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Stability of an HTLV-HIV coinfection model with multiple delays and CTL-mediated immunity

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Abstract

In the literature, several mathematical models have been formulated and developed to describe the within-host dynamics of either human immunodeficiency virus (HIV) or human T-lymphotropic virus type I (HTLV-I) monoinfections. In this paper, we formulate and analyze a novel within-host dynamics model of HTLV-HIV coinfection taking into consideration the response of cytotoxic T lymphocytes (CTLs). The uninfected $CD4^{+}T$ cells can be infected via HIV by two mechanisms, free-to-cell and infected-to-cell. On the other hand, the HTLV-I has two modes for transmission, (i) horizontal, via direct infected-to-cell touch, and (ii) vertical, by mitotic division of active HTLV-infected cells. It is well known that the intracellular time delays play an important role in within-host virus dynamics. In this work, we consider six types of distributed-time delays. We investigate the fundamental properties of solutions. Then, we calculate the steady states of the model in terms of threshold parameters. Moreover, we study the global stability of the steady states by using the Lyapunov method. We conduct numerical simulations to illustrate and support our theoretical results. In addition, we discuss the effect of multiple time delays on stability of the steady states of the system.

MSC: 34D20; 34D23; 37N25; 92B05

Keywords: HTLV-HIV coinfection; CTL immune response; Intracellular delay; Mitotic transmission; Global stability; Lyapunov function

1 Introduction

During the past several decades many human viruses and their associated diseases, such as human immunodeficiency virus (HIV), hepatitis C virus (HCV), hepatitis B virus (HBV), dengue virus, human T-lymphotropic virus type I (HTLV-I), and recently coronavirus, have been recognized. Human body can be infected by more that one virus at the same time such as HTLV-HIV, coronavirus/influenza, HCV-HIV, HBV-HIV, HCV-HBV, and malaria-HIV. HTLV and HIV are universal public health matters. HTLV and HIV are two viruses which infect most effective immune cells, $CD4^{+}T$ cells. Adult T-cell leukemia (ATL) and HTLV-I-associated myelopathy/tropical spastic paraparesis (HAM/TSP) are the last stage of HTLV-I infection. Chronic HIV infection leads to acquired immunodeficiency syndrome (AIDS). Both HTLV and HIV have the same ways of transmission such as sharing contaminated needles and unprotected sexual contact with infected partners. Over the last 10 years HTLV-HIV coinfection has been widely documented (see e.g. [1–3], and [4]).

1.1 Mathematical models

Mathematical models of HIV and HTLV-I dynamics have become efficient tools to biological and medical scientists. These models can provide a deeper understanding of within-host virus dynamics and assist in predicting *the impact of antiviral drug efficacy on viral infection progression* (see e.g. [5–16]).

- *HIV monoinfection model*: The standard HIV dynamics model under the effect of cytotoxic T lymphocytes (CTLs) has been formulated by Nowak and Bangham [17] as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \eta - \varrho S(t) - \vartheta_1 S(t)V(t), \\ \frac{dI(t)}{dt} = \vartheta_1 S(t)V(t) - aI(t) - \mu_1 C^I(t)I(t), \\ \frac{dV(t)}{dt} = bI(t) - \varepsilon V(t), \\ \frac{dC^I(t)}{dt} = \sigma_1 C^I(t)I(t) - \pi_1 C^I(t), \end{cases} \tag{1}$$

  where $S(t)$, $I(t)$, $V(t)$, and $C^I(t)$ are the concentrations of uninfected CD4$^+$T cells, active HIV-infected cells, free HIV particles, and HIV-specific CTLs, respectively, and t is the time. η refers to the generation rate of the uninfected CD4$^+$T cells. The uninfected CD4$^+$T cells are infected via HIV particles (free-to-cell infection) at rate $\vartheta_1 SV$. The HIV-infected cells produce HIV particles at rate bI. The stimulation rate of effective HIV-specific CTLs due to the presence of HIV-infected cells is defined by $\sigma_1 C^I I$. The term $\mu_1 C^I I$ accounts for the killing rate of HIV-infected cells due to its specific CTLs. The four compartments S, I, V, and C^I have normal death rates ϱS, aI, εV, and $\pi_1 C^I$, respectively. Several extensions on model (1) have been accomplished (see e.g. [18–20]).
- *HTLV-I monoinfection model*: The within-host dynamics of HTLV-I has been mathematically modeled in several papers [21–24]. CTL immunity has been included into the HTLV-I dynamics models in many works (see e.g. [25–33]). Lim and Maini [28] have formulated a model for HTLV-I dynamics under the consideration of CTL immunity and mitotic division of active HTLV-infected cells as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \eta - \varrho S(t) - \vartheta_3 S(t)Y(t), \\ \frac{dE(t)}{dt} = \vartheta_3 S(t)Y(t) + \mathcal{K} r^* Y(t) - (\psi + \omega)E(t), \\ \frac{dY(t)}{dt} = \psi E(t) - \delta^* Y(t) - \mu_2 C^Y(t)Y(t), \\ \frac{dC^Y(t)}{dt} = \sigma_2 Y(t) - \pi_2 C^Y(t), \end{cases} \tag{2}$$

  where $E(t)$, $Y(t)$, and $C^Y(t)$ are the concentrations of latent HTLV-infected cells, active HTLV-infected cells, and HTLV-specific CTLs at time t, respectively. The term $\vartheta_3 SY$ denotes the infected-to-cell contact rate between HTLV-infected cells and uninfected CD4$^+$T cells (horizontal transmission). The active HTLV-infected cells transmit vertically to latent compartment at rate $\mathcal{K}r^*Y$ (mitotic transmission), where $\mathcal{K} \in (0,1)$. The HTLV-specific CTLs kill the active HTLV-infected cells at rate $\mu_2 C^Y Y$ and are stimulated at rate $\sigma_2 Y$. The term ψE denotes the activation rate of latent HTLV-infected cells. The death rates of E, Y, and C^Y are given by ωE, $\delta^* Y$, and $\pi_2 C^Y$, respectively.

- *HTLV-HIV coinfection model*: Elaiw and AlShamrani [34] have recently formulated an HTLV-HIV coinfection model as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \eta - \varrho S(t) - \vartheta_1 S(t)V(t) - \vartheta_2 S(t)I(t) - \vartheta_3 S(t)Y(t), \\ \frac{dL(t)}{dt} = (1-\beta)(\vartheta_1 S(t)V(t) + \vartheta_2 S(t)I(t)) - (\lambda + \gamma)L(t), \\ \frac{dI(t)}{dt} = \beta(\vartheta_1 S(t)V(t) + \vartheta_2 S(t)I(t)) + \lambda L(t) - aI(t) - \mu_1 C^I(t)I(t), \\ \frac{dE(t)}{dt} = \varphi\vartheta_3 S(t)Y(t) - (\psi + \omega)E(t), \\ \frac{dY(t)}{dt} = \psi E(t) - \delta^* Y(t) - \mu_2 C^Y(t)Y(t), \\ \frac{dV(t)}{dt} = bI(t) - \varepsilon V(t), \\ \frac{dC^I(t)}{dt} = \sigma_1 C^I(t)I(t) - \pi_1 C^I(t), \\ \frac{dC^Y(t)}{dt} = \sigma_2 C^Y(t)Y(t) - \pi_2 C^Y(t), \end{cases} \tag{3}$$

where $L(t)$ is the concentration of latent HIV-infected cells. The term $\vartheta_2 SI$ describes the infection rate of uninfected $CD4^+T$ cells by HIV-infected cells. λL and γL are the activation and death rates of latent HIV-infected cells. The parameter $\beta \in (0,1)$ represents the part of newly HIV-infected cells that becomes active, and the other part $1-\beta$ enters a latent stage. The parameter $\varphi \in (0,1)$ refers to the part of newly HTLV-infected cells that become latent.

Intracellular delay plays a crucial role in within-host virus dynamics and is defined as the time lapse between viral entry a cell and its production. In case of HIV, it has been estimated that the time between the HIV enters a target cell until producing new HIV particles is about 0.9 days [35]. Time delay has also an important effect in HTLV-I infection. Several works have been devoted to developing mathematical models with time delays to describe the dynamics of HIV (see e.g. [36–43]) and HTLV (see e.g. [44–53]).

Our aim is to take model (3) to further destination by incorporating multiple intracellular time delays and mitotic transmission. We study the fundamental and global properties of the system, then we present numerical simulation. The outcomes of this paper will help clinicians to estimate the suitable time to start the treatment. Our model may be helpful to study different coinfections such as influenza-coronavirus, HCV-HIV, HBV-HIV, and malaria-HIV. *It is interesting to note that fractional-order differential equations (FODEs) have been widely studied in several works (see e.g. [54–57]). Modeling and analysis of HIV dynamics with FODEs have been investigated in many papers (see e.g. [58–60]). Clinicians can use the information (in terms of behavior predictions) of fractional-order systems to fit patients' data with the most appropriate noninteger-order index. As a future work, our coinfection model can be formulated as a system of FODEs.*

2 The multiple delays model

In this section, we extend system (3) by taking under consideration multiple types of distributed-time delays and mitosis of active HTLV-infected cells. We achieve this goal by considering the following system of delay differential equations (DDEs):

$$\begin{cases} \frac{dS(t)}{dt} = \eta - \varrho S(t) - \vartheta_1 S(t)V(t) - \vartheta_2 S(t)I(t) - \vartheta_3 S(t)Y(t), \\ \frac{dL(t)}{dt} = (1-\beta)\int_0^{\kappa_1} \Lambda_1(\ell)e^{-\hbar_1 \ell} S(t-\ell)[\vartheta_1 V(t-\ell) + \vartheta_2 I(t-\ell)]\,d\ell - (\lambda+\gamma)L(t), \\ \frac{dI(t)}{dt} = \beta \int_0^{\kappa_2} \Lambda_2(\ell)e^{-\hbar_2 \ell} S(t-\ell)[\vartheta_1 V(t-\ell) + \vartheta_2 I(t-\ell)]\,d\ell \\ \qquad + \lambda \int_0^{\kappa_3} \Lambda_3(\ell)e^{-\hbar_3 \ell} L(t-\ell)\,d\ell - aI(t) - \mu_1 C^I(t)I(t), \\ \frac{dE(t)}{dt} = \varphi\vartheta_3 \int_0^{\kappa_4} \Lambda_4(\ell)e^{-\hbar_4 \ell} S(t-\ell)Y(t-\ell)\,d\ell + \mathcal{K} r^* Y(t) - (\psi+\omega)E(t), \\ \frac{dY(t)}{dt} = \psi \int_0^{\kappa_5} \Lambda_5(\ell)e^{-\hbar_5 \ell} E(t-\ell)\,d\ell + (1-\mathcal{K})r^* Y(t) - \delta^* Y(t) - \mu_2 C^Y(t)Y(t), \\ \frac{dV(t)}{dt} = b\int_0^{\kappa_6} \Lambda_6(\ell)e^{-\hbar_6 \ell} I(t-\ell)\,d\ell - \varepsilon V(t), \\ \frac{dC^I(t)}{dt} = \sigma_1 C^I(t)I(t) - \pi_1 C^I(t), \\ \frac{dC^Y(t)}{dt} = \sigma_2 C^Y(t)Y(t) - \pi_2 C^Y(t). \end{cases} \tag{4}$$

The factor $\Lambda_1(\ell)e^{-\hbar_1\ell}$ represents the probability that uninfected CD4$^+$T cells contacted by HIV particles or active HIV-infected cells at time $t-\ell$ survived ℓ time units and become latent infected at time t. The term $\Lambda_2(\ell)e^{-\hbar_2\ell}$ is the probability that uninfected CD4$^+$T cells contacted by HIV particles or active HIV-infected cells at time $t-\ell$ survived ℓ time units and become actively infected at time t. The term $\Lambda_3(\ell)e^{-\hbar_3\ell}$ is the probability that latent HIV-infected CD4$^+$T cells survived ℓ time units before transmitted to be active at time t. Moreover, the factor $\Lambda_4(\ell)e^{-\hbar_4\ell}$ demonstrates the probability that the initial infection of uninfected CD4$^+$T cells and the HTLV-infected cells at time $t-\ell$ completing all the intracellular processes that are required for it to become latent HTLV-infected CD4$^+$T cells at time t. Further, the probability that latent HTLV-infected CD4$^+$T cells survived ℓ time units before transmitted to active HTLV-infected cells at time t is given by the factor $\Lambda_5(\ell)e^{-\hbar_5\ell}$. Furthermore, the term $\Lambda_6(\ell)e^{-\hbar_6\ell}$ refers to the probability that new immature HIV particles at time $t-\ell$ lost ℓ time units and become mature at time t. Here $\hbar_i$, $i = 1,2,\dots,6$, are positive constants. The delay parameter ℓ is randomly taken from a probability distribution function $\Lambda_i(\ell)$ over the time interval $[0,\kappa_i]$, $i = 1,2,\dots,6$, where κ_i is the limit superior of this delay period. The function $\Lambda_i(\ell)$, $i = 1,2,\dots,6$ satisfies $\Lambda_i(\ell) > 0$ and

$$\int_0^{\kappa_i} \Lambda_i(\ell)\,d\ell = 1 \quad \text{and} \quad \int_0^{\kappa_i} \Lambda_i(\ell)e^{-u\ell}\,d\ell < \infty,$$

where $u > 0$. Let us denote

$$\bar{\mathcal{H}}_i(\ell) = \Lambda_i(\ell)e^{-\hbar_i\ell} \quad \text{and} \quad \mathcal{H}_i = \int_0^{\kappa_i} \bar{\mathcal{H}}_i(\ell)\,d\ell,$$

where $i = 1,2,\dots,6$. Thus $0 < \mathcal{H}_i \le 1$, $i = 1,2,\dots 6$.

