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Well-posedness results and blow-up for a class of semilinear heat equations

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Abstract

This paper considers the initial value problem for nonlinear heat equation in the whole space $\mathbb{R}^N$. The local existence theory related to the finite time blow-up is also obtained for the problem with nonlinearity source (like polynomial types).

MSC: 35K05; 35K15; 37K40

Keywords: Nonlinear heat equation; Existence; Blow-up

1 Introduction

We consider the following initial value problem for $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$:

$$\begin{cases} u_t = \Delta u + \lambda u - \rho u^m, & \text{in } \mathbb{R}^N \times [0,T], \\ u(x,0) = f(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{$\mathbb{P}$}$$

where $\lambda, \rho \in \mathbb{R}$ and $m, N \in \mathbb{N}^*$ are parameters; Δ is the standard Laplacian with Dirichlet boundary conditions in $L^2(\mathbb{R}^N)$; $u = u(x,t)$ is the state of the unknown function and f is given function. Mathematical formulations modeled by problem ($\mathbb{P}$) appear in many other practical applications of mathematics and engineering science models [5–7, 12]. The goal of this paper is the study of the local existence, unique continuation and a finite time blow-up of solution to Problem ($\mathbb{P}$).

For the homogeneous linear case of problem ($\mathbb{P}$)

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \mathbb{R}^N \times [0,T], \\ u(x,0) = f(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

this is the well-known classical heat equation and these problems have been studied for decades and much work on this topic has been published.

In recent years, results on inhomogeneous partial differential equations have been extensively studied. Unlike linear source functions, nonlinear source functions describe more complex systems and have many applications in the real world. For nonlinear problems that appear in some physical phenomena there are many results devoted to nonlinear heat equations; see [2–11, 13] and the references therein. Studies of well-posedness and asymptotic behavior of solutions for semilinear heat equations have been performed by many authors; see [1, 7, 14] and the references therein. Equations of the form

$$u_t = \Delta u + \lambda\left(u - u^3\right)$$

on a bounded one-dimensional domain were studied by Chafee and Infante (1974) (so this equation is sometimes called the Chafee–Infante equation). Although there has been published much work on the semilinear cases, the literature on the case of the form of Problem ($\mathbb{P}$) is quite scarce. In this work, we consider the special case with

$$\phi_m(u) := \lambda u - \rho u^m, \tag{1.2}$$

so that we can focus on the essential ideas. Our goal is to establish the local well-posedness results for a nonlinearity source like the form (1.2) which causes difficulties. However, our study here is new in the sense that we consider Sobolev spaces $H^{2s}(\mathbb{R}^N)$. The key ideas for the existence of solutions in $H^{2s}(\mathbb{R}^N)$ is to apply the Banach fixed point theorem. Based on the conditions of the constants s depending on the dimensions $N \geq 1$, we set up the Sobolev embeddings $H^{2s}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$. Moreover, we also establish the results on the continuation and finite time blow-up of solutions in $H^{2s}(\mathbb{R}^N)$.

We assume that the following hypotheses hold:

$$\left|\phi_m(u)(t) - \phi_m(v)(t)\right| \leq C|u-v|\left(1 + |u|^{m-1} + |v|^{m-1}\right), \tag{$\boxed{H_1}$}$$

$$\left|\phi_m(u)(t)\right| \leq C|u|\left(1 + |u|^{m-1}\right), \tag{$\boxed{H_2}$}$$

for $C > 0$, $m > 1$.

The notation and the functional setting are introduced in Sect. 2.1 and in Sect. 2.2 we give some related results. The main results of this paper are in Sect. 3; we present local existence in Sect. 3.1, uniqueness continuation of the solution is discussed in Sect. 3.2 and a finite time blow-up result is demonstrated in Sect. 3.3.

2 Preliminaries

2.1 Functional setting and notation

The notation $\|\cdot\|_B$ stands for the norm in the Banach space B. For $1 \leq p \leq \infty$, $T > 0$, consider the Banach space of real-valued measurable functions $f : (0,T) \to B$ with the following norm:

$$\|f\|_{L^p(0,T;B)} = \left(\int_0^T \|f(t)\|_B^p \, dt\right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty, \tag{2.1}$$

$$\|f\|_{L^\infty(0,T;B)} = \operatorname*{ess\,sup}_{t \in (0,T)} \|f(t)\|_B, \quad \text{for } p = \infty. \tag{2.2}$$

Given a Banach space B, let $C([0,T];B)$ be the set of all continuous functions which map $[0,T]$ into B. The norm of the function space $C^k([0,T];B)$, for $0 \leq k \leq \infty$ is denoted

$$\|f\|_{C^k([0,T];B)} = \sum_{i=0}^{k} \sup_{t \in [0,T]} \left\|f^{(i)}(t)\right\|_B < \infty. \tag{2.3}$$

