varepsilon_n \overset{q}{\bullet} \epsilon_n)(x)\, d_q x = \left(\int_0^\infty \epsilon_n(t)\, d_q t \right) \left(\int_0^\infty \varepsilon_n(y)\, d_q y \right) = 1.$$

This proves the Δ_q^1 part. The proof of the Δ_q^2 part follows from similar techniques, whereas the Δ_q^3 part is clearly valid, by conducting the fact

$$\operatorname{supp}(\varepsilon_n \overset{q}{\bullet} \epsilon_n) \subset \operatorname{supp}(\varepsilon_n) + \operatorname{supp}(\epsilon_n) \quad \text{for } (\varepsilon_n), (\epsilon_n) \in \Delta_q.$$

This ends the proof of the lemma. □

Lemma 3, hence, displays that every sequence in Δ_q forms, to a great extent, the q-delta sequence concept.

Lemma 4 *Let $g_1, g_2 \in L_q^1(\mathbb{R}_{q,+})$, $\kappa_1, \kappa_2 \in \mathbb{D}_q$, and $\alpha \in \mathbb{C}$. Then the following assertions are valid*:

$$\text{(i)}\quad \kappa_1 \overset{q}{\bullet} \kappa_2 = \kappa_2 \overset{q}{\bullet} \kappa_1, \qquad \text{(ii)}\quad (g_1 + g_2) \overset{q}{\bullet} \kappa_1 = g_1 \overset{q}{\bullet} \kappa_1 + g_2 \overset{q}{\bullet} \kappa_1,$$

$$\text{(iii)}\quad (\alpha g_1) \overset{q}{\bullet} \kappa_1 = \alpha (g_1 \overset{q}{\bullet} \kappa_1), \qquad \text{(iv)}\quad g_1 \overset{q}{\bullet} (\kappa_1 \overset{q}{\bullet} \kappa_2) = (g_1 \overset{q}{\bullet} \kappa_1) \overset{q}{\bullet} \kappa_2.$$

Proof (i) As the convolution product of the functions κ_1 and κ_2 in $\mathbb{D}_q$ is exceptionally given by

$$(\kappa \overset{q}{\bullet} \kappa_2)(x) = \int_0^\infty t^{-1} \kappa_1\bigl(t^{-1}x\bigr) \kappa_2(t)\, d_q t, \tag{15}$$

the change of variables $t^{-1}x = y$ reveals us to write (15) into the form

$$(\kappa_1 \overset{q}{\bullet} \kappa_2)(x) = \int_0^\infty y^{-1} \kappa_2\bigl(x^{-1}y\bigr) \kappa_1(y)\, d_q y.$$

Hence (i) follows. To prove (ii) and (iii), we merely follow simple integral calculus. To prove (iv), we employ the definition of the product $\overset{q}{\bullet}$ to get

$$\begin{aligned}\bigl(g_1 \overset{q}{\bullet} (\kappa_1 \overset{q}{\bullet} \kappa_2)\bigr)(x) &= \int_0^\infty t^{-1} g_1\bigl(t^{-1}x\bigr)(\kappa_1 \overset{q}{\bullet} \kappa_2)(t)\, d_q t \\ &= \int_0^\infty t^{-1} g_1\bigl(t^{-1}x\bigr)\left(\int_0^\infty y^{-1} \kappa_1\bigl(y^{-1}t\bigr)\kappa_2(y)\, d_q y\right) d_q t.\end{aligned}$$

That is,

$$\bigl(g_1 \overset{q}{\bullet} (\kappa_1 \overset{q}{\bullet} \kappa_2)\bigr)(x) = \int_0^\infty y^{-1}\left(\int_0^\infty t^{-1} g_1\bigl(t^{-1}x\bigr)\kappa_1\bigl(y^{-1}t\bigr)\, d_q t\right)\kappa_2(y)\, d_q y. \tag{16}$$

Now, by employing the change of variables $y^{-1}t = z$, we write down equation (16) into the form

$$\begin{aligned}\bigl(g_1 \overset{q}{\bullet} (\kappa_1 \overset{q}{\bullet} \kappa_2)\bigr)(x) &= \int_0^\infty y^{-1}\left(\int_0^\infty z^{-1} g_1\bigl(z^{-1}\bigl(y^{-1}x\bigr)\bigr)\kappa_1(z)\, d_q z\right)\kappa_2(y)\, d_q y \\ &= \int_0^\infty y^{-1}(g_1 \overset{q}{\bullet} \kappa_1)\bigl(y^{-1}x\bigr)\kappa_2(y)\, d_q y.\end{aligned}$$

This ends the proof of the lemma. □

To proceed in our construction, we establish the following lemma.