According to [28], we assume that $r^* < \min\{\varrho, \omega, \delta^*\}$. This yields $\delta^* - (1-\mathcal{K})r^* > 0$. Let $r = \mathcal{K}r^*$ and $\delta = \delta^* - (1-\mathcal{K})r^*$. Then system (4) becomes

$$\begin{cases} \frac{dS(t)}{dt} = \eta - \varrho S(t) - \vartheta_1 S(t)V(t) - \vartheta_2 S(t)I(t) - \vartheta_3 S(t)Y(t), \\ \frac{dL(t)}{dt} = (1-\beta)\int_0^{\kappa 1} \bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})[\vartheta_1 V(t-\boldsymbol{\ell}) + \vartheta_2 I(t-\boldsymbol{\ell})]\,d\boldsymbol{\ell} - (\lambda+\gamma)L(t), \\ \frac{dI(t)}{dt} = \beta\int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})[\vartheta_1 V(t-\boldsymbol{\ell}) + \vartheta_2 I(t-\boldsymbol{\ell})]\,d\boldsymbol{\ell} \\ \qquad + \lambda\int_0^{\kappa 3} \bar{\mathcal{H}}_3(\boldsymbol{\ell})L(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - aI(t) - \mu_1 C^I(t)I(t), \\ \frac{dE(t)}{dt} = \varphi\vartheta_3\int_0^{\kappa 4} \bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + rY(t) - (\psi+\omega)E(t), \\ \frac{dY(t)}{dt} = \psi\int_0^{\kappa 5} \bar{\mathcal{H}}_5(\boldsymbol{\ell})E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \delta Y(t) - \mu_2 C^Y(t)Y(t), \\ \frac{dV(t)}{dt} = b\int_0^{\kappa 6} \bar{\mathcal{H}}_6(\boldsymbol{\ell})I(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \varepsilon V(t), \\ \frac{dC^I(t)}{dt} = \sigma_1 C^I(t)I(t) - \pi_1 C^I(t), \\ \frac{dC^Y(t)}{dt} = \sigma_2 C^Y(t)Y(t) - \pi_2 C^Y(t). \end{cases} \tag{5}$$

The initial conditions of system (5) are given by:

$$\begin{aligned} &S(x) = \epsilon_1(x), \quad L(x) = \epsilon_2(x), \quad I(x) = \epsilon_3(x), \quad E(x) = \epsilon_4(x), \\ &Y(x) = \epsilon_5(x), \quad V(x) = \epsilon_6(x), \quad C^I(x) = \epsilon_7(x), \quad C^Y(x) = \epsilon_8(x), \\ &\epsilon_j(x) \geq 0, \quad x \in [-\kappa, 0], j = 1, 2, \ldots, 8, \kappa = \max\{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6\}, \end{aligned} \tag{6}$$

where $\epsilon_j(x) \in \mathcal{C}([-\kappa, 0], \mathbb{R}_{\geq 0})$, $j = 1, 2, \ldots, 8$, and $\mathcal{C} = \mathcal{C}([-\kappa, 0], \mathbb{R}_{\geq 0})$ is the Banach space of continuous functions mapping the interval $[-\kappa, 0]$ into $\mathbb{R}_{\geq 0}$ with norm $\|\epsilon_j\| = \sup_{-\kappa \leq q \leq 0} |\epsilon_j(q)|$ for $\epsilon_j \in \mathcal{C}$. Therefore, system (5) with initial conditions (6) has a unique solution by using the standard theory of functional differential equations [61, 62].

3 Well-posedness of solutions

Proposition 1 *All solutions of system* (5) *with initial conditions* (6) *are nonnegative and ultimately bounded.*

Proof From the first equation of system (5), we have $\frac{dS(t)}{dt}|_{S=0} = \eta > 0$, then $S(t) > 0$ for all $t \geq 0$. Moreover, the rest of equations of system (5) give us the following:

$$\begin{aligned} L(t) &= \epsilon_2(0)e^{-(\lambda+\gamma)t} \\ &\quad + (1-\beta)\int_0^t e^{-(\lambda+\gamma)(t-\varkappa)}\int_0^{\kappa 1} \bar{\mathcal{H}}_1(\boldsymbol{\ell})S(\varkappa-\boldsymbol{\ell})\big[\vartheta_1 V(\varkappa-\boldsymbol{\ell}) + \vartheta_2 I(\varkappa-\boldsymbol{\ell})\big]\,d\boldsymbol{\ell}\,d\varkappa, \\ I(t) &= \epsilon_3(0)e^{-\int_0^t (a+\mu_1 C^I(y))\,dy} \\ &\quad + \int_0^t e^{-\int_\varkappa^t (a+\mu_1 C^I(y))\,dy}\bigg[\beta\int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})S(\varkappa-\boldsymbol{\ell})\big[\vartheta_1 V(\varkappa-\boldsymbol{\ell}) + \vartheta_2 I(\varkappa-\boldsymbol{\ell})\big]\,d\boldsymbol{\ell} \\ &\quad + \lambda\int_0^{\kappa 3} \bar{\mathcal{H}}_3(\boldsymbol{\ell})L(\varkappa-\boldsymbol{\ell})\,d\boldsymbol{\ell}\bigg]\,d\varkappa, \\ E(t) &= \epsilon_4(0)e^{-(\psi+\omega)t} + \int_0^t e^{-(\psi+\omega)(t-\varkappa)}\bigg[\varphi\vartheta_3\int_0^{\kappa 4} \bar{\mathcal{H}}_4(\boldsymbol{\ell})S(\varkappa-\boldsymbol{\ell})Y(\varkappa-\boldsymbol{\ell})\,d\boldsymbol{\ell} + rY(\varkappa)\bigg]\,d\varkappa, \\ Y(t) &= \epsilon_5(0)e^{-\int_0^t (\delta+\mu_2 C^Y(y))\,dy} + \psi\int_0^t e^{-\int_\varkappa^t (\delta+\mu_2 C^Y(y))\,dy}\int_0^{\kappa 5} \bar{\mathcal{H}}_5(\boldsymbol{\ell})E(\varkappa-\boldsymbol{\ell})\,d\boldsymbol{\ell}\,d\varkappa, \end{aligned}$$

$$V(t) = \epsilon_6(0)e^{-\varepsilon t} + b\int_0^t e^{-\varepsilon(t-\varkappa)}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})I(\varkappa-\boldsymbol{\ell})\,d\boldsymbol{\ell}\,d\varkappa,$$

$$C^I(t) = \epsilon_7(0)e^{-\int_0^t(\pi_1-\sigma_1 I(y))\,dy},$$

$$C^Y(t) = \epsilon_8(0)e^{-\int_0^t(\pi_2-\sigma_2 Y(y))\,dy}.$$

Therefore, $L(t), I(t), E(t), Y(t), V(t), C^I(t), C^Y(t) \geq 0$ for all $t \in [0,\kappa]$. Thus, by a recursive argument, we get $S(t), L(t), I(t), E(t), Y(t), V(t), C^I(t), C^Y(t) \geq 0$ for all $t \geq 0$. Hence, the solutions of system (5) with initial conditions (6) satisfy $(S(t), L(t), I(t), E(t), Y(t), V(t), C^I(t), C^Y(t)) \in \mathbb{R}^8_{\geq 0}$ for all $t \geq 0$. Next, we establish the boundedness of the model's solutions. The nonnegativity of the model's solution implies that $\limsup_{t\to\infty} S(t) \leq \frac{\eta}{\varrho}$. To show the ultimate boundedness of $L(t)$, we let

$$\Psi_1(t) = (1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + L(t).$$

Then

$$\begin{aligned}
\frac{d\Psi_1(t)}{dt} &= (1-\beta)\left[\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\{\eta-\varrho S(t-\boldsymbol{\ell})\}\,d\boldsymbol{\ell} - \vartheta_3\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}\right] \\
&\quad - (\lambda+\gamma)L(t) \\
&= (1-\beta)\left[\eta\mathcal{H}_1 - \varrho\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \vartheta_3\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}\right] \\
&\quad - (\lambda+\gamma)L(t) \\
&\leq \eta\mathcal{H}_1(1-\beta) - \varrho(1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - (\lambda+\gamma)L(t) \\
&\leq \eta(1-\beta) - \phi_1\left[(1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + L(t)\right] = \eta(1-\beta) - \phi_1\Psi_1(t),
\end{aligned}$$

where $\phi_1 = \min\{\varrho, \lambda+\gamma\}$. It follows that $\limsup_{t\to\infty}\Psi_1(t) \leq \Omega_1$, where $\Omega_1 = \frac{\eta(1-\beta)}{\phi_1}$. Since $\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}$ and $L(t)$ are nonnegative, then $\limsup_{t\to\infty} L(t) \leq \Omega_1$. Further, we let

$$\Psi_2(t) = \beta\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + I(t) + \frac{\mu_1}{\sigma_1}C^I(t).$$

