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Fixed point problems for generalized contractions with applications

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Abstract

In this paper, we investigate the conditions on the control mappings $\psi, \varphi : (0,\infty) \to \mathbb{R}$ that guarantee the existence of the fixed points of the mapping $T : X \to P(X)$ satisfying the following inequalities:

$$\psi(H(Tx,Ty)) \leq \varphi(d(x,y)) \quad \forall x,y \in X, \text{provided that } H(Tx,Ty) > 0,$$

and

$$\psi(H(Tx,Ty)) \leq \varphi(A(x,y)) \quad \forall x,y \in X, \text{provided that } H(Tx,Ty) > 0,$$

where $A(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), (d(x,Ty) + d(Tx,y))/2\}$, and (X,d) is a metric space. The obtained fixed point results improve many earlier results on the set-valued contractions. As an application, we consider the existence of the solutions of an FDE.

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1 Introduction and preliminaries

The metric fixed point theory was initiated by Banach (1922) [15]. This theory has been enriched with well-known structures and metric generalizations. Recently, Chifu [20] established common fixed point theorems endowed with directed graphs in extended b-metric spaces. Ozturk [31] proved a fixed point theorem involving a simulation function and F-contraction. Ozyurt [33] presented fixed point theorems covering a comparison function and large contractions. In [32] the author proved some results on $\alpha - \varphi$ contractions in Branciari b-metric spaces. The study of F-metric spaces attracted attention of many researchers, and in this direction, several papers were published (see [11, 12, 16]). The existence of solutions of FDEs, IEs, ODEs, and PDEs was investigated by applying various known fixed point results; see [4, 6, 9, 13, 17, 30, 38, 39] for details. A useful generalization of the Banach contraction principle is an F-contraction presented by Wordowski [40]; a survey on this contraction is given in [25]. The notion of F-contraction was generalized using various structures; see [1, 25, 30] for details. Boyd and Wong [18] introduced mapping $\psi : [0,\infty) \to [0,\infty)$ that satisfy the following conditions to generalize the Banach contraction principle:

(1) $\psi(y) < y$ for all $y > 0$,
(2) $\lim_{x\to y+} \psi(x) < y$ for all $y > 0$.

Boyd and Wong [18] used such mapping to present the following result.

Theorem 1.1 *Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping satisfying the following contractive condition*:

$$d(Tx,Ty) \le \psi\bigl(d(x,y)\bigr) \quad \text{for all } x,y \in X,$$

where $\psi : [0,\infty) \to [0,\infty)$ satisfies (1)–(2). *Then T admits a unique fixed point u in X, and $(T^n x) \to u$ for all $x \in X$.*

Note that Theorem 1.1 improves the fixed point theorems given by Rakotch [35] and Browder [19]. Recently, Proinov [34] presented a generalization of F-contraction, JS-contraction [24], and many recent results by introducing (ϕ,ψ)-generalized contractions. Using the mappings $\psi,\varphi : (0,\infty) \to (-\infty,\infty)$, he introduced the following contractive-type conditions:

$$\psi\bigl(d(Tx,Ty)\bigr) \le \varphi\bigl(d(x,y)\bigr) \quad \forall x,y \in X, \text{provided that } \min\bigl\{d(x,y), d(Tx,Ty)\bigr\} > 0, \tag{1.1}$$

and

$$\psi\bigl(d(Tx,Ty)\bigr) \le \varphi\bigl(A(x,y)\bigr) \quad \forall x,y \in X, \text{provided that } \min\bigl\{A(x,y), d(Tx,Ty)\bigr\} > 0, \tag{1.2}$$

where $A(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), (d(x,Ty) + d(Tx,y))/2\}$ and $T : X \to X$. In this paper, we call them (ψ,φ)-contractions.

Proinov [34] established the following fixed point results.

Theorem 1.2 *Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping satisfying* (1.1). *Suppose the mappings $\psi,\varphi : (0,\infty) \to (-\infty,\infty)$ satisfy the following conditions*:

(i) *ψ is nondecreasing*;
(ii) *$\varphi(y) < \psi(y)$ for all $y > 0$*;
(iii) *$\limsup_{y\to r+} \varphi(y) < \psi(r+)$ for all $r > 0$.*

Then T has a unique fixed point $p \in X$, and the iterative sequence $(T^n x)$ converges to p for all $x \in X$.

Theorem 1.3 ([34]) *Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping satisfying* (1.1). *Suppose the mappings $\psi,\varphi : (0,\infty) \to (-\infty,\infty)$ satisfy the following conditions*:

(i) *$\varphi(y) < \psi(y)$ for all $y > 0$*;
(ii) *$\inf_{y>\epsilon} \psi(y) > -\infty$ for all $\epsilon > 0$*;

(iii) *if* $(\psi(y_n))$ *and* $(\varphi(y_n))$ *are convergent sequences such that* $\lim_{n\to\infty}\psi(y_n) = \lim_{n\to\infty}\varphi(y_n)$ *and* $(\psi(y_n))$ *is strictly decreasing, then* $\lim_{n\to\infty} y_n = 0$;

(iv) $\limsup_{y\to\epsilon+}\varphi(y) < \liminf_{y\to\epsilon}\psi(y)$ *or* $\limsup_{y\to\epsilon}\varphi(y) < \liminf_{y\to\epsilon+}\psi(y)$ *for all* $\epsilon > 0$;

(v) T *has a closed graph, or* $\limsup_{y\to 0+}\varphi(y) < \liminf_{y\to\epsilon}\psi(y)$ *for all* $\epsilon > 0$.

Then T *has a unique fixed point* $u \in X$, *and the iterative sequence* $(T^n x)$ *converges to* u *for all* $x \in X$.

Remark 1.4 Theorems 1.2 and 1.3 also hold if we replace contractive condition (1.1) with (1.2).

The following lemma is often seen in different papers (see [18, 34]) and provides a method to prove that a sequence to be Cauchy.

Lemma 1.5 *Let* (X,d) *be a metric space, and let* $\{q_n\} \subset X$ *be a sequence such that* $\lim_{n\to\infty} d(q_n,q_{n+1}) = 0$. *If the sequence* $\{q_n\}$ *is not Cauchy, then there exist two subsequences* $\{q_{n_k}\}$ *and* $\{q_{m_k}\}$ *and* $\epsilon > 0$ *such that*

$$\lim_{k\to\infty} d(q_{n_k+1},q_{m_k+1}) = \epsilon+ \tag{1.3}$$

and

$$\lim_{k\to\infty} d(q_{n_k},q_{m_k}) = d(q_{n_k+1},q_{m_k}) = d(q_{n_k},q_{m_k+1}) = \epsilon. \tag{1.4}$$

For the control mappings $\psi,\varphi : (0,\infty) \to \mathbb{R}$, the following conditions are needed for the upcoming results:

(i) $\inf_{x>\epsilon}\psi(x) > -\infty$ for any $\epsilon > 0$.

