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Lower and upper solutions method to the fully elastic cantilever beam equation with support

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Abstract

The aim of this paper is to consider a fully cantilever beam equation with one end fixed and the other connected to a resilient supporting device, that is,

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & t \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = g(u(1)), \end{cases}$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Under the assumption of monotonicity, two existence results for solutions are acquired with the monotone iterative technique and the auxiliary truncated function method.

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Keywords: Fully fourth-order boundary value problem; Cantilever beam equation; Nonlinear boundary condition; Monotone iterative technique; Lower and upper solutions

1 Introduction

In this paper, we investigate a fully fourth-order differential equation with nonlinear boundary condition

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & t \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = g(u(1)), \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are continuous functions.

The two-point boundary value problems (BVPs) of fourth-order differential equations are the mathematical models for describing the states of elastic beams. It is generally known that the elastic beams are one of the basic structures of modern architecture, aircraft, and ships. Due to their practical mathematical models and extensive application

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background, they have attracted the general attention of researchers, see [1–21] and the references therein.

The solvability for the cantilever beam equations with zero boundary conditions $u(0) = u'(0) = u''(1) = u'''(1) = 0$ has been researched by many scholars (see [5, 10–12, 18, 20, 21]). Specially, in [10–12], Li considered the cantilever beam equation

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & t \in [0, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.2}$$

and studied the existence of solutions or positive solutions by means of the fixed point theorem of completely continuous operators, the fixed point index theory in cones, and the lower and upper solutions method, respectively. It should be noted that in the above works, all the lower-order derivatives of unknown function are involved in the nonlinear function, which greatly generalizes and extends many early results.

It is common knowledge that in the force analysis of beams, the physical meaning of the derivatives $u'(t)$, $u''(t)$, $u'''(t)$, and $u^{(4)}(t)$ of $u(t)$ are slope, bending moment, shear force, and load density, respectively (see [1, 6, 7, 11–14]). Thus, the nonlinear boundary condition

$$u(0) = u'(0) = u''(1) = 0, \qquad u'''(1) = g\big(u(1)\big), \tag{1.3}$$

indicates that the shear force is equal to $g(u(1))$, which implies that there may be a nonlinear relationship between the displacement $u(1)$ and the vertical force, as well as $u''(1) = 0$ denotes that the beam has no bending moment at $t = 1$, so it is supported on the bearing g. Therefore, BVP (1.1) simulates the static deformation for a spring beam of length 1 in which the left end is fixed and the right end is linked together with an elastic bearing device described by g, which is the cantilever beam equation with support in engineering and mechanics.

Owing to its specific boundary condition and realistic physical meaning, the solvability for cantilever equation (1.1) has been studied by some scholars, see [2–4, 7, 8, 13, 14, 17, 19]. In [4, 13, 14, 17, 19]. The existence theory of solutions for the cantilever beam equation with nonlinear boundary condition (1.3) was studied under the condition that the nonlinear term f does not involve the derivative terms of deformation function u. In [2, 7, 8], the solvability conclusions were obtained by some fixed point theorems and monotone iterative method under the conditions that the nonlinear function f only contains the first-order derivative term. Recently, Azarnavid et al. [3] used the reproducing kernel space method to construct an analytical approximate solution for BVP (1.1). However, because of the influence of fully derivative terms in the nonlinear function f and the nonlinearity of the boundary condition, the solvability for BVP (1.1) has not been studied extensively. In particular, as far as we know, there are fewer results on the equations of fully cantilever beam with nonlinear boundary conditions by using the method involving lower and upper solutions.

Inspired by the above literature, in the present paper, we utilize the monotone iterative technique involving lower and upper solutions and the auxiliary truncation function method to discuss the existence of solutions for BVP (1.1) between lower solution and upper solution. In order to study BVP (1.1), we put forward some reasonable monotonicity

assumptions for the nonlinear functions f and g. Indeed, in order to reasonably introduce the definition of lower or upper solutions, we also consider the function g. Therefore, our results generalize and improve many results in the existing literature, which are new and meaningful. Our main results and proof processes are presented in Sect. 3, and two examples are given to verify our results in Sect. 4. In the following section, we introduce the definitions of lower and upper solutions and provide some preliminary results, which are useful in the proof.

2 Preliminaries

Denote $I = [0,1]$. Let $C(I)$ be a continuous function space with the norm $\|u\|_C = \max_{t \in I} |u(t)|$, and $C^n(I)$ ($n = 1, 2, 3, 4$) be an n-order continuous differentiable function space with the norm $\|u\|_{C^n} = \max\{\|u\|_C, \|u'\|_C, \dots, \|u^{(n)}\|_C\}$. Set $C^+(I) = \{u \in C(I) \mid u(t) \geq 0, t \in I\}$, which is a positive cone of $C(I)$.

