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# Nonexistence results of Caputo-type fractional problem

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## Abstract

In this paper, we deal with Caputo-type fractional differential inequality where there is a low-order fractional derivative with the term polynomial source. We investigate the nonexistence of nontrivial global solutions in a suitable space via the test function technique and some properties of fractional integrals. Finally, we demonstrate three examples to illustrate our results. The presented results are more general than those in the literature, which can be obtained as particular cases.

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## 1 Introduction

Fractional-order differential equations have more benefits in contrast with integer-order differential equations. These equations are adaptable and exact in portraying the changing law of things. Hence fractional-order differential equations are broadly utilized in real life [1–6]. Nonetheless, some physical meanings of the fractional differential equations (FDEs) are yet to be generally perceived because of the intricacy of its initial values; consequently, the improvement of the theory of FDEs is as yet in its early stages. However, these equations have become a significant point among numerous researchers in light of their wide practical applications and theoretical importance.

Although fractional calculus was proposed 300 years ago, the scientists and researchers are still developing and building up this field significantly, as it is closely related to many other disciplines. Because of the significance of fractional calculus in applications, in the previous few decades, there has been a developing interest in the investigation of FDEs. Specifically, from the theoretical perspective, the existence of solutions for various classes of FDEs was discussed in numerous contributions (see, e.g., [7–19]). As applications, there are many recent models and numerical results regarding several classes of FDEs involving various types of fractional derivatives (FDs) [20–26]. Regarding the problem of the nonexistence of solutions for FDEs, we refer to [27–36]. In this regard, we consider the following

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problem:

$$\begin{cases} {}^C D_0^\lambda \varkappa(t) + {}^C D_0^\gamma \varkappa(t) = \mathfrak{F}(t, \varkappa(t)), & t > 0, \\ \varkappa^{(i)}(0) = b_i, & i = 0, 1, \dots, n-1, n = -[-\lambda], \end{cases} \tag{1}$$

where  ${}^C D_a^\sigma$  is the Caputo derivative of order  $\sigma (> 0) \in \{\lambda, \gamma\}$ ,  $n < \lambda, \gamma < n + 1$  ( $n \in \mathbb{N}$ ).

We will highlight the nonexistence result of nontrivial global solutions for (1) along with the following condition:

$$\mathfrak{F}(t, \varkappa(t)) \geq t^\eta |{}^C D_0^\vartheta \varkappa(t)|^m, \quad \lambda = \kappa + 1, \quad \gamma = \nu + 1,$$

for some  $m > 1$ ,  $\eta \in \mathbb{R}$ , and  $n - 1 < \lambda, \gamma, \vartheta < n$  ( $n \in \mathbb{N}$ ). That is, we regard the following problem:

$$\begin{cases} {}^C D_0^{\kappa+1} \varkappa(t) + {}^C D_0^{\nu+1} \varkappa(t) \geq t^\eta |{}^C D_0^\vartheta \varkappa(t)|^m, & t > 0, m > 1, \\ \varkappa^{(i)}(0) = b_i, & i = 0, 1, \dots, n, n = -[-\kappa], \end{cases} \tag{2}$$

where  $\kappa, \nu, \vartheta \in (n - 1, n)$ ,  $n \in \mathbb{N}$ , and we show that there are no solutions for specific values of  $\eta$  and  $m$ . Specifically, we discover the range of values of  $m$  for which solutions do not exist globally. Obviously, sufficient conditions for nonexistence give necessary conditions for the existence of solutions.

*Remark 1* The existence and uniqueness of solutions for problem (1) was discussed in [4].

*Remark 2* In the case  $\kappa = \nu = \vartheta = 0$  in (2), we obtain the problem

$$\varkappa'(t) = t^\eta \varkappa^m(t), \quad \varkappa(0) = b,$$

which has a solution

$$\varkappa(t) = \left[ \frac{1 - m}{1 + \eta} t^{1+\eta} + b^{1-m} \right]^{1/(1-m)} \quad \text{for } m > 1.$$

Notice that the solution blows up in finite time for  $m > 1$ .