Then we obtain

$$\begin{aligned}
\frac{d\Psi_2(t)}{dt} &= \beta\left[\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\{\eta-\varrho S(t-\boldsymbol{\ell})\}\,d\boldsymbol{\ell} - \vartheta_3\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}\right] \\
&\quad + \lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})L(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - aI(t) - \frac{\mu_1\pi_1}{\sigma_1}C^I(t) \\
&= \beta\left[\eta\mathcal{H}_2 - \varrho\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \vartheta_3\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}\right] \\
&\quad + \lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})L(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - aI(t) - \frac{\mu_1\pi_1}{\sigma_1}C^I(t) \\
&\leq \eta\beta\mathcal{H}_2 + \lambda\Omega_1\mathcal{H}_3 - \varrho\beta\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - aI(t) - \frac{\mu_1\pi_1}{\sigma_1}C^I(t)
\end{aligned}$$

$$
\begin{aligned}
&\leq \eta\beta + \lambda\Omega_1 - \phi_2\left[\beta\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + I(t) + \frac{\mu_1}{\sigma_1}C^I(t)\right] \\
&= \eta\beta + \lambda\Omega_1 - \phi_2\Psi_2(t),
\end{aligned}
$$

where $\phi_2 = \min\{\varrho, a, \pi_1\}$. It follows that $\limsup_{t\to\infty}\Psi_2(t) \leq \Omega_2$, where $\Omega_2 = \frac{\eta\beta+\lambda\Omega_1}{\phi_2}$. Since $\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}$, $I(t)$ and $C^I(t)$ are nonnegative, then $\limsup_{t\to\infty} I(t) \leq \Omega_2$ and $\limsup_{t\to\infty} C^I(t) \leq \Omega_3$, where $\Omega_3 = \frac{\sigma_1\Omega_2}{\mu_1}$. Furthermore, we let

$$
\begin{aligned}
\Psi_3(t) = {}& \int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + \frac{1}{\varphi}\big[E(t)+Y(t)\big] \\
&+ \frac{\psi}{\varphi}\int_0^{\kappa_5}\Lambda_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} e^{-\hbar_5(t-\varkappa)}E(\varkappa)\,d\varkappa\,d\boldsymbol{\ell} + \frac{\mu_2}{\sigma_2\varphi}C^Y(t).
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{d\Psi_3(t)}{dt} = {}& \int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\big[\eta - \varrho S(t-\boldsymbol{\ell}) - S(t-\boldsymbol{\ell})\big\{\vartheta_1 V(t-\boldsymbol{\ell}) + \vartheta_2 I(t-\boldsymbol{\ell}) + \vartheta_3 Y(t-\boldsymbol{\ell})\big\}\big]\,d\boldsymbol{\ell} \\
&+ \vartheta_3\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + \frac{r}{\varphi}Y(t) - \frac{\psi+\omega}{\varphi}E(t) \\
&+ \frac{\psi}{\varphi}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \frac{\delta}{\varphi}Y(t) \\
&- \hbar_5\frac{\psi}{\varphi}\int_0^{\kappa_5}\Lambda_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} e^{-\hbar_5(t-\varkappa)}E(\varkappa)\,d\varkappa\,d\boldsymbol{\ell} \\
&+ \frac{\psi}{\varphi}E(t)\int_0^{\kappa_5}\Lambda_5(\boldsymbol{\ell})\,d\boldsymbol{\ell} - \frac{\psi}{\varphi}\int_0^{\kappa_5}\Lambda_5(\boldsymbol{\ell})e^{-\hbar_5\boldsymbol{\ell}}E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \frac{\mu_2\pi_2}{\sigma_2\varphi}C^Y(t) \\
\leq {}& \int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\big[\eta - \varrho S(t-\boldsymbol{\ell})\big]\,d\boldsymbol{\ell} - \frac{\omega}{\varphi}E(t) + \left(\frac{r}{\varphi} - \frac{\delta}{\varphi}\right)Y(t) \\
&- \hbar_5\frac{\psi}{\varphi}\int_0^{\kappa_5}\Lambda_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} e^{-\hbar_5(t-\varkappa)}E(\varkappa)\,d\varkappa\,d\boldsymbol{\ell} - \frac{\mu_2\pi_2}{\sigma_2\varphi}C^Y(t) \\
\leq {}& \eta - \varrho\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \frac{\omega}{\varphi}E(t) - \frac{\delta - r}{\varphi}Y(t) \\
&- \hbar_5\frac{\psi}{\varphi}\int_0^{\kappa_5}\Lambda_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} e^{-\hbar_5(t-\varkappa)}E(\varkappa)\,d\varkappa\,d\boldsymbol{\ell} - \frac{\mu_2\pi_2}{\sigma_2\varphi}C^Y(t).
\end{aligned}
$$

Since $\delta - r = \delta^* - r^* > 0$, then

$$
\begin{aligned}
\frac{d\Psi_3(t)}{dt} \leq {}& \eta - \varrho\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \frac{\omega}{\varphi}E(t) - \frac{\delta^* - r^*}{\varphi}Y(t) \\
&- \hbar_5\frac{\psi}{\varphi}\int_0^{\kappa_5}\Lambda_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} e^{-\hbar_5(t-\varkappa)}E(\varkappa)\,d\varkappa\,d\boldsymbol{\ell} - \frac{\mu_2\pi_2}{\sigma_2\varphi}C^Y(t) \\
\leq {}& \eta - \phi_3\bigg[\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + \frac{1}{\varphi}\big\{E(t)+Y(t)\big\} \\
&+ \frac{\psi}{\varphi}\int_0^{\kappa_5}\Lambda_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} e^{-\hbar_5(t-\varkappa)}E(\varkappa)\,d\varkappa\,d\boldsymbol{\ell} + \frac{\mu_2}{\sigma_2\varphi}C^Y(t)\bigg] \\
= {}& \eta - \phi_3\Psi_3(t),
\end{aligned}
$$

where $\phi_3 = \min\{\varrho, \omega, \delta^* - r^*, \hbar_5, \pi_2\}$. It follows that $\limsup_{t\to\infty} \Psi_3(t) \leq \frac{\eta}{\phi_3}$. Since $\int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell) S(t-\ell)\,d\ell \geq 0$, $E(t) \geq 0$, $Y(t) \geq 0$, and $C^Y(t) \geq 0$, then $\limsup_{t\to\infty} E(t) \leq \Omega_4$, $\limsup_{t\to\infty} Y(t) \leq \Omega_4$, and $\limsup_{t\to\infty} C^Y(t) \leq \Omega_5$, where $\Omega_4 = \frac{\eta\varphi}{\phi_3}$ and $\Omega_5 = \frac{\eta\sigma_2\varphi}{\mu_2\phi_3}$. Finally, from the sixth equation of system (5), we have

$$\frac{dV(t)}{dt} = b\int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell) I(t-\ell)\,d\ell - \varepsilon V(t) \leq b\mathcal{H}_6\Omega_2 - \varepsilon V(t) \leq b\Omega_2 - \varepsilon V(t).$$

This implies that $\limsup_{t\to\infty} V(t) \leq \Omega_6$, where $\Omega_6 = \frac{b\Omega_2}{\varepsilon}$. □

According to Proposition 1, we can show that the region

$$\Theta = \{(S,L,I,E,Y,V,C^I,C^Y) \in \mathcal{C}^8_{\geq 0} : \|S\| \leq \Omega_1, \|L\| \leq \Omega_1, \|I\| \leq \Omega_2,$$
$$\|E\| \leq \Omega_4, \|Y\| \leq \Omega_4, \|V\| \leq \Omega_6, \|C^I\| \leq \Omega_3, \|C^Y\| \leq \Omega_5\}$$

is positively invariant with respect to system (5).

4 Steady states analysis

In this section, we calculate all possible steady states of the model and derive the threshold parameters which guarantee the existence of the steady states. Let us define

$$\mathcal{P} = \lambda\mathcal{H}_1\mathcal{H}_3(1-\beta) + \beta\mathcal{H}_2(\gamma+\lambda), \tag{7}$$

which will be used throughout the paper. Let (S,L,I,E,Y,V,C^I,C^Y) be any steady state of system (5) satisfying the following equations:

$$\begin{aligned}
0 &= \eta - \varrho S - \vartheta_1 SV - \vartheta_2 SI - \vartheta_3 SY,\\
0 &= \mathcal{H}_1(1-\beta)(\vartheta_1 SV + \vartheta_2 SI) - (\lambda+\gamma)L,\\
0 &= \beta\mathcal{H}_2(\vartheta_1 SV + \vartheta_2 SI) + \lambda\mathcal{H}_3 L - aI - \mu_1 C^I I,\\
0 &= \varphi\vartheta_3\mathcal{H}_4 SY + rY - (\psi+\omega)E,\\
0 &= \psi\mathcal{H}_5 E - \delta Y - \mu_2 C^Y Y,\\
0 &= b\mathcal{H}_6 I - \varepsilon V,\\
0 &= (\sigma_1 I - \pi_1)C^I,\\
0 &= (\sigma_2 Y - \pi_2)C^Y.
\end{aligned}$$

We find that system (5) has eight possible steady states.

(i) Infection-free steady state, $Đ_0 = (S_0,0,0,0,0,0,0,0)$, where $S_0 = \eta/\varrho$. In this case, the body is free from HTLV and HIV.

(ii) Persistent HIV monoinfection steady state with ineffective immune response, $Đ_1 = (S_1,L_1,I_1,0,0,V_1,0,0)$, where

$$\begin{aligned}
S_1 &= \frac{S_0}{\Re_1}, \qquad L_1 = \frac{a\varepsilon\varrho\mathcal{H}_1(1-\beta)}{\mathcal{P}(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2)}(\Re_1 - 1),\\
I_1 &= \frac{\varepsilon\varrho}{b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2}(\Re_1 - 1), \qquad V_1 = \frac{\varrho b\mathcal{H}_6}{b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2}(\Re_1 - 1),
\end{aligned} \tag{8}$$

and $\Re_1$ is the basic HIV monoinfection reproduction number for system (5) and is defined as follows:

$$\Re_1 = \frac{\mathcal{P}S_0(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2)}{a\varepsilon(\gamma + \lambda)} = \Re_{11} + \Re_{12},$$

where

$$\Re_{11} = \frac{\mathcal{P}S_0 b\vartheta_1\mathcal{H}_6}{a\varepsilon(\gamma + \lambda)}, \qquad \Re_{12} = \frac{\mathcal{P}S_0\vartheta_2}{a(\gamma + \lambda)}.$$

The parameter $\Re_1$ determines whether or not a persistent HIV infection can be established. In fact, $\Re_{11}$ measures the average number of secondary HIV infected generation caused by an existing free HIV particle due to free-to-cell transmission, while $\Re_{12}$ measures the average numbers of secondary HIV infected generation caused by living active HIV-infected cells due to infected-to-cell transmission. The steady state $Ð_1$ describes the case of persistent HIV monoinfection without immune response.

(iii) Persistent HTLV monoinfection steady state with ineffective immune response, $Ð_2 = (S_2, 0, 0, E_2, Y_2, 0, 0, 0)$, where

$$S_2 = \frac{S_0}{\Re_2}, \qquad E_2 = \frac{\varrho\delta}{\vartheta_3\psi\mathcal{H}_5}(\Re_2 - 1), \qquad Y_2 = \frac{\varrho}{\vartheta_3}(\Re_2 - 1),$$

and $\Re_2$ is the basic HTLV monoinfection reproduction number for system (5) and is defined as follows:

$$\Re_2 = \frac{\varphi\vartheta_3\psi\mathcal{H}_4\mathcal{H}_5 S_0}{(\delta - r\mathcal{H}_5)\psi + \delta\omega}.$$

The parameter $\Re_2$ decides whether or not a persistent HTLV infection can be established. The steady state $Ð_2$ describes a persistent HTLV monoinfection without immune response.

We mention that $\Re_1$ and $\Re_2$ state the threshold dynamics of infection-free equilibrium $Ð_0$ and can be calculated by different methods such as (a) the next-generation matrix method of van den Driessche and Watmough [63], (b) local stability of the infection-free equilibrium $Ð_0$, and (c) the existence of the chronic HIV and HTLV monoinfection equilibria with inactive immune response. In the present paper, we derive $\Re_1$ and $\Re_2$ by method (c).

(iv) Persistent HIV monoinfection steady state with only effective HIV-specific CTL, $Ð_3 = (S_3, L_3, I_3, 0, 0, V_3, C_3^I, 0)$, where

$$S_3 = \frac{\varepsilon\sigma_1\eta}{\pi_1(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2) + \varrho\varepsilon\sigma_1}, \qquad L_3 = \frac{\eta\pi_1\mathcal{H}_1(1-\beta)(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2)}{(\gamma + \lambda)[\pi_1(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2) + \varrho\varepsilon\sigma_1]},$$
$$I_3 = \frac{\pi_1}{\sigma_1}, \qquad V_3 = \frac{b\mathcal{H}_6}{\varepsilon}I_3 = \frac{b\pi_1\mathcal{H}_6}{\varepsilon\sigma_1}, \qquad C_3^I = \frac{a}{\mu_1}(\Re_3 - 1),$$

and

$$\Re_3 = \frac{\sigma_1\eta\mathcal{P}(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2)}{a(\gamma + \lambda)[\pi_1(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2) + \varrho\varepsilon\sigma_1]},$$

is the HIV-specific CTL immunity reproduction number in case of HIV monoinfection. The parameter $\Re_3$ determines whether or not the HIV-specific CTL immune response is effective in the absence of HTLV.

