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Asymptotic behavior of third order delay difference equations with a negative middle term

S.H. Saker[1,2], S. Selvarangam[3], S. Geetha[3], E. Thandapani[4] and J. Alzabut[5,6*]

*Correspondence:
jalzabut@psu.edu.sa
[5]Department of Mathematics and General Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia
[6]Group of Mathematics, Faculty of Engineering, Ostim Technical University, Ankara 06374, Turkey
Full list of author information is available at the end of the article

Abstract

In this paper, we establish some sufficient conditions which ensure that the solutions of the third order delay difference equation with a negative middle term

$$\Delta(a_n\Delta(\Delta w_n)^\alpha) - p_n(\Delta w_{n+1})^\alpha - q_n h(w_{n-l}) = 0, \quad n \geq n_0,$$

are oscillatory. Moreover, we study the asymptotic behavior of the nonoscillatory solutions. Two illustrative examples are included for illustration.

MSC: 39A10

Keywords: Third order delay difference equations; Comparison method; Oscillation; Nonoscillation

1 Introduction

In this paper, we are concerned with the asymptotic behavior of solutions of third order delay difference equations with a negative middle term of the form

$$\Delta\big(a_n\Delta(\Delta w_n)^\alpha\big) - p_n(\Delta w_{n+1})^\alpha - q_n h(w_{n-l}) = 0, \quad n \geq n_0, \tag{1.1}$$

where n_0 is a nonnegative integer and α is a quotient of odd positive integers. Throughout this paper, we assume without further mention that: $\{a_n\}$ is a positive real sequence, $\{p_n\}$ is a nonnegative real sequence, and $\{q_n\}$ is a positive real sequence for all $n \geq n_0$, l is a positive integer, h is a continuous, nondecreasing real-valued function such that $\eta h(\eta) > 0$ for $\eta \neq 0$, and $h(\eta\xi) \geq h(\eta)h(\xi)$ for $\eta\xi > 0$.

By a solution of equation (1.1) we mean a nontrivial real sequence $\{w_n\}$ that is defined for all $n \geq n_0 - l$ and satisfies equation (1.1) for all $n \geq n_0$. A nontrivial solution $\{w_n\}$ of equation (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and oscillatory otherwise. A difference equation is called nonoscillatory (oscillatory) if all its solutions are nonoscillatory (oscillatory). Following the terms used in [6], we define

$$L_0 w_n = w_n, \qquad L_1 w_n = (\Delta w_n)^\alpha, \qquad L_2 w_n = a_n \Delta(L_1 w_n) \quad \text{and}$$
$$L_3 w_n = \Delta(L_2 w_n), \quad \forall n \geq n_0.$$

Using these notations, one can write equation (1.1) as

$$L_3 w_n - p_n L_1 w_n - q_n h(w_{n-l}) = 0.$$

We introduce the following class of nonoscillatory (without loss of generality we say positive) solutions which give the sign structure of possible nonoscillatory solutions to equation (1.1):

$$w_n \in W_1 \quad \Leftrightarrow \quad w_n > 0, \qquad L_1 w_n > 0, \qquad L_2 w_n < 0, \qquad L_3 w_n > 0;$$
$$w_n \in W_3 \quad \Leftrightarrow \quad w_n > 0, \qquad L_1 w_n > 0, \qquad L_2 w_n > 0, \qquad L_3 w_n > 0,$$

for all $n \geq n_1 \geq n_0$. In Lemma 2.3, we will prove that if

$$\sum_{n=n_1}^{\infty} P_n^{\frac{1}{\alpha}} = \infty, \quad \text{where } P_n = \prod_{s=n_1}^{n-1} \left(1 - \frac{1}{a_s} \sum_{t=s}^{\infty} p_t\right), \tag{1.2}$$

then the set W of all positive solutions of equation (1.1) has the following decomposition:

$$W = W_1 \cup W_3.$$

According to the well-known results in [1, 2], the oscillation criteria are often accomplished by introducing the concepts having property (A) and/or (B). Equation (1.1) is said to have property (B) if $W = W_3$.