(ii) $\liminf_{x\to\epsilon+}\psi(x) > -\infty$ for any $\epsilon > 0$.

(iii) $\lim_{n\to\infty}\psi(x_n) = -\infty$ implies $\lim_{n\to\infty} x_n = 0$.

(iv) $\lim_{n\to\infty}\varphi(x_n) = 0$ implies $\lim_{n\to\infty} x_n = 0$, where $\{x_n\}$ is a bounded sequence.

(v) $\liminf_{x\to\epsilon}\varphi(x) > 0$ for all $\epsilon > 0$.

(vi) $\limsup_{x\to\epsilon}\varphi(x) < \liminf_{x\to\epsilon}\psi(x)$ for all $\epsilon > 0$.

(vii) If $\{x_n\}$ is a positive bounded sequence and if $\{\psi(x_n)\}$ and $\{\varphi(x_n)\}$ are two convergent sequences having the same limit, then $\lim_{n\to\infty} x_n = 0$.

In the following conditions, we let $\varphi : (0,\infty) \to (0,\infty)$.

(viii) If $\lim_{n\to\infty} x_n = \epsilon > 0$, then $\liminf_{n\to\infty}\varphi(x_n) > 0$.

(ix) $\liminf_{x\to\epsilon}\varphi(x) > 0$ for all $\epsilon > 0$.

(x) $\limsup_{x\to\epsilon}\varphi(x) > \limsup_{x\to\epsilon}\psi(x) - \liminf_{x\to\epsilon}\psi(x)$.

Using conditions (i)–(x), Proinov [34] obtained the following lemma.

Lemma 1.6 ([34])

(1) *Let* $\psi : (0,\infty) \to \mathbb{R}$. *Then conditions* (i), (ii), *and* (iii) *are equivalent*.

(2) *Let* $\varphi : (0,\infty) \to \mathbb{R}$. *Then condition* (iv) *implies* (v).

(3) *Let* $\varphi : (0,\infty) \to (0,\infty)$. *Then conditions* (viii), (vi), *and* (ix) *are equivalent*.

(4) *Let* $\psi,\varphi : (0,\infty) \to \mathbb{R}$ *be two mappings satisfying conditions* (vi) *and* (vii). *Then* $\lim_{n\to\infty} x_n = 0$.

(5) *Let* $\varphi : (0,\infty) \to (0,\infty)$ *and* $\psi : (0,\infty) \to \mathbb{R}$. *Then condition* (x) *implies* (iv).

Let $(\mathcal{A}, d)$ be a metric space, let $P(\mathcal{A})$ denote the set of all nonempty subsets of $\mathcal{A}$, let $P_{cb}(\mathcal{A})$ denote the set of all nonempty closed bounded subsets of $\mathcal{A}$, and let $C(\mathcal{A})$ denote the compact subsets of $\mathcal{A}$.

Let $d(q, A) = \inf_{a \in A} d(q, a)$, and let the mapping

$$H : P(\mathcal{A}) \times P(\mathcal{A}) \to [0, \infty)$$

be defined by

$$H(A, B) = \max\Bigl\{\sup_{q \in A} D(q, B), \sup_{b \in B} D(b, A)\Bigr\}.$$

The mapping H satisfies all the axioms of metric and is known as the Hausdorff metric induced by the metric d.

Definition 1.7 Let $T : \mathcal{A} \to P(\mathcal{A})$ be a set-valued mapping. A point $\sigma \in \mathcal{A}$ is said to be a fixed point of T if $\sigma \in T(\sigma)$.

Definition 1.8 Let $T : \mathcal{A} \to P(\mathcal{A})$ and $f : \mathcal{A} \times \mathcal{A} \to [0, \infty)$. The mapping T is said to be strictly f-admissible if for all $q \in \mathcal{A}$ and $\varsigma \in T(q)$ with $f(q, \varsigma) > 1$, there exists $\omega \in T(\varsigma)$ such that $f(\varsigma, \omega) > 1$.

Definition 1.9 Let $(\mathcal{A}, d)$ be a metric space, and let $f : \mathcal{A} \times \mathcal{A} \to [0, \infty)$. The space $(\mathcal{A}, d)$ is said to be strictly f-regular if for any sequence $\{q_n\} \subset \mathcal{A}$ such that $f(q_n, q_{n+1}) > 1$ for all $n \in \mathbb{N}$ and $q_n \to q$ as $n \to \infty$, we have $f(q_n, q) > 1$ for all $n \in \mathbb{N}$.

Definition 1.10 A mapping $T : (X, d) \to (X, d)$ is said to be asymptotically regular at a point x of X if

$$\lim_{n \to \infty} d\bigl(T^n x, T^{n+1} x\bigr) = 0.$$

If T is asymptotically regular at every point of X, then it is called an asymptotically regular mapping.

Lemma 1.11 plays a key role in the upcoming results.

Lemma 1.11 ([29]) *Let A and B be nonempty closed bounded subsets of a metric space $(\mathcal{A}, d)$, and let $q > 1$. Then for all $a \in A$, there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.*

2 Set-valued $(\psi, \varphi)_f$-contractions and related fixed point problems

In this section, we introduce set-valued $(\psi, \varphi)_f$-contractions. We discuss their nature and generality. We investigate various conditions for the existence of fixed points of set-valued $(\psi, \varphi)_f$-contractions.

Definition 2.1 Let $(\mathcal{A}, d)$ be a metric space. A mapping $T : \mathcal{A} \to P_{cb}(\mathcal{A})$ is said to be a set-valued $(\psi, \varphi)_f$-contraction if there exists $f : \mathcal{A} \times \mathcal{A} \to [0, \infty)$, such that

$$\psi\bigl(f(q, \varsigma) H\bigl(T(q), T(\varsigma)\bigr)\bigr) \leq \varphi\bigl(d(q, \varsigma)\bigr) \tag{2.1}$$

for all $q, \varsigma \in \mathcal{A}$ with $f(q, \varsigma) > 1$ and $H(T(q), T(\varsigma)) > 0$.

Remark 2.2 Inequality (2.1) reduces to multivalued F-contraction [2] if $\varphi(\sigma) = \psi(\sigma) - \tau$ for all $\sigma \in (0,\infty)$. Moreover, it turns into Nadler contraction [29] for $\psi(\sigma) = \ln(\sigma)$. Let $\psi : (0,\infty) \to (0,\infty)$ be a nondecreasing mapping, and let $\beta : (0,\infty) \to (0,1)$ be a mapping satisfying $\limsup_{y\to\epsilon+} \beta(y) < 1$ for any $\epsilon > 0$. Then substituting $\varphi(y) = \beta(y)\psi(y)$ and $\psi(y) = y$ for all $y > 0$, we obtain a very famous multivalued Geraghty's contraction discussed in [5].

The following theorem suggests a set of conditions for the existence of a fixed point of mapping T.