Firstly, with regard to the linear boundary value problem (LBVP)

$$\begin{cases} u^{(4)}(t) = h(t), & t \in I, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{2.1}$$

By [10], we know that, for given $h \in C(I)$, LBVP (2.1) has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)\,ds := Sh(t), \quad t \in I, \tag{2.2}$$

where the Green function is defined by

$$G(t,s) = \frac{1}{6}\begin{cases} t^2(3s - t), & 0 \leq t \leq s \leq 1, \\ s^2(3t - s), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.3}$$

From (2.2) and (2.3), it is easy to find that $S : C(I) \to C^4(I)$ is a bounded linear operator. Moreover, $S : C(I) \to C^3(I)$ is completely continuous based on the compactness of imbedding $C^4(I) \hookrightarrow C^3(I)$.

Next, we consider the LBVP corresponding to BVP (1.1)

$$\begin{cases} u^{(4)}(t) = h(t), & t \in I, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, & u'''(1) = \gamma, \end{cases} \tag{2.4}$$

where $\gamma \leq 0$ is a constant.

With a simple calculation, for each $h \in C(I)$,

$$u(t) = \int_0^1 G(t,s)h(s)\,ds - \gamma\varphi(t), \quad t \in I, \tag{2.5}$$

is the unique solution of LBVP (2.4), where $G(t,s)$ is defined by (2.3) and

$$\varphi(t) = \frac{1}{2}t^2 - \frac{1}{6}t^3, \quad t \in I. \tag{2.6}$$

As a matter of fact, by (2.4) and (2.5), one can find that the solutions of BVP (1.1) can be expressed by

$$u(t) = \int_0^1 G(t,s)f\big(t,u(t),u'(t),u''(t),u'''(t)\big)\,ds - g\big(u(1)\big)\varphi(t) := Tu(t), \tag{2.7}$$

which suggests that the solutions of BVP (1.1) are equivalent to the fixed points of the operator T. For $u \in C^3(I)$, define operators $B : C^3(I) \to C^3(I)$ and $F : C^3(I) \to C(I)$ by

$$B(u)(t) = -g\big(u(1)\big)\varphi(t), \quad t \in I, \tag{2.8}$$

$$F(u)(t) = f\big(t,u(t),u'(t),u''(t),u'''(t)\big), \quad t \in I. \tag{2.9}$$

Thus, for every $u \in C^3(I)$,

$$Tu = (S \circ F)(u) + Bu. \tag{2.10}$$

Lemma 2.1 *For every $u \in C^3(I)$, $B : C^3(I) \to C^3(I)$ is a completely continuous operator and $\|Bu\|_{C^3} = |g(u(1))|$.*

Proof By (2.6), for any $t \in I$,

$$\varphi'(t) = t\left(1 - \frac{1}{2}t\right) \geq 0, \qquad \varphi''(t) = 1 - t \geq 0, \qquad \varphi'''(t) = -1 < 0.$$

Hence, we can conclude that

$$\max_{t \in I}\big|\varphi(t)\big| = \varphi(1) = \frac{1}{3}, \qquad \max_{t \in I}\big|\varphi'(t)\big| = \varphi'(1) = \frac{1}{2},$$

$$\max_{t \in I}\big|\varphi''(t)\big| = \varphi''(0) = 1, \qquad \max_{t \in I}\big|\varphi'''(t)\big| = 1.$$

Then, from (2.8), it follows that, for any $u \in C^3(I)$,

$$\|Bu\|_C = \max_{t \in I}\big|-g\big(u(1)\big)\varphi(t)\big| = \frac{1}{3}\big|g\big(u(1)\big)\big|,$$

$$\big\|(Bu)'\big\|_C = \max_{t \in I}\big|-g\big(u(1)\big)\varphi'(t)\big| = \frac{1}{2}\big|g\big(u(1)\big)\big|,$$

$$\big\|(Bu)''\big\|_C = \max_{t \in I}\big|-g\big(u(1)\big)\varphi''(t)\big| = \big|g\big(u(1)\big)\big|,$$

$$\big\|(Bu)'''\big\|_C = \max_{t \in I}\big|-g\big(u(1)\big)\varphi'''(t)\big| = \big|g\big(u(1)\big)\big|,$$

and $B : C^3(I) \to C^3(I)$ is completely continuous. Evidently,

$$\|Bu\|_{C^3} = \max\big\{\|Bu\|_C, \big\|(Bu)'\big\|_C, \big\|(Bu)''\big\|_C, \big\|(Bu)'''\big\|_C\big\} = \big|g\big(u(1)\big)\big|.$$

The proof is completed. □

Now, we establish the following comparison principle.