For $s \in \mathbb{R}^N$, the Sobolev space $H^s(\mathbb{R}^N)$ consists of all tempered distributions $w \in \mathbb{S}'(\mathbb{R}^N)$ whose Fourier transform $\widehat{w}$ is a regular distribution such that

$$\int_{\mathbb{R}^N} \left(1 + |\xi|^2\right)^s \left|\widehat{w}(\xi)\right|^2 d\xi < \infty.$$

The inner product and norm of $w, v \in H^s(\mathbb{R}^N)$ are defined by

$$(w, v)_{H^s(\mathbb{R}^N)} = (2\pi)^N \int_{\mathbb{R}^N} \left(1 + |\xi|^2\right)^s \widehat{w}(\xi)\overline{\widehat{v}}(\xi)\, d\xi$$

and

$$\|w\|_{H^s(\mathbb{R}^N)} = (2\pi)^N \left(\int_{\mathbb{R}^N} \left(1 + |\xi|^2\right)^s \left|\widehat{w}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}}.$$

Let $\mathcal{A}$ denote the negative Laplacian operator in $L^2(\mathbb{R}^N)$,

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2\left(\mathbb{R}^N\right) \to L^2\left(\mathbb{R}^N\right), \qquad \mathcal{A} = -\Delta, \qquad \mathcal{D}(\mathcal{A}) = H^2\left(\mathbb{R}^N\right). \tag{2.4}$$

We define $\mathcal{A}$ as an operator acting in $L^2(\mathbb{R}^N)$ because we can study it explicitly by use of the Fourier transform. As is well known, $\mathcal{A}$ is a closed, densely defined positive operator, and $-\mathcal{A}$ is the generator of a strongly continuous contraction semigroup

$$\left\{e^{-t\mathcal{A}} : t \geq 0\right\} \quad \text{on } L^2\left(\mathbb{R}^N\right).$$

The Fourier representation of the semigroup operators is

$$e^{-t\mathcal{A}} : L^2\left(\mathbb{R}^N\right) \to L^2\left(\mathbb{R}^N\right), \qquad \left(\widehat{e^{-t\mathcal{A}}w}\right)(\xi) = e^{-t|\xi|^2}\widehat{w}(\xi). \tag{2.5}$$

If $t > 0$ we have for any $s > 0$

$$e^{-t\mathcal{A}} : L^2\left(\mathbb{R}^N\right) \to H^{2s}\left(\mathbb{R}^N\right).$$

We define the nonlinear function

$$\phi_m : H^{2s}\left(\mathbb{R}^N\right) \to L^2\left(\mathbb{R}^N\right), \qquad \phi_m(w)(x) = \lambda w(x) - \rho w^m(x). \tag{2.6}$$

2.2 Some related results

We define mild $H^{2s}(\mathbb{R}^N)$-value solutions of ($\mathbb{P}$) as follows.

Definition 2.1 Suppose that $T > 0$, $4s > N$, and $f \in H^{2s}(\mathbb{R}^N)$. A mild H^{2s}-value solution of ($\mathbb{P}$) on $[0, T]$ is a function

$$u \in C\left([0, T]; H^{2s}\left(\mathbb{R}^N\right)\right),$$

such that

$$u(t) = e^{-t\mathcal{A}}f + \int_0^t e^{-(t-\tau)\mathcal{A}}\phi_m\left(u(\tau)\right) d\tau, \quad \text{for every } 0 \leq t \leq T, \tag{2.7}$$

where $e^{-t\mathcal{A}}$ is given by (2.5), and ϕ_m is given by (1.2).

Lemma 2.2 *Let $e^{-t\mathcal{A}}$ be the semigroup operator defined in* (2.5) *and* $s > 0$. *If* $t > 0$, *then*

$$e^{-t\mathcal{A}} : L^2(\mathbb{R}^N) \to H^{2s}(\mathbb{R}^N),$$

and there is a constant $C = C(s,N)$ *such that*

$$\left\| e^{-t\mathcal{A}} w \right\|_{H^{2s}(\mathbb{R}^N)} \le \frac{Ce^t}{t^s} \|w\|_{L^2(\mathbb{R}^N)}. \tag{2.8}$$