Theorem 2.3 *Let $(\mathcal{A}, d)$ be an f-regular complete metric space. Let $T : \mathcal{A} \to P_{cb}(\mathcal{A})$ be an f-admissible mapping satisfying* (2.1). *Suppose the mappings $\psi, \varphi : (0,\infty) \to (-\infty,\infty)$ satisfy the following conditions*:

(i) *for any $q_0 \in \mathcal{A}$, there exists $q_1 \in T(q_0)$ such that $f(q_0, q_1) \geq 1$;*

(ii) *ψ is nondecreasing, and $\varphi(y) < \psi(y)$ for all $y > 0$;*

(iii) *$\limsup_{y\to r+} \varphi(y) < \psi(r+)$ for all $r > 0$.*

Then T admits a fixed point in $\mathcal{A}$.

Proof Step 1. By assumption (i), for any $q_0 \in \mathcal{A}$, there exists $q_1 \in T(q_0)$ such that $f(q_0, q_1) > 1$. Since T is an f-admissible mapping, there exists $q_2 \in T(q_1)$ such that $f(q_1, q_2) > 1$ and $q_3 \in T(q_2)$ such that $f(q_2, q_3) > 1$. In general, there exist $q_{n+1} \in T(q_n)$ such that $f(q_n, q_{n+1}) > 1$ for all $n \geq 0$. Note that if $q_n \in T(q_n)$, then q_n is a fixed point of T for all $n \geq 0$. So we assume that $q_n \notin T(q_n)$ for all $n \geq 0$. Thus $H(Tq_{n-1}, Tq_n) > 0$; otherwise, $q_n \in Tq_n$. Since $f(q_n, q_{n+1}) > 1$ and $T(q_n)$, $T(q_{n+1})$ are closed and bounded sets for all $n \geq 0$, by Lemma 1.11 there exist $q_{n+1} \in T(q_n)$ ($q_n \neq q_{n+1}$) such that $d(q_n, q_{n+1}) \leq f(q_{n-1}, q_n)H(T(q_{n-1}), T(q_n))$ for all $n \geq 1$. By first part of (ii) and (2.1) we have

$$\psi\big(d(q_n, q_{n+1})\big) \leq \psi\big(f(q_{n-1}, q_n)H\big(T(q_{n-1}), T(q_n)\big)\big) \leq \varphi\big(d(q_{n-1}, q_n)\big).$$

By the second part of assumption (ii) we have

$$\psi\big(d(q_n, q_{n+1})\big) \leq \varphi\big(d(q_{n-1}, q_n)\big) < \psi\big(d(q_{n-1}, q_n)\big). \tag{2.2}$$

Since ψ is a nondecreasing mapping, $d(q_n, q_{n+1}) < d(q_{n-1}, q_n)$ for every $n \geq 1$. This shows that the sequence $\{d(q_{n-1}, q_n)\}$ is positively decreasing. Thus there exists $L \geq 0$ such that $\lim_{n\to\infty} d(q_{n-1}, q_n) = L+$. If $L > 0$, then by (2.2) we obtain a contradiction to assumption (iii) as follows:

$$\psi(L+) = \lim_{n\to\infty} \psi\big(d(q_n, q_{n+1})\big) \leq \lim_{n\to\infty} \sup \varphi\big(d(q_{n-1}, q_n)\big) \leq \lim_{\sigma\to L+} \sup \varphi(\sigma).$$

Hence $L = 0$, and, consequently, T is an asymptotically regular mapping.

Step 2. We show that $\{q_n\}$ is a Cauchy sequence. Assume on the contrary that $\{q_n\}$ is not a Cauchy sequence. In this case, by Lemma 1.5 there exist two subsequences $\{q_{n_k}\}$, $\{q_{m_k}\}$ of $\{q_n\}$ and $\epsilon > 0$ such that (1.3) and (1.4) hold. By (1.3) we infer that $d(q_{n_k+1}, q_{m_k+1}) > \epsilon$ and $f(q_{n_k}, q_{m_k}) > 1$ for all $k \geq 1$. Letting $q = q_{n_k}$ and $\varsigma = q_{m_k}$ in (2.1), we have

$$\psi\big(d(q_{n_k+1}, q_{m_k+1})\big) \leq \psi\big(f(q_{n_k}, q_{m_k})H(Tq_{n_k}, Tq_{m_k})\big) \leq \varphi\big(d(q_{n_k}, q_{m_k})\big) \quad \text{for all } k \geq 1,$$

since if $a_k = d(q_{n_k+1}, q_{m_k+1})$ and $b_k = d(q_{n_k}, q_{m_k})$, then

$$\psi(a_k) \leq \varphi(b_k) < \psi(b_k) \quad \text{for any } k \geq 1 \text{ implies that } a_k < b_k.$$

Since $\lim_{k\to\infty} a_k = \epsilon+$, we also have $\lim_{k\to\infty} b_k = \epsilon+$. Thus

$$\psi(\epsilon+) = \lim_{k\to\infty} \psi(a_k) \leq \lim_{k\to\infty} \sup \varphi(b_k) \leq \lim_{\sigma\to\epsilon+} \varphi(\sigma).$$

This is a contradiction to assumption (iii), and, consequently, $\{q_n\}$ is a Cauchy sequence in $(\mathcal{A}, d)$. Since $(\mathcal{A}, d)$ is a complete metric space, there exists $q^* \in \mathcal{A}$ such that $q_n \to q^*$ as $n \to \infty$, and the f-regularity of the space $(\mathcal{A}, d)$ implies $f(q_n, q^*) > 1$. We claim that $d(q^*, T(q^*)) = 0$. On the contrary, assume that $d(q^*, T(q^*)) > 0$. Then there exists $n_1 \in \mathbb{N}$ such that $d(q_n, T(q^*)) > 0$ for each $n \geq n_1$. By (2.1)

$$\psi(d(q_{n+1}, T(q^*))) \leq \psi(f(q_n, q^*)H(T(q_n), T(q^*))) \leq \varphi(d(q_n, q^*)) < \psi(d(q_n, q^*)).$$

By the first part of assumption (ii) we have $d(q_{n+1}, T(q^*)) < d(q_n, q^*)$. Taking the limit on both sides of the last inequality as $n \to \infty$, we have $d(q^*, T(q^*)) < 0$. This implies $d(q^*, T(q^*)) = 0$. Since $T(q^*)$ is closed, $q^* \in T(q^*)$. The uniqueness of q^* is obvious from the contractive condition (2.1). □

The following theorem suggests another set of conditions for the existence of a fixed point of a self-mapping T satisfying (2.1).