Lemma 5 (i) *Let g_1 and g_2 be integrable functions in $L_q^1(\mathbb{R}_{q,+})$ and (ε_n) be a delta sequence in the set Δ_q such that $g_1 \overset{q}{\bullet} \varepsilon_n = g_2 \overset{q}{\bullet} \varepsilon_n$. Then $g_1 = g_2$ in $L_q^1(\mathbb{R}_{q,+})$ for every $n \in \mathbb{N}$.*

(ii) *Let g and (g_n) be integrable functions in $L_q^1(\mathbb{R}_{q,+})$ such that $g_n \to g$ as $n \to \infty$ in $L_q^1(\mathbb{R}_{q,+})$. Then*

$$g_n \overset{q}{\bullet} \psi \to g \overset{q}{\bullet} \psi \quad \text{for every } \psi \in \mathbb{D}_q \text{ as } n \to \infty.$$

Proof To prove (i), we merely need to show that $g_1 \overset{q}{\bullet} \varepsilon_n = g_1 \in L_q^1(\mathbb{R}_{q,+})$. By using Δ_q^1 and Δ_q^3, we obtain

$$\begin{aligned}\int_0^\infty \left|(g_1 \overset{q}{\bullet} \varepsilon_n)(x) - g_1(x)\right| d_q x &\le \int_0^\infty \int_0^\infty \left|t^{-1} g_1\left(t^{-1}x\right) - g_1(x)\right|\left|\varepsilon_n(t)\right| d_q t\, d_q x \\ &= \int_0^\infty \int_{a_n}^{b_n} \left|t^{-1} g_1\left(t^{-1}x\right) - g_1(x)\right|\left|\varepsilon_n(t)\right| d_q t\, d_q x.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^\infty \left|(g_1 \overset{q}{\bullet} \varepsilon_n)(x) - g_1(x)\right| d_q x \\ &\quad \le \int_0^\infty \int_{a_n}^{b_n} \left|t^{-1} g_1\left(t^{-1}x\right)\right|\left|\varepsilon_n(t)\right| d_q t\, d_q x \\ &\qquad + \int_0^\infty \int_{a_n}^{b_n} \left|g_1(x)\right|\left|\varepsilon_n(t)\right| d_q t\, d_q x. \end{aligned} \tag{17}$$

Hence, for $g_1 \in L_q^1(\mathbb{R}_{q,+})$, by using (17) we turn to write

$$\int_0^\infty \left|(g_1 \overset{q}{\bullet} \varepsilon_n)(x) - g_1(x)\right| d_q x \le A \int_0^{b_n} \left|t^{-1}\right|\left|\varepsilon_n(t)\right| d_q t + A \int_0^{b_n} \left|\varepsilon_n(t)\right| d_q t.$$

Therefore, by the properties of the delta sequences Δ_q^2 and Δ_q^3, we conclude that

$$\int_0^\infty \left|(g_1 \overset{q}{\bullet} \varepsilon_n)(x) - g_1(x)\right| d_q x \le AM \ln(b_n) + AM(b_n) \to 0$$

as $n \to \infty$.

Proof of (ii) follows from simple integration. We therefore omit the details. Hence the proof of this lemma is ended. □

Lemma 6 *Let g_1 be an integrable function in the space $L_q^1(\mathbb{R}_{q,+})$. Then $g_1 \overset{q}{\bullet} \varepsilon_n \to g_1$ as $n \to \infty$ for every $(\varepsilon_n) \in \Delta_q$.*

The proof of this lemma is a straightforward conclusion from the proof of Lemma 4. Hence, we delete the details.

Thus, the space $\mathbb{B}$ with $(L_q^1(\mathbb{R}_{q,+}), \overset{q}{\bullet})$, $(\mathbb{D}_q, \overset{q}{\bullet})$, and Δ_q is defined. The canonical embedding of $L_q^1(\mathbb{R}_{q,+})$ in $\mathbb{B}$ is given by

$$g \to \frac{g \overset{q}{\bullet} \varepsilon_n}{\varepsilon_n}. \tag{18}$$

That is, every element in the space $L_q^1(\mathbb{R}_{q,+})$ can be identified as a member of the space $\mathbb{B}$. Addition, scalar multiplication, differentiation, Δ_q and δ_q convergence are defined in a natural way as follows:

If $(\varphi_n) \in L_q^1(\mathbb{R}_{q,+})$ and $(\varepsilon_n) \in \Delta_q$, then the pair $(\varphi_n, \varepsilon_n)$ (or $\frac{\varphi_n}{\varepsilon_n}$) is said to be a q-quotient of sequences if $\varphi_n \overset{q}{\bullet} \varepsilon_m = \varphi_m \overset{q}{\bullet} \varepsilon_n, \forall n, m \in \mathbb{N}$. Therefore, if $\frac{\varphi_n}{\epsilon_n}$ and $\frac{g_n}{\varepsilon_n}$ are q-quotients of sequences and $g \in L_q^1(\mathbb{R}_{q,+})$, then it is easy to see that

$$\frac{g \overset{q}{\bullet} \epsilon_n}{\epsilon_n} \quad \text{and} \quad \frac{\varphi_n \overset{q}{\bullet} \epsilon_n + g_n \overset{q}{\bullet} \epsilon_n}{\epsilon_n \overset{q}{\bullet} \varepsilon_n}$$

are q-quotients of sequences. Two q-quotients of sequences $\frac{\varphi_n}{\epsilon_n}$ and $\frac{g_n}{\varepsilon_n}$ are said to be equivalent if

$$\varphi_n \overset{q}{\bullet} \varepsilon_m = g_m \overset{q}{\bullet} \epsilon_n, \quad \forall n, m \in \mathbb{N}.$$

We can easily check the following equivalence relations:

$$\frac{\varphi_n}{\epsilon_n \overset{q}{\bullet} g} \sim \frac{\varphi_n \overset{q}{\bullet} g}{\epsilon_n} \quad \text{and} \quad \frac{\varphi_n}{\epsilon_n \overset{q}{\bullet} g_n} \sim \frac{\varphi_n \overset{q}{\bullet} g_n}{\epsilon_n}.$$

The equivalent class $\breve{w} = (\frac{\varphi_n}{\epsilon_n})$ of q-quotients of sequences containing $\frac{\varphi_n}{\epsilon_n}$ is said to be a q-Boehmian. The space of such q-Boehmians is denoted by $\mathbb{B}$.

Remark 7 For two q-Boehmians $\breve{w} = (\frac{\varphi_n}{\epsilon_n})$ and $\breve{z} = (\frac{g_n}{\varepsilon_n})$ in $\mathbb{B}$, we have the following identities:

(i) $\breve{w} + \breve{z} = \left(\dfrac{\varphi_n \overset{q}{\bullet} \epsilon_n + g_n \overset{q}{\bullet} \epsilon_n}{\epsilon_n \overset{q}{\bullet} \varepsilon_n}\right)$,

(ii) $\beta \breve{w} = \left(\dfrac{\beta \varphi_n}{\epsilon_n}\right)$,

(iii) $\breve{w} \overset{q}{\bullet} \breve{z} = \left(\dfrac{\varphi_n \overset{q}{\bullet} g_n}{\epsilon_n \overset{q}{\bullet} \varepsilon_n}\right)$,

(iv) $D^k \breve{w} = \left(\dfrac{D^k \varphi_n}{\epsilon_n}\right)$,

(v) $\breve{w} \overset{q}{\bullet} g = \left(\dfrac{\varphi_n \overset{q}{\bullet} g}{\epsilon_n}\right)$,

where $k \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $D^k \breve{w}$ is the kth derivative of $\breve{w}$, and $\psi \in L_q^1(\mathbb{R}_{q,+})$.

Definition 8 (i) For $n = 1, 2, 3, \ldots$ and $\breve{w}_n, \breve{w} \in \mathbb{B}$, the sequence $(\breve{w}_n)$ is δ_q-convergent to $\breve{w}$, denoted by $\delta_q - \lim_{n\to\infty} \breve{w}_n = \breve{w}$, provided there can be found a q-delta sequence (ϵ_n) such that

$$(\breve{w}_n \overset{q}{\bullet} \epsilon_k), (\breve{w} \overset{q}{\bullet} \epsilon_k) \quad \text{in } L_q^1(\mathbb{R}_{q,+}) \quad \text{and} \quad \lim_{n\to\infty} \breve{w}_n \overset{q}{\bullet} \epsilon_k = \breve{w} \overset{q}{\bullet} \epsilon_k \quad \text{in } L_q^1(\mathbb{R}_{q,+}) \ (\forall\, k \in \mathbb{N}).$$

(ii) For $n = 1, 2, 3, \ldots$ and $\breve{w}_n, \breve{w} \in \mathbb{B}$, the sequence $(\breve{w}_n)$ is said to be Δ_q-convergent to $\breve{w}$, denoted by $\Delta_q\text{-}\lim_{n\to\infty} \breve{w}_n = \breve{w}$, provided there can be found a q-delta sequence (ϵ_n) such that

$$(\breve{w}_n - \breve{w}) \overset{q}{\bullet} \epsilon_n \in L_q^1(\mathbb{R}_{q,+}) \quad (\forall n \in \mathbb{N}) \quad \text{and} \quad \lim_{n\to\infty} (\breve{w}_n - \breve{w}) \overset{q}{\bullet} \epsilon_n = 0 \quad \text{in } L_q^1(\mathbb{R}_{q,+}).$$

Now we have the following few corollaries.