*Remark 3* In case  $\lambda = 1, \gamma = 0$ , and  $\mathfrak{F}(t, \varkappa(t)) = \varkappa^m(t)$ ,  $m > 1$  in (1), we obtain the Bernoulli differential problem

$$\varkappa'(t) + \varkappa(t) = \varkappa^m(t), \quad \varkappa(0) = b,$$

which has a solution

$$\varkappa(t) = \left[ 1 + (b^{1-m} - 1) \exp(m - 1)t \right]^{1/(1-m)}.$$

Obviously,  $\varkappa(t)$  blows up in the finite time  $c = \frac{1}{1-m} \ln(1 - b^{1-m})$  for  $m, b > 1$ .

*Remark 4* In the case  $\lambda = \gamma = \kappa$  and  $\mathfrak{F}(t, \varkappa(t)) \geq t^\vartheta |\varkappa(t)|^m$  in (1), we obtain

$$\begin{cases} 2D_0^\kappa \varkappa(t) \geq t^\vartheta |\varkappa(t)|^m, & t > 0, \\ I_0^{1-\kappa} \varkappa(t)|_{t=0} = b. \end{cases} \tag{3}$$

Problem (3) was taken into consideration by Laskri and Tatar [29]. It turns out that if  $\vartheta > -\kappa$  and  $1 < m \leq \frac{\vartheta+1}{1-\kappa}$ , then problem (3) admits no global nontrivial solutions if  $b \geq 0$ .

Kassim et al. [35] studied the problem

$$\begin{cases} {}^C D_0^\kappa \varkappa(t) + {}^C D_0^\nu \varkappa(t) \geq t^\vartheta |\varkappa(t)|^m, & t > 0, \\ \varkappa^{(k)}(0) = b_k, & k = 0, 1, \dots, n-1, \end{cases} \tag{4}$$

where  $m > 1, n \geq 1$  is an integer,  $n-1 < \nu \leq \kappa < n$ , and  $b_k \geq 0$ . It turns out that if  $m(1-\nu) - 1 < \vartheta < m-1$ , then problem (4) admits no nontrivial global solutions.

In case  $0 < \kappa, \nu, \vartheta < 1$  and  $\eta = 0$  in (2), we obtain

$$\begin{cases} {}^C D_0^{\kappa+1} \varkappa(t) + {}^C D_0^{\nu+1} \varkappa(t) \geq |{}^C D_0^\vartheta \varkappa(t)|^m, & t > 0, \\ \varkappa(0) = b_0, & \varkappa'(0) = b_1. \end{cases} \tag{5}$$

Not long ago, Jleli and Samet [37] studied (5). They proved that the problem admits no global solutions if  $b_1 > 0$ .

In this work, we investigate the case of a lower order FD in the inequality (or equation). It is clear that for hyperbolic equations, for example, the wave equation with an interval fractional damping represented by the first derivative (i.e.,  $\kappa = 1, \nu = 0$ ), this damping process has a squandering effect. It will contend with the polynomial source and may take care of this blowing-up term under certain conditions. Besides, in the telegraphing problem [38], the solutions approach the solution of the same problem without the  $n$ th derivative as  $t \rightarrow \infty$  (i.e., the parabolic equation). This result has been summed up and generalized to the FD case in [38] and [30]. For our concern with a problem (2), we might want to see how effective  $D_0^\nu \varkappa$  will be on the blowup phenomenon, specifically, how the range of values  $m$  guaranteeing to blowup in finite time would be influenced. We arrived at the conclusion that here it is the lower-order derivative (i.e.,  $\nu$ ), which determines the range of blowup much the same as the parabolic portion and hyperbolic problem.

In Sect. 2, we give some notations, definitions, and lemmas required later in our analysis. Sections 3 and 4 are committed to the test function and the nonexistence result. In Sect. 5, we provide some examples to justify the preceding results. In the final section, we close our work with concluding remarks.

## 2 Preliminaries

In this section, we recall some primary facts utilized in our outcomes. We refer the reader to [4–6] for additional insights about FDs.

**Definition 1** We denote by  $AC[0, \infty)$  the space of absolutely continuous functions on  $[0, \infty)$  and by  $AC^n[0, \infty)$  the space of functions  $\varpi$  that have continuous derivatives up to order  $n-1$  on  $[0, \infty)$  such that  $\varpi^{(n-1)} \in AC[0, \infty)$ , where  $\varpi^{(n-1)}$  denotes the derivative of order  $n-1$  of  $\varpi$ .