In recent years the asymptotic behavior of nonoscillatory solutions and the oscillatory behavior of solutions to different classes of third order difference equations have been the interest of many researchers, see for example [3–8, 10, 12–19] and the references cited therein. Recently, in the papers [6, 16], the authors used the comparison method and the summation averaging technique to establish some sufficient conditions for oscillation of all solutions of the third order delay difference equation

$$\Delta\big(a_n \Delta\big(b_n (\Delta w_n)^\alpha\big)\big) + p_n (\Delta w_{n+1})^\alpha + q_n h(w_{n-l}) = 0, \tag{1.3}$$

where $\{p_n\}$ and $\{q_n\}$ are positive real sequences, and the auxiliary equation of second order

$$\Delta(a_n \Delta z_n) + \frac{p_n}{b_{n+1}} z_{n+1} = 0$$

is nonoscillatory. In [11] the authors used the oscillation of a third order difference equation of the form

$$\Delta\big(a_n \Delta(d_n \Delta w_n)^\gamma\big) + q_n h(d_{n-\tau} w_{n-\tau+1}) = 0 \tag{1.4}$$

to obtain oscillation conditions for solutions of second order neutral delay difference equations.

Following this trend, in this paper, we study the asymptotic behavior of solutions of equation (1.1). Our approach depends on the application of a technique imposing one restrictive condition on the coefficients of the corresponding auxiliary equation. We show that any nonoscillatory solution $\{w_n\}$ of equation (1.1) satisfies $w_n \Delta w_n > 0$. Further, we obtain new sufficient conditions for all solutions of equation (1.1) to have property (B). Two examples are provided to illustrate the main results.

2 Main results

For the sake of simplicity, we define the following:

$$\bar{P}_n = \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} q_t, \qquad R_1(n,n_1) = \sum_{s=n_1}^{n-1} \frac{1}{a_s}, \qquad Q_n = p_n \bar{P}_n + q_n,$$

$$\bar{Q}_n = \frac{1}{a_n} \sum_{s=n}^{\infty} Q_s, \qquad B(n,n_1) = \sum_{s=n_1}^{n-1} R_1^{\frac{1}{\alpha}}(s,n_1)$$

for $s \geq n \geq n_1$, where $n_1 \geq n_0$. Throughout we assume that $R_1(n,n_1) \to \infty$ as $n \to \infty$. To make sense of the definitions P_n and $\bar{P}_n$, we also assume that

$$\sum_{n=n_0}^{\infty} p_n < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} q_n < \infty.$$

In the sequel, and without loss of generality, we can deal only with the positive solutions of equation (1.1), since the proof for the opposite case is similar. From our technique, which will be described later, we will see that the properties of solutions to equation (1.1) are closely related to nonoscillatory solutions of an auxiliary second order difference equation

$$\Delta(a_n \Delta z_n) - p_n z_{n+1} = 0. \tag{2.1}$$

First, we prove the following lemmas which will be used in the proofs of the main results.

Lemma 2.1 *Let $\{z_n\}$ be a positive solution of (2.1) for all $n \geq n_0$. Then (1.1) can be written in the form*

$$\Delta\left(a_n z_n z_{n+1} \Delta\left(\frac{1}{z_n}(\Delta w_n)^{\alpha}\right)\right) - q_n z_{n+1} h(w_{n-l}) = 0 \tag{2.2}$$

for all $n \geq n_0$.

Proof It is easy to see that

$$\frac{1}{z_{n+1}} \Delta\left(a_n z_n z_{n+1} \Delta\left(\frac{1}{z_n}(\Delta w_n)^{\alpha}\right)\right)$$
$$= \frac{1}{z_{n+1}} \Delta\big(a_n z_{n+1} \Delta\big((\Delta w_n)^{\alpha}\big) - a_n (\Delta w_{n+1})^{\alpha} \Delta z_n\big)$$

$$= \Delta\left(a_n\Delta\left((\Delta w_n)^\alpha\right)\right) + \frac{1}{z_{n+1}}a_{n+1}\Delta\left((\Delta w_{n+1})^\alpha\right)\Delta z_{n+1} - \frac{1}{z_{n+1}}a_{n+1}\Delta\left((\Delta w_{n+1})^\alpha\right)\Delta z_{n+1} - \frac{1}{z_{n+1}}(\Delta w_{n+1})^\alpha\Delta(a_n\Delta z_n)$$
$$= \Delta\left(a_n\Delta\left((\Delta w_n)^\alpha\right)\right) - p_n(\Delta w_{n+1})^\alpha,$$

where we have used (2.1). Using the above equality in equation (1.1) and rearranging, we obtain equation (2.2). This completes the proof. □