(v) Persistent HTLV monoinfection steady state with only effective HTLV-specific CTL, $Đ_4 = (S_4, 0, 0, E_4, Y_4, 0, 0, C_4^Y)$, where

$$\begin{aligned}
S_4 &= \frac{\sigma_2 \eta}{\pi_2 \vartheta_3 + \varrho \sigma_2}, \\
Y_4 &= \frac{\pi_2}{\sigma_2}, \\
E_4 &= \frac{\pi_2 [r(\pi_2 \vartheta_3 + \varrho \sigma_2) + \vartheta_3 \eta \varphi \sigma_2 \mathcal{H}_4]}{\sigma_2 (\psi + \omega)(\pi_2 \vartheta_3 + \varrho \sigma_2)}, \\
C_4^Y &= \frac{(\delta - r\mathcal{H}_5)\psi + \delta \omega}{\mu_2 (\psi + \omega)} (\Re_4 - 1),
\end{aligned}$$

and $\Re_4$ is the HTLV-specific CTL immunity reproduction number in case of HTLV monoinfection and is stated as follows:

$$\Re_4 = \frac{\sigma_2 \eta \varphi \vartheta_3 \psi \mathcal{H}_4 \mathcal{H}_5}{(\pi_2 \vartheta_3 + \varrho \sigma_2)[(\delta - r\mathcal{H}_5)\psi + \delta \omega]}.$$

The parameter $\Re_4$ determines whether or not the HTLV-specific CTL immune response is effective in the absence of HIV.

(vi) Persistent HTLV-HIV coinfection steady state with only effective HIV-specific CTL, $Đ_5 = (S_5, L_5, I_5, E_5, Y_5, V_5, C_5^I, 0)$, where

$$\begin{aligned}
S_5 &= \frac{(\delta - r\mathcal{H}_5)\psi + \delta \omega}{\varphi \vartheta_3 \psi \mathcal{H}_4 \mathcal{H}_5} = S_2, \\
I_5 &= \frac{\pi_1}{\sigma_1} = I_3, \\
V_5 &= \frac{b \pi_1 \mathcal{H}_6}{\varepsilon \sigma_1} = V_3, \\
L_5 &= \frac{\pi_1 \mathcal{H}_1 (1 - \beta)(b \vartheta_1 \mathcal{H}_6 + \varepsilon \vartheta_2)[(\delta - r\mathcal{H}_5)\psi + \delta \omega]}{\varepsilon \vartheta_3 \sigma_1 \varphi \psi \mathcal{H}_4 \mathcal{H}_5 (\gamma + \lambda)}, \\
E_5 &= \frac{\delta [\pi_1 (b \vartheta_1 \mathcal{H}_6 + \varepsilon \vartheta_2) + \varrho \varepsilon \sigma_1]}{\varepsilon \vartheta_3 \sigma_1 \psi \mathcal{H}_5} (\Re_5 - 1), \\
Y_5 &= \frac{\pi_1 (b \vartheta_1 \mathcal{H}_6 + \varepsilon \vartheta_2) + \varrho \varepsilon \sigma_1}{\varepsilon \vartheta_3 \sigma_1} (\Re_5 - 1), \\
C_5^I &= \frac{a}{\mu_1} \left(\frac{\Re_1}{\Re_2} - 1 \right),
\end{aligned}$$

where

$$\Re_5 = \frac{\eta \varphi \varepsilon \vartheta_3 \sigma_1 \psi \mathcal{H}_4 \mathcal{H}_5}{[(\delta - r\mathcal{H}_5)\psi + \delta \omega][\pi_1 (b \vartheta_1 \mathcal{H}_6 + \varepsilon \vartheta_2) + \varrho \varepsilon \sigma_1]}.$$

Here, the parameter $\Re_5$ is the HTLV infection reproduction number in the presence of HIV infection and determines whether or not HIV-infected patients could be dually infected with HTLV.

(vii) Persistent HTLV-HIV coinfection steady state with only effective HTLV-specific CTL, $Đ_6 = (S_6, L_6, I_6, E_6, Y_6, V_6, 0, C_6^Y)$, where

$$
\begin{aligned}
S_6 &= \frac{a\varepsilon(\gamma+\lambda)}{\mathcal{P}(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)} = S_1,\\
Y_6 &= \frac{\pi_2}{\sigma_2} = Y_4,\\
L_6 &= \frac{a\varepsilon\mathcal{H}_1(1-\beta)(\pi_2\vartheta_3+\varrho\sigma_2)}{\sigma_2\mathcal{P}(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)}(\Re_6-1),\\
I_6 &= \frac{\varepsilon(\pi_2\vartheta_3+\varrho\sigma_2)}{\sigma_2(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)}(\Re_6-1),\\
E_6 &= \frac{\pi_2[r\mathcal{P}(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)+a\varepsilon\vartheta_3\varphi\mathcal{H}_4(\gamma+\lambda)]}{\sigma_2\mathcal{P}(\psi+\omega)(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)},\\
V_6 &= \frac{b\mathcal{H}_6(\pi_2\vartheta_3+\varrho\sigma_2)}{\sigma_2(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)}(\Re_6-1),\\
C_6^Y &= \frac{(\delta-r\mathcal{H}_5)\psi+\delta\omega}{\mu_2(\psi+\omega)}\left(\frac{\Re_2}{\Re_1}-1\right),
\end{aligned}
$$

and

$$
\Re_6 = \frac{\eta\sigma_2\mathcal{P}(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)}{a\varepsilon(\gamma+\lambda)(\pi_2\vartheta_3+\varrho\sigma_2)},
$$

is the HIV infection reproduction number in the presence of HTLV infection. It is clear that $\Re_6$ determines whether or not HTLV-infected patients could be dually infected with HIV.

(viii) Persistent HTLV-HIV coinfection steady state with effective HIV-specific CTL and HTLV-specific CTL, $Đ_7 = (S_7, L_7, I_7, E_7, Y_7, V_7, C_7^I, C_7^Y)$, where

$$
\begin{aligned}
S_7 &= \frac{\varepsilon\sigma_1\sigma_2\eta}{\pi_1\sigma_2(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)+\varepsilon\sigma_1(\pi_2\vartheta_3+\varrho\sigma_2)},\\
L_7 &= \frac{\pi_1\sigma_2\eta\mathcal{H}_1(1-\beta)(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)}{(\gamma+\lambda)[\pi_1\sigma_2(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)+\varepsilon\sigma_1(\pi_2\vartheta_3+\varrho\sigma_2)]},\\
E_7 &= \frac{\pi_2[\vartheta_3\varepsilon\sigma_1\sigma_2\eta\varphi\mathcal{H}_4+r\{\pi_1\sigma_2(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)+\varepsilon\sigma_1(\pi_2\vartheta_3+\varrho\sigma_2)\}]}{\sigma_2(\psi+\omega)[\pi_1\sigma_2(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)+\varepsilon\sigma_1(\pi_2\vartheta_3+\varrho\sigma_2)]},\\
I_7 &= \frac{\pi_1}{\sigma_1} = I_3 = I_5,\\
Y_7 &= \frac{\pi_2}{\sigma_2} = Y_4 = Y_6,\\
V_7 &= \frac{b\pi_1\mathcal{H}_6}{\varepsilon\sigma_1} = V_3 = V_5,\\
C_7^I &= \frac{a}{\mu_1}(\Re_7-1),\\
C_7^Y &= \frac{(\delta-r\mathcal{H}_5)\psi+\delta\omega}{\mu_2(\psi+\omega)}(\Re_8-1),
\end{aligned}
$$

and

$$\Re_7 = \frac{\sigma_1\sigma_2\eta\mathcal{P}(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2)}{a(\gamma + \lambda)[\pi_1\sigma_2(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2) + \varepsilon\sigma_1(\pi_2\vartheta_3 + \varrho\sigma_2)]},$$
$$\Re_8 = \frac{\varepsilon\vartheta_3\eta\sigma_1\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}{[\pi_1\sigma_2(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2) + \varepsilon\sigma_1(\pi_2\vartheta_3 + \varrho\sigma_2)][(\delta - r\mathcal{H}_5)\psi + \delta\omega]}.$$

The parameter $\Re_7$ is the competed HIV-specific CTL immunity reproduction number in case of HTLV-HIV coinfection. The parameter $\Re_8$ is the competed HTLV-specific CTL immunity reproduction number in case of HTLV-HIV coinfection. Clearly, $Đ_7$ exists when $\Re_7 > 1$ and $\Re_8 > 1$.

5 Global stability analysis

In this section, we use the Lyapunov method to show the global asymptotic stability of the model's steady states. For formation of Lyapunov functionals, we follow the works [64, 65]. Denote $U = U(t)$, where $U \in (S, L, I, E, Y, V, C^I, C^Y)$.

Let a function $\Phi_j(S, L, I, E, Y, V, C^I, C^Y)$ and Υ_j' be the largest invariant subset of

$$\Upsilon_j = \left\{(S, L, I, E, Y, V, C^I, C^Y) : \frac{d\Phi_j}{dt} = 0\right\}, \quad j = 0, 1, 2, \ldots, 7.$$

We define a function $\digamma(x) = x - 1 - \ln x$.

Theorem 1 *If* $\Re_1 \leq 1$ *and* $\Re_2 \leq 1$, *then* $Đ_0$ *is globally asymptotically stable* (*GAS*).

Proof We define a Lyapunov functional as follows:

$$\begin{aligned}
\Phi_0 &= \mathcal{P}S_0\digamma\left(\frac{S}{S_0}\right) + \lambda\mathcal{H}_3 L + (\gamma + \lambda)I + \frac{\mathcal{P}}{\varphi\mathcal{H}_4}E + \frac{\mathcal{P}(\psi + \omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y + \frac{\mathcal{P}\vartheta_1 S_0}{\varepsilon}V \\
&\quad + \frac{\mu_1(\gamma + \lambda)}{\sigma_1}C^I + \frac{\mu_2\mathcal{P}(\psi + \omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y + \lambda\mathcal{H}_3(1 - \beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} S(\varkappa) \\
&\quad \times \left[\vartheta_1 V(\varkappa) + \vartheta_2 I(\varkappa)\right] d\varkappa\, d\boldsymbol{\ell} \\
&\quad + \beta(\gamma + \lambda)\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} S(\varkappa)\left[\vartheta_1 V(\varkappa) + \vartheta_2 I(\varkappa)\right] d\varkappa\, d\boldsymbol{\ell} \\
&\quad + \lambda(\gamma + \lambda)\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} L(\varkappa)\, d\varkappa\, d\boldsymbol{\ell} + \frac{\mathcal{P}\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} S(\varkappa)Y(\varkappa)\, d\varkappa\, d\boldsymbol{\ell} \\
&\quad + \frac{\mathcal{P}(\psi + \omega)}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} E(\varkappa)\, d\varkappa\, d\boldsymbol{\ell} + \frac{b\mathcal{P}\vartheta_1 S_0}{\varepsilon}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} I(\varkappa)\, d\varkappa\, d\boldsymbol{\ell}.
\end{aligned}$$