Lemma 2.2 *Let $\gamma \leq 0$ be a constant and $u \in C^{(4)}(I)$ satisfy*

$$\begin{cases} u^{(4)}(t) \geq 0, & t \in I, \\ u(0) \geq 0, \quad u'(0) \geq 0, \quad u''(1) \geq 0, \quad u'''(1) \leq \gamma, \end{cases}$$

then for any $t \in I$, $u(t) \geq 0$, $u'(t) \geq 0$, $u''(t) \geq 0$, $u'''(t) \leq \gamma$.

Proof By the definition of integral, for any $t \in I$,

$$u'''(t) = u'''(1) - \int_t^1 u^{(4)}(s)\,ds \leq \gamma,$$

$$u''(t) = u''(1) - \int_t^1 u'''(s)\,ds \geq -\gamma(1-t) \geq 0,$$

$$u'(t) = u'(0) + \int_0^t u''(s)\,ds \geq 0,$$

$$u(t) = u(0) + \int_0^t u'(s)\,ds \geq 0.$$

The proof is completed. □

Definition 2.3 If $\alpha(t) \in C^4(I)$ satisfies

$$\begin{cases} \alpha^{(4)}(t) \leq f(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t)), & t \in I, \\ \alpha(0) \leq 0, \quad \alpha'(0) \leq 0, \quad \alpha''(1) \leq 0, \quad \alpha'''(1) \geq g(\alpha(1)), \end{cases}$$

then $\alpha(t)$ is named a lower solution of BVP (1.1). And if the inequalities in the above are all reversed, then $\alpha(t)$ is named an upper solution of BVP (1.1).

Lemma 2.4 *Let $\alpha(t)$ and $\beta(t)$ be a pair of lower and upper solutions for BVP (1.1) with $\alpha'''(t) \geq \beta'''(t)$ for every $t \in I$. Then*

$$\alpha(t) \leq \beta(t), \quad \alpha'(t) \leq \beta'(t), \quad \alpha''(t) \leq \beta''(t), \quad t \in I.$$

Proof Denote $u(t) = \beta(t) - \alpha(t)$, then $u'''(t) \leq 0$. By Definition 2.3,

$$u(0) \geq 0, \quad u'(0) \geq 0, \quad u''(1) \geq 0.$$

So, for any $t \in I$, we can figure out that

$$u''(t) = u''(1) - \int_t^1 u'''(s) \geq 0,$$

$$u'(t) = u'(0) + \int_0^t u''(s) \geq 0,$$

$$u(t) = u(0) + \int_0^t u'(s) \geq 0.$$

At this point, the proof is finished. □

For convenience, we recommend a semi-ordering "$\preceq$" in $C^3(I)$:

$$\alpha \preceq \beta \quad \Leftrightarrow \quad \alpha \leq \beta, \qquad \alpha' \leq \beta', \qquad \alpha'' \leq \beta'', \qquad \alpha''' \geq \beta''',$$

we note that $\beta \succeq \alpha$ is equal to $\alpha \preceq \beta$. Let $\alpha, \beta \in C^3(I)$ and $\alpha \preceq \beta$, we also introduce the order-interval in $C^3(I)$:

$$\mathcal{D}_\alpha^\beta = \left\{ u \in C^3(I) \mid \alpha \preceq u \preceq \beta \right\}, \tag{2.11}$$

then $\mathcal{D}_\alpha^\beta \subset C^3(I)$ is a nonempty bounded convex closed set.

At the end of this section, we present an important result, which will be used in Sect. 3.

Lemma 2.5 *Let $f : I \times \mathbb{R}^4 \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be boundary continuous functions, then BVP (1.1) has a solution $u \in C^4(I)$.*

Proof Obviously, $F : C^3(I) \to C(I)$ is bounded and continuous. Thus, $S \circ F : C^3(I) \to C^3(I)$ is completely continuous, which implies that $T : C^3(I) \to C^3(I)$ is a completely continuous operator. Here, we apply the Schauder fixed point theorem to testify that there is a fixed point for the operator T in $C^3(I)$.