We recall that equation (2.1) (see Theorem 6.3.4 of [1]) always has a couple of nonoscillatory solutions $\{z_n\}$ such that either

$$z_n\Delta z_n > 0 \tag{2.3}$$

or

$$z_n\Delta z_n < 0 \tag{2.4}$$

for all $n \geq n_0$.

To find the structure of positive nonoscillatory solutions of equation (1.1), the following property of a nonoscillatory solution $\{z_n\}$ satisfying (2.4) plays a crucial role.

Lemma 2.2 *If* (1.2) *holds, then* (2.1) *has a positive solution* $\{z_n\}$ *satisfying*

$$\sum_{n=n_1}^{\infty}\frac{1}{a_nz_nz_{n+1}} = \sum_{n=n_1}^{\infty} z_n^{\frac{1}{\alpha}} = \infty. \tag{2.5}$$

Proof Let $\{z_n\}$ be a positive solution of equation (2.1) such that (2.4) holds for all $n \geq n_1 \geq n_0$. It is clear from the fact that $\Delta z_n < 0$, there is a constant $M > 0$ such that $z_n \leq M$. Hence

$$\sum_{n=n_1}^{\infty}\frac{1}{a_nz_nz_{n+1}} = \infty.$$

On the other hand, since

$$\Delta(a_n\Delta z_n) = p_nz_{n+1} \geq 0,$$

then $a_n\Delta z_n$ is increasing and there exists a constant $c \leq 0$ such that $\lim_{n\to\infty} a_n\Delta z_n = c$. We claim that $c = 0$, if not, then

$$z_n \leq z_{n_1} + c\sum_{s=n_1}^{n-1}\frac{1}{a_s} \to -\infty, \quad \text{as } n \to \infty,$$

a contradiction. Hence $c = 0$. Summing (2.1) from n to ∞, we have

$$-a_n\Delta z_n = \sum_{s=n}^{\infty} p_sz_{s+1} \leq z_n\sum_{s=n}^{\infty} p_s$$

or

$$\frac{z_{n+1}}{z_n} \geq 1 - \frac{1}{a_n} \sum_{s=n}^{\infty} p_s. \tag{2.6}$$

Then from (2.6) we obtain

$$z_n \geq z_{n_1} \prod_{s=n_1}^{n-1} \left(1 - \frac{1}{a_s} \sum_{t=s}^{\infty} p_t\right),$$

which yields

$$z_n^{\frac{1}{\alpha}} \geq z_{n_1}^{\frac{1}{\alpha}} P_n^{\frac{1}{\alpha}}. \tag{2.7}$$

Now summing (2.7) from n_1 to $n-1$ and then combining with (1.2) implies that the second summation in (2.5) is divergent. This completes the proof. □

Lemma 2.3 *Assume that condition* (1.2) *holds. If* $\{w_n\}$ *is a positive solution of* (1.1) *for all* $n \geq n_0$*, then there is an integer* n_1 *such that either* $w_n \in W_1$ *or* $w_n \in W_3$ *for all* $n \geq n_1 \geq n_0$.