Clearly, $\Phi_0(S, L, I, E, Y, V, C^I, C^Y) > 0$ for all $S, L, I, E, Y, V, C^I, C^Y > 0$, and $\Phi_0(S_0, 0, 0, 0, 0, 0, 0, 0) = 0$. We calculate $\frac{d\Phi_0}{dt}$ along the solutions of model (5) as follows:

$$\begin{aligned}
\frac{d\Phi_0}{dt} &= \mathcal{P}\left(1 - \frac{S_0}{S}\right)(\eta - \varrho S - \vartheta_1 SV - \vartheta_2 SI - \vartheta_3 SY) + \lambda\mathcal{H}_3\left[(1 - \beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t - \boldsymbol{\ell})\right. \\
&\quad \left. \times \left\{\vartheta_1 V(t - \boldsymbol{\ell}) + \vartheta_2 I(t - \boldsymbol{\ell})\right\} d\boldsymbol{\ell} - (\lambda + \gamma)L\right] + (\gamma + \lambda)\left[\beta\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t - \boldsymbol{\ell})\right. \\
&\quad \left. \times \left\{\vartheta_1 V(t - \boldsymbol{\ell}) + \vartheta_2 I(t - \boldsymbol{\ell})\right\} d\boldsymbol{\ell} + \lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})L(t - \boldsymbol{\ell})\, d\boldsymbol{\ell} - aI - \mu_1 C^I I\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\mathcal{P}}{\varphi\mathcal{H}_4}\left[\varphi\vartheta_3\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + rY - (\psi+\omega)E\right] \\
&+ \frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left[\psi\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \delta Y - \mu_2 C^Y Y\right] \\
&+ \frac{\mathcal{P}\vartheta_1 S_0}{\varepsilon}\left[b\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})I(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \varepsilon V\right] + \frac{\mu_1(\gamma+\lambda)}{\sigma_1}\left(\sigma_1 C^I I - \pi_1 C^I\right) \\
&+ \frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(\sigma_2 C^Y Y - \pi_2 C^Y\right) + \mathcal{P}(\vartheta_1 SV + \vartheta_2 SI) - \lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell}) \\
&\times S(t-\boldsymbol{\ell})\left[\vartheta_1 V(t-\boldsymbol{\ell}) + \vartheta_2 I(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} - \beta(\gamma+\lambda)\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell}) \\
&\times \left[\vartheta_1 V(t-\boldsymbol{\ell}) + \vartheta_2 I(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} + \lambda(\gamma+\lambda)\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\left[L - L(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} \\
&+ \frac{\mathcal{P}\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\left[SY - S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} + \frac{\mathcal{P}(\psi+\omega)}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell}) \\
&\times \left[E - E(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} + \frac{b\mathcal{P}\vartheta_1 S_0}{\varepsilon}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\left[I - I(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell}. \qquad (9)
\end{aligned}
$$

Summing the terms of Eq. (9), we obtain

$$
\begin{aligned}
\frac{d\Phi_0}{dt} &= \mathcal{P}\left(1-\frac{S_0}{S}\right)(\eta - \varrho S) + \mathcal{P}\vartheta_2 S_0 I - a(\lambda+\gamma)I + \frac{b\mathcal{P}\vartheta_1\mathcal{H}_6 S_0}{\varepsilon}I + \mathcal{P}\vartheta_3 S_0 Y \\
&\quad - \frac{\mathcal{P}[(\delta - r\mathcal{H}_5)\psi + \delta\omega]}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y - \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1}C^I - \frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y.
\end{aligned}
$$

Using $S_0 = \eta/\varrho$, we obtain

$$
\begin{aligned}
\frac{d\Phi_0}{dt} &= -\varrho\mathcal{P}\frac{(S-S_0)^2}{S} + a(\lambda+\gamma)(\Re_1 - 1)I + \frac{\mathcal{P}[(\delta - r\mathcal{H}_5)\psi + \delta\omega]}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}(\Re_2 - 1)Y \\
&\quad - \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1}C^I - \frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y.
\end{aligned}
$$

Since $r < \delta$ and $0 < \mathcal{H}_5 \le 1$, then $\delta - r\mathcal{H}_5 > 0$. Therefore, $\frac{d\Phi_0}{dt} \le 0$ for all $S, I, Y, C^I, C^Y > 0$; moreover, $\frac{d\Phi_0}{dt} = 0$ when $(S(t), I(t), Y(t), C^I(t), C^Y(t)) = (S_0, 0, 0, 0, 0)$. The solutions of system (5) converge to Υ_0'. The set Υ_0' includes elements with $(S(t), I(t), Y(t), C^I(t), C^Y(t)) = (S_0, 0, 0, 0, 0)$. Then $\frac{dS(t)}{dt} = \frac{dY(t)}{dt} = 0$ and the first and fifth equations of system (5) become

$$
\begin{aligned}
0 &= \frac{dS(t)}{dt} = \eta - \varrho S_0 - \vartheta_1 S_0 V(t), \\
0 &= \frac{dY(t)}{dt} = \psi\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell},
\end{aligned}
$$

which give $V(t) = E(t) = 0$ for all t. In addition, we have $\frac{dI(t)}{dt} = 0$, and from the third equation of system (5) we have

$$
0 = \frac{dI(t)}{dt} = \lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})L(t-\boldsymbol{\ell})\,d\boldsymbol{\ell},
$$

which yields $L(t) = 0$ for all t and hence $\Upsilon_0' = \{\text{Đ}_0\}$. Applying Lyapunov–LaSalle asymptotic stability (LLAS) theorem [66–68], we get that Đ_0 is GAS. □

The following equalities are needed in the next theorems:

$$\begin{aligned}
\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) &= \left[\ln\left(\frac{S(t-\ell)V(t-\ell)L_n}{S_nV_nL}\right) + \ln\left(\frac{S_n}{S}\right) + \ln\left(\frac{V_nL}{VL_n}\right)\right], \\
\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) &= \left[\ln\left(\frac{S(t-\ell)V(t-\ell)I_n}{S_nV_nI}\right) + \ln\left(\frac{S_n}{S}\right) + \ln\left(\frac{V_nI}{VI_n}\right)\right], \\
\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right) &= \left[\ln\left(\frac{S(t-\ell)I(t-\ell)L_n}{S_nI_nL}\right) + \ln\left(\frac{S_n}{S}\right) + \ln\left(\frac{I_nL}{IL_n}\right)\right], \\
\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right) &= \left[\ln\left(\frac{S(t-\ell)I(t-\ell)}{S_nI}\right) + \ln\left(\frac{S_n}{S}\right)\right], \\
\ln\left(\frac{L(t-\ell)}{L}\right) &= \ln\left(\frac{L(t-\ell)I_n}{L_nI}\right) + \ln\left(\frac{L_nI}{LI_n}\right), \\
\ln\left(\frac{I(t-\ell)}{I}\right) &= \ln\left(\frac{I(t-\ell)V_n}{I_nV}\right) + \ln\left(\frac{I_nV}{IV_n}\right), \quad \text{where } n = 1,3,5,6,7.
\end{aligned} \tag{10}$$

Further,

$$\begin{aligned}
\ln\left(\frac{S(t-\ell)Y(t-\ell)}{SY}\right) &= \ln\left(\frac{S(t-\ell)Y(t-\ell)E_m}{S_mY_mE}\right) + \ln\left(\frac{S_m}{S}\right) + \ln\left(\frac{Y_mE}{YE_m}\right), \\
\ln\left(\frac{E(t-\ell)}{E}\right) &= \ln\left(\frac{E(t-\ell)Y_m}{E_mY}\right) + \ln\left(\frac{YE_m}{Y_mE}\right), \quad \text{where } m = 2,4,5,6,7.
\end{aligned} \tag{11}$$

Theorem 2 *Let* $\Re_1 > 1$, $\Re_2/\Re_1 \le 1$, *and* $\Re_3 \le 1$, *then* $Đ_1$ *is GAS.*

Proof Define a functional as follows:

$$\begin{aligned}
\Phi_1 = {} & \mathcal{P}S_1F\left(\frac{S}{S_1}\right) + \lambda\mathcal{H}_3L_1F\left(\frac{L}{L_1}\right) + (\gamma+\lambda)I_1F\left(\frac{I}{I_1}\right) + \frac{\mathcal{P}}{\varphi\mathcal{H}_4}E + \frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y \\
& + \frac{\mathcal{P}\vartheta_1S_1}{\varepsilon}V_1F\left(\frac{V}{V_1}\right) + \frac{\mu_1(\gamma+\lambda)}{\sigma_1}C^I + \frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y \\
& + \vartheta_1\lambda\mathcal{H}_3(1-\beta)S_1V_1\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)V(\varkappa)}{S_1V_1}\right)d\varkappa\,d\ell \\
& + \vartheta_2\lambda\mathcal{H}_3(1-\beta)S_1I_1\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)I(\varkappa)}{S_1I_1}\right)d\varkappa\,d\ell \\
& + \vartheta_1\beta(\gamma+\lambda)S_1V_1\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)V(\varkappa)}{S_1V_1}\right)d\varkappa\,d\ell \\
& + \vartheta_2\beta(\gamma+\lambda)S_1I_1\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)I(\varkappa)}{S_1I_1}\right)d\varkappa\,d\ell \\
& + \lambda(\gamma+\lambda)L_1\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\ell)\int_{t-\ell}^{t}F\left(\frac{L(\varkappa)}{L_1}\right)d\varkappa\,d\ell \\
& + \frac{\mathcal{P}\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\ell)\int_{t-\ell}^{t}S(\varkappa)Y(\varkappa)\,d\varkappa\,d\ell \\
& + \frac{\mathcal{P}(\psi+\omega)}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\ell)\int_{t-\ell}^{t}E(\varkappa)\,d\varkappa\,d\ell \\
& + \frac{b\mathcal{P}\vartheta_1S_1I_1}{\varepsilon}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\ell)\int_{t-\ell}^{t}F\left(\frac{I(\varkappa)}{I_1}\right)d\varkappa\,d\ell.
\end{aligned}$$

Calculate $\frac{d\Phi_1}{dt}$ as follows:

$$\begin{aligned}
\frac{d\Phi_1}{dt} &= \mathcal{P}\left(1-\frac{S_1}{S}\right)(\eta-\varrho S-\vartheta_1 SV-\vartheta_2 SI-\vartheta_3 SY)+\lambda\mathcal{H}_3\left(1-\frac{L_1}{L}\right) \\
&\quad\times\left[(1-\beta)\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\bigl\{\vartheta_1 V(t-\boldsymbol{\ell})+\vartheta_2 I(t-\boldsymbol{\ell})\bigr\}\,d\boldsymbol{\ell}-(\lambda+\gamma)L\right] \\
&\quad+(\gamma+\lambda)\left(1-\frac{I_1}{I}\right)\left[\beta\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\bigl\{\vartheta_1 V(t-\boldsymbol{\ell})+\vartheta_2 I(t-\boldsymbol{\ell})\bigr\}\,d\boldsymbol{\ell}\right. \\
&\quad\left.+\lambda\int_0^{\kappa 3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})L(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}-aI-\mu_1 C^I I\right] \\
&\quad+\frac{\mathcal{P}}{\varphi\mathcal{H}_4}\left[\varphi\vartheta_3\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}\right. \\
&\quad\left.+rY-(\psi+\omega)E\right]+\frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left[\psi\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}-\delta Y-\mu_2 C^Y Y\right] \\
&\quad+\frac{\mathcal{P}\vartheta_1 S_1}{\varepsilon}\left(1-\frac{V_1}{V}\right)\left[b\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})I(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}-\varepsilon V\right] \\
&\quad+\frac{\mu_1(\gamma+\lambda)}{\sigma_1}\left(\sigma_1 C^I I-\pi_1 C^I\right)+\frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(\sigma_2 C^Y Y-\pi_2 C^Y\right) \\
&\quad+\vartheta_1\lambda\mathcal{H}_3(1-\beta)S_1V_1 \\
&\quad\times\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\left[\frac{SV}{S_1V_1}-\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{S_1V_1}+\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)\right]d\boldsymbol{\ell} \\
&\quad+\vartheta_2\lambda\mathcal{H}_3(1-\beta)S_1I_1 \\
&\quad\times\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\left[\frac{SI}{S_1I_1}-\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_1I_1}+\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)\right]d\boldsymbol{\ell} \\
&\quad+\vartheta_1\beta(\gamma+\lambda)S_1V_1 \\
&\quad\times\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{SV}{S_1V_1}-\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{S_1V_1}+\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)\right]d\boldsymbol{\ell} \\
&\quad+\vartheta_2\beta(\gamma+\lambda)S_1I_1 \\
&\quad\times\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{SI}{S_1I_1}-\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_1I_1}+\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)\right]d\boldsymbol{\ell} \\
&\quad+\lambda(\gamma+\lambda)L_1\int_0^{\kappa 3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\left[\frac{L}{L_1}-\frac{L(t-\boldsymbol{\ell})}{L_1}+\ln\left(\frac{L(t-\boldsymbol{\ell})}{L}\right)\right]d\boldsymbol{\ell} \\
&\quad+\frac{\mathcal{P}\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\bigl[SY-S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\bigr]\,d\boldsymbol{\ell} \\
&\quad+\frac{\mathcal{P}(\psi+\omega)}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\bigl[E-E(t-\boldsymbol{\ell})\bigr]\,d\boldsymbol{\ell} \\
&\quad+\frac{b\mathcal{P}\vartheta_1 S_1 I_1}{\varepsilon}\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\left[\frac{I}{I_1}-\frac{I(t-\boldsymbol{\ell})}{I_1}+\ln\left(\frac{I(t-\boldsymbol{\ell})}{I}\right)\right]d\boldsymbol{\ell}. \qquad (12)
\end{aligned}$$