Corollary 9 (i) *Let $g \in L_q^1(\mathbb{R}_{q,+})$ and $(\epsilon_n) \in \Delta_q$ be fixed. Then the mapping*

$$g \to \breve{w},$$

where $\breve{w} = \frac{g \overset{q}{\bullet} \epsilon_n}{\epsilon_n}$ is an injective mapping from $L_q^1(\mathbb{R}_{q,+})$ into $\mathbb{B}$.
(ii) *Let $(\epsilon_n) \in \Delta_q$. Then, if $g_n \to g$ in $L_q^1(\mathbb{R}_{q,+})$ as $n \to \infty$, then for all $k \in \mathbb{N}$,*

$$g_n \overset{q}{\bullet} \epsilon_k \to g \overset{q}{\bullet} \epsilon_k \quad \textit{and} \quad \breve{w}_n \to \breve{w} \quad \textit{in } \mathbb{B} \textit{ as } n \to \infty.$$

Therefore, it can be easily checked that $L_q^1(\mathbb{R}_{q,+})$ can be mathematically identified as a subspace of $\mathbb{B}$.

The above corollary leads to the following corollary.

Corollary 10 *The q-embedding, $g \to \breve{w}$, $\breve{w} = \frac{g \overset{q}{\bullet} \epsilon_n}{\epsilon_n}$, of the space $L_q^1(\mathbb{R}_{q,+})$ into the space $\mathbb{B}$ is continuous.*

3 The *q*-ultraBoehmian space $\mathbb{B}_{\mathbb{M}}$

In this section, we provide sufficient axioms to define the q-ultraBoehmian space $\mathbb{B}_{\mathbb{M}}$ with the set $(L_{\mathbb{M}}, \circ)$, the subset $(\mathbb{D}_{\mathbb{M}}, \circ)$, the set $(\Delta_{q,\mathbb{M}}, \circ)$ of q-delta sequences, and the product $\circ$, where $L_{\mathbb{M}}, \mathbb{D}_{\mathbb{M}}$, and $\Delta_{q,\mathbb{M}}$ are the q-Mellin transforms of the sets $L_q^1(\mathbb{R}), \mathbb{D}_q$, and Δ_q respectively. To this end, we introduce the following convolution operation.

Definition 11 Let ω_1 and ω_2 be in $\mathbb{B}_{\mathbb{M}}$. For ω_1 and ω_2, we define a product $\circ$ as

$$(\omega_1 \circ \omega_2)(t) = \omega_1(t)\omega_2(t). \tag{19}$$

The following assertion holds in the space $L_{\mathbb{M}}$.

Theorem 12 *Let ω_1 be in $L_{\mathbb{M}}$. Then $\omega_1 \circ \eta \in L_{\mathbb{M}}$ for all $\eta \in \mathbb{D}_{\mathbb{M}}$.*

Proof Let $\omega_1 \in L_{\mathbb{M}}$. Then, by the definition of the space $L_{\mathbb{M}}$ and the definition of the product $\circ$, we write

$$(\omega_1 \circ \omega_2)(t) = \omega_1(t)\omega_2(t) = M_q(g_1)M_q(g_2) \tag{20}$$

for some $g_1, g_2 \in L_q^1(\mathbb{R}_{q,+})$. Hence, by virtue of Def. 11, (20) can be written in the form

$$(\omega_1 \circ \omega_2)(t) = M_q(g_1 \overset{q}{\bullet} g_2). \tag{21}$$

Therefore, as $g_1 \circ g_2 \in L_q^1(\mathbb{R}_{q,+})$, it follows from (21) that $\omega_1 \circ \eta \in L_{\mathbb{M}}$. This ends the proof of the theorem. □

Theorem 13 *Let ω be an integrable function in $\mathbb{L}_{\mathbb{M}}$. Then $\omega \circ (\eta_1 \circ \eta_2) = (\omega \circ \eta_1) \circ \eta_2$ for all $\eta_1, \eta_2 \in \mathbb{D}_q$.*

Proof By the concept of the convolution $\circ$, we get

$$\bigl(\omega \circ (\eta_1 \circ \eta_2)\bigr)(t) = \omega(t)(\eta_1 \circ \eta_2)(t) = \omega(t)\eta_1(t)\eta_2(t).$$

By using Def. 11 twice, we write the preceding equation as

$$\bigl(\omega \circ (\eta_1 \circ \eta_2)\bigr)(t) = (\omega \circ \eta_1)(t)\eta_2(t) = \bigl((\omega \circ \eta_1) \circ \eta_2\bigr)(t).$$

This ends the proof of the theorem. □

The following axioms are straightforward.