Proof Assume that $\{w_n\}$ is a positive solution of equation (1.1) for all $n \geq n_0$. By Lemma 2.1, we may write (1.1) in an equivalent form (2.2). From Lemma 2.2, there is a positive sequence $\{z_n\}$ of (2.1) which satisfies (2.5), and so we see that

$$\Delta\left(a_n z_n z_{n+1} \Delta\left(\frac{1}{z_n}(\Delta w_n)^{\alpha}\right)\right) > 0.$$

Then, by discrete Kneser's theorem [1], we have

$$w_n > 0, \qquad (\Delta w_n)^{\alpha} > 0, \qquad \Delta\left(\frac{1}{z_n}(\Delta w_n)^{\alpha}\right) < 0,$$

or

$$w_n > 0, \qquad (\Delta w_n)^{\alpha} > 0, \qquad \Delta\left(\frac{1}{z_n}(\Delta w_n)^{\alpha}\right) > 0,$$

for all $n \geq n_1 \geq n_0$. Note that in both cases we have $\Delta w_n > 0$, and by virtue of (1.1) we see that $L_3 w_n > 0$. The rest sign properties of $L_i w_n$, $i = 1, 2$, immediately follow from discrete Kneser's theorem. The proof is now complete. □

Next, we state and prove some useful estimates which will play an important role in the proofs of our main results.

Lemma 2.4 *Let* $w_n \in W_1$ *be a positive solution of* (1.1) *for all* $n \geq n_1 \geq n_0$. *Then*

$$\frac{w_n}{(n-n_1)} \text{ is nonincreasing,} \tag{2.8}$$

and there is an integer $n_2 > n_1$ *such that*

$$L_1 w_n \geq \bar{P}_n h(w_{n-l}) \quad \text{for all } n > n_2. \tag{2.9}$$

Proof Let $w_n \in W_1$ be a positive solution of equation (1.1) for $n \geq n_1$. From the monotonicity of $L_1 w_n$, we have

$$w_n \geq w_n - w_{n_1} = \sum_{s=n_1}^{n-1} (L_1 w_s)^{1/\alpha} \geq (n - n_1) L_1^{1/\alpha} w_n. \tag{2.10}$$

Therefore

$$\Delta\left(\frac{w_n}{n - n_1}\right) = \frac{(n - n_1) L_1^{1/\alpha} w_n - w_n}{(n - n_1)(n + 1 - n_1)} \leq 0,$$

and so $w_n/(n - n_1)$ is nonincreasing. Next, summing (1.1) from n to ∞, we obtain

$$-L_2 w_n \geq \sum_{s=n}^{\infty} p_s L_1 w_s + \sum_{s=n}^{\infty} q_s h(w_{s-l}) \geq h(w_{n-l}) \sum_{s=n}^{\infty} q_s.$$

Again summing, we obtain

$$L_1 w_n \geq \sum_{s=n}^{\infty} \frac{h(w_{s-l})}{a_s} \sum_{t=s}^{\infty} q_t \geq \bar{P}_n h(w_{n-l}).$$

This completes the proof. □

Lemma 2.5 *Let $w_n \in W_3$ be a positive solution of* (1.1) *for all $n \geq n_1 \geq n_0$. If*

$$\sum_{n=n_1}^{\infty} \bigl[p_n R_1(s, n_1) + q_n h\bigl(B(n - l, n_1)\bigr)\bigr] = \infty, \tag{2.11}$$

and there is an integer $n_2 > n_1$ such that

$$\frac{w_n}{B(n, n_1)} \text{ is nondecreasing} \quad \text{for all } n \geq n_2. \tag{2.12}$$

Proof Let $w_n \in W_3$ be a positive solution of (1.1) for all $n \geq n_1$. Since $L_2 w_n$ is increasing, there is a constant $M > 0$ such that $L_2 w_n \geq M$ for all $n \geq n_1$. Clearly,

$$L_1 w_n \geq M R_1(n, n_1) \quad \text{and} \quad w_n \geq M^{1/\alpha} B(n, n_1) \quad \text{for } n \geq n_1.$$

We claim that condition (2.11) implies $\lim_{n\to\infty} L_2 w_n = \infty$. Using the above estimates into (1.1), we obtain

$$L_3 w_n \geq M p_n R_1(n, n_1) + h\bigl(M^{1/\alpha}\bigr) q_n h\bigl(B(n - l, n_1)\bigr). \tag{2.13}$$