Proof Suppose that $w \in L^2(\mathbb{R}^N)$. Using the Fourier representation (2.5) of $e^{-t\mathcal{A}}$ as multiplication by $e^{-t|\xi|^2}$ and the definition of the H^{2s}-norm, we get

$$\begin{aligned} \left\| e^{-t\mathcal{A}} w \right\|^2_{H^{2s}(\mathbb{R}^N)} &= (2\pi)^N \int_{\mathbb{R}^N} \left(1 + |\xi|^2\right)^{2s} e^{-2t|\xi|^2} \left|\widehat{w}(\xi)\right|^2 d\xi \\ &\le (2\pi)^N \sup_{\xi \in \mathbb{R}^N} \left(\left(1 + |\xi|^2\right)^{2s} e^{-2t|\xi|^2}\right) \int_{\mathbb{R}^N} \left|\widehat{w}(\xi)\right|^2 d\xi. \end{aligned}$$

Hence, by Parseval's theorem, we have

$$\left\| e^{-t\mathcal{A}} w \right\|_{H^{2s}(\mathbb{R}^N)} \le M \|w\|_{L^2(\mathbb{R}^N)},$$

where

$$M = (2\pi)^{\frac{N}{2}} \sup_{\xi \in \mathbb{R}^N} \left(\left(1 + |\xi|^2\right)^{2s} e^{-2t|\xi|^2}\right)^{\frac{1}{2}}.$$

Writing $1 + |\xi|^2 = z$, we have

$$M = (2\pi)^{\frac{N}{2}} e^t \sup_{z \ge 1} \left(z^s e^{-tz}\right) \le \frac{Ce^t}{t^s}, \tag{2.9}$$

and the result of this lemma follows. □

3 Local well-posedness results

3.1 Local existence of the solutions to problem ($\mathbb{P}$)

Theorem 3.1 (Local existence) *Let* $N \in [1,3]$, *and* $s \in (\frac{N}{4}, 1)$. *Let* $f \in H^{2s}(\mathbb{R}^N)$, *then there is a time constant* $T > 0$ (*depending only on* N, s, f) *such that Problem* ($\mathbb{P}$) *has a unique mild solution belonging to* $C([0,T]; H^{2s}(\mathbb{R}^N))$ *in the sense of Definition* 2.1.

Proof We write (2.7) as

$$\begin{aligned} &u = \mathbf{J}(u), \\ &\mathbf{J} : C\left([0,T]; H^{2s}(\mathbb{R}^N)\right) \to C\left([0,T]; H^{2s}(\mathbb{R}^N)\right), \\ &\mathbf{J}(u)(t) = e^{-t\mathcal{A}} f + \int_0^t e^{-(t-\tau)\mathcal{A}} \phi_m(u)(\tau)\, d\tau. \end{aligned} \tag{3.1}$$

We will show that $\mathbf{J}$ defined in (3.1) is a contraction mapping on a suitable ball in $C([0,T]; H^{2s}(\mathbb{R}^N))$. We write $\mathbf{J}$ in (3.1) as

$$\mathbf{J}(u)(t) = e^{-t\mathcal{A}} f + \mathbf{F}(u)(t), \qquad \mathbf{F}(u)(t) = \int_0^t e^{-(t-\tau)\mathcal{A}} \phi_m(u)(\tau)\, d\tau.$$

Since $f \in H^{2s}(\mathbb{R}^N)$ and $\{e^{-t\mathcal{A}} : t \geq 0\}$ is a strongly continuous semigroup on $H^{2s}(\mathbb{R}^N)$, the map

$$t \mapsto e^{-t\mathcal{A}} \quad \text{belongs to} \quad C\big([0,T]; H^{2s}\big(\mathbb{R}^N\big)\big).$$

Thus we only need to prove the result for $\mathbf{J}$. The fact that $\mathbf{F}(u) \in C([0,T]; H^{2s}(\mathbb{R}^N))$ if $u \in C([0,T]; H^{2s}(\mathbb{R}^N))$ follows from the Lipschitz continuity of $\mathbf{F}$ and a density argument. Thus, we only need to prove the Lipschitz estimate.

If $u, v \in C([0,T]; H^{2s}(\mathbb{R}^N))$, then using Lemma 2.2 we find that

$$\begin{aligned}
&\big\|\mathbf{F}(u)(t) - \mathbf{F}(u)(t)\big\|_{H^{2s}(\mathbb{R}^N)} \\
&\quad \leq C\int_0^t \frac{e^{t-\tau}}{|t-\tau|^s}\big\|\phi_m(u)(\tau) - \phi_m(v)(\tau)\big\|_{L^2(\mathbb{R}^N)}\,d\tau \\
&\quad \leq Ce^T \sup_{0\leq\tau\leq T}\big\|\phi_m(u)(\tau) - \phi_m(v)(\tau)\big\|_{L^2(\mathbb{R}^N)}\int_0^t |t-\tau|^{-s}\,d\tau.
\end{aligned}$$