Summing the terms of Eq. (12), we get

$$\frac{d\Phi_1}{dt} = \mathcal{P}\left(1-\frac{S_1}{S}\right)(\eta-\varrho S)+\mathcal{P}\vartheta_2 S_1 I+\mathcal{P}\vartheta_3 S_1 Y$$

$$
\begin{aligned}
& - \vartheta_1 \lambda \mathcal{H}_3 (1-\beta) \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) \frac{S(t-\boldsymbol{\ell}) V(t-\boldsymbol{\ell}) L_1}{L} \, d\boldsymbol{\ell} \\
& - \vartheta_2 \lambda \mathcal{H}_3 (1-\beta) \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) \frac{S(t-\boldsymbol{\ell}) I(t-\boldsymbol{\ell}) L_1}{L} \, d\boldsymbol{\ell} + \lambda \mathcal{H}_3 (\lambda+\gamma) L_1 - a(\lambda+\gamma) I \\
& - \vartheta_1 \beta (\gamma+\lambda) \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) \frac{S(t-\boldsymbol{\ell}) V(t-\boldsymbol{\ell}) I_1}{I} \, d\boldsymbol{\ell} \\
& - \vartheta_2 \beta (\gamma+\lambda) \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) \frac{S(t-\boldsymbol{\ell}) I(t-\boldsymbol{\ell}) I_1}{I} \, d\boldsymbol{\ell} \\
& - \lambda (\lambda+\gamma) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\boldsymbol{\ell}) \frac{L(t-\boldsymbol{\ell}) I_1}{I} \, d\boldsymbol{\ell} + a(\lambda+\gamma) I_1 \\
& + \mu_1 (\lambda+\gamma) C^I I_1 + \frac{\mathcal{P} r}{\varphi \mathcal{H}_4} Y - \frac{\mathcal{P} \delta (\psi+\omega)}{\varphi \psi \mathcal{H}_4 \mathcal{H}_5} Y \\
& - \frac{\mathcal{P} b \vartheta_1 S_1}{\varepsilon} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\boldsymbol{\ell}) \frac{I(t-\boldsymbol{\ell}) V_1}{V} \, d\boldsymbol{\ell} + \mathcal{P} \vartheta_1 S_1 V_1 \\
& - \frac{\mu_1 \pi_1 (\lambda+\gamma)}{\sigma_1} C^I - \frac{\mu_2 \pi_2 \mathcal{P} (\psi+\omega)}{\sigma_2 \varphi \psi \mathcal{H}_4 \mathcal{H}_5} C^Y \\
& + \lambda \mathcal{H}_3 (1-\beta) \vartheta_1 S_1 V_1 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) \ln\left(\frac{S(t-\boldsymbol{\ell}) V(t-\boldsymbol{\ell})}{SV} \right) d\boldsymbol{\ell} \\
& + \lambda \mathcal{H}_3 (1-\beta) \vartheta_2 S_1 I_1 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) \ln\left(\frac{S(t-\boldsymbol{\ell}) I(t-\boldsymbol{\ell})}{SI} \right) d\boldsymbol{\ell} \\
& + \beta (\gamma+\lambda) \vartheta_1 S_1 V_1 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) \ln\left(\frac{S(t-\boldsymbol{\ell}) V(t-\boldsymbol{\ell})}{SV} \right) d\boldsymbol{\ell} \\
& + \beta (\gamma+\lambda) \vartheta_2 S_1 I_1 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) \ln\left(\frac{S(t-\boldsymbol{\ell}) I(t-\boldsymbol{\ell})}{SI} \right) d\boldsymbol{\ell} \\
& + \lambda (\gamma+\lambda) L_1 \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\boldsymbol{\ell}) \ln\left(\frac{L(t-\boldsymbol{\ell})}{L} \right) d\boldsymbol{\ell} \\
& + \frac{b \mathcal{P} \vartheta_1 S_1 \mathcal{H}_6}{\varepsilon} I + \frac{\mathcal{P} b \vartheta_1 S_1 I_1}{\varepsilon} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\boldsymbol{\ell}) \ln\left(\frac{I(t-\boldsymbol{\ell})}{I} \right) d\boldsymbol{\ell}.
\end{aligned}
$$

The steady state conditions for $Đ_1$ are given by

$$
\begin{aligned}
& \eta = \varrho S_1 + \vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1, \qquad \mathcal{H}_1 (1-\beta)(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) = (\lambda+\gamma) L_1, \\
& \beta \mathcal{H}_2 (\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) + \lambda \mathcal{H}_3 L_1 = a I_1 \qquad V_1 = \frac{b \mathcal{H}_6 I_1}{\varepsilon}.
\end{aligned}
$$

Then we get

$$
\mathcal{P}(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) = a(\lambda+\gamma) I_1.
$$

Further, we obtain

$$
\begin{aligned}
\frac{d\Phi_1}{dt} = {} & \mathcal{P}\left(1 - \frac{S_1}{S}\right)(\varrho S_1 - \varrho S) + \mathcal{P}(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1)\left(1 - \frac{S_1}{S}\right) + \mathcal{P} \vartheta_3 S_1 Y \\
& - \lambda \mathcal{H}_3 (1-\beta) \vartheta_1 S_1 V_1 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) \frac{S(t-\boldsymbol{\ell}) V(t-\boldsymbol{\ell}) L_1}{S_1 V_1 L} \, d\boldsymbol{\ell}
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_1 I_1 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\frac{S(t-\ell)I(t-\ell)L_1}{S_1 I_1 L}\,d\ell \\
&+ \lambda\mathcal{H}_1\mathcal{H}_3(1-\beta)(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) \\
&- \beta(\gamma+\lambda)\vartheta_1 S_1 V_1 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\frac{S(t-\ell)V(t-\ell)I_1}{S_1 V_1 I}\,d\ell \\
&- \beta(\gamma+\lambda)\vartheta_2 S_1 I_1 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\frac{S(t-\ell)I(t-\ell)}{S_1 I}\,d\ell \\
&- \lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell)\frac{L(t-\ell)I_1}{L_1 I}\,d\ell \\
&+ \mathcal{P}(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) + \mu_1(\lambda+\gamma)C^I I_1 - \frac{\mathcal{P}[(\delta - r\mathcal{H}_5)\psi + \delta\omega]}{\varphi\psi\mathcal{H}_4\mathcal{H}_5} Y \\
&- \frac{\mathcal{P}\vartheta_1 S_1 V_1}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell)\frac{I(t-\ell)V_1}{I_1 V}\,d\ell \\
&+ \mathcal{P}\vartheta_1 S_1 V_1 - \frac{\mu_1\pi_1(\lambda+\gamma)}{\sigma_1} C^I - \frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5} C^Y \\
&+ \lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_1 V_1 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right)d\ell \\
&+ \lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_1 I_1 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right)d\ell \\
&+ \beta(\gamma+\lambda)\vartheta_1 S_1 V_1 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right)d\ell \\
&+ \beta(\gamma+\lambda)\vartheta_2 S_1 I_1 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right)d\ell \\
&+ \lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell)\ln\left(\frac{L(t-\ell)}{L}\right)d\ell \\
&+ \frac{\mathcal{P}\vartheta_1 S_1 V_1}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell)\ln\left(\frac{I(t-\ell)}{I}\right)d\ell.
\end{aligned}
$$

Using the equalities given by (10) in case of $n = 1$, we get

$$
\begin{aligned}
\frac{d\Phi_1}{dt} = {} & -\varrho\mathcal{P}\frac{(S-S_1)^2}{S} - \mathcal{P}(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1)\left[\frac{S_1}{S} - 1 - \ln\left(\frac{S_1}{S}\right)\right] \\
&- \lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_1 V_1 \\
&\times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\left[\frac{S(t-\ell)V(t-\ell)L_1}{S_1 V_1 L} - 1 - \ln\left(\frac{S(t-\ell)V(t-\ell)L_1}{S_1 V_1 L}\right)\right]d\ell \\
&- \lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_1 I_1 \\
&\times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\left[\frac{S(t-\ell)I(t-\ell)L_1}{S_1 I_1 L} - 1 - \ln\left(\frac{S(t-\ell)I(t-\ell)L_1}{S_1 I_1 L}\right)\right]d\ell \\
&- \beta(\gamma+\lambda)\vartheta_1 S_1 V_1 \\
&\times \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\left[\frac{S(t-\ell)V(t-\ell)I_1}{S_1 V_1 I} - 1 - \ln\left(\frac{S(t-\ell)V(t-\ell)I_1}{S_1 V_1 I}\right)\right]d\ell \\
&- \beta(\gamma+\lambda)\vartheta_2 S_1 I_1 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\left[\frac{S(t-\ell)I(t-\ell)}{S_1 I} - 1 - \ln\left(\frac{S(t-\ell)I(t-\ell)}{S_1 I}\right)\right]d\ell
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda \mathcal{H}_1(1-\beta)(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) \\
&\times \int_0^{\kappa 3} \bar{\mathcal{H}}_3(\ell)\left[\frac{L(t-\ell)I_1}{L_1 I} - 1 - \ln\left(\frac{L(t-\ell)I_1}{L_1 I}\right)\right] d\ell \\
&- \frac{\mathcal{P}\vartheta_1 S_1 V_1}{\mathcal{H}_6} \int_0^{\kappa 6} \bar{\mathcal{H}}_6(\ell)\left[\frac{I(t-\ell)V_1}{I_1 V} - 1 - \ln\left(\frac{I(t-\ell)V_1}{I_1 V}\right)\right] d\ell \\
&+ \frac{\mathcal{P}[(\delta - r\mathcal{H}_5)\psi + \delta\omega]}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(\frac{\vartheta_3\varphi\psi\mathcal{H}_4\mathcal{H}_5 S_1}{(\delta - r\mathcal{H}_5)\psi + \delta\omega} - 1\right) Y \\
&+ \mu_1(\lambda+\gamma)\left(I_1 - \frac{\pi_1}{\sigma_1}\right) C^I - \frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5} C^Y.
\end{aligned} \tag{13}
$$

Therefore, Eq. (13) becomes

$$
\begin{aligned}
\frac{d\Phi_1}{dt} &= -\varrho\mathcal{P}\frac{(S-S_1)^2}{S} - \mathcal{P}(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) F\left(\frac{S_1}{S}\right) \\
&- \lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_1 V_1 \int_0^{\kappa 1} \bar{\mathcal{H}}_1(\ell) F\left(\frac{S(t-\ell)V(t-\ell)L_1}{S_1 V_1 L}\right) d\ell \\
&- \lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_1 I_1 \int_0^{\kappa 1} \bar{\mathcal{H}}_1(\ell) F\left(\frac{S(t-\ell)I(t-\ell)L_1}{S_1 I_1 L}\right) d\ell \\
&- \beta(\gamma+\lambda)\vartheta_1 S_1 V_1 \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\ell) F\left(\frac{S(t-\ell)V(t-\ell)I_1}{S_1 V_1 I}\right) d\ell \\
&- \beta(\gamma+\lambda)\vartheta_2 S_1 I_1 \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\ell) F\left(\frac{S(t-\ell)I(t-\ell)}{S_1 I}\right) d\ell \\
&- \lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_1 V_1 + \vartheta_2 S_1 I_1) \int_0^{\kappa 3} \bar{\mathcal{H}}_3(\ell) F\left(\frac{L(t-\ell)I_1}{L_1 I}\right) d\ell \\
&- \frac{\mathcal{P}\vartheta_1 S_1 V_1}{\mathcal{H}_6} \int_0^{\kappa 6} \bar{\mathcal{H}}_6(\ell) F\left(\frac{I(t-\ell)V_1}{I_1 V}\right) d\ell + \frac{\mathcal{P}[(\delta - r\mathcal{H}_5)\psi + \delta\omega]}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(\frac{\Re_2}{\Re_1} - 1\right) Y \\
&+ \frac{\mu_1(\gamma+\lambda)[\pi_1(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2) + \varrho\varepsilon\sigma_1]}{\sigma_1(b\vartheta_1\mathcal{H}_6 + \varepsilon\vartheta_2)}(\Re_3 - 1) C^I - \frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5} C^Y.
\end{aligned}
$$