Theorem 2.4 *Let $(\mathcal{A}, d)$ be an f-regular complete metric space. Let $T : \mathcal{A} \to P_{cb}(\mathcal{A})$ be an f-admissible mapping satisfying (2.1). Suppose mappings $\psi, \varphi : (0,\infty) \to (-\infty,\infty)$ satisfy the following conditions:*

(i) *for all $\sigma_0 \in \mathcal{A}$, there exists $\sigma_1 \in T(\sigma_0)$ such that $f(\sigma_0, \sigma_1) \geq 1$;*
(ii) *ψ is nondecreasing, and $\varphi(y) < \psi(y)$ for all $y > 0$;*
(iii) $\inf_{\sigma>\epsilon} \psi(\sigma) > -\infty$;
(iv) *if the sequences $\{\psi(\sigma_n)\}$ and $\{\varphi(\sigma_n)\}$ converge to the same limit and $\{\psi(\sigma_n)\}$ is strictly decreasing, then $\lim_{n\to\infty} \sigma_n = 0$;*
(v) $\limsup_{\sigma\to\epsilon} \varphi(\sigma) < \liminf_{\sigma\to\epsilon+} \psi(\sigma)$ *for all $\epsilon > 0$;*
(vi) $\limsup_{\sigma\to\epsilon_1} \varphi(\sigma) < \liminf_{\sigma\to\epsilon} \varphi(\sigma)$ *for all $\epsilon, \epsilon_1 > 0$.*

Then T has a unique fixed point in $\mathcal{A}$.

Proof For the proof, the first four conditions (i)–(iv) are needed to prove that T is asymptotically regular. Condition (v) is required to prove that $\{q_n\}$ is a Cauchy sequence, and condition (vi) is helpful to show the existence of a fixed point.

By assumption (i), for any $\sigma_0 \in \mathcal{A}$, there exists $\sigma_1 \in T(\sigma_0)$ such that $f(\sigma_0, \sigma_1) > 1$. Since T is an f-admissible mapping, there exist $\sigma_2 \in T(\sigma_1)$ such that $f(\sigma_1, \sigma_2) > 1$ and $\sigma_3 \in T(\sigma_2)$ such that $f(\sigma_2, \sigma_3) > 1$. In general, there exist $\sigma_{n+1} \in T(\sigma_n)$ such that $f(\sigma_n, \sigma_{n+1}) > 1$ for all $n \geq 0$. Note that if $\sigma_n \in T(\sigma_n)$, then σ_n is a fixed point of T for all $n \geq 0$. We assume that $\sigma_n \notin T(\sigma_n)$ for all $n \geq 0$. Thus $H(T\sigma_{n-1}, T\sigma_n) > 0$; otherwise, $\sigma_n \in T\sigma_n$. Since $f(\sigma_n, \sigma_{n+1}) > 1$ and $T(\sigma_n)$, $T(\sigma_{n+1})$ are closed bounded sets for all $n \geq 0$, by Lemma 1.11 there exists

$\sigma_{n+1} \in T(\sigma_n)$ ($\sigma_n \neq \sigma_{n+1}$) such that $d(\sigma_n,\sigma_{n+1}) \leq f(\sigma_{n-1},\sigma_n)H(T(\sigma_{n-1}),T(\sigma_n))$ for all $n \geq 1$. By the first part of (ii) and (2.1) we have that for all $n \geq 1$,

$$\begin{aligned}\psi\bigl(d(\sigma_n,\sigma_{n+1})\bigr) &\leq \psi\bigl(f(\sigma_{n-1},\sigma_n)H\bigl(T(\sigma_{n-1}),T(\sigma_n)\bigr)\bigr)\\ &\leq \varphi\bigl(d(\sigma_{n-1},\sigma_n)\bigr) < \psi\bigl(d(\sigma_{n-1},\sigma_n)\bigr).\end{aligned} \tag{2.3}$$

Inequality (2.3) shows that $\{\psi(d(\sigma_{n-1},\sigma_n))\}$ is a strictly decreasing sequence. Then it is either bounded below or not. If it is not bounded below, then by assumption (iii) and Lemma 1.6(1) we infer that $\lim_{n\to\infty} d(\sigma_{n-1},\sigma_n) = 0$. If it bounded below, then $\{\psi(d(\sigma_{n-1},\sigma_n))\}$ is a convergent sequence, and by (2.3) the sequence $\{\varphi(d(\sigma_{n-1},\sigma_n))\}$ also converges, and both have the same point of convergence. Thus by assumption (iv) we have $\lim_{n\to\infty} d(\sigma_{n-1},\sigma_n) = 0$. Hence T is asymptotically regular.

Following Step 2 of the proof of Theorem 2.3, we have

$$\psi(a_k) \leq \varphi(b_k), \quad \text{for any } k \geq 1. \tag{2.4}$$

By (1.3) and (1.4) we have $\lim_{k\to\infty} a_k = \epsilon+$ and $\lim_{k\to\infty} b_k = \epsilon$. By (2.4) we infer that

$$\liminf_{\sigma\to\epsilon+} \psi(\sigma) \leq \liminf_{k\to\infty} \psi(a_k) \leq \limsup_{k\to\infty} \varphi(b_k) \leq \limsup_{\sigma\to\epsilon} \varphi(\sigma).$$

This is a contradiction to (v), and hence $\{\sigma_n\}$ is a Cauchy sequence in $(\mathcal{A},d)$. Since $(\mathcal{A},d)$ is a complete metric space, there exists $\sigma^* \in \mathcal{A}$ such that $\sigma_n \to \sigma^*$ as $n \to \infty$.

Now we have to prove that the point of convergence σ^* is a fixed point of T. We consider two cases.

Case 1. If $d(\sigma_{n+1},T\sigma^*) = 0$ for some $n \geq 0$, then by the triangle property of d we obtain

$$d\bigl(\sigma^*,T\sigma^*\bigr) \leq d\bigl(\sigma^*,\sigma_{n+1}\bigr) + d\bigl(\sigma_{n+1},T\sigma^*\bigr) = d\bigl(\sigma^*,\sigma_{n+1}\bigr).$$

Taking the limit as $n \to \infty$ on both sides, we have $d(\sigma^*,T\sigma^*) \leq 0$. This implies $d(\sigma^*,T(\sigma^*)) = 0$. Since $T(\sigma^*)$ is closed, $\sigma^* \in T(\sigma^*)$.