Evaluating the τ-integral, with $s < 1$, and taking the supremum of the result over $0 \leq t \leq T$, we get

$$\big\|\mathbf{F}(u) - \mathbf{F}(u)\big\|_{L^\infty(0,T;H^{2s}(\mathbb{R}^N))} \leq Ce^T T^{1-s}\big\|\phi_m(u) - \phi_m(v)\big\|_{L^\infty(0,T;L^2(\mathbb{R}^N))}. \tag{3.2}$$

From (1.2), if $u, v \in C_0(\mathbb{R}^N) \subset H^{2s}(\mathbb{R}^N)$ we have

$$\big\|\phi_m(u) - \phi_m(v)\big\|_{L^2(\mathbb{R}^N)} \leq |\lambda|\|u-v\|_{L^2(\mathbb{R}^N)} + |\gamma|\big\|u^m - v^m\big\|_{L^2(\mathbb{R}^N)}$$

and

$$\big\|u^m - v^m\big\|_{L^2(\mathbb{R}^N)} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^N)}^{m-1} + \|v\|_{L^\infty(\mathbb{R}^N)}^{m-1}\big)\|u-v\|_{L^2(\mathbb{R}^N)}.$$

For $s \geq \frac{N}{4}$, we have the Sobolev embeddings

$$H^{2s}\big(\mathbb{R}^N\big) \hookrightarrow C_0\big(\mathbb{R}^N\big),$$

and using the facts that $\|u\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|_{H^{2s}(\mathbb{R}^N)}$ and $\|u\|_{L^2(\mathbb{R}^N)} \leq C\|u\|_{H^{2s}(\mathbb{R}^N)}$, we get

$$\big\|\phi_m(u) - \phi_m(v)\big\|_{L^2(\mathbb{R}^N)} \leq C\big(1 + \|u\|_{H^{2s}(\mathbb{R}^N)}^{m-1} + \|v\|_{H^{2s}(\mathbb{R}^N)}^{m-1}\big)\|u-v\|_{H^{2s}(\mathbb{R}^N)},$$

which means that $\phi_m : H^{2s}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is locally Lipschitz continuous.

If $\|f\|_{H^{2s}(\mathbb{R}^N)} = R$, then

$$\big\|e^{-t\mathcal{A}}f\big\|_{H^{2s}(\mathbb{R}^N)} \leq R, \quad \text{for every } 0 \leq t \leq T,$$

since $\{e^{-t\mathcal{A}}\}$ is a contraction semigroup on $H^{2s}(\mathbb{R}^N)$. Therefore, if we choose

$$B_R = \big\{u \in C\big([0,T]; H^{2s}\big(\mathbb{R}^N\big)\big) : \|u\|_{C([0,T];H^{2s}(\mathbb{R}^N))} \leq 2R\big\},$$

we see that $\mathbf{J} : B_R \to B_R$ if we choose $T > 0$ such that

$$Ce^T T^{1-s}\left(1 + 2R^{m-1}\right) = \rho R,$$

where $0 < \rho < 1$. Moreover, in that case

$$\left\|\mathbf{J}(u) - \mathbf{J}(v)\right\|_{C([0,T];H^{2s}(\mathbb{R}^N))} \le \rho R\|u - v\|_{C([0,T];H^{2s}(\mathbb{R}^N))}, \quad \text{for every } u, v \in B_R.$$

The contraction mapping theorem then implies the existence of a unique solution $u \in B_R$. □

3.2 Uniqueness continuation of the solution

Since we already know that the mild solution of Problem ($\mathbb{P}$) does exist, the question is whether it will continue (continuation to a bigger interval of existence) and in what situation it is non-continuation by blow-up.

Definition 3.2 Given a mild solution $u \in C([0,T];H^2(\mathbb{R}^N))$ of ($\mathbb{P}$), we say that $u^\star$ is a *continuation* of u in $[0,T^\star]$ for $T^\star > T$ if it is satisfies

$$\begin{cases} u^\star \in C([0,T^\star];H^{2s}(\mathbb{R}^N)) \text{ is a mild solution of } (\mathbb{P}) \quad \forall t \in [T,T^\star], \\ u^\star(\cdot,t) = u(\cdot,t) \quad \text{whenever } t \in [0,T]. \end{cases} \tag{3.3}$$

Theorem 3.3 (Continuation) *Suppose that the assumptions of Theorem 3.1 are satisfied. Then the mild solution* (*unique*) *on* $[0,T]$ *of Problem* ($\mathbb{P}$) *can be extended to the interval* $[0,T^\star]$, *for some* $T^\star > T$, *so that the extended function is also the mild solution* (*unique*) *of Problem* ($\mathbb{P}$) *on* $[0,T^\star]$.