By the boundedness of f and g, there exist constants $M_1, M_2 > 0$ such that

$$\|Fu\|_C \leq M_1, \qquad \left| g\big(u(1)\big) \right| \leq M_2, \quad u \in C^3(I).$$

Choose $R \geq M_1 \|S\| + M_2$, and define a closed bounded convex set $\Omega = \{ u \in C^3(I) : \|u\|_{C^3} \leq R \}$, where $\|S\|$ is the norm of operator S in $C^3(I)$. For $u \in \Omega$, one can find

$$\begin{aligned}
\|Tu\|_{C^3} &\leq \left\| SF(u) \right\|_{C^3} + \|Bu\|_{C^3} \\
&\leq \|S\| \cdot \|Fu\|_C + \left| g\big(u(1)\big) \right| \\
&\leq M_1 \|S\| + M_2 \leq R,
\end{aligned}$$

it shows that $T(\Omega) \subset \Omega$. Hence, the Schauder fixed point theorem guarantees that the operator T has a fixed point $u \in \Omega$, which is a solution of BVP (1.1). $\qquad\square$

3 Main results

Theorem 3.1 *Let $\alpha(t)$ and $\beta(t)$ be a pair of lower and upper solutions of BVP (1.1) with $\alpha'''(t) \geq \beta'''(t)$ for every $t \in I$. If the following conditions are established:*

(H1) *$f : I \times \mathbb{R}^4 \to \mathbb{R}$ is continuous and satisfies:*

    (i) *$f(t, x_0, x_1, x_2, x_3)$ is increasing with respect to x_0, x_1, and x_2 in $[\alpha(t), \beta(t)] \times [\alpha'(t), \beta'(t)] \times [\alpha''(t), \beta''(t)]$ for every $t \in I$, $x_3 \in [\beta'''(t), \alpha'''(t)]$;*

    (ii) *$f(t, x_0, x_1, x_2, x_3)$ is decreasing with respect to x_3 in $[\beta'''(t), \alpha'''(t)]$ for every $t \in I$, $x_i \in [\alpha^{(i)}(t), \beta^{(i)}(t)]$ $(i = 0, 1, 2)$.*

(H2) *$g : \mathbb{R} \to \mathbb{R}$ is continuous and decreasing with respect to x in $[\alpha(t), \beta(t)]$ for every $t \in I$.*

Then there exists maximal solution $\overline{u}$ and minimal solution $\underline{u}$ for BVP (1.1) in $\mathcal{D}_\alpha^\beta$.

Proof It is easy to know that $\alpha \preceq \beta$ by Lemma 2.4. Let $F : C^3(I) \to C(I)$ be defined by (2.9), then F is continuous. Based on (H1), we can certify that

$$u_1 \preceq u_2 \quad \Rightarrow \quad F(u_1) \le F(u_2), \quad u_1, u_2 \in \mathcal{D}_\alpha^\beta. \tag{3.1}$$

Since S is completely continuous, thus $S \circ F : \mathcal{D}_\alpha^\beta \to C^3(I)$ is completely continuous. Therefore, $T : \mathcal{D}_\alpha^\beta \to C^3(I)$ is a completely continuous operator.

Next, we will accomplish the proof in three steps:

I. We prove that $T : \mathcal{D}_\alpha^\beta \to \mathcal{D}_\alpha^\beta$ is an order-increasing operator under the semi-ordering "$\preceq$", namely

$$\alpha \preceq u_1 \preceq u_2 \preceq \beta \quad \Rightarrow \quad \alpha \preceq Tu_1 \preceq Tu_2 \preceq \beta. \tag{3.2}$$

To this end, let $x = Tu = (S \circ F)(u) + Bu$ for $u \in \mathcal{D}_\alpha^\beta$. Denote $h = F(u)$, $\gamma = g(u(1))$, then $x = Tu$ is the solution of LBVP (2.4) and satisfies

$$\begin{cases} x^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & t \in I, \\ x(0) = x'(0) = 0, \\ x''(1) = 0, \qquad x'''(1) = g(u(1)). \end{cases} \tag{3.3}$$

Using the definition of lower solution together with (H1),

$$(x - \alpha)^{(4)}(t) \ge F(u)(t) - F(\alpha)(t) \ge 0.$$

By the boundary conditions and (H2), one can find

$$(x - \alpha)(0) \ge 0, \qquad (x - \alpha)'(0) \ge 0,$$
$$(x - \alpha)''(1) \ge 0, \qquad (x - \alpha)'''(1) \le g\big(u(1)\big) - g\big(\alpha(1)\big) \le 0.$$

Thus, according to Lemma 2.2, for every $t \in I$,

$$(x - \alpha)(t) \ge 0, \qquad (x - \alpha)'(t) \ge 0,$$
$$(x - \alpha)''(t) \ge 0, \qquad (x - \alpha)'''(t) \le g\big(u(1)\big) - g\big(\alpha(1)\big) \le 0,$$

which means $\alpha \preceq x$. Similarly, one can see $x \preceq \beta$. Therefore, $\alpha \preceq x \preceq \beta$, that is, $T : \mathcal{D}_\alpha^\beta \to \mathcal{D}_\alpha^\beta$.