Theorem 14 (i) *Let ω_1 and ω_2 be in $L_{\mathbb{M}}$. Then $(\omega_1 + \omega_2) \circ \eta = \omega_1 \circ \eta + \omega_2 \circ \eta$ for all $\eta \in \mathbb{D}_q$.*
(ii) *Let ω_1 be in $L_{\mathbb{M}}$. Then $(\alpha\omega_1 \circ \eta) = \alpha(\omega_1 \circ \eta)$ for all $\eta \in \mathbb{D}_q$ and $\alpha \in \mathbb{C}$.*

Proof (i) Let ω_1 and ω_2 be in $L_{\mathbb{M}}$. Then, by Def. 11, we write

$$\bigl((\omega_1 + \omega_2) \circ \eta\bigr)(t) = (\omega_1 + \omega_2)(t)\eta(t) = \omega_1(t)\eta(t) + \omega_2(t)\eta(t) = (\omega_1 \circ \eta)(t) + (\omega_2 \circ \eta)(t).$$

The proof of the first part is finished. The proof of the second part is trivial. This completes the proof of the theorem. □

Theorem 15 (i) *Let ω_1 and (ω_n) be members of the space $L_{\mathbb{M}}$ and $\eta \in \mathbb{D}_{\mathbb{M}}$. If $\omega_n \to \omega_1$ in $L_{\mathbb{M}}$ as $n \to \infty$, then $\omega_n \circ \eta \to \omega_1 \circ \eta$ as $n \to \infty$.*
(ii) *Let ω_1 and ω_2 be in $L_{\mathbb{M}}$ and $(\upsilon_n) \in \Delta_{q,\mathbb{M}}$. If $\omega_1 \circ \upsilon_n = \omega_2 \circ \upsilon_n$, then $\omega_1 = \omega_2$ in $L_{\mathbb{M}}$.*
(iii) *Let ω_1 be an integrable function in $L_{\mathbb{M}}$ and $(\upsilon_n) \in \Delta_{q,\mathbb{M}}$, $\upsilon_n(t) \neq 0$ for all $t \in \mathbb{R}_{q,+}$. Then $\omega_1 \circ \upsilon_n \to 0$ in $L_{\mathbb{M}}$ as $n \to \infty$.*

Proof To prove (i), let ω_1 and (ω_n) be members of $L_{\mathbb{M}}$ and $\eta \in \mathbb{D}_{\mathbb{M}}$. If $\omega_n \to \omega_1$ in $L_{\mathbb{M}}$ as $n \to \infty$, then by Def. 11 and Theo. 14, we have

$$(\omega_n \circ \eta - \omega_1 \circ \eta)(t) = \bigl((\omega_n - \omega_1) \circ \eta\bigr)(t) = (\omega_n - \omega_1)(t)\eta(t) = \omega_n(t)\eta(t) - \omega_1(t)\eta(t).$$

Hence, by the hypothesis of the theorem, we obtain

$$\omega_n \circ \eta - \omega_1 \circ \eta \to \omega_1 \circ \eta - \omega_1 \circ \eta \to 0 \quad \text{as } n \to \infty.$$

Hence, the first part of the theorem is completely proved. To prove (ii), let ω_1 and ω_2 be in $L_{\mathbb{M}}$ and $(\upsilon_n) \in \Delta_{q,\mathbb{M}}$. If $\omega_1 \circ \upsilon_n = \omega_2 \circ \upsilon_n$, then $\omega_1(t)\upsilon_n(t) = \omega_2(t)\upsilon_n(t)$. Hence,

$$(\omega_1 - \omega_2)(t)\upsilon_n(t) = 0 \quad \text{for all } t \in \mathbb{R}_{q,+}.$$

Therefore, $(\omega_1 - \omega_2)(t) = 0$ for all $\mathbb{R}_{q,+}$. Thus, $\omega_1 = \omega_2$ in $L_{\mathbb{M}}$. The proof of (iii) is similar. Hence, the theorem is completely proved.