**Definition 2** We denote by  $L_p(a, b)$ ,  $p \geq 1$ , the spaces of Lebesgue-integrable functions on  $(a, b)$ .

**Definition 3** Let  $a < t < b$  and  $\kappa > 0$ , and let  $\varpi \in L_1(a, b)$ . Then the left- and right-sided Riemann–Liouville fractional integrals of order  $\kappa$  of  $\varpi$  are given by

$$(I_a^\kappa \varpi)(t) := \frac{1}{\Gamma(\kappa)} \int_a^t \frac{\varpi(r)}{(t-r)^{1-\kappa}} dr \tag{6}$$

and

$$(I_{b-}^\kappa \varpi)(t) := \frac{1}{\Gamma(\kappa)} \int_t^b \frac{\varpi(r)}{(r-t)^{1-\kappa}} dr, \tag{7}$$

respectively, where  $\Gamma$  is the gamma function. Note that if  $\kappa = 0$ , then  $I_a^0 \varpi = I_{b-}^0 \varpi = \varpi$ .

**Definition 4** Let  $\varpi \in AC^n[0, \infty)$ . The expression

$${}^C D_a^\kappa \varpi(t) = I_a^{n-\kappa} \varpi^{(n)}(t) = \frac{1}{\Gamma(n-\kappa)} \int_a^t \frac{\varpi^{(n)}(s)}{(t-s)^{\kappa+1-n}} ds \tag{8}$$

is called left-sided Caputo FD of order  $\kappa$  of  $\varpi$ .

**Lemma 1** ([6]) Let  $\kappa > 0$ ,  $p \geq 1$ ,  $q \geq 1$ , and  $\frac{1}{p} + \frac{1}{q} \leq 1 + \kappa$  ( $p \neq 1$  and  $q \neq 1$  in the case where  $\frac{1}{p} + \frac{1}{q} = 1 + \kappa$ ). If  $\Theta_1 \in L_p(a, b)$  and  $\Theta_2 \in L_q(a, b)$ , then

$$\int_a^b \Theta_1(t)(I_a^\kappa \Theta_2)(t) dt = \int_a^b \Theta_2(t)(I_{b-}^\kappa \Theta_1)(t) dt. \tag{9}$$

**Lemma 2** If  $\kappa \geq 0$  and  $\nu > 0$ , then

$$I_T^\kappa (T-t)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\kappa+1)} (T-t)^{\nu+\kappa},$$

$$D_T^\kappa (T-t)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\kappa+1)} (T-t)^{\nu-\kappa}.$$

**Lemma 3** Let  $\kappa > 0$  and  $\nu > 0$ . If  $\varpi \in L_p(a, b)$ , then

$$I_a^\nu I_a^\kappa \varpi(t) = I_a^{\nu+\kappa} \varpi(t), \quad t > a,$$

$$I_b^\nu I_b^\kappa \varpi(t) = I_b^{\nu+\kappa} \varpi(t), \quad t < b.$$

**Lemma 4** Let  $\kappa \in (n-1, n)$ ,  $n \in \mathbb{N}$ , and  $\varpi \in L_p(0, b)$ . Then the IVP

$${}^C D_0^\kappa \varkappa(t) = \varpi(t), \quad t > 0,$$

$$\varkappa^{(i)}(0) = c_i, \quad i = 0, 1, \dots, n-1,$$

has the solution

$$\varkappa(t) = c_0 + c_1 t + \dots + \frac{c_{n-1}}{(n-1)!} t^{n-1} + \frac{1}{\Gamma(\kappa)} \int_0^t (t-r)^{\kappa-1} \varpi(r) dr.$$

### 3 The test function

We use the test function

$$\Theta(t) = \begin{cases} T^{-\varsigma}(T-t)^\varsigma, & 0 \leq t \leq T, \varsigma > 0, \\ 0, & t > T. \end{cases} \tag{10}$$

This test function has the following properties.

**Lemma 5** *Let  $\eta \geq 0$  and  $\Theta$  be as in (10). Then*

$$\begin{aligned} I_T^\eta \Theta(t) &= \frac{\Gamma(\varsigma + 1)}{\Gamma(\eta + \varsigma + 1)} T^{-\varsigma} (T-t)^{\varsigma+\eta}, \\ I_T^\eta \Theta(0) &= \frac{\Gamma(\varsigma + 1)}{\Gamma(\eta + \varsigma + 1)} T^\eta \quad \text{and} \quad I_T^\eta \Theta(T) = 0, \\ \frac{d^2}{dt^2} I_T^\eta \Theta(t) &= \frac{\Gamma(\varsigma + 1)}{\Gamma(\eta + \varsigma - 1)} T^{-\varsigma} (T-t)^{\varsigma+\eta-2}. \end{aligned}$$