Case 2. If $d(\sigma_{n+1},T\sigma^*) > 0$ for all $n \geq 0$, then by the f-regularity of the space $(\mathcal{A},d)$ we have $f(\sigma_n,\sigma^*) > 1$. By contractive condition (2.1) we have

$$\psi\bigl(d\bigl(\sigma_{n+1},T\sigma^*\bigr)\bigr) \leq \psi\bigl(f\bigl(\sigma_n,\sigma^*\bigr)H\bigl(T\sigma_n,T\sigma^*\bigr)\bigr) \leq \varphi\bigl(d\bigl(\sigma_n,\sigma^*\bigr)\bigr) \quad \text{for all } n \geq 0.$$

Let $a_n = d(\sigma_{n+1},T\sigma^*)$ and $b_n = d(\sigma_n,\sigma^*)$. Then the last inequality reduces to

$$\psi(a_n) \leq \varphi(b_n) \quad \text{for all } n \geq 0. \tag{2.5}$$

Let $\epsilon = d(\sigma^*,T\sigma^*)$. Then we observe that $a_n \to \epsilon$ and $b_n \to 0$ as $n \to \infty$. Applying the limits on (2.5), we have

$$\liminf_{\sigma\to\epsilon} \psi(\sigma) \leq \liminf_{n\to\infty} \psi(a_n) \leq \limsup_{n\to\infty} \varphi(b_n) \leq \liminf_{\sigma\to 0} \varphi(\sigma).$$

The last inequality is a contradiction to assumption (vi) if $\epsilon > 0$. Thus we have $d(\sigma^*,T\sigma^*) = 0$. Hence $\sigma^* \in T\sigma^*$, that is, σ^* is a fixed point of T. The uniqueness of σ^* is obvious from the contractive condition (2.1). □

Note that Theorems 2.3 and 2.4 reduce to the Nadler fixed point theorem [15] if $\psi(y) = y$ and $\varphi(y) = \lambda y$ for all $y > 0$ and $0 \le \lambda < 1$. If $\psi(y) = y$ for all $y > 0$, then they reduce to the multivalued version of the Boyd–Wong fixed point theorem (Theorem 1.1). By substituting $\varphi(y) = \psi(y) - \tau$ into Theorems 2.3 and 2.4 we obtain an improvement of fixed point theorems established in [2, 22] and of the results presented by Secelean [36] and Lukacs and Kajanto [27] as follows.

Corollary 2.5 *Let $(\mathcal{A}, d)$ be an f-regular complete metric space, and let $T : \mathcal{A} \to P_{cb}(\mathcal{A})$ be a set-valued strictly f-admissible mapping satisfying the following inequality*:

$$\psi\big(f(x,y)H(Tx,Ty)\big) \le \psi\big(d(x,y)\big) - \tau \quad \forall x,y \in \mathcal{A}, \textit{provided that } H(Tx,Ty) > 0,$$

where $\psi : (0,\infty) \to \mathbb{R}$ is a nondecreasing mapping, and $\tau > 0$. If for any initial guess $\sigma_0 \in \mathcal{A}$, there exists $\sigma_1 \in T(\sigma_0)$ such that $f(\sigma_0,\sigma_1) \ge 1$, then T has a unique fixed point in $\mathcal{A}$.

If ψ is lower semicontinuous and φ is upper semicontinuous, then Theorem 2.4 is an improvement of the Amini–Harandi–Petrusel fixed point theorem [10]. If we take $\varphi(y) = h(\psi(y))$ in Theorem 2.3, we obtain the following improvement of Moradi's theorem [28].

Corollary 2.6 *Let $(\mathcal{A}, d)$ be a f-regular complete metric space, and let $T : \mathcal{A} \to P_{cb}(\mathcal{A})$ be a set-valued strictly f-admissible mapping satisfying the following inequality*:

$$\psi\big(f(x,y)H(Tx,Ty)\big) \le h\big(\psi\big(d(x,y)\big)\big) \quad \forall x,y \in \mathcal{A}, \textit{provided } H(Tx,Ty) > 0,$$

where

(i) *$h : I \to [0,\infty)$ is an upper semicontinuous mapping such that $h(y) < y$ for all $y \in I \subset \mathbb{R}$;*

(ii) *$\psi : (0,\infty) \to I$ is nondecreasing.*

If for any initial guess $\sigma_0 \in \mathcal{A}$, there exists $\sigma_1 \in T(\sigma_0)$ such that $f(\sigma_0,\sigma_1) \ge 1$, then T has a unique fixed point in $\mathcal{A}$.

Taking $h(y) = y^r$ with $r \in (0,1)$ in Corollary 2.6, we obtain the following result.

Corollary 2.7 *Let $(\mathcal{A}, d)$ be an f-regular complete metric space, and let $T : \mathcal{A} \to P_{cb}(\mathcal{A})$ be a set-valued strictly f-admissible mapping satisfying the following inequality*:

$$\psi\big(f(x,y)H(Tx,Ty)\big) \le \big(\psi\big(d(x,y)\big)\big)^r \quad \forall x,y \in \mathcal{A}, \textit{provided that } H(Tx,Ty) > 0,$$

where, $\psi : (0,\infty) \to (0,1)$ is a nondecreasing mapping. If for any initial guess $\sigma_0 \in \mathcal{A}$, there exists $\sigma_1 \in T(\sigma_0)$ such that $f(\sigma_0,\sigma_1) \ge 1$. Then T has a unique fixed point in $\mathcal{A}$.

It is obvious that Corollary 2.7 improves the Jleli–Samet fixed point theorem [24] and the results presented by Ahmad et al. [7] and Li and Jiang [26].

We also note that an improvement of particular case of the Skof fixed point theorem [37] can be obtained by taking $\varphi(y) = \lambda\psi(y)$ in Theorems 2.3 and 2.4 as follows.

Corollary 2.8 *Let $(\mathcal{A}, d)$ be an f-regular complete metric space, and let $T : \mathcal{A} \to P_{cb}(\mathcal{A})$ be a set-valued strictly f-admissible mapping satisfying the following inequality*:

$$\psi\big(f(x,y)H(Tx,Ty)\big) \le \lambda\psi\big(d(x,y)\big) \quad \forall x,y \in \mathcal{A}, \textit{provided that } H(Tx,Ty) > 0,$$

where $\psi : (0,\infty) \to (0,\infty)$ is a nondecreasing mapping, and $\lambda \in (0,1)$. If for any initial guess $\sigma_0 \in \mathcal{A}$, there exists $\sigma_1 \in T(\sigma_0)$ such that $f(\sigma_0,\sigma_1) \ge 1$, then T has a unique fixed point in $\mathcal{A}$.

Let us consider a nondecreasing mapping $\psi : (0,\infty) \to (0,\infty)$ and a mapping $\beta : (0,\infty) \to (0,1)$ satisfying $\limsup_{y\to\epsilon+} \beta(y) < 1$ for any $\epsilon > 0$. Then taking $\varphi(y) = \beta(y)\psi(y)$ and $\psi(y) = y$ for all $y > 0$ in Theorem 2.3, we obtain an improvement of the well-known Geraghty fixed point theorem [23].

3 Theorems on generalized $(\psi,\varphi)_f$-contractions

Since the generalized $(\psi,\varphi)_f$-contractions are not $(\psi,\varphi)_f$-contractions in general, in this section, we give some fixed-point results for the class of generalized $(\psi,\varphi)_f$-contractions defined below.