Furthermore, for every $u_1, u_2 \in \mathcal{D}_\alpha^\beta$ with $u_1 \preceq u_2$, denote $x_1 = Tu_1$, $x_2 = Tu_2$, then x_1, x_2 satisfy equation (3.3), respectively. Combining with (H1) and (H2), we obtain that

$$(x_2 - x_1)^{(4)}(t) = F(u_2)(t) - F(u_1)(t) \ge 0, \quad t \in I,$$
$$(x_2 - x_1)(0) \ge 0, \qquad (x_2 - x_1)'(0) \ge 0,$$
$$(x_2 - x_1)''(1) \ge 0, \qquad (x_2 - x_1)'''(1) = g\big(u_2(1)\big) - g\big(u_1(1)\big) \le 0.$$

By Lemma 2.2, for every $t \in I$, we have

$$(x_2 - x_1)(t) \geq 0, (x_2 - x_1)'(t) \geq 0, \qquad (x_2 - x_1)''(t) \geq 0, (x_2 - x_1)'''(t) \leq 0,$$

which implies that $Tu_1 \preceq Tu_2$.

II. We testify that BVP (1.1) has a solution in $\mathcal{D}_\alpha^\beta$.

Taking $\alpha_0 = \alpha$, $\beta_0 = \beta$, establish iterative sequences

$$\alpha_n = T\alpha_{n-1}, \qquad \beta_n = T\beta_{n-1}, \quad n = 1, 2, \dots. \tag{3.4}$$

Using (3.2), we can easily confirm that α_n and β_n satisfy the monotone conditions:

$$\alpha_0 \preceq \alpha_1 \preceq \cdots \preceq \alpha_n \preceq \beta_n \preceq \cdots \preceq \beta_1 \preceq \beta_0, \quad n = 1, 2, \dots. \tag{3.5}$$

From the compactness of T, it follows that $\{\alpha_n\}, \{\beta_n\} \subset T(\mathcal{D}_\alpha^\beta)$ are relatively compact in $C^3(I)$. Hence, by (3.5), $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly convergent in $C^3(I)$, which means that there exist $\underline{u}, \overline{u} \in C^3(I)$ such that $\alpha_n \to \underline{u}$, $\beta_n \to \overline{u}$. With the convexity and closeness of $\mathcal{D}_\alpha^\beta$, $\underline{u}, \overline{u} \in \mathcal{D}_\alpha^\beta$. By (3.4) and the continuity of T, one can deduce $\underline{u} = T\underline{u}$, $\overline{u} = T\overline{u}$. Thus, $\underline{u}$ and $\overline{u}$ are the solutions of BVP (1.1) in $\mathcal{D}_\alpha^\beta$.

III. We demonstrate that $\underline{u}$ and $\overline{u}$ are the minimal and maximal solutions of BVP (1.1) in $\mathcal{D}_\alpha^\beta$, respectively.

Let $u \in \mathcal{D}_\alpha^\beta$ be a solution of BVP (1.1), then $\alpha \preceq u \preceq \beta$, by (3.2),

$$\alpha_n \preceq u \preceq \beta_n, \quad n = 1, 2, \dots.$$

Setting $n \to \infty$, then

$$\underline{u} \preceq u \preceq \overline{u}.$$

Consequently, $\underline{u}$ and $\overline{u}$ are the minimal and maximal solutions of BVP (1.1) in $\mathcal{D}_\alpha^\beta$, respectively.

The proof is finished. □

On the basis of the above argument, it is easy to draw the following corollary.

Corollary 3.2 *Suppose that the conditions of Theorem 3.1 are established. Constructing iterative sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ by the iterative equation*

$$\begin{cases} u_n^{(4)}(t) = f(t, u_{n-1}(t), u_{n-1}'(t), u_{n-1}''(t), u_{n-1}'''(t)), & t \in I, \\ u_n(0) = u_n'(0) = 0, \\ u_n''(1) = 0, \qquad u_n'''(1) = g(u_{n-1}(1)), \end{cases} \tag{3.6}$$

then (3.5) *holds and*

$$\lim_{n \to \infty} \alpha_n^{(i)}(t) = \underline{u}^{(i)}(t), \qquad \lim_{n \to \infty} \beta_n^{(i)}(t) = \underline{u}^{(i)}(t), \quad i = 0, 1, 2, 3, \tag{3.7}$$

uniformly for $t \in I$, where $\underline{u}$ and $\overline{u}$ are the minimal and maximal solutions of BVP (1.1) in $\mathcal{D}_\alpha^\beta$, respectively.