Proof Let $u : [0,T] \to H^{2s}(\mathbb{R}^N)$ be a mild solution of Problem ($\mathbb{P}$) (T is the time from Theorem 3.1). Fix $R^\star > 0$, and for $T^\star > T$ ($T^\star$ depending on $R^\star$), we shall prove that $u^\star : [0,T^\star] \to H^{2s}(\mathbb{R}^N)$ is a mild solution of Problem ($\mathbb{P}$). Assume the following estimates hold: $\max\{\theta_i\} \le \frac{R^\star}{4}$, $\theta_i > 0$, $i = 1,\dots,4$, with the constants θ_i ($i = 1,\dots,4$) defined as follows:

$$\frac{Ce^T}{T^s}\|f\|_{H^{2s}(\mathbb{R}^N)} =: \theta_1; \tag{3.4a}$$

$$Ce^{T^\star} Q_{T,R^\star}\left(1 + Q_{T,R^\star}^{m-1}\right)\left(T^\star\right)^{1-s} =: \theta_2; \tag{3.4b}$$

$$\frac{Ce^T}{T^s} Q_{T,R^\star}\left(1 + Q_{T,R^\star}^{m-1}\right)\sqrt{T^\star} =: \theta_3; \tag{3.4c}$$

$$Ce^{T^\star}\left(T^\star\right)^{1-s}\left(1 + 2Q_{T,R^\star}^{m-1}\right) =: \theta_4, \tag{3.4d}$$

where $Q_{T,R^\star} = R^\star + \|u(\cdot,T)\|_{H^{2s}(\mathbb{R}^n)}$. For $T^\star \ge T > 0$ and $R^\star > 0$, let us define

$$\widetilde{B}_{R^\star} := \left\{ u^\star \in C\left(\left[0,T^\star\right];H^{2s}\left(\mathbb{R}^N\right)\right) : \left| \begin{array}{ll} u^\star(\cdot,t) = u(\cdot,t), & \forall t \in [0,T], \\ \left\|u^\star(\cdot,t) - u(\cdot,T)\right\|_{C([T,T^\star];H^{2s}(\mathbb{R}^N))} \le R^\star, & \forall t \in \left[T,T^\star\right] \end{array} \right. \right\}. \tag{3.5}$$

• *Step I: We show that* $\mathbf{J}$ *defined as in* (3.1) *is the operator on* $\widetilde{B}_{R^\star}$. Let $u^\star \in \widetilde{B}_{R^\star}$ and we consider two cases.

Case 1: If $t \in [0, T]$, then by virtue of Theorem 3.1, we have the Problem ($\mathbb{P}$) has a unique solution and we also have $u^\star(\cdot, t) = u(\cdot, t)$. Thus $\mathbf{J}u^\star(t) = \mathbf{J}u(t) = u(\cdot, t)$ for all $t \in [0, T]$.

Case 2: If $t \in [T, T^\star]$, we have

$$\begin{aligned} \mathbf{J}u^\star(t) - u(\cdot, T) \leq{}& \big(e^{-(t-T)\mathcal{A}} - I\big)e^{-T\mathcal{A}}f \\ &+ \int_T^t e^{-(t-\tau)\mathcal{A}}\phi_m\big(u^\star\big)(\tau)\,d\tau + \big(e^{-(t-T)\mathcal{A}} - I\big)\int_0^T e^{-(T-\tau)\mathcal{A}}\phi_m\big(u^\star\big)(\tau)\,d\tau \\ =:{}& (I) + (II) + (III). \end{aligned} \tag{3.6}$$

Estimating the term (I), using (2.9) we have, for all $t \in [T, T^\star]$,

$$\begin{aligned} \big\|(I)\big\|^2_{H^{2s}(\mathbb{R}^N)} &= \big\|\big(e^{-(t-T)\mathcal{A}} - I\big)e^{-T\mathcal{A}}f\big\|^2_{H^{2s}(\mathbb{R}^N)} \\ &= (2\pi)^N \int_{\mathbb{R}^N}\big(1 + |\xi|^2\big)^{2s}\big(e^{-(t-T)|\xi|^2} - 1\big)^2 e^{-2T|\xi|^2}\big|\widehat{f}(\xi)\big|^2\,d\xi \\ &\leq (2\pi)^N \sup_{\xi\in\mathbb{R}^N}\Big(\big(1 + |\xi|^2\big)^{2s}e^{-2T|\xi|^2}\Big)\int_{\mathbb{R}^N}\big|\widehat{f}(\xi)\big|^2\,d\xi \leq \frac{Ce^{2T}}{T^{2s}}\|f\|^2_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Hence, we get

$$\big\|(I)\big\|_{H^{2s}(\mathbb{R}^N)} \leq \frac{Ce^T}{T^s}\|f\|_{H^{2s}(\mathbb{R}^N)}.$$