By summing (2.13) from n_1 to ∞, we see that the claim holds. Therefore, for any $n \geq n_2 \geq n_1$, we have

$$\begin{aligned} L_1 w_n &= L_1 w_{n_2} + \sum_{s=n_2}^{n-1} \frac{L_2 w_s}{a_s} \leq L_1 w_{n_2} + R_1(n,n_2) L_2 w_n \\ &= L_1 w_{n_2} - R_1(n_2,n_1) L_2 w_n + R_1(n,n_1) L_2 w_n \\ &\leq R_1(n,n_1) L_2 w_n, \end{aligned}$$

which yields

$$\Delta\left(\frac{L_1 w_n}{R_1(n,n_1)}\right) = \frac{R_1(n,n_1) L_2 w_n - L_1 w_n}{a_n R_1(n,n_1) R_1(n+1,n_1)} \geq 0,$$

and hence $L_1 w_n / R_1(n,n_2)$ is nondecreasing for all $n \geq n_2$. Again for any $n \geq n_3 \geq n_2$, we have

$$\begin{aligned} w_n &= w_{n_3} + \sum_{s=n_3}^{n-1} \left(\frac{R_1(s,n_1) L_1 w_s}{R_1(s,n_1)}\right)^{\frac{1}{\alpha}} \leq w_{n_3} + B(n,n_3)\left(\frac{L_1 w_n}{R_1(n,n_1)}\right)^{\frac{1}{\alpha}} \\ &\leq w_{n_3} - B(n_3,n_1)\left(\frac{L_1 w_n}{R_1(n,n_1)}\right)^{\frac{1}{\alpha}} + B(n,n_1)\left(\frac{L_1 w_n}{R_1(n,n_1)}\right)^{\frac{1}{\alpha}}. \end{aligned}$$

It follows from discrete L'Hospital rule [1] that

$$\lim_{n\to\infty} \frac{L_1 w_n}{R_1(n,n_1)} = \lim_{n\to\infty} L_2 w_n = \infty,$$

and so we have

$$w_n \leq B(n,n_1)\left(\frac{L_1 w_n}{R_1(n,n_1)}\right)^{\frac{1}{\alpha}}, \quad n \geq n_3.$$

Then

$$\Delta\left(\frac{w_n}{B(n,n_1)}\right) = \frac{B(n,n_1)(L_1 w_n)^{1/\alpha} - R_1^{1/\alpha}(n,n_1) w_n}{B(n,n_1) B(n+1,n_1)} \geq 0.$$

Thus $w_n / B(n,n_1)$ is nondecreasing for all $n \geq n_3$. The proof is complete. □

We conclude this section with the following remark.

Remark 2.6 It is easy to see that from Lemma 2.5, if (1.1) has property (B), then any positive solution of (1.1) satisfies

$$\lim_{n\to\infty} \frac{w_n}{B(n,n_1)} = \infty,$$

which gives us information about the rate of convergence of possible positive solutions.

In the following, we present some sufficient conditions which ensure that equation (1.1) has property (B).

Theorem 2.7 *Let condition* (1.2) *hold for all* $n \geq n_1$. *If the first order delay difference equation*

$$\Delta x_n + \bar{Q}_n h(n - l - n_1) h\bigl(x_{n-l}^{1/\alpha}\bigr) = 0, \quad n \geq n_1, \tag{2.14}$$

is oscillatory, then (1.1) *has property* (B).

Proof Let $\{w_n\}$ be a positive solution of (1.1) for $n \geq n_0$. From Lemma 2.3 there exists an integer $n_1 \geq n_0$ such that either $w_n \in W_1$ or $w_n \in W_3$ for all $n \geq n_1$. If $w_n \in W_1$, then by equation (1.1) and (2.9), we have

$$L_3 w_n \geq (p_n \bar{P}_n + q_n) h(w_{n-l}) = Q_n h(w_{n-l}). \tag{2.15}$$

Summing (2.15) from n to ∞, we find

$$-L_2 w_n \geq \sum_{s=n}^{\infty} Q_s h(w_{s-l}) \geq \left(\sum_{s=n}^{\infty} Q_s\right) h(w_{n-l}). \tag{2.16}$$

Using (2.10) in the above inequality, we obtain

$$-\Delta(L_1 w_n) \geq \frac{1}{a_n} \left(\sum_{s=n}^{\infty} Q_s\right) h(n - l - n_1) h\bigl(L_1^{1/\alpha} w_{n-l}\bigr).$$