Theorem 3.1 guarantees the existence results for BVP (1.1) in the order-interval $\mathcal{D}_\alpha^\beta$ under the assumptions that f and g are monotonic. Now, we weaken the monotonicity assumption and prove the existence results for BVP (1.1) between lower and upper solutions by using the classical auxiliary truncation function method.

Theorem 3.3 *Let $\alpha(t)$ and $\beta(t)$ be a pair of lower and upper solutions of BVP (1.1) with $\alpha'''(t) \geq \beta'''(t)$ for every $t \in I$. If the following conditions are established*:

(H3) $f : I \times \mathbb{R}^4 \to \mathbb{R}$ *is continuous, and for any* $t \in I$,
$$(x_0, x_1, x_2) \in [\alpha(t), \beta(t)] \times [\alpha'(t), \beta'(t)] \times [\alpha''(t), \beta''(t)],$$

$$f\big(t, x_0, x_1, x_2, \alpha'''(t)\big) \geq f\big(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t)\big),$$
$$f\big(t, x_0, x_1, x_2, \beta'''(t)\big) \leq f\big(t, \beta(t), \beta'(t), \beta''(t), \beta'''(t)\big);$$

(H4) $g : \mathbb{R} \to \mathbb{R}$ *is continuous, and for any* $t \in I$, $x \in [\alpha(t), \beta(t)]$,

$$g\big(\beta(t)\big) \leq g(x) \leq g\big(\alpha(t)\big).$$

Then there is at least one solution $u \in \mathcal{D}_\alpha^\beta$ for BVP (1.1).

Proof By Lemma 2.4, we know that $\alpha \preceq \beta$. For any $t \in I$, $x \in \mathbb{R}$, let

$$\eta_i(t, x) = \min\big\{\max\big\{\alpha^{(i)}(t), x\big\}, \beta^{(i)}(t)\big\}, \quad i = 0, 1, 2,$$
$$\eta_3(t, x) = \min\big\{\max\big\{\beta'''(t), x\big\}, \alpha'''(t)\big\}.$$

Then $\eta_i : I \times \mathbb{R} \to \mathbb{R}$ $(i = 0, 1, 2, 3)$ are continuous functions and satisfy

$$\alpha^{(i)}(t) \leq \eta_i(t, x) \leq \beta^{(i)}(t), \quad i = 0, 1, 2,$$
$$\beta'''(t) \leq \eta_3(t, x) \leq \alpha'''(t)$$

for any $t \in I$, $x \in \mathbb{R}$. Now, we construct a truncating function of f as follows:

$$f^*(t, x_0, x_1, x_2, x_3) = f\big(t, \eta_0(t, x_0), \eta_1(t, x_1), \eta_2(t, x_2), \eta_3(t, x_3)\big) + \frac{x_3 - \eta_3(t, x_3)}{1 + x_3^2}$$

for any $t \in I$ and $x_i \in \mathbb{R}$ $(i = 0, 1, 2, 3)$. According to the definition of $\eta_i(t, x)$ $(i = 0, 1, 2, 3)$, $f^* : I \times \mathbb{R}^4 \to \mathbb{R}$ is a bounded continuous function. Hence, by Lemma 2.5, there exists a solution $u_0 \in C^4(I)$ for BVP

$$\begin{cases} u^{(4)}(t) = f^*(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in I, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = g(u(1)). \end{cases} \qquad (3.8)$$

Here, we show that u_0 is also a solution of BVP (1.1) in $\mathcal{D}_\alpha^\beta$. Firstly, we check that

$$\beta''' \leq u_0''' \leq \alpha'''. \qquad (3.9)$$

Suppose on the contrary, then there exists $t_0 \in [0, 1]$ such that

$$u_0'''(t_0) - \alpha'''(t_0) = \max_{t \in [0,1]} \left(u_0'''(t) - \alpha'''(t) \right) > 0. \tag{3.10}$$