From (3.4a), this implies that the following estimate holds:

$$\big\|(I)\big\|_{C([0,T^\star];H^{2s}(\mathbb{R}^N))} \leq \frac{R^\star}{4}. \tag{3.7}$$

From Lemma 2.2, (H_2), we have the following estimate for all $t \in [T, T^\star]$:

$$\begin{aligned} \big\|(II)\big\|_{H^{2s}(\mathbb{R}^N)} &\leq \int_T^t \big\|e^{-(t-\tau)\mathcal{A}}\phi_m\big(u^\star\big)(\tau)\big\|_{H^{2s}(\mathbb{R}^N)}\,d\tau \\ &\leq Ce^{T^\star}\int_T^t (t-\tau)^{-s}\big\|\phi_m\big(u^\star\big)(\cdot,\tau)\big\|_{L^2(\mathbb{R}^N)}\,d\tau \\ &\leq Ce^{T^\star}\int_T^t (t-\tau)^{-s}\big(1 + \big\|u^\star(\cdot,\tau)\big\|^{m-1}_{H^{2s}(\mathbb{R}^N)}\big)\big\|u^\star(\cdot,\tau)\big\|_{H^{2s}(\mathbb{R}^N)}\,d\tau, \end{aligned}$$

and from (3.5), for all $t \in [T, T^\star]$, we have used that

$$\big\|u^\star(\cdot,t)\big\|_{H^{2s}(\mathbb{R}^N)} \leq R^\star + \big\|u(\cdot,T)\big\|_{H^{2s}(\mathbb{R}^N)} = Q_{T,R^\star}.$$

Using (3.4b), we obtain

$$\big\|(II)\big\|_{H^{2s}(\mathbb{R}^N)} \leq Ce^{T^\star}Q_{T,R^\star}\big(1 + Q^{m-1}_{T,R^\star}\big)\big(T^\star\big)^{1-s} \leq \frac{R^\star}{4}. \tag{3.8}$$

From (2.9) and (H_2), we have the following estimate for all $t \in [T, T^\star]$:

$$\begin{aligned}
&\|(III)\|^2_{H^{2s}(\mathbb{R}^N)} \\
&\leq (2\pi)^N \int_0^T \int_{\mathbb{R}^N} (1+|\xi|^2)^{2s} \big(e^{-(t-T)|\xi|^2} - 1\big)^2 e^{-2T|\xi|^2} \big|\widehat{\phi}_m(u^\star)(\xi,\tau)\big|^2 \, d\xi \, d\tau \\
&\leq \frac{Ce^{2T}}{T^{2s}} \int_0^T \big\|\phi_m(u^\star)(\cdot,\tau)\big\|^2_{L^2(\mathbb{R}^N)} \, d\tau \leq \frac{Ce^{2T}}{T^{2s}} \int_0^T \big(1 + \|u^\star\|^{m-1}_{H^{2s}(\mathbb{R}^N)}\big)^2 \|u^\star\|^2_{H^{2s}(\mathbb{R}^N)} \\
&\leq \frac{Ce^{2T}}{T^{2s}} Q^2_{T,R^\star} \big(1 + Q^{m-1}_{T,R^\star}\big)^2 T^\star.
\end{aligned} \tag{3.9}$$

Using (3.4c), we infer that

$$\|(III)\|_{C([0,T^\star];H^{2s}(\mathbb{R}^N))} \leq \frac{Ce^T}{T^s} Q_{T,R^\star} \big(1 + Q^{m-1}_{T,R^\star}\big) \sqrt{T^\star} \leq \frac{R^\star}{4}. \tag{3.10}$$

It follows from (3.7), (3.8), (3.10) that, for every $t \in [0, T^\star]$

$$\big\|\mathbf{J}u^\star - u(\cdot,T)\big\|_{C([0,T^\star];H^{2s}(\mathbb{R}^N))} \leq \frac{R^\star}{4} + \frac{R^\star}{4} + \frac{R^\star}{4} = \frac{3R^\star}{4} \leq R^\star.$$

We have shown that $\mathbf{J}$ is a map $\widetilde{B}_{R^\star}$ into $\widetilde{B}_{R^\star}$.