*Proof* It follows from Lemma 2. □

**Lemma 6** *Let  $\Theta$  be as in (10) with  $\varsigma + p(v + \kappa - 2) + 1 > 0$ ,  $\vartheta(1 - p) + 1 > 0$ ,  $v, \kappa \geq 0$ , and  $p > 1$ . Then*

$$\int_0^T t^{\vartheta(1-p)} \Theta^{1-p}(t) \left( I_T^v \frac{d^2}{dt^2} I_T^\kappa \Theta(t) \right)^p dt = K_{v,\kappa}^{\vartheta,p,\varsigma} T^{\vartheta(1-p)+p(v+\kappa-2)+1},$$

where

$$K_{v,\kappa}^{\vartheta,p,\varsigma} = \left[ \frac{\Gamma(\varsigma + 1)}{\Gamma(v + \kappa + \varsigma - 1)} \right]^p \frac{\Gamma(\vartheta(1 - p) + 1) \Gamma(\varsigma + p(v + \kappa - 2) + 1)}{\Gamma(\vartheta(1 - p) + \varsigma + p(v + \kappa - 2) + 2)}.$$

*Proof* By Lemma 5 we have

$$\begin{aligned} I_T^v \frac{d^2}{dt^2} I_T^\kappa \Theta(t) &= \frac{\Gamma(\varsigma + 1)}{\Gamma(\kappa + \varsigma - 1)} T^{-\varsigma} I_T^v (T-t)^{\varsigma+\kappa-2} \\ &= \frac{\Gamma(\varsigma + 1)}{\Gamma(v + \kappa + \varsigma - 1)} T^{-\varsigma} (T-t)^{v+\varsigma+\kappa-2}. \end{aligned}$$

Then

$$\begin{aligned} &\int_0^T t^{\vartheta(1-p)} \Theta^{1-p}(t) \left( I_T^v \frac{d^2}{dt^2} I_T^\kappa \Theta(t) \right)^p dt \\ &= \left[ \frac{\Gamma(\varsigma + 1)}{\Gamma(v + \kappa + \varsigma - 1)} \right]^p T^{-p\varsigma} \int_0^T t^{\vartheta(1-p)} [T^{-\varsigma}(T-t)^\varsigma]^{1-p} (T-t)^{p(v+\kappa-2)} dt \\ &= \left[ \frac{\Gamma(\varsigma + 1)}{\Gamma(v + \kappa + \varsigma - 1)} \right]^p T^{-\varsigma} \int_0^T t^{\vartheta(1-p)} (T-t)^{\varsigma+p(v+\kappa-2)} dt. \end{aligned}$$

Let  $t = sT$ . Then

$$\int_0^T t^{\vartheta(1-p)} \Theta^{1-p}(t) \left( I_T^v \frac{d^2}{dt^2} I_T^\kappa \Theta(t) \right)^p dt$$

$$\begin{aligned}
 &= \left[ \frac{\Gamma(\zeta + 1)}{\Gamma(\nu + \kappa + \zeta - 1)} \right]^p T^{\vartheta(1-p)+p(\nu+\kappa-2)+1} \int_0^1 s^{\vartheta(1-p)} (1-s)^{\zeta+p(\nu+\kappa-2)} ds \\
 &= \left[ \frac{\Gamma(\zeta + 1)}{\Gamma(\nu + \kappa + \zeta - 1)} \right]^p \frac{\Gamma(\vartheta(1-p) + 1)\Gamma(\zeta + p(\nu + \kappa - 2) + 1)}{\Gamma(\vartheta(1-p) + \zeta + p(\nu + \kappa - 2) + 2)} T^{\vartheta(1-p)+p(\nu+\kappa-2)+1}. \quad \square
 \end{aligned}$$

*Remark 5* For the rest of the paper, we will utilize the following equivalency. If  $m, m' > 1$ , and  $\frac{1}{m} + \frac{1}{m'} = 1$ , then:

1.  $m' = \frac{m}{m-1}$ .
2.  $\frac{m'}{m} = m' - 1$ .
3.  $m(\kappa - 1) + 1 > 0 \iff m'\kappa > 1$  for  $\kappa > 0$ .