We can prove that $t_0 \neq 1$. In fact, since $\eta_i(t, \alpha^{(i)}(t)) = \alpha^{(i)}(t)$ $(i = 0, 1, 2, 3)$ for any $t \in I$, we infer that

$$
\begin{aligned}
f^* &\left(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t) \right) \\
&= f\left(t, \eta_0(t, \alpha(t)), \eta_1(t, \alpha'(t)), \eta_2(t, \alpha''(t)), \eta_3(t, \alpha'''(t)) \right) + \frac{\alpha'''(t) - \eta_3(t, \alpha'''(t))}{1 + [\alpha'''(t)]^2} \\
&= f\left(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t) \right) \\
&\geq \alpha^{(4)}(t),
\end{aligned}
$$

which means that $\alpha(t)$ is a lower solution of BVP (3.8). Thus, $\alpha(1) \leq u_0(1)$, combining with (H4), it is quite evident that

$$u_0'''(1) = g\left(u_0(1) \right) \leq g\left(\alpha(1) \right) \leq \alpha'''(1).$$

Hence, by (3.10), there exists $t_0 \in [0, 1)$ such that

$$u_0'''(t_0) - \alpha'''(t_0) > 0, \tag{3.11}$$

$$u_0^{(4)}(t_0) - \alpha^{(4)}(t_0) \leq 0. \tag{3.12}$$

In addition, by the definition of η_3, we can obtain that

$$\eta_3\left(t_0, u_0'''(t_0) \right) = \alpha'''(t_0). \tag{3.13}$$

Then, by BVP (3.8), we can deduce that

$$
\begin{aligned}
u^{(4)}(t_0) &= f^*\left(t_0, u_0(t_0), u_0'(t_0), u_0''(t_0), u_0'''(t_0) \right) \\
&= f\left(t_0, \eta_0(t_0, u_0(t_0)), \eta_1(t_0, u_0'(t_0)), \eta_2(t_0, u_0''(t_0)), \alpha'''(t_0) \right) \\
&\quad + \frac{u_0'''(t_0) - \alpha'''(t_0)}{1 + [u_0'''(t_0)]^2} \\
&\geq f\left(t_0, \alpha(t_0), \alpha'(t_0), \alpha''(t_0), \alpha'''(t_0) \right) + \frac{u_0'''(t_0) - \alpha'''(t_0)}{1 + [u_0'''(t_0)]^2} \\
&> f\left(t_0, \alpha(t_0), \alpha'(t_0), \alpha''(t_0), \alpha'''(t_0) \right) \\
&\geq \alpha^{(4)}(t_0).
\end{aligned}
$$

Therefore, we get a contradiction. Hence $u_0''' \leq \alpha'''$ holds.

Using an analogous technique, $\beta''' \leq u_0'''$ holds, and so (3.9) is valid. It follows from Lemma 2.4 that $u_0 \in \mathcal{D}_\alpha^\beta$. By the definition of $\eta_i(t, x)$,

$$\eta_i\left(t, u_0^{(i)}(t) \right) = u_0^{(i)}(t), \quad t \in I, i = 0, 1, 2, 3.$$

Therefore, by BVP (3.8), we can get that, for any $t \in I$,

$$
\begin{aligned}
u_0^{(4)}(t) &= f^*\big(t, u_0(t), u_0'(t), u_0''(t), u_0'''(t)\big) \\
&= f\big(t, \eta_0\big(t, u_0(t)\big), \eta_1\big(t, u_0'(t)\big), \eta_2\big(t, u_0''(t)\big), \eta_3\big(t, u_0'''(t)\big)\big) + \frac{u_0'''(t) - \eta_3}{1 + [u_0'''(t)]^2} \\
&= f\big(t, u_0(t), u_0'(t), u_0''(t), u_0'''(t)\big).
\end{aligned}
$$

It means that $u_0 \in \mathcal{D}_\alpha^\beta$ is a solution of BVP (1.1). $\qquad\square$

4 Example

Example 4.1 Consider the following nonlinear BVP:

$$
\begin{cases}
u^{(4)}(t) = \frac{\sin t}{9} u(t) + \frac{1}{8} u'(t) u''(t) + \frac{\cos t}{9} u''(t) - \frac{1}{4} u'''(t) + \frac{1}{3} e^{-t}, & t \in I, \\
u(0) = u'(0) = 0, \\
u''(1) = 0, \qquad u'''(1) = \frac{1 - u(1)}{5 + u(1)} - \frac{1}{5}.
\end{cases}
\tag{4.1}
$$

Obviously, $\alpha(t) \equiv 0$ is a lower solution of BVP (4.1). Now we prove that BVP (4.1) has an upper solution

$$
\beta(t) = e^{-t} + t - 1 \geq 0, \quad t \in I.
$$

Apparently,

$$
\beta'(t) = 1 - e^{-t} \geq 0, \qquad \beta''(t) = e^{-t} \geq 0, \qquad \beta'''(t) = -e^{-t} \leq 0.
$$