Letting $x_n = L_1 w_n$, we see that the difference inequality

$$\Delta x_n + \bar{Q}_n h(n - l - n_1) h\bigl(x_{n-l}^{1/\alpha}\bigr) \leq 0$$

has a positive solution. By Lemma 2.7 of [19], we see that equation (2.14) also has a positive solution, which is a contradiction. Therefore $w_n \in W_3$, which implies that equation (1.1) has property (B). This completes the proof. □

Applying some known criteria for oscillation of first order delay difference equation (2.14), one can easily obtain criteria for equation (1.1) to have property (B). The following one is given in [9].

Corollary 2.8 *Assume that* $h(u) = u^{\alpha}$ *and condition* (1.2) *hold. If*

$$\lim_{n\to\infty} \inf \sum_{s=n-l}^{n-1} \bar{Q}_s (s - l - n_1)^{1/\alpha} > \left(\frac{l}{l+1}\right)^{l+1}, \tag{2.17}$$

then (1.1) *has property* (B).

Finally, we present another result for equation (1.1) to have property (B) which is applicable even to the ordinary equation.

Theorem 2.9 *Let condition* (1.2) *hold for all* $n \geq n_1$. *Assume that*

$$\sum_{n=n_1}^{\infty}\left(\sum_{s=n}^{\infty}\frac{1}{a_s}\sum_{t=s}^{\infty}Q_t\right)^{\frac{1}{\alpha}}=\infty, \tag{2.18}$$

and the function h *satisfies*

$$\lim_{u=\pm\infty}\frac{u}{h^{1/\alpha}(u)}=M<\infty. \tag{2.19}$$

If

$$\lim_{n\to\infty}\sup\left\{h^{1/\alpha}\left(\frac{1}{n-n_1}\right)\sum_{s=n_1}^{n-1}\left(h(s-l-n_1)\sum_{t=s}^{\infty}\bar{Q}_t\right)^{\frac{1}{\alpha}}\right\}>M, \tag{2.20}$$

then (1.1) *has property* (B).

Proof Let $\{w_n\}$ be a positive solution of (1.1) for all $n \geq n_0$. Then from Lemma 2.3 there exists an integer $n_1 \geq n_0$ such that either $w_n \in W_1$ or $w_n \in W_3$ for all $n \geq n_1$. If $w_n \in W_1$, then as in the proof of Theorem 2.6 we obtain (2.16), and by summing it from n to ∞, we find that

$$L_1 w_n \geq \sum_{s=n}^{\infty}h(w_{s-l})\sum_{t=s}^{\infty}Q_t \geq \left(\sum_{s=n}^{\infty}\bar{Q}_s\right)h(w_{n-l}). \tag{2.21}$$

Now, summing (2.21) from n_1 to $n-1$, one can easily see that

$$w_n \geq \sum_{s=n_1}^{n-1}\left(h(w_{s-l})\sum_{t=s}^{\infty}\bar{Q}_t\right)^{\frac{1}{\alpha}}. \tag{2.22}$$

Using the monotonicity property (2.8) in (2.22), we have

$$\begin{aligned} w_n &\geq h^{1/\alpha}\left(\frac{w_{n-l}}{(n-l-n_1)}\right)\sum_{s=n_1}^{n-1}\left(h(s-l-n_1)\sum_{t=s}^{\infty}\bar{Q}_t\right)^{1/\alpha} \\ &\geq h^{1/\alpha}\left(\frac{w_n}{(n-n_1)}\right)\sum_{s=n_1}^{n-1}\left(h(s-l-n_1)\sum_{t=s}^{\infty}\bar{Q}_t\right)^{1/\alpha}. \end{aligned}$$

Applying hypothesis (H_3) assumed on the function h and then dividing both sides of the last inequality by $h^{1/\alpha}(w_n)$, we see that

$$\frac{w_n}{h^{1/\alpha}(w_n)} \geq h^{1/\alpha}\left(\frac{1}{(n-n_1)}\right)\sum_{s=n_1}^{n-1}\left(h(s-l-n_1)\sum_{t=s}^{\infty}\bar{Q}_t\right)^{1/\alpha}. \tag{2.23}$$

It follows from (2.18) that $\lim_{n\to\infty} w_n = \infty$. Taking the limit supremum on both sides of (2.23), we are led to a contradiction with (2.20). Thus $w_n \in W_3$, which means that equation (1.1) has property (B). This completes the proof. □

We conclude this section with the following remark.