If $(\omega_n) \in L_{\mathbb{M}}$ and $(\upsilon_n) \in \Delta_{q,\mathbb{M}}$, then the pair (ω_n, υ_n) (or $\frac{\omega_n}{\upsilon_n}$) is said to be a q-quotient of sequences if

$$\omega_n \circ \upsilon_m = \omega_m \circ \upsilon_n, \quad \forall n, m \in \mathbb{N}.$$

Therefore, if $\frac{\omega_n}{\epsilon_n}$ and $\frac{g_n}{\upsilon_n}$ are q-quotients of sequences and $\omega \in L_{\mathbb{M}}$, then it is easy to see that

$$\frac{\omega \circ \epsilon_n}{\epsilon_n} \quad \text{and} \quad \frac{\omega_n \circ \epsilon_n + g_n \circ \epsilon_n}{\epsilon_n \circ \upsilon_n}$$

are q-quotients of sequences. Furthermore, it is easy to see the following equivalence relations:

$$\frac{\omega_n}{\epsilon_n \circ \omega} \sim \frac{\omega_n \circ \omega}{\epsilon_n} \quad \text{and} \quad \frac{\omega_n}{\epsilon_n \circ g_n} \sim \frac{\omega_n \circ g_n}{\epsilon_n}.$$

Two q-quotients of sequences $\frac{\omega_n}{\epsilon_n}$ and $\frac{g_n}{\upsilon_n}$ are said to be equivalent if $\omega_n \circ \upsilon_m = g_m \circ \epsilon_n, \forall n, m \in \mathbb{N}$. The equivalent class $\breve{w} = (\frac{\omega_n}{\epsilon_n})$ of q-quotients of sequences containing $\frac{\varphi_n}{\epsilon_n}$ is said to be a q-Boehmian. The space of such q-Boehmians is denoted by $\mathbb{B}_{\mathbb{M}}$. □

Remark 16 For two q-Boehmians $\breve{w} = (\frac{\omega_n}{\epsilon_n})$ and $\breve{z} = (\frac{g_n}{\upsilon_n})$ in $\mathbb{B}_{\mathbb{M}}$, the following are well defined on $\mathbb{B}_{\mathbb{M}}$:

(i) $\breve{w} + \breve{z} = \left(\frac{\omega_n \circ \epsilon_n + g_n \circ \epsilon_n}{\epsilon_n \circ \upsilon_n}\right)$,

(ii) $\beta \breve{w} = \left(\frac{\beta \omega_n}{\epsilon_n}\right)$,

(iii) $\breve{w} \circ \breve{z} = \left(\frac{\omega_n \circ g_n}{\epsilon_n \circ \upsilon_n}\right)$,

(iv) $D^k \breve{w} = \left(\frac{D^k \omega_n}{\epsilon_n}\right)$,

(v) $\breve{w} \circ \omega = \left(\frac{\omega_n \circ \omega}{\epsilon_n}\right)$,

where $k \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $D^k \breve{w}$ is the kth derivative of $\breve{w}$, and $\psi \in L_{\mathbb{M}}$.

Definition 17 (i) For $n = 1, 2, 3, \ldots$ and $\breve{w}_n, \breve{w} \in \mathbb{B}_{\mathbb{M}}$, the sequence $(\breve{w}_n)$ is said to be δ_q-convergent to $\breve{w}$, denoted by $\delta_q - \lim_{n \to \infty} \breve{w}_n = \breve{w}$, provided there can be found a q-delta sequence (υ_n) such that

$$(\breve{w}_n \circ \upsilon_k), (\breve{w} \circ \upsilon_k) \quad \text{in } L_{\mathbb{M}}\ (\forall n, k \in \mathbb{N}) \quad \text{and} \quad \lim_{n \to \infty} \breve{w}_n \circ \upsilon_k = \breve{w} \circ \upsilon_k \quad \text{in } L_{\mathbb{M}}\ (\forall\, k \in \mathbb{N}).$$

(ii) For $n = 1, 2, 3, \ldots$ and $\breve{w}_n, \breve{w} \in \mathbb{B}_{\mathbb{M}}$, the sequence $(\breve{w}_n)$ is said to be Δ_q-convergent to $\breve{w}$, denoted by Δ_q-$\lim_{n \to \infty} \breve{w}_n = \breve{w}$, provided there can be found a q-delta q-sequence (υ_n) such that

$$(\breve{w}_n - \breve{w}) \circ \upsilon_n \in L_{\mathbb{M}} \quad (\forall n \in \mathbb{N}) \quad \text{and} \quad \lim_{n \to \infty} (\breve{w}_n - \breve{w}) \circ \upsilon_n = 0 \quad \text{in } L_{\mathbb{M}}.$$

Corollary 18 (i) *Let $\omega \in L_{\mathbb{M}}$ and $(\upsilon_n) \in \Delta_q$ be fixed. Then the mapping*

$$\omega \to \breve{w},$$

where $\breve{w} = \frac{\omega \circ \upsilon_n}{\upsilon_n}$ is an injective mapping from $L_{\mathbb{M}}$ into $\mathbb{B}_{\mathbb{M}}$.