• *Step II: We show that* $\mathbf{J}$ *is a contraction on* $\widetilde{B}_{R^\star}$. Let $u, v \in \widetilde{B}_{R^\star}$, and we have, for $0 \leq t \leq T^\star$,

$$\mathbf{J}u(t) - \mathbf{J}v(t) = \int_0^t e^{-(t-\tau)\mathcal{A}} \big(\phi_m(u)(\tau) - \phi_m(v)(\tau)\big) \, d\tau, \tag{3.11}$$

where we note that $\mathbf{J}u(t) - \mathbf{J}v(t) = 0$, vanishes in $\widetilde{B}_{R^\star}$ for all $t \in (0, T]$. Then, for all $t \in [0, T^\star]$, proceeding as in the proof of the last theorem, we have

$$\big\|\mathbf{J}u(t) - \mathbf{J}v(t)\big\|_{H^{2s}(\mathbb{R}^N)} \leq Ce^{T^\star} \big(T^\star\big)^{1-s} \big(1 + 2Q^{m-1}_{T,R^\star}\big) \|u - v\|_{C([0,T^\star];H^{2s}(\mathbb{R}^N))},$$

and from (3.4d), we infer that

$$\|\mathbf{J}u - \mathbf{J}v\|_{C([0,T^\star];H^{2s}(\mathbb{R}^N))} \leq \frac{R^\star}{4} \|u - v\|_{C([0,T^\star];H^{2s}(\mathbb{R}^N))}. \tag{3.12}$$

So without loss of generality, we may assume that $0 \leq R^\star < \frac{1}{4}$, this implies that $\mathbf{J}$ is a $\frac{R^\star}{4}$-contraction. By the Banach contraction principle it follows that $\mathbf{J}$ has a unique fixed point $u^\star$ of $\mathbf{J}$ in $\widetilde{B}_{R^\star}$, which is a continuation of u. This finishes the proof. □

3.3 Finite time blow-up

The next results are on global existence or non-continuation by a blow-up.

Definition 3.4 Let $u(x,t)$ be a solution of ($\mathbb{P}$). We define the maximal existence time $T_{\max}$ of $u(x,t)$ as follows:

(i) If $u(x,t)$ exists for all $0 \leq t < \infty$, then $T_{\max} = \infty$.

(ii) If there exists $T \in [0,\infty)$ such that $u(x,t)$ exists for $0 \leq t < T$, but does not exist at $t = T$, then $T_{\max} = T$.

Definition 3.5 Let $u(x,t)$ be a solution of (ℙ). We say $u(x,t)$ blows up in finite time if the maximal existence time $T_{\max}$ is finite and

$$\lim_{t\to T_{\max}^-}\sup \|u(\cdot,t)\|_{H^{2s}(\mathbb{R}^N)} = \infty. \tag{3.13}$$

Theorem 3.6 (Global existence or finite time blow-up) *Let $N \in [1,3]$, and $s \in (\frac{N}{4},1)$. Then there exists a maximal time $T_{\max} > 0$ such that $u \in C([0,T_{\max}];H^{2s}(\mathbb{R}^N))$ is a mild solution of (ℙ). Thus, either Problem (ℙ) has a unique global mild solution on $[0,\infty)$, or there exists a maximal time $T_{\max} < \infty$ such that*

$$\lim_{t\to T_{\max}^-}\sup \|u(\cdot,t)\|_{H^{2s}(\mathbb{R}^N)} = \infty.$$

Proof Define

$$T_{\max} := \sup\{T > 0 : \text{there exits a solution on } [0,T]\}.$$

Assume that $T_{\max} < \infty$, and $\|u(\cdot,t)\|_{H^{2s}(\mathbb{R}^N)} \le R_0$, for some $R_0 > 0$. Now suppose there exists a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0,T_{\max})$ such that $t_k \to T_{\max}$ and $\{u(\cdot,t_k)\}_{k\in\mathbb{N}} \subset H^{2s}(\mathbb{R}^N)$. Let us show that $\{u(\cdot,t_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $H^{2s}(\mathbb{R}^N)$. Indeed, given $\epsilon > 0$, fix $N \in \mathbb{N}$ such that, for all $k,n > N$, $0 < t_k < t_n < T_{\max}$, we have