### 4 Nonexistence result

In this part, we discuss the problem

$$\begin{cases} {}^C D_0^{\kappa+1} \varkappa(t) + {}^C D_0^{\nu+1} \varkappa(t) \geq t^\eta |{}^C D_0^\vartheta \varkappa(t)|^m, & t > 0, m > 1, 0 < \nu < \kappa, \\ \varkappa^{(i)}(0) = b_i, & i = 0, 1, \dots, n, n = -[\kappa]. \end{cases} \tag{11}$$

**Theorem 1** *Let  $m(\vartheta - n) + n - \nu - 1 < \eta < m - 1$  and  $m > 1$ . If  $b_n > 0$ , then problem (11) admits no global nontrivial solutions in  $AC^{n+1}[0, \infty)$ .*

*Proof* Let  $\varkappa \in AC^{n+1}[0, \infty)$  be a global solution to (11). Let  $\Theta$  be as in (10) with  $\zeta > \frac{m}{m-1}(\kappa + 1 - \vartheta) - 1$ . Multiplying both sides of (11) by  $\Theta(t)$  and integrating over  $[0, T]$ , we get

$$I = \int_0^T \Theta(t) t^\eta |{}^C D_0^\vartheta \varkappa(t)|^m dt \leq I_1 + I_2, \tag{12}$$

where

$$I_1 = \int_0^T \Theta(t) {}^C D_0^{\kappa+1} \varkappa(t) dt$$

and

$$I_2 = \int_0^T \Theta(t) {}^C D_0^{\nu+1} \varkappa(t) dt.$$

From the definition of  ${}^C D_0^\vartheta \varkappa(t)$  and Lemma 1 we have

$$I_1 = \int_0^T \Theta(t) I_0^{n-\kappa} \varkappa^{(n+1)}(t) dt = \int_0^T \varkappa^{(n+1)}(t) I_T^{n-\kappa} \Theta(t) dt. \tag{13}$$

Integrating by parts and using Lemma 5, we get

$$\begin{aligned}
 I_1 &= \varkappa^{(n)}(t) I_T^{n-\kappa} \Theta(t) \Big|_{t=0}^T - \int_0^T \varkappa^{(n)}(t) \frac{d}{dt} I_T^{n-\kappa} \Theta(t) dt \\
 &= -b_n I_T^{n-\kappa} \Theta(0) - \int_0^T \varkappa^{(n)}(t) \frac{d}{dt} I_T^{n-\kappa} \Theta(t) dt. \tag{14}
 \end{aligned}$$

On the other hand, by Lemma 3 we obtain

$$\begin{aligned} \varkappa^{(n)}(t) &= \frac{d}{dt} \int_0^t \varkappa^{(n)}(s) ds = \frac{d}{dt} (I_0^1 \varkappa^{(n)}(t)) \\ &= \frac{d}{dt} (I_0^{\vartheta+1-n} I_0^{n-\vartheta} \varkappa^{(n)}(t)) = \frac{d}{dt} (I_0^{\vartheta+1-n} {}^C D_0^\vartheta \varkappa(t)). \end{aligned}$$

Then

$$I_1 = -b_n I_T^{n-\kappa} \Theta(0) - \int_0^T \frac{d}{dt} (I_0^{\vartheta+1-n} {}^C D_0^\vartheta \varkappa(t)) \frac{d}{dt} I_T^{n-\kappa} \Theta(t) dt.$$

Using integration by parts and Lemma 1, we have that

$$\begin{aligned} I_1 &= -b_n I_T^{n-\kappa} \Theta(0) - I_0^{\vartheta+1-n} {}^C D_0^\vartheta \varkappa(t) \frac{d}{dt} I_T^{n-\kappa} \Theta(t) \Big|_{t=0}^T \\ &\quad + \int_0^T I_0^{\vartheta+1-n} {}^C D_0^\vartheta \varkappa(t) \frac{d^2}{dt^2} I_T^{n-\kappa} \Theta(t) dt \\ &= -b_n I_T^{n-\kappa} \Theta(0) + \int_0^T I_0^{\vartheta+1-n} {}^C D_0^\vartheta \varkappa(t) \frac{d^2}{dt^2} I_T^{n-\kappa} \Theta(t) dt \\ &= -b_n I_T^{n-\kappa} \Theta(0) + \int_0^T {}^C D_0^\vartheta \varkappa(t) I_T^{\vartheta+1-n} \frac{d^2}{dt^2} I_T^{n-\kappa} \Theta(t) dt. \end{aligned} \tag{15}$$