(ii) *Let $(\upsilon_n) \in \Delta_{q,\mathbb{M}}$. Then, if $\omega_n \to \omega$ in $L_{\mathbb{M}}$ as $n \to \infty$, then for all $k \in \mathbb{N}$,*

$$\omega_n \circ \upsilon_k \to \omega \circ \upsilon_k \quad \text{and} \quad \breve{w}_n \to \breve{w} \quad \text{in } \mathbb{B}_{\mathbb{M}} \text{ as } n \to \infty. \tag{22}$$

Therefore, it can be easily checked that $L_{\mathbb{M}}$ may be identified as a subspace of $\mathbb{B}_{\mathbb{M}}$.

The above corollary can be stated as follows.

Corollary 19 *The q-embedding $\psi \to \breve{w}$, $\breve{w} = \frac{\omega \circ \upsilon_n}{\upsilon_n}$, of the space $L_{\mathbb{M}}$ into the space $\mathbb{B}_{\mathbb{M}}$ is continuous.*

4 The q-Mellin transform of the generalized q-theory

This section aims to discuss a definition and some basic properties of the generalized q-Mellin transform in a context of the new q-theory. All results are brief and concise, and may give the reader a general overview of the generalized q-theory of the Mellin operator. However, by virtue of the preceding analysis, we introduce the following definition.

Definition 20 Let $\frac{g_n}{\varepsilon_n} \in \mathbb{B}$, then we define the q-Mellin transform of the q-Boehmian $\frac{g_n}{\varepsilon_n}$ as

$$\mathbb{M}_q \frac{g_n}{\varepsilon_n} = \tilde{\omega}_n, \tag{23}$$

where $\tilde{\omega}_n = \frac{\omega_n}{\upsilon_n}, \omega_n = M_q g$, and $\upsilon_n = M_q \varepsilon_n$. Indeed $\tilde{\omega}_n$ belongs to $\mathbb{B}_{\mathbb{M}}$.

Theorem 21 *The operator $\mathbb{M}_q : \mathbb{B} \to \mathbb{B}_{\mathbb{M}}$ is sequentially continuous, i.e., if $\Delta_q - \lim_{k\to\infty} \tilde{\omega}_{n,k} = \tilde{\omega}_n$ in $\mathbb{B}$, then $\Delta_{q,\mathbb{M}} - \lim_{n\to\infty} \mathbb{M}_q \tilde{\omega}_{n,k} = \mathbb{M}_q \tilde{\omega}_n$ in $\mathbb{B}_{\mathbb{M}}$.*

Proof Let $\Delta_q - \lim_{k\to\infty} \tilde{\omega}_{n,k} = \tilde{\omega}_n$ in $\mathbb{B}$, then there is $(\varepsilon_n) \in \Delta_q$ such that

$$\Delta_q - \lim_{n\to\infty} (\tilde{\omega}_{n,k} - \tilde{\omega}_n) \overset{q}{\bullet} \varepsilon_n = 0 \quad \text{in } \mathbb{B}.$$

The continuity of the integral operator gives

$$\Delta_{q,\mathbb{M}} - \lim_{n\to\infty} \mathbb{M}_q \big((\tilde{\omega}_{n,k} - \tilde{\omega}_n) \overset{q}{\bullet} \varepsilon_n \big) = \Delta - \lim_{n\to\infty} \big((\mathbb{M}_q \tilde{\omega}_{n,k} - \mathbb{M}_q \tilde{\omega}_n) \circ \upsilon_n \big) = 0,$$

where $\mathbb{M}_q \varepsilon_n = \upsilon_n$. Thus, we have $\Delta_{q,\mathbb{M}} - \lim_{n\to\infty} \mathbb{M}_q \tilde{\omega}_{n,k} = \mathbb{M}_q \tilde{\omega}_n$ in $\mathbb{B}_{\mathbb{M}}$.