Definition 3.1 Let $(\mathcal{A}, d)$ be a metric space. A mapping $T : \mathcal{A} \to P(\mathcal{A})$ is said to be a set-valued generalized $(\psi,\varphi)_f$-contraction if there exists $f : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ such that

$$\psi\big(f(q,\varsigma)H\big(T(q),T(\varsigma)\big)\big) \le \varphi\big(A(q,\varsigma)\big) \tag{3.1}$$

for all $q,\varsigma \in \mathcal{A}$ with $f(q,\varsigma) > 1$ and $H(T(q),T(\varsigma)) > 0$, where

$$A(q,\varsigma) = \max\big\{d(q,\varsigma), d(q,Tq), d(\varsigma,T\varsigma), \big(d(q,T\varsigma) + d(Tq,\varsigma)\big)/2\big\}.$$

The following theorems generalize many fixed point theorems involving Ciric type contractions. For Ćirić contraction and related fixed-point results, see ([3, 21, 41]).

Theorem 3.2 *Let $(\mathcal{A}, d)$ be an f-regular complete metric space. Let $T : \mathcal{A} \to C(\mathcal{A})$ be an f-admissible mapping satisfying (3.1). Suppose mappings $\psi,\varphi : (0,\infty) \to (-\infty,\infty)$ satisfy the following conditions*:

(i) *for all $q_0 \in \mathcal{A}$, there exists $q_1 \in T(q_0)$ such that $f(q_0,q_1) \ge 1$;*
(ii) *ψ is nondecreasing, and $\varphi(y) < \psi(y)$ for all $y > 0$;*
(iii) *$\limsup_{y\to r+} \varphi(y) < \psi(r+)$ for all $r > 0$.*

Then T admits a fixed point in $\mathcal{A}$.

Proof Let $q_0 \in \mathcal{A}$ be an arbitrary initial guess. Following the arguments in Step 1 of the proof of Theorem 2.3, we have $d(q_n,q_{n+1}) \le f(q_{n-1},q_n)H(T(q_{n-1}),T(q_n))$ for all $n \ge 1$. By the first part of (ii) and (3.1) we have

$$\psi\big(d(q_n,q_{n+1})\big) \le \psi\big(f(q_{n-1},q_n)H\big(T(q_{n-1}),T(q_n)\big)\big) \le \varphi\big(A(q_{n-1},q_n)\big).$$

Since $T(x)$ is compact for all $x \in \mathcal{A}$, there exists $q_n \in T(q_{n-1})$ such that $d(q_{n-1}, q_n) = d(q_{n-1}, T(q_{n-1}))$ for all $n \geq 1$ and

$$\begin{aligned}
&\psi\big(d(q_n, q_{n+1})\big)\\
&\quad \leq \varphi\big(A(q_{n-1}, q_n)\big)\\
&\quad = \varphi\big(\max\big\{d(q_{n-1}, q_n), d\big(q_{n-1}, T(q_{n-1})\big), d\big(q_n, T(q_n)\big), d\big(q_{n-1}, T(q_n)\big)\\
&\qquad + d\big(q_n, T(q_{n-1})\big)/2\big\}\big)\\
&\quad = \varphi\big(\max\big\{d(q_{n-1}, q_n), d(q_n, q_{n+1})\big\}\big).
\end{aligned}$$

If $d(q_{n-1}, q_n) < d(q_n, q_{n+1})$, then $\psi(d(q_n, q_{n+1})) \leq \varphi(d(q_n, q_{n+1}))$, which is a contradiction to the second part of assumption (ii). Thus we have $d(q_{n-1}, q_n) > d(q_n, q_{n+1})$ and

$$\psi\big(d(q_n, q_{n+1})\big) \leq \varphi\big(d(q_{n-1}, q_n)\big).$$

By the second part of assumption (ii) we have

$$\psi\big(d(q_n, q_{n+1})\big) \leq \varphi\big(d(q_{n-1}, q_n)\big) < \psi\big(d(q_{n-1}, q_n)\big). \tag{3.2}$$

Since ψ is a nondecreasing mapping, $d(q_n, q_{n+1}) < d(q_{n-1}, q_n)$ for every $n \geq 1$. This shows that the sequence $\{d(q_{n-1}, q_n)\}$ is positively decreasing. Thus there exists $L \geq 0$ such that $\lim_{n\to\infty} d(q_{n-1}, q_n) = L+$. If $L > 0$, then by (3.2) we obtain a contradiction to assumption (iii) as follows:

$$\psi(L+) = \lim_{n\to\infty} \psi\big(d(q_n, q_{n+1})\big) \leq \lim_{n\to\infty} \sup \varphi\big(d(q_{n-1}, q_n)\big) \leq \lim_{\sigma\to L+} \sup \varphi(\sigma).$$

Hence $L = 0$, and, consequently, T is an asymptotically regular mapping.

Now we show that $\{q_n\}$ is a Cauchy sequence. Assume on the contrary that the sequence $\{q_n\}$ is not Cauchy. In this case, by Lemma 1.5 there exist two subsequences $\{q_{n_k}\}$, $\{q_{m_k}\}$ of $\{q_n\}$ and $\epsilon > 0$ such that (1.3) and (1.4) hold. By (1.3) we infer that $d(q_{n_k+1}, q_{m_k+1}) > \epsilon$ and $f(q_{n_k}, q_{m_k}) > 1$ for all $k \geq 1$. Letting $q = q_{n_k}$ and $\varsigma = q_{m_k}$ in (3.1), we have

$$\psi\big(d(q_{n_k+1}, q_{m_k+1})\big) \leq \psi\big(f(q_{n_k}, q_{m_k}) H(Tq_{n_k}, Tq_{m_k})\big) \leq \varphi\big(A(q_{n_k}, q_{m_k})\big) \quad \text{for all } k \geq 1.$$

If $a_k = d(q_{n_k+1}, q_{m_k+1})$ and $b_k = A(q_{n_k}, q_{m_k})$, then

$$\psi(a_k) \leq \varphi(b_k) < \psi(b_k) \quad \text{for any } k \geq 1 \text{ implies that } a_k < b_k.$$

Since $\lim_{k\to\infty} a_k = \epsilon+$, $\lim_{k\to\infty} b_k = \epsilon+$. Thus

$$\psi(\epsilon+) = \lim_{k\to\infty} \psi(a_k) \leq \lim_{k\to\infty} \sup \varphi(b_k) \leq \lim_{\sigma\to\epsilon+} \varphi(\sigma).$$

This is a contradiction to assumption (iii), and, consequently, $\{q_n\}$ is a Cauchy sequence in $(\mathcal{A}, d)$. Since $(\mathcal{A}, d)$ is a complete metric space, there exists $q^* \in \mathcal{A}$ such that $q_n \to q^*$ as $n \to \infty$, and the f-regularity of the space $(\mathcal{A}, d)$ implies $f(q_n, q^*) > 1$. We claim that

$d(q^*, T(q^*)) = 0$. On the contrary, assume that $d(q^*, T(q^*)) > 0$. Then there exists $n_1 \in \mathbb{N}$ such that $d(q_n, T(q^*)) > 0$ for each $n \geq n_1$. By (3.1)

$$\psi\left(d\left(q_{n+1}, T\left(q^*\right)\right)\right) \leq \psi\left(f\left(q_n, q^*\right)H\left(T\left(q_n\right), T\left(q^*\right)\right)\right) \leq \varphi\left(A\left(q_n, q^*\right)\right) < \psi\left(A\left(q_n, q^*\right)\right).$$