Since $\Re_2/\Re_1 \leq 1$ and $\Re_3 \leq 1$, then $\frac{d\Phi_1}{dt} \leq 0$ for all $S, L, I, Y, V, C^I, C^Y > 0$. Moreover, $\frac{d\Phi_1}{dt} = 0$ when $S = S_1$ and $Y = C^I = C^Y = F = 0$. The solutions of system (5) converge to Υ_1' which includes elements that satisfy $S(t) = S_1$ and $F = 0$ i.e.

$$
\begin{aligned}
\frac{S(t-\ell)V(t-\ell)L_1}{S_1 V_1 L} &= \frac{S(t-\ell)I(t-\ell)L_1}{S_1 I_1 L} = \frac{S(t-\ell)V(t-\ell)I_1}{S_1 V_1 I} \\
&= \frac{S(t-\ell)I(t-\ell)}{S_1 I} = \frac{L(t-\ell)I_1}{L_1 I} = \frac{I(t-\ell)V_1}{I_1 V} = 1
\end{aligned} \tag{14}
$$

for all $t \in [0, \kappa]$. If $S(t) = S_1$, then from Eq. (14) we get $L(t) = L_1$, $I(t) = I_1$, and $V(t) = V_1$ for all t. Further, for each element of Υ_1', we have $Y(t) = 0$ and then $\frac{dY(t)}{dt} = 0$. The fifth equation of system (5) becomes

$$
0 = \frac{dY(t)}{dt} = \psi \int_0^{\kappa 5} \bar{\mathcal{H}}_5(\ell) E(t-\ell)\, d\ell,
$$

which provides $E(t) = 0$ for all t, and hence $\Upsilon_1' = \{Đ_1\}$. Therefore, using LLAS theorem we get that $Đ_1$ is GAS. □

Theorem 3 *If* $\Re_2 > 1$, $\Re_1/\Re_2 \leq 1$, *and* $\Re_4 \leq 1$, *then* $Đ_2$ *is GAS.*

Proof Define

$$\begin{aligned}
\Phi_2 ={}& \mathcal{P}S_2F\left(\frac{S}{S_2}\right) + \lambda\mathcal{H}_3L + (\gamma+\lambda)I + \frac{\mathcal{P}}{\varphi\mathcal{H}_4}E_2F\left(\frac{E}{E_2}\right) + \frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y_2F\left(\frac{Y}{Y_2}\right) \\
&+ \frac{\mathcal{P}\vartheta_1S_2}{\varepsilon}V + \frac{\mu_1(\gamma+\lambda)}{\sigma_1}C^I + \frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y + \lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^t S(\varkappa) \\
&\times\left[\vartheta_1V(\varkappa)+\vartheta_2I(\varkappa)\right]d\varkappa\, d\boldsymbol{\ell} \\
&+ \beta(\gamma+\lambda)\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^t S(\varkappa)\left[\vartheta_1V(\varkappa)+\vartheta_2I(\varkappa)\right]d\varkappa\, d\boldsymbol{\ell} \\
&+ \lambda(\gamma+\lambda)\int_0^{\kappa 3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^t L(\varkappa)\,d\varkappa\, d\boldsymbol{\ell} \\
&+ \frac{\mathcal{P}\vartheta_3S_2Y_2}{\mathcal{H}_4}\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^t F\left(\frac{S(\varkappa)Y(\varkappa)}{S_2Y_2}\right)d\varkappa\, d\boldsymbol{\ell} \\
&+ \frac{\mathcal{P}(\psi+\omega)E_2}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^t F\left(\frac{E(\varkappa)}{E_2}\right)d\varkappa\, d\boldsymbol{\ell} \\
&+ \frac{b\mathcal{P}\vartheta_1S_2}{\varepsilon}\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^t I(\varkappa)\,d\varkappa\, d\boldsymbol{\ell}.
\end{aligned}$$

We calculate $\frac{d\Phi_2}{dt}$ as follows:

$$\begin{aligned}
\frac{d\Phi_2}{dt} ={}& \mathcal{P}\left(1-\frac{S_2}{S}\right)(\eta - \varrho S - \vartheta_1SV - \vartheta_2SI - \vartheta_3SY) \\
&+ \lambda\mathcal{H}_3\left[(1-\beta)\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\left\{\vartheta_1V(t-\boldsymbol{\ell})+\vartheta_2I(t-\boldsymbol{\ell})\right\}d\boldsymbol{\ell} - (\lambda+\gamma)L\right] \\
&+ (\gamma+\lambda)\left[\beta\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\left\{\vartheta_1V(t-\boldsymbol{\ell})+\vartheta_2I(t-\boldsymbol{\ell})\right\}d\boldsymbol{\ell}\right. \\
&\left.+ \lambda\int_0^{\kappa 3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})L(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - aI - \mu_1C^II\right] \\
&+ \frac{\mathcal{P}}{\varphi\mathcal{H}_4}\left(1-\frac{E_2}{E}\right)\left[\varphi\vartheta_3\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} + rY - (\psi+\omega)E\right] \\
&+ \frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(1-\frac{Y_2}{Y}\right)\left[\psi\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \delta Y - \mu_2C^YY\right] + \frac{\mathcal{P}\vartheta_1S_2}{\varepsilon} \\
&\times\left[b\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})I(t-\boldsymbol{\ell})\,d\boldsymbol{\ell} - \varepsilon V\right] + \frac{\mu_1(\gamma+\lambda)}{\sigma_1}\left(\sigma_1C^II - \pi_1C^I\right) + \frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5} \\
&\times\left(\sigma_2C^YY - \pi_2C^Y\right) + \mathcal{P}(\vartheta_1SV+\vartheta_2SI) - \lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell}) \\
&\times\left[\vartheta_1V(t-\boldsymbol{\ell})+\vartheta_2I(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} - \beta(\gamma+\lambda)\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\left[\vartheta_1V(t-\boldsymbol{\ell})\right. \\
&\left.+ \vartheta_2I(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} + \lambda(\gamma+\lambda)\int_0^{\kappa 3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\left[L - L(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} \\
&+ \frac{\mathcal{P}\vartheta_3S_2Y_2}{\mathcal{H}_4}\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\left[\frac{SY}{S_2Y_2}\right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{S_2Y_2} + \ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY}\right)\bigg] d\boldsymbol{\ell} \\
&+ \frac{\mathcal{P}(\psi+\omega)E_2}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\left[\frac{E}{E_2} - \frac{E(t-\boldsymbol{\ell})}{E_2} + \ln\left(\frac{E(t-\boldsymbol{\ell})}{E}\right)\right] d\boldsymbol{\ell} \\
&+ \frac{b\mathcal{P}\vartheta_1S_2}{\varepsilon}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\big[I - I(t-\boldsymbol{\ell})\big] d\boldsymbol{\ell}.
\end{aligned} \tag{15}
$$

By collecting the terms of Eq. (15), we get

$$
\begin{aligned}
\frac{d\Phi_2}{dt} = \mathcal{P}\bigg[&\left(1-\frac{S_2}{S}\right)(\eta - \varrho S) + \vartheta_2S_2I + \vartheta_3S_2Y - \frac{a(\lambda+\gamma)}{\mathcal{P}}I + \frac{r}{\varphi\mathcal{H}_4}Y \\
&- \frac{\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_2}{E} d\boldsymbol{\ell} - \frac{r}{\varphi\mathcal{H}_4}\frac{YE_2}{E} + \frac{\psi+\omega}{\varphi\mathcal{H}_4}E_2 - \frac{\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y \\
&- \frac{\psi+\omega}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\frac{E(t-\boldsymbol{\ell})Y_2}{Y} d\boldsymbol{\ell} + \frac{\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y_2 + \frac{\mu_2(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^YY_2 \\
&- \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1\mathcal{P}}C^I - \frac{\mu_2\pi_2(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y \\
&+ \frac{\vartheta_3S_2Y_2}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY}\right) d\boldsymbol{\ell} \\
&+ \frac{(\psi+\omega)E_2}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\ln\left(\frac{E(t-\boldsymbol{\ell})}{E}\right) d\boldsymbol{\ell} + \frac{b\vartheta_1S_2\mathcal{H}_6}{\varepsilon}I\bigg].
\end{aligned}
$$

Using the steady state conditions for $Đ_2$

$$
\eta = \varrho S_2 + \vartheta_3S_2Y_2, \qquad \vartheta_3S_2Y_2 + \frac{rY_2}{\varphi\mathcal{H}_4} = \frac{(\psi+\omega)E_2}{\varphi\mathcal{H}_4} = \frac{\delta(\psi+\omega)Y_2}{\varphi\psi\mathcal{H}_4\mathcal{H}_5},
$$

we obtain

$$
\begin{aligned}
\frac{d\Phi_2}{dt} = \mathcal{P}\bigg[&\left(1-\frac{S_2}{S}\right)(\varrho S_2 - \varrho S) + \vartheta_3S_2Y_2\left(1-\frac{S_2}{S}\right) \\
&- \frac{\vartheta_3S_2Y_2}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_2}{S_2Y_2E} d\boldsymbol{\ell} - \frac{rY_2}{\varphi\mathcal{H}_4}\frac{YE_2}{Y_2E} \\
&+ \vartheta_3S_2Y_2 + \frac{rY_2}{\varphi\mathcal{H}_4} - \left(\frac{\vartheta_3S_2Y_2}{\mathcal{H}_5} + \frac{rY_2}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\frac{E(t-\boldsymbol{\ell})Y_2}{E_2Y} d\boldsymbol{\ell} \\
&+ \vartheta_3S_2Y_2 + \frac{rY_2}{\varphi\mathcal{H}_4} + \frac{\mu_2(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^YY_2 - \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1\mathcal{P}}C^I - \frac{\mu_2\pi_2(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y \\
&+ \frac{\vartheta_3S_2Y_2}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY}\right) d\boldsymbol{\ell} \\
&+ \left(\frac{\vartheta_3S_2Y_2}{\mathcal{H}_5} + \frac{rY_2}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\ln\left(\frac{E(t-\boldsymbol{\ell})}{E}\right) d\boldsymbol{\ell} \\
&+ \frac{a(\lambda+\gamma)}{\mathcal{P}}\left\{\frac{\mathcal{P}S_2(\varepsilon\vartheta_2 + b\vartheta_1\mathcal{H}_6)}{a\varepsilon(\lambda+\gamma)} - 1\right\}I\bigg].
\end{aligned}
$$

Using the equalities given by (11) in case of $m = 2$, we get

$$
\frac{d\Phi_2}{dt} = -\mathcal{P}\left[\varrho\frac{(S-S_2)^2}{S} + \vartheta_3S_2Y_2\left\{\frac{S_2}{S} - 1 - \ln\left(\frac{S_2}{S}\right)\right\} + \frac{rY_2}{\varphi\mathcal{H}_4}\left\{\frac{YE_2}{Y_2E} - 1 - \ln\left(\frac{YE_2}{Y_2E}\right)\right\}\right.
$$

$$
\begin{aligned}
&+\frac{\vartheta_3 S_2 Y_2}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\ell)\left\{\frac{S(t-\ell)Y(t-\ell)E_2}{S_2Y_2E}-1-\ln\left(\frac{S(t-\ell)Y(t-\ell)E_2}{S_2Y_2E}\right)\right\}d\ell\\
&+\left(\frac{\vartheta_3 S_2 Y_2}{\mathcal{H}_5}+\frac{rY_2}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\ell)\left\{\frac{E(t-\ell)Y_2}{E_2Y}-1-\ln\left(\frac{E(t-\ell)Y_2}{E_2Y}\right)\right\}d\ell\\
&-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1\mathcal{P}}C^I+\frac{\mu_2(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(Y_2-\frac{\pi_2}{\sigma_2}\right)C^Y\Bigg]\\
&+a(\lambda+\gamma)\left(\frac{\mathcal{P}S_2(\varepsilon\vartheta_2+b\vartheta_1\mathcal{H}_6)}{a\varepsilon(\lambda+\gamma)}-1\right)I.
\end{aligned}
\tag{16}
$$