Corresponding to BVP (1.1),

$$
f(t, x_0, x_1, x_2, x_3) = \frac{\sin t}{9} x_0 + \frac{1}{8} x_1 x_2 + \frac{\cos t}{9} x_2 - \frac{1}{4} x_3 + \frac{1}{3} e^{-t},
$$

$$
g(x) = \frac{1 - x}{5 + x} - \frac{1}{5}.
$$

Then, we can obtain that, for every $t \in I$,

$$
\begin{aligned}
f\big(t, \beta, \beta', \beta'', \beta'''\big) &= \frac{\sin t}{9} \beta(t) + \frac{1}{8} \beta'(t) \beta''(t) + \frac{\cos t}{9} \beta''(t) - \frac{1}{4} \beta'''(t) + \frac{1}{3} e^{-t} \\
&= \frac{\sin t}{9}(t - 1) + \frac{\sin t}{9} e^{-t} + \frac{e^{-t}}{8}\big(1 - e^{-t}\big) + \frac{\cos t}{9} e^{-t} + \frac{1}{4} e^{-t} + \frac{1}{3} e^{-t} \\
&\leq \frac{\sqrt{2}}{9} \sin\Big(t + \frac{\pi}{4}\Big) e^{-t} + \frac{2}{3} \cdot \frac{e^{-t}}{8} + \frac{1}{4} e^{-t} + \frac{1}{3} e^{-t} \\
&< e^{-t} = \beta^{(4)}(t),
\end{aligned}
$$

and

$$
\beta(0) = 0, \qquad \beta'(0) = 0, \qquad \beta''(1) = \frac{1}{e} > 0,
$$

$$g\big(\beta(1)\big) - \beta'''(1) = \frac{1 - e^{-1}}{5 + e^{-1}} - \frac{1}{5} + \frac{1}{e} = \frac{19 + 5e^{-1}}{5e(5 + e^{-1})} > 0,$$

that is, $\beta'''(1) < g(\beta(1))$. Those imply that $\beta(t)$ is an upper solution of BVP (4.1) with $\beta'''(t) \le \alpha'''(t)$.

Moreover, it is quite obvious that $f(t, x_0, x_1, x_2, x_3)$ is increasing on x_0, x_1, and x_2 in $[0, \beta(t)] \times [0, \beta'(t)] \times [0, \beta''(t)]$ and decreasing on x_3 in $[\beta'''(t), 0]$; $g(x)$ is decreasing on x in $[0, \beta(t)]$, which indicate that f and g satisfy (H1) and (H2). Then, by Theorem 3.1, BVP (4.1) has maximal solution $\overline{u}$ and minimal solution $\underline{u}$ in $\mathcal{D}_0^\beta$.

Example 4.2 Consider the following nonlinear BVP:

$$\begin{cases} u^{(4)}(t) = \frac{\sin t}{9} u(t) + \frac{1}{8} u'(t) u''(t) + \frac{\cos t}{9} u''(t) + \frac{1}{4}(u'''(t))^3 + \frac{1}{3} e^{-t}, \quad t \in I, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = \frac{1 - u(1)}{5 + u(1)} - \frac{1}{5}. \end{cases} \tag{4.2}$$

We can check that $\alpha(t) \equiv 0$ and $\beta(t) = e^{-t} + t - 1$ are a pair of lower and upper solutions for BVP (4.2) with $\beta'''(t) \le \alpha'''(t)$ for every $t \in I$. In view of the fact that

$$f(t, x_0, x_1, x_2, x_3) = \frac{\sin t}{9} x_0 + \frac{1}{8} x_1 x_2 + \frac{\cos t}{9} x_2 + \frac{1}{4} x_3^3 + \frac{1}{3} e^{-t}$$

is nondecreasing on x_3, Theorem 3.1 does not apply to BVP (4.2). However, it is easy to see that $f(t, x_0, x_1, x_2, x_3)$ is increasing on x_0, x_1, and x_2 in $\mathbb{R}^3$, and

$$g(x) = \frac{1 - x}{5 + x} - \frac{1}{5}$$

is decreasing on x, then conditions (H3) and (H4) are valid. Therefore, by Theorem 3.3, BVP (4.2) has at least one solution in $\mathcal{D}_0^\beta$.

Availability of data and materials
Not applicable.

Authors' contributions
MW carried out the first draft of this manuscript. All authors read and approved the final manuscript.

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