By the first part of assumption (ii) we have $d(q_{n+1}, T(q^*)) < A(q_n, q^*)$. Applying the limit as $n \to \infty$ on both sides of the last inequality, we have $d(q^*, T(q^*)) < d(q^*, T(q^*))$, a contradiction, and thus $d(q^*, T(q^*)) = 0$. Since $T(q^*)$ is compact, $q^* \in T(q^*)$. □

Theorem 3.3 *Let $(\mathcal{A}, d)$ be an f-regular complete metric space. Let $T : \mathcal{A} \to C(\mathcal{A})$ be an f-admissible mapping satisfying (3.1). Suppose the mappings $\psi, \varphi : (0, \infty) \to (-\infty, \infty)$ satisfy the following conditions*:

(i) *for all $\sigma_0 \in \mathcal{A}$, there exists $\sigma_1 \in T(\sigma_0)$ such that $f(\sigma_0, \sigma_1) \geq 1$;*
(ii) *ψ is nondecreasing, and $\varphi(y) < \psi(y)$ for all $y > 0$;*
(iii) $\inf_{\sigma > \epsilon} \psi(\sigma) > -\infty$;
(iv) *if the sequences $\{\psi(\sigma_n)\}$ and $\{\varphi(\sigma_n)\}$ converge to the same limit and $\{\psi(\sigma_n)\}$ is strictly decreasing, then $\lim_{n\to\infty} \sigma_n = 0$;*
(v) $\limsup_{\sigma\to\epsilon} \varphi(\sigma) < \liminf_{\sigma\to\epsilon+} \psi(\sigma)$ *for all $\epsilon > 0$;*
(vi) $\limsup_{\sigma\to\epsilon_1} \varphi(\sigma) < \liminf_{\sigma\to\epsilon} \varphi(\sigma)$ *for all $\epsilon, \epsilon_1 > 0$.*

Then T has a fixed point in $\mathcal{A}$.

Proof This proof can be obtained by following the proofs of Theorems 2.4 and 3.2. We omit the details. □

For single-valued mappings, we have the following result.

Theorem 3.4 *Let $(\mathcal{A}, d)$ be an f-regular complete metric space, and let $T : X \to X$ be a strictly f-admissible mapping satisfying following inequality*:

$$\tau + \psi\left(f(\sigma, \varsigma)\, d\left(T(\sigma), T(\varsigma)\right)\right) \leq \psi\left(A(\sigma, \varsigma)\right) \tag{3.3}$$

for all $\sigma, \varsigma \in \mathcal{A}$ with $d(T(\sigma), T(\varsigma)) > 0$, where $\psi : (0, \infty) \to \mathbb{R}$ is a nondecreasing mapping, and $\tau > 0$. If for any initial guess $\sigma_0 \in \mathcal{A}$, there exists $\sigma_1 = T(\sigma_0)$ such that $f(\sigma_0, \sigma_1) \geq 1$, then T admits a unique fixed point.

Proof Setting $\varphi(y) = \psi(y) - \tau$ for all $y > 0$ and letting $T(x)$ to be a singleton set for all $x \in \mathcal{A}$ in Theorem 3.2, we have required result. □

Remark 3.5 It is noted in [27] that the Riech and Hardy–Roger contractions are reducible to the Ćirić contraction (also called generalized contraction). Thus Theorems 3.2, 3.3, and 3.4 remains true if we replace $A(\sigma, \varsigma)$ by anyone of the following:

(1) $\max\{d(\sigma, \varsigma), d(\sigma, T(\sigma)), d(\varsigma, T(\varsigma))\}$,
(2) $\max\{d(\sigma, T(\sigma)), d(\varsigma, T(\varsigma))\}$,
(3) $\max\{d(\sigma, \varsigma), \frac{d(\sigma, T(\sigma)) + d(\varsigma, T(\varsigma))}{2}, \frac{d(\varsigma, T(\sigma)) + d(\sigma, T(\varsigma))}{2}\}$,
(4) $ad(\sigma, \varsigma) + b(d(\sigma, T(\sigma)) + d(\varsigma, T(\varsigma))) + c(d(\varsigma, T(\sigma)) + d(\sigma, T(\varsigma)))$ with $a + b + c < 1$,
(5) $ad(\sigma, \varsigma) + bd(\sigma, T(\sigma)) + cd(\varsigma, T(\varsigma))$ with $a + b + c < 1$.

4 Applications to fractional differential equations

Lacroix (1819) introduced and investigated several applicable properties of fractional differentials. Recently, various new models involving the Caputo–Fabrizio derivative (CFD) were discovered and analyzed [8, 14, 38, 39]. In the following, we investigate one of these models in metric spaces. We introduce some notations for this purpose.

Let $\mathcal{C}_{0,1}$ be the space of continuous functions $w:[0,1]\to\mathbb{R}$. Define the metric $d:\mathcal{C}_{0,1}\times\mathcal{C}_{0,1}\to[0,\infty)$ by

$$d(w,g)=\|w-g\|_\infty=\max_{\nu\in[0,1]}\bigl|w(\nu)-g(\nu)\bigr| \quad \text{for } w,g\in\mathcal{C}_{0,1}.$$

Then the space $(\mathcal{C}_{0,1},d)$ is a complete metric space. Let $f:\mathcal{C}_{0,1}\times\mathcal{C}_{0,1}\to(1,\infty)$ be defined by

$$f(r,t)=e^{\|r+t\|_\infty} \quad \text{for } r,t\in\mathcal{C}_{0,1}.$$

Let $K_1:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous mapping. We will investigate the CFDE

$$^{C}D^{\beta}q(\nu)=K_1\bigl(\nu,q(\nu)\bigr) \tag{4.1}$$

with boundary conditions

$$q(0)=0, \qquad Iq(1)=q'(0).$$

Here $^{C}D^{\beta}$ denotes the CFD of order β defined by

$$^{C}D^{\beta}K_1(\nu)=\frac{1}{\Gamma(n-\beta)}\int_0^{\nu}(\nu-\eta)^{n-\beta-1}K_1^{n}(\eta))\,d\eta,$$

where

$$n-1<\beta<n \quad \text{and} \quad n=[\beta]+1,$$

and $I^{\beta}K_1$ is given by

$$I^{\beta}K_1(\nu)=\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta)\,d\eta \quad \text{with } \beta>0.$$

Then equation (4.1) can be modified to

$$q(\nu)=\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1\bigl(\eta,q(\eta)\bigr)\,d\eta+\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1\bigl(u,q(u)\bigr)\,du\,d\eta.$$

Theorem 4.1 *Equation (4.1) admits a solution in $\mathcal{C}_{0,1}$ provided that:*