Note that

$$I_0^{\vartheta+1-n} {}^C D_0^\vartheta \varkappa(t) \frac{d}{dt} I_T^{n-\kappa} \Theta(t) \Big|_{t=T} = 0, \quad \varsigma > 1 + \kappa - n,$$

and

$$\begin{aligned} I_0^{\vartheta+1-n} {}^C D_0^\vartheta \varkappa(t) \frac{d}{dt} I_T^{n-\kappa} \Theta(t) \Big|_{t=0} &= I_0^{\vartheta+1-n} I_0^{n-\vartheta} \varkappa^{(n)}(t) \frac{d}{dt} I_T^{n-\kappa} \Theta(t) \Big|_{t=0} \\ &= I_0^1 \varkappa^{(n)}(t) \frac{d}{dt} I_T^{n-\kappa} \Theta(t) \Big|_{t=0} \\ &= (\varkappa^{(n-1)}(t) - \varkappa^{(n-1)}(0)) \frac{d}{dt} I_T^{n-\kappa} \Theta(t) \Big|_{t=0} = 0. \end{aligned}$$

Next, we insert  $t^{\eta/m} \Theta(t)^{1/m} t^{-\eta/m} \Theta(t)^{-1/m}$  inside the integral of (15):

$$I_1 = -b_n I_T^{n-\kappa} \Theta(0) + \int_0^T {}^C D_0^\vartheta \varkappa(t) \Theta(t)^{1/m} t^{\eta/m} \Theta(t)^{-1/m} t^{-\eta/m} I_T^{\vartheta+1-n} \frac{d^2}{dt^2} I_T^{n-\kappa} \Theta(t) dt.$$

Using the  $\varepsilon$ -Young inequality with  $0 < \varepsilon < 1/2$ , we obtain

$$\begin{aligned} I_1 &\leq -b_n I_T^{n-\kappa} \Theta(0) + \varepsilon \int_0^T |{}^C D_0^\vartheta \varkappa(t)|^m \Theta(t) t^\eta dt \\ &\quad + K(\varepsilon, m) \int_0^T \Theta(t)^{-m'/m} t^{-\eta m'/m} \left( I_T^{\vartheta+1-n} \frac{d^2}{dt^2} I_T^{n-\kappa} \Theta(t) \right)^{m'} dt, \quad K(\varepsilon, m) > 0. \end{aligned}$$

By Lemmas 5 and 6 we have

$$I_1 \leq -b_n \frac{\Gamma(\zeta + 1)}{\Gamma(n - \kappa + \zeta + 1)} T^{n-\kappa} + \varepsilon \int_0^T |{}^C D_0^\vartheta \varkappa(t)|^m \Theta(t) t^\eta dt + K(\varepsilon, m) K_{\vartheta+1-n, n-\kappa}^{\eta, m', \zeta} T^{\eta(1-m') + m'(\vartheta-\kappa-1) + 1}. \tag{16}$$

Similarly,

$$I_2 \leq -b_n \frac{\Gamma(\zeta + 1)}{\Gamma(n - \nu + \zeta + 1)} T^{n-\nu} + \varepsilon \int_0^T |{}^C D_0^\vartheta \varkappa(t)|^m \Theta(t) t^\eta dt + K(\varepsilon, m) K_{\vartheta+1-n, n-\nu}^{\eta, m', \zeta} T^{\eta(1-m') + m'(\vartheta-\nu-1) + 1}. \tag{17}$$

Hence from (12), (16), and (17) it follows that

$$(1 - 2\varepsilon)I + \Gamma(\zeta + 1)b_n T^{n-\nu} \left( \frac{T^{\nu-\kappa}}{\Gamma(n - \kappa + \zeta + 1)} + \frac{1}{\Gamma(n - \nu + \zeta + 1)} \right) \leq K(\varepsilon, m) K_{\vartheta+1-n, n-\kappa}^{\eta, m', \zeta} T^{\eta(1-m') + m'(\vartheta-\kappa-1) + 1} + K(\varepsilon, m) K_{\vartheta+1-n, n-\nu}^{\eta, m', \zeta} T^{\eta(1-m') + m'(\vartheta-\nu-1) + 1}. \tag{18}$$