$$\begin{aligned}
u(\cdot,t_n) - u(\cdot,t_k) &= \big(e^{-(t_n-t_k)\mathcal{A}} - I\big)e^{-t_k\mathcal{A}}f + \int_{t_k}^{t_n} e^{-(t_n-\tau)\mathcal{A}}\phi_m(u)(\tau)\,d\tau \\
&\quad + \big(e^{-(t_n-t_k)\mathcal{A}} - I\big)\int_0^{t_k} e^{-(t_k-\tau)\mathcal{A}}\phi_m(u)(\tau)\,d\tau \\
&= \big(e^{-(t_n-t_k)\mathcal{A}} - I\big)\bigg(e^{-t_k\mathcal{A}}f + \int_0^{t_k} e^{-(t_k-\tau)\mathcal{A}}\phi_m(u)(\tau)\,d\tau\bigg) \\
&\quad + \int_{t_k}^{t_n} e^{-(t_n-\tau)\mathcal{A}}\phi_m(u)(\tau)\,d\tau \\
&= \big(e^{-(t_n-t_k)\mathcal{A}} - I\big)u(\cdot,t_k) + \int_{t_k}^{t_n} e^{-(t_n-\tau)\mathcal{A}}\phi_m(u)(\tau)\,d\tau.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|u(\cdot,t_n) - u(\cdot,t_k)\|_{H^{2s}(\mathbb{R}^N)} &\le \big\|\big(e^{-(t_n-t_k)\mathcal{A}} - I\big)u(\cdot,t_k)\big\|_{H^{2s}(\mathbb{R}^N)} \\
&\quad + \int_{t_k}^{t_n} \big\|e^{-(t_n-\tau)\mathcal{A}}\phi_m(u)(\tau)\big\|_{H^{2s}(\mathbb{R}^N)}\,d\tau \\
&=: (O_1) + (O_2).
\end{aligned} \tag{3.14}$$

First, we estimate the term (O_1), we have

$$\begin{aligned}
\|(O_1)\|_{H^{2s}(\mathbb{R}^N)} &= \big\|\big(e^{-(t_n-t_k)\mathcal{A}} - I\big)u(\cdot,t_k)\big\|_{H^{2s}(\mathbb{R}^N)} \\
&\le \big\|e^{-(t_n-t_k)\mathcal{A}} - I\big\|_{\mathcal{L}(H^{2s}(\mathbb{R}^N))}\|u(\cdot,t_k)\|_{H^{2s}(\mathbb{R}^N)}.
\end{aligned} \tag{3.15}$$

From Lemma 2.2 and (H_2), we have the following estimate:

$$\begin{aligned}\|(O_2)\|_{H^{2s}(\mathbb{R}^N)} &\le Ce^{T_{\max}}\int_{t_k}^{t_n}(t_n-\tau)^{-s}\|\phi_m(u)(\cdot,\tau)\|_{L^2(\mathbb{R}^N)}\,d\tau\\ &\le Ce^{T_{\max}}\int_{t_k}^{t_n}(t_n-\tau)^{-s}\big(1+\|u(\cdot,\tau)\|_{H^{2s}(\mathbb{R}^N)}^{m-1}\big)\|u(\cdot,\tau)\|_{H^{2s}(\mathbb{R}^N)}\,d\tau\\ &\le Ce^{T_{\max}}\big(1+R_0^{m-1}\big)R_0\int_{\frac{t_k}{t_n}}^{1}(1-\eta)^{-s}d\eta.\end{aligned}$$

Thus, since $\{t_k\}_{k\in\mathbb{N}^*}$ is convergent we can take $N:=N(\epsilon)\in\mathbb{N}^*$ with $n\ge k\ge N$ such that $|t_n-t_k|$ is as small as we want. Since the semigroup $\{e^{-t\mathcal{A}}\}_{t\ge0}$ is strongly continuous in $H^{2s}(\mathbb{R}^N)$, given $\epsilon>0$, we have

$$R_0\|e^{-(t_n-t_k)\mathcal{A}}-I\|_{\mathcal{L}(H^{2s}(\mathbb{R}^N))}<\frac{\epsilon}{2}$$

and

$$Ce^{T_{\max}}\big(1+R_0^{m-1}\big)R_0\int_{\frac{t_k}{t_n}}^{1}(1-\eta)^{-s}\,d\eta<\frac{\epsilon}{2}.$$

Hence, given $\epsilon>0$ there exists $N\in\mathbb{N}$ such that

$$\|u(\cdot,t_n)-u(\cdot,t_k)\|_{H^{2s}(\mathbb{R}^N)}<\epsilon,\quad \text{for } n,k\ge N. \tag{3.16}$$

It follows that $\{u(\cdot,t_k)\}_{k\in\mathbb{N}}\subset H^{2s}(\mathbb{R}^N)$ is a Cauchy sequence and for $\{t_k\}_{k\in\mathbb{N}^*}$ arbitrary we have proved the existence of the limit

$$\lim_{t\to T_{\max}^-}\|u(\cdot,t)\|_{H^{2s}(\mathbb{R}^N)}<\infty.$$

From our previous result we deduce that the solution can extended to some larger interval (u can be continued beyond $T_{\max}$), and this contradicts the definition of $T_{\max}$. Thus, either $T_{\max}=\infty$ or if $T_{\max}<\infty$ then $\lim_{t\to T_{\max}^-}\|u(\cdot,t)\|_{H^{2s}(\mathbb{R}^N)}=\infty$. The proof of Theorem 3.6 is finished. □

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