Remark 2.10 If all conditions of Theorem 2.7 (Theorem 2.9) are satisfied, then one can conclude that all bounded solutions of equation (1.1) are oscillatory.

3 Applications

In the following, we present two examples to illustrate the main results.

Example 3.1 Consider the third order delay difference equation

$$\Delta^3 w_n - \frac{1}{(n+1)(n+2)}\Delta w_{n+1} - \frac{1}{n(n+1)} w_{n-1} = 0, \quad n \geq 1. \tag{3.1}$$

Here

$$p_n = \frac{1}{(n+1)(n+2)}, \qquad q_n = \frac{1}{n(n+1)}, \qquad \alpha = 1, \quad \text{and} \quad l = 1.$$

A simple calculation shows that

$$P_n = \frac{1}{n}, \qquad \bar{P}_n = \sum_{s=n}^{\infty} \frac{1}{s} > \frac{1}{n},$$

$$Q_n > \frac{1}{n(n+1)(n+2)} + \frac{1}{n(n+1)}, \qquad \bar{Q}_n > \frac{1}{2n(n+1)} + \frac{1}{n},$$

and

$$\lim_{n\to\infty} \inf \sum_{s=n-1}^{n-1} \bar{Q}_s(s-2) > 1 > \frac{1}{4}.$$

Hence all conditions of Corollary 2.8 are satisfied, and therefore (3.1) has property (B).

Example 3.2 Consider the third order delay difference equation

$$\Delta^2\big((\Delta w_n)^3\big) - \frac{(\Delta w_{n+1})^3}{(n+1)(n+2)} - \frac{w_{n-1}^3}{n(n+1)(n+2)} = 0, \quad n \geq 1. \tag{3.2}$$

Here

$$p_n = \frac{1}{(n+1)(n+2)}, \qquad q_n = \frac{2}{n(n+1)(n+2)}, \qquad \alpha = 3, \quad \text{and} \quad l = 1.$$

A simple calculation shows that

$$P_n = \frac{1}{n}, \qquad \bar{P}_n = \frac{1}{n}, \qquad Q_n = \frac{3}{n(n+1)(n+2)}, \qquad \bar{Q}_n = \frac{3}{2}\left(\frac{1}{n(n+1)}\right), \qquad M = 1.$$

Since

$$\sum_{n=1}^{\infty}\left(\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}Q_t\right)^{\frac{1}{3}} = \sum_{n=1}^{\infty}\left(\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\frac{3}{t(t+1)(t+2)}\right)^{\frac{1}{3}}$$
$$= \left(\frac{3}{2}\right)^{\frac{1}{3}}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{\frac{1}{3}} = \infty$$

and

$$\lim_{n\to\infty}\sup\left\{\frac{1}{n-1}\sum_{s=1}^{n-1}(s-2)\left(\sum_{t=s}^{\infty}\frac{3}{2}\left(\frac{1}{t(t+1)}\right)\right)^{\frac{1}{3}}\right\}$$
$$= \lim_{n\to\infty}\sup\left\{\frac{1}{(n-1)}\sum_{s=1}^{n-1}\left(\frac{3}{2}\right)^{\frac{1}{3}}\frac{(s-2)}{s^{1/3}}\right\} = \infty > 1,$$

we see that all conditions of Theorem 2.9 are satisfied, and hence (3.2) has property (B).

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Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, Galala University, Galala New City, Suez, Egypt. [2]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt. [3]Department of Mathematics, Presidency College (Autonomous), Chennai, 600 005, India. [4]Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai, 600 005, India. [5]Department of Mathematics and General Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia. [6]Group of Mathematics, Faculty of Engineering, Ostim Technical University, Ankara 06374, Turkey.

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