Since  $b_n > 0$  and  $0 < \varepsilon < 1/2$ , we find

$$\Gamma(\zeta + 1)b_n T^{n-\nu} \left( \frac{T^{\nu-\kappa}}{\Gamma(n - \kappa + \zeta + 1)} + \frac{1}{\Gamma(n - \nu + \zeta + 1)} \right) \geq \frac{\Gamma(\zeta + 1)}{\Gamma(n - \nu + \zeta + 1)} b_n T^{n-\nu}$$

and

$$(1 - 2\varepsilon) \int_0^T \Theta(t) t^\eta |{}^C D_0^\vartheta \varkappa(t)|^m dt + \Gamma(\zeta + 1)b_n T^{n-\nu} \left( \frac{T^{\nu-\kappa}}{\Gamma(n - \kappa + \zeta + 1)} + \frac{1}{\Gamma(n - \nu + \zeta + 1)} \right) \geq \Gamma(\zeta + 1)b_n T^{n-\nu} \left( \frac{T^{\nu-\kappa}}{\Gamma(n - \kappa + \zeta + 1)} + \frac{1}{\Gamma(n - \nu + \zeta + 1)} \right) \geq \frac{\Gamma(\zeta + 1)}{\Gamma(n - \nu + \zeta + 1)} b_n T^{n-\nu}.$$

Therefore

$$C_1 b_n T^{n-\nu} \leq C_2 T^{\eta(1-m') + m'(\vartheta-\kappa-1) + 1} + C_3 T^{\eta(1-m') + m'(\vartheta-\nu-1) + 1},$$

where

$$C_1 = \frac{\Gamma(\zeta + 1)}{\Gamma(n - \nu + \zeta + 1)}, \quad C_2 = K(\varepsilon, m) K_{\vartheta+1-n, n-\kappa}^{\eta, m', \zeta}, \quad C_3 = K(\varepsilon, m) K_{\vartheta+1-n, n-\nu}^{\eta, m', \zeta},$$

or

$$b_n \leq \frac{1}{C_1} T^{\nu-n} (C_2 T^{\eta(1-m') + m'(\vartheta-\kappa-1) + 1} + C_3 T^{\eta(1-m') + m'(\vartheta-\nu-1) + 1})$$



$$= \frac{1}{C_1} T^{\nu-n+\eta(1-m')+m'(\vartheta-\nu-1)+1} (C_2 T^{m'(\nu-\kappa)} + C_3).$$

Note that  $\nu - n + \eta(1 - m') + m'(\vartheta - \nu - 1) + 1 < 0$  and  $m'(\nu - \kappa) < 0$ , and, consequently,  $T^{\nu-n+\eta(1-m')+m'(\vartheta-\nu-1)+1}, T^{m'(\nu-\kappa)} \rightarrow 0$  as  $T \rightarrow \infty$ . Therefore

$$b_n \leq 0.$$

We get a contradiction since  $b_n > 0$ . □

**Theorem 2** *Let  $m(\vartheta - \nu) - 1 < \eta < m - 1, m > 1,$  and  $b_n = 0$ . Then the only global solution to problem (11) is*

$$\varkappa(t) = c_0 + c_1 t + \dots + \frac{c_{n-1}}{(n-1)!} t^{n-1}, \quad t > 0.$$

*Proof* Taking  $b_n = 0$  in (18), we find

$$(1 - 2\varepsilon)I \leq C_2 T^{\eta(1-m')+m'(\vartheta-\kappa-1)+1} + C_3 T^{\eta(1-m')+m'(\vartheta-\nu-1)+1},$$

or

$$I \leq C_4 T^{\eta(1-m')+m'(\vartheta-\nu-1)+1} (C_2 T^{m'(\nu-\kappa)} + C_3), \tag{19}$$

where  $C_4 = \frac{1}{1-2\varepsilon}$  and  $0 < \varepsilon < 1/2$ . Using (10) and (12), we get

$$\int_0^T \left(1 - \frac{t}{T}\right)^5 t^\eta |{}^C D_0^\vartheta \varkappa(t)|^m dt \leq C_4 T^{\eta(1-m')+m'(\vartheta-\nu-1)+1} \times (C_2 T^{m'(\nu-\kappa)} + C_3). \tag{20}$$