This finishes the proof of the theorem. □

Theorem 22 (i) *$\mathbb{M}_q$ is a linear isomorphism from the space $\mathbb{B}$ onto the space $\mathbb{B}_{\mathbb{M}}$.*

(ii) *$\mathbb{M}_q$ is continuous with respect to δ_q and Δ_q-convergence.*

(iii) *The operator $\mathbb{M}_q$ coincides with the operator M_q.*

Proof We prove Part (iii) since similar proofs for Part (i)–Part (ii) are available in literature. Let $g \in L_q^1(\mathbb{R}_{q,+})$ and $\frac{g \overset{q}{\bullet} \varepsilon_n}{\varepsilon_n}$ be its representative in $\mathbb{B}$, where $(\varepsilon_n) \in \Delta_q$ $(\forall n \in \mathbb{N})$. Clearly, for all $n \in \mathbb{N}$, (ε_n) is independent from the representative. Let $\mathbb{M}_q \varepsilon_n = \upsilon_n$, then, by the q-convolution theorem, we get

$$\mathbb{M}_q \frac{g \overset{q}{\bullet} \varepsilon_n}{\varepsilon_n} = \mathbb{M}_q \frac{g \overset{q}{\bullet} \varepsilon_n}{\varepsilon_n} = \frac{M_q g \circ M_q \varepsilon_n}{M_q \varepsilon_n} = M_q g \circ \frac{M_q \varepsilon_n}{M_q \varepsilon_n} = \omega \circ \frac{\upsilon_n}{\upsilon_n}.$$

Hence, the q-Boehmian $\frac{\omega \circ \upsilon_n}{\upsilon_n}$ is the representative of $\mathbb{M}_q$ in the space $L_{\mathbb{M}}$, where $\omega = M_q g$.

The proof is, therefore, ended. □

We introduce the inverse transform of $\mathbb{M}_q$ as follows.

Definition 23 We define the inverse integral operator of $\mathbb{M}_q$ of a q-Boehmian $\frac{\omega_n}{\upsilon_n}$ in $\mathbb{B}_{\mathbb{M}}$ as follows:

$$\mathbb{N}_q \frac{\omega_n}{\upsilon_n} = \frac{g_n}{\varepsilon_n} \in \mathbb{B},$$

where $\upsilon_n = \mathbb{M}_q \varepsilon_n$ and $\omega_n = M_q g_n$ for some $(\varepsilon_n) \in \Delta_q$ and $\{g_n\} \in L_q^1(\mathbb{R}_{q,+})$.

Theorem 24 *Let* $\frac{\omega_n}{\upsilon_n} \in \mathbb{B}_{\mathbb{M}}$ *and* $\omega \in L_{\mathbb{M}}$. *Then we have*

$$\mathbb{N}_q \left(\frac{\omega_n}{\upsilon_n} \circ \omega \right) = \frac{g_n}{\varepsilon_n} \overset{q}{\bullet} g \quad \textit{and} \quad \mathbb{M}_q \left(\frac{g_n}{\varepsilon_n} \overset{q}{\bullet} g \right) = \frac{\omega_n}{\upsilon_n} \circ \omega.$$

Proof Assume $\frac{\omega_n}{\upsilon_n} \in \mathbb{B}_{\mathbb{M}}$ where $\omega_n = M_q g_n$. Then, for every $\omega = M_q g \in L_{\mathbb{M}}$ and $\upsilon_n = M_q \varepsilon_n$, we have

$$\mathbb{N}_q \left(\frac{\omega_n}{\upsilon_n} \circ M_q g \right) = \mathbb{N}_q \frac{\omega_n \circ \omega}{\upsilon_n} = \mathbb{N}_q \frac{M_q (g_n \overset{q}{\bullet} g)}{\upsilon_n} = \frac{g_n \overset{q}{\bullet} g}{\varepsilon_n} = \frac{g_n}{\varepsilon_n} \overset{q}{\bullet} g.$$

The proof of the first part is finished. The proof of the second part is almost similar. Hence, we omit the details.

This completely ends the proof of the theorem. □

5 Conclusion

This paper has given an extension of the quantum theory of the q-Mellin transform operator [40] to sets of q-generalized functions named q-Boehmians and q-ultraBoehmians. Every element g in the function space $L_q^1(\mathbb{R}_{q,+})$ is identified as a member in the generalized space $\mathbb{B}$ by the identification formula

$$g \to \frac{g \overset{q}{\bullet} \varepsilon_n}{\varepsilon_n},$$

where (ε_n) is an arbitrary delta sequence. It also shows that the q-embedding

$$g \to \breve{w}, \; \breve{w} = \frac{g \overset{q}{\bullet} \varepsilon_n}{\varepsilon_n}$$

of the space $L_q^1(\mathbb{R}_{q,+})$ into the space $\mathbb{B}$ is continuous, (ε_n) being an arbitrary q-delta sequence. The q-Mellin transform operator is extended to the generalized q-calculus theory, and many properties are discussed. Further, the inversion of the q-Mellin transform operator is also discussed.

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