(I) *there exists $\tau>0$ such that for all $q,\varsigma\in\mathcal{C}_{0,1}$, we have*

$$\begin{aligned}&\bigl|K_1\bigl(\eta,q(\eta)\bigr)-K_1\bigl(\eta,\varsigma(\eta)\bigr)\bigr|\\&\quad\le\frac{e^{-\tau}\Gamma(\beta+1)}{4M}\bigl|q(\eta)-\varsigma(\eta)\bigr|\bigl(M=\min\bigl\{f(q,\varsigma)|q,\varsigma\in\mathcal{C}_{0,1}\bigr\}\bigr);\end{aligned}$$

(II) *there exists* $q_0 \in \mathcal{C}_{0,1}$ *such that for all* $\nu \in [0,1]$, *we have*

$$q_0(\nu) \leq \frac{1}{\Gamma(\beta)} \int_0^\nu (\nu - \eta)^{\beta-1} K_1\big(\eta, q_0(\eta)\big)\, d\eta + \frac{2\nu}{\Gamma(\beta)} \int_0^1 \int_0^\eta (\eta - u)^{\beta-1} K_1\big(u, q_0(u)\big)\, du\, d\eta.$$

Proof Consistently with the notations introduced, define the mapping $R : \mathcal{C}_{0,1} \to \mathcal{C}_{0,1}$ by

$$R\big(q(\nu)\big) = \frac{1}{\Gamma(\beta)} \int_0^\nu (\nu - \eta)^{\beta-1} K_1\big(\eta, q(\eta)\big)\, d\eta + \frac{2\nu}{\Gamma(\beta)} \int_0^1 \int_0^\eta (\eta - u)^{\beta-1} K_1\big(u, q(u)\big)\, du\, d\eta.$$

By (II) there exists $q_0 \in \mathcal{C}_{0,1}$ such that $q_n = R^n(q_0))$. The continuity of the mapping K_1 leads to the continuity of the mapping R on $\mathcal{C}_{0,1}$. It is easy to verify the assumptions of Theorem 3.4. Let us verify the contractive condition (3.3) of Theorem 3.4.

$$\big|R\big(q(\nu)\big) - R\big(\varsigma(\nu)\big)\big| = \left|\begin{array}{l} \frac{1}{\Gamma(\beta)} \int_0^\nu (\nu-\eta)^{\beta-1} K_1(\eta, q(\eta))\, d\eta \\ \quad - \frac{1}{\Gamma(\beta)} \int_0^\nu (\nu-\eta)^{\beta-1} K_1(\eta, \varsigma(\eta))\, d\eta \\ \quad + \frac{2\nu}{\Gamma(\beta)} \int_0^1 \int_0^\eta (\eta-u)^{\beta-1} K_1(u, q(u))\, du\, d\eta \\ \quad - \frac{2\nu}{\Gamma(\beta)} \int_0^1 \int_0^\eta (\eta-u)^{\beta-1} K_1(u, \varsigma(u))\, du\, d\eta \end{array}\right| \quad \text{implies}$$

$$\begin{aligned}
&\big|R\big(q(\nu)\big) - R\big(\varsigma(\nu)\big)\big| \\
&\quad\leq \left| \int_0^\nu \left(\frac{1}{\Gamma(\beta)} (\nu - \eta)^{\beta-1} K_1\big(\eta, q(\eta)\big) - \frac{1}{\Gamma(\beta)} (\nu - \eta)^{\beta-1} K_1\big(\eta, \varsigma(\eta)\big) \right) d\eta \right| \\
&\qquad + \left| \int_0^1 \int_0^\eta \left(\frac{2}{\Gamma(\beta)} (\eta - u)^{\beta-1} K_1\big(\eta, q(\eta)\big) - \frac{2}{\Gamma(\beta)} (\eta - u)^{\beta-1} K_1\big(u, \varsigma(u)\big) \right) du\, d\eta \right| \\
&\quad\leq \frac{1}{\Gamma(\beta)} \frac{e^{-\tau}\Gamma(\beta+1)}{4M} \cdot \int_0^\nu (\nu - \eta)^{\beta-1} \big(q(\eta) - \varsigma(\eta)\big)\, d\eta \\
&\qquad + \frac{2}{\Gamma(\beta)} \frac{e^{-\tau}\Gamma(\beta+1)}{4M} \cdot \int_0^1 \int_0^\eta (\eta - u)^{\beta-1} \big(\varsigma(u) - q(u)\big)\, du\, d\eta \\
&\quad\leq \frac{1}{\Gamma(\beta)} \frac{e^{-\tau}\Gamma(\beta+1)}{4M} \cdot d(q,\varsigma) \cdot \int_0^\nu (\nu - \eta)^{\beta-1}\, d\eta \\
&\qquad + \frac{2}{\Gamma(\beta)} \frac{e^{-\tau}\Gamma(\beta) \cdot \Gamma(\beta+1)}{4M\Gamma(s) \cdot \Gamma(\beta+1)} \cdot d(q,\varsigma) \cdot \int_0^1 \int_0^\eta (\eta - u)^{\beta-1}\, du\, d\eta \\
&\quad\leq \left(\frac{e^{-\tau}\Gamma(\beta) \cdot \Gamma(\beta+1)}{4M\Gamma(\beta) \cdot \Gamma(\beta+1)} \right) \cdot d(q,\varsigma) + 2e^{-\tau} B(\beta+1, 1) \frac{\Gamma(\beta) \cdot \Gamma(\beta+1)}{4M\Gamma(\beta) \cdot \Gamma(\beta+1)} \cdot d(q,\varsigma) \\
&\quad\leq \frac{e^{-\tau}}{4M} d(q,\varsigma) + \frac{e^{-\tau}}{2M} d(q,\varsigma) < \frac{e^{-\tau}}{M} d(q,\varsigma)
\end{aligned}$$

where B is the beta mapping. The last inequality can be written as

$$Md\big(R(q), R(\varsigma)\big) \leq f(q,\varsigma)\, d\big(R(q), R(\varsigma)\big) \leq e^{-\tau} \psi(q,\varsigma). \tag{4.2}$$

Define the mapping $\psi(q(\nu)) = \ln(q(\nu))$ for $q, \varsigma \in \mathcal{C}_{0,1}$. Then inequality (4.2) can be written as

$$\tau + \psi\big(f(q,\varsigma)\,d\big(R(q),R(\varsigma)\big)\big) \leq \psi\big(\psi(q,\varsigma)\big).$$

By Theorem 3.4 the self-mapping R admits a fixed point, and hence equation (4.1) has a solution. □

5 Conclusion

The $(\psi,\varphi)_f$-contractions are general enough to contain famous contractions. The theorems give a general criterion for the existence of unique fixed points of the self-mappings satisfying $(\psi,\varphi)_f$-contractions. We investigated the existence of a solution to a fractional differential equation through fixed point methodology.

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