Since  $\nu \leq \kappa$  and  $m(\vartheta - \nu) - 1 < \eta$ , we have  $m'(\nu - \kappa) \leq 0$  and  $\eta(1 - m') + m'(\vartheta - \nu - 1) + 1 < 0$ . Taking the limit as  $T \rightarrow \infty$  in (20) and using Fatou’s lemma, we obtain

$$\int_0^T t^\eta |{}^C D_0^\vartheta \varkappa(t)|^m dt = 0,$$

which yields

$${}^C D_0^\vartheta \varkappa(t) = 0, \quad t > 0.$$

Then by Lemma 4 we have

$$\varkappa(t) = c_0 + c_1 t + \dots + \frac{c_{n-1}}{(n-1)!} t^{n-1}, \quad t > 0. \tag{□}$$

### 5 Examples

In this section, we give some examples to justify the preceding results.

*Example 1* The fractional differential problem

$$\begin{aligned}
 {}^C D_0^{1.8} \varkappa(t) + {}^C D_0^{1.5} \varkappa(t) &\geq t^{0.5} |{}^C D_0^{0.6} \varkappa(t)|^2, \quad t > 0, \\
 \varkappa(0) = 0, \quad \varkappa'(0) &= 1,
 \end{aligned}
 \tag{21}$$

is a particular case of (11) where  $\kappa = 0.8, \nu = 0.5, \eta = 0.5, \vartheta = 0.6, m = 2, b_0 = 0,$  and  $b_1 = 1.$  Therefore by Theorem 1 the fractional differential problem (21) has no nontrivial global solutions in  $AC^2[0, \infty).$

*Example 2* The fractional differential problem

$$\begin{aligned}
 {}^C D_0^{2.7} \varkappa(t) + {}^C D_0^{2.4} \varkappa(t) &\geq t^{-0.4} |{}^C D_0^{1.5} \varkappa(t)|^3, \quad t > 0, \\
 \varkappa(0) = 1, \quad \varkappa'(0) = -2, \quad \varkappa''(0) &= 3,
 \end{aligned}
 \tag{22}$$

is a particular case of (11) where  $\kappa = 1.7, \nu = 1.4, \eta = -0.4, \vartheta = 1.5, m = 3$  and  $b_0 = 1, b_1 = -2, b_2 = 3.$  Therefore, by Theorem 1, the fractional differential problem (22) has no nontrivial global solution in  $AC^3[0, \infty).$

*Example 3* The fractional differential problem

$$\begin{aligned}
 {}^C D_0^{3.7} \varkappa(t) + {}^C D_0^{3.5} \varkappa(t) &\geq t^{0.8} |{}^C D_0^{2.6} \varkappa(t)|^3, \quad t > 0, \\
 \varkappa(0) = 4, \quad \varkappa'(0) = 1, \quad \varkappa''(0) = 2, \quad \varkappa'''(0) &= 0,
 \end{aligned}
 \tag{23}$$

is a special case of (11) when  $\kappa = 2.7, \nu = 2.5, \eta = 0.8, \vartheta = 2.6, m = 3, b_0 = 4, b_1 = 1, b_2 = 2,$  and  $b_3 = 0.$  Therefore by Theorem 2 the only global solution to problem (23) is

$$\varkappa(t) = 4 + t + t^2, \quad t > 0.$$

### 6 Concluding remarks

In this paper, we studied a new class of fractional differential inequalities involving the Caputo fractional derivatives depending on two different orders. With the aid of the test function technique and some properties of fractional integrals, we investigated the nonexistence of nontrivial global solutions in a suitable space. Three simulation examples were presented to illustrate our acquired results. Moreover, for the telegraphing problem [38], it was deduced that the solutions approach the solution of the same problem without the  $n$ th derivative as  $t \rightarrow \infty.$  This result was summed up and generalized to the FD. We investigated the case in which there is a lower-order FD in the inequality (or equation), for example, problem (2). We realized how effective  $D_0^\nu \varkappa$  would be on the blow-up phenomenon. Specifically, it affected the range of values  $m$  which guaranteed blow-ups in finite time. Based on this, we arrived at the conclusion that the lower-order derivative (i.e.,  $\nu$ ) determines the range of blow-up much the same as the parabolic portion and hyperbolic problem. The presented results are more general than those in the literature, which can be obtained as particular cases: for more detail, see Remarks 2, 3, and 4.

In future work, many cases can be established for more general operators containing another function, for instance, the generalized Caputo [39] or Hilfer [40] fractional operator. Also, it will be of interest to study the problem of this paper for the Mittag-Leffler power law [41].

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