Therefore, Eq. (16) becomes

$$
\begin{aligned}
\frac{d\Phi_2}{dt}={}&-\mathcal{P}\Bigg[\varrho\frac{(S-S_2)^2}{S}+\frac{rY_2}{\varphi\mathcal{H}_4}F\left(\frac{YE_2}{Y_2E}\right)\\
&+\frac{\vartheta_3 S_2 Y_2}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\ell)\left\{F\left(\frac{S(t-\ell)Y(t-\ell)E_2}{S_2Y_2E}\right)+F\left(\frac{S_2}{S}\right)\right\}d\ell\\
&+\left(\frac{\vartheta_3 S_2 Y_2}{\mathcal{H}_5}+\frac{rY_2}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\ell)F\left(\frac{E(t-\ell)Y_2}{E_2Y}\right)d\ell\\
&-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1\mathcal{P}}C^I+\frac{\mu_2(\psi+\omega)(\varrho\sigma_2+\vartheta_3\pi_2)}{\vartheta_3\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}(\Re_4-1)C^Y\Bigg]\\
&+a(\lambda+\gamma)\left(\frac{\Re_1}{\Re_2}-1\right)I.
\end{aligned}
$$

Thus, if $\Re_1/\Re_2 \le 1$ and $\Re_4 \le 1$, then $\frac{d\Phi_2}{dt} \le 0$ for all $S, I, E, Y, C^I, C^Y > 0$. Moreover, $\frac{d\Phi_2}{dt} = 0$ when $(S, E, Y, I, C^I, C^Y) = (S_2, E_2, Y_2, 0, 0, 0)$. The solutions of system (5) converge to Υ_2' which includes elements with $(S(t), E(t), Y(t), I(t), C^I(t), C^Y(t)) = (S_2, E_2, Y_2, 0, 0, 0)$. Then we have $\frac{dS(t)}{dt} = 0$, and the first equation of system (5) becomes

$$
0=\frac{dS(t)}{dt}=\eta-\varrho S_2-\vartheta_1 S_2 V(t)-\vartheta_3 S_2 Y_2,
$$

which yields $V(t) = 0$ for all t. Moreover, we have $\frac{dI(t)}{dt} = 0$ and from the third equation of system (5) we get

$$
0=\frac{dI(t)}{dt}=\lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\ell)L(t-\ell)\,d\ell,
$$

which implies that $L(t) = 0$ for all t. Therefore, $\Upsilon_2' = \{Đ_2\}$. Applying LLAS theorem, we get $Đ_2$ is GAS. □

Theorem 4 *Let $\Re_3 > 1$ and $\Re_5 \le 1$, then $Đ_3$ is GAS.*

Proof Define a functional as follows:

$$
\begin{aligned}
\Phi_3={}&\mathcal{P}S_3F\left(\frac{S}{S_3}\right)+\lambda\mathcal{H}_3L_3F\left(\frac{L}{L_3}\right)+(\gamma+\lambda)I_3F\left(\frac{I}{I_3}\right)+\frac{\mathcal{P}}{\varphi\mathcal{H}_4}E+\frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y\\
&+\frac{\mathcal{P}\vartheta_1 S_3}{\varepsilon}V_3F\left(\frac{V}{V_3}\right)+\frac{\mu_1(\gamma+\lambda)}{\sigma_1}C_3^IF\left(\frac{C^I}{C_3^I}\right)+\frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y
\end{aligned}
$$

$$
\begin{aligned}
&+\vartheta_1\lambda\mathcal{H}_3(1-\beta)S_3V_3\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)V(\varkappa)}{S_3V_3}\right)d\varkappa\,d\ell+\vartheta_2\lambda\mathcal{H}_3(1-\beta)S_3I_3\\
&\times\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)I(\varkappa)}{S_3I_3}\right)d\varkappa\,d\ell+\vartheta_1\beta(\gamma+\lambda)S_3V_3\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)\\
&\times\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)V(\varkappa)}{S_3V_3}\right)d\varkappa\,d\ell\\
&+\vartheta_2\beta(\gamma+\lambda)S_3I_3\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)I(\varkappa)}{S_3I_3}\right)d\varkappa\,d\ell\\
&+\lambda(\gamma+\lambda)L_3\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\ell)\int_{t-\ell}^{t}F\left(\frac{L(\varkappa)}{L_3}\right)d\varkappa\,d\ell\\
&+\frac{\mathcal{P}\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\ell)\int_{t-\ell}^{t}S(\varkappa)Y(\varkappa)\,d\varkappa\,d\ell\\
&+\frac{\mathcal{P}(\psi+\omega)}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\ell)\int_{t-\ell}^{t}E(\varkappa)\,d\varkappa\,d\ell\\
&+\frac{b\mathcal{P}\vartheta_1S_3I_3}{\varepsilon}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\ell)\int_{t-\ell}^{t}F\left(\frac{I(\varkappa)}{I_3}\right)d\varkappa\,d\ell.
\end{aligned}
\tag{17}
$$

We calculate $\frac{d\Phi_3}{dt}$ as follows:

$$
\begin{aligned}
\frac{d\Phi_3}{dt}={}&\mathcal{P}\left(1-\frac{S_3}{S}\right)(\eta-\varrho S-\vartheta_1SV-\vartheta_2SI-\vartheta_3SY)+\lambda\mathcal{H}_3\left(1-\frac{L_3}{L}\right)\\
&\times\left[(1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)S(t-\ell)\left\{\vartheta_1V(t-\ell)+\vartheta_2I(t-\ell)\right\}d\ell-(\lambda+\gamma)L\right]\\
&+(\gamma+\lambda)\left(1-\frac{I_3}{I}\right)\left[\beta\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)S(t-\ell)\left\{\vartheta_1V(t-\ell)+\vartheta_2I(t-\ell)\right\}d\ell\right.\\
&\left.+\lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\ell)L(t-\ell)\,d\ell-aI-\mu_1C^II\right]\\
&+\frac{\mathcal{P}}{\varphi\mathcal{H}_4}\left[\varphi\vartheta_3\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\ell)S(t-\ell)Y(t-\ell)\,d\ell+rY-(\psi+\omega)E\right]\\
&+\frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left[\psi\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\ell)E(t-\ell)\,d\ell-\delta Y-\mu_2C^YY\right]\\
&+\frac{\mathcal{P}\vartheta_1S_3}{\varepsilon}\left(1-\frac{V_3}{V}\right)\left[b\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\ell)I(t-\ell)\,d\ell-\varepsilon V\right]+\frac{\mu_1(\gamma+\lambda)}{\sigma_1}\\
&\times\left(1-\frac{C_3^I}{C^I}\right)\left(\sigma_1C^II-\pi_1C^I\right)+\frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(\sigma_2C^YY-\pi_2C^Y\right)\\
&+\vartheta_1\lambda\mathcal{H}_3(1-\beta)S_3V_3\\
&\times\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\left[\frac{SV}{S_3V_3}-\frac{S(t-\ell)V(t-\ell)}{S_3V_3}+\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right)\right]d\ell\\
&+\vartheta_2\lambda\mathcal{H}_3(1-\beta)S_3I_3\\
&\times\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\left[\frac{SI}{S_3I_3}-\frac{S(t-\ell)I(t-\ell)}{S_3I_3}+\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right)\right]d\ell\\
&+\vartheta_1\beta(\gamma+\lambda)S_3V_3
\end{aligned}
$$

$$\times \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\ell)\left[\frac{SV}{S_3V_3} - \frac{S(t-\ell)V(t-\ell)}{S_3V_3} + \ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right)\right] d\ell$$
$$+ \vartheta_2\beta(\gamma+\lambda)S_3I_3$$
$$\times \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\ell)\left[\frac{SI}{S_3I_3} - \frac{S(t-\ell)I(t-\ell)}{S_3I_3} + \ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right)\right] d\ell$$
$$+ \lambda(\gamma+\lambda)L_3 \int_0^{\kappa 3} \bar{\mathcal{H}}_3(\ell)\left[\frac{L}{L_3} - \frac{L(t-\ell)}{L_3} + \ln\left(\frac{L(t-\ell)}{L}\right)\right] d\ell$$
$$+ \frac{\mathcal{P}\vartheta_3}{\mathcal{H}_4} \int_0^{\kappa 4} \bar{\mathcal{H}}_4(\ell)\left[SY - S(t-\ell)Y(t-\ell)\right] d\ell$$
$$+ \frac{\mathcal{P}(\psi+\omega)}{\varphi\mathcal{H}_4\mathcal{H}_5} \int_0^{\kappa 5} \bar{\mathcal{H}}_5(\ell)\left[E - E(t-\ell)\right] d\ell$$
$$+ \frac{b\mathcal{P}\vartheta_1S_3I_3}{\varepsilon} \int_0^{\kappa 6} \bar{\mathcal{H}}_6(\ell)\left[\frac{I}{I_3} - \frac{I(t-\ell)}{I_3} + \ln\left(\frac{I(t-\ell)}{I}\right)\right] d\ell. \tag{18}$$

Collecting the terms of Eq. (18), we derive

$$\frac{d\Phi_3}{dt} = \mathcal{P}\left(1 - \frac{S_3}{S}\right)(\eta - \varrho S) + \mathcal{P}\vartheta_2S_3I + \mathcal{P}\vartheta_3S_3Y - \vartheta_1\lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa 1} \bar{\mathcal{H}}_1(\ell)$$
$$\times \frac{S(t-\ell)V(t-\ell)L_3}{L}\, d\ell - \vartheta_2\lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa 1} \bar{\mathcal{H}}_1(\ell)\frac{S(t-\ell)I(t-\ell)L_3}{L}\, d\ell$$
$$+ \lambda\mathcal{H}_3(\lambda+\gamma)L_3 - a(\lambda+\gamma)I - \vartheta_1\beta(\gamma+\lambda)\int_0^{\kappa 2} \bar{\mathcal{H}}_2(\ell)\frac{S(t-\ell)V(t-\ell)I_3}{I}\, d\ell$$
$$- \vartheta_2\beta(\gamma+\lambda)\int_0^{\kappa 2} \bar{\mathcal{H}}_2(\ell)\frac{S(t-\ell)I(t-\ell)I_3}{I}\, d\ell$$
$$- \lambda(\lambda+\gamma)\int_0^{\kappa 3} \bar{\mathcal{H}}_3(\ell)\frac{L(t-\ell)I_3}{I}\, d\ell + a(\lambda+\gamma)I_3 + \mu_1(\lambda+\gamma)C^I I_3$$
$$+ \frac{\mathcal{P}r}{\varphi\mathcal{H}_4}Y - \frac{\mathcal{P}\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y - \frac{\mathcal{P}b\vartheta_1S_3}{\varepsilon}\int_0^{\kappa 6} \bar{\mathcal{H}}_6(\ell)\frac{I(t-\ell)V_3}{V}\, d\ell$$
$$+ \mathcal{P}\vartheta_1S_3V_3 - \frac{\mu_1\pi_1(\lambda+\gamma)}{\sigma_1}C^I - \mu_1(\lambda+\gamma)C_3^I I$$
$$+ \frac{\mu_1\pi_1(\lambda+\gamma)}{\sigma_1}C_3^I - \frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y$$
$$+ \lambda\mathcal{H}_3(1-\beta)\vartheta_1S_3V_3\int_0^{\kappa 1} \bar{\mathcal{H}}_1(\ell)\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) d\ell$$
$$+ \lambda\mathcal{H}_3(1-\beta)\vartheta_2S_3I_3\int_0^{\kappa 1} \bar{\mathcal{H}}_1(\ell)\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right) d\ell$$
$$+ \beta(\gamma+\lambda)\vartheta_1S_3V_3\int_0^{\kappa 2} \bar{\mathcal{H}}_2(\ell)\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) d\ell$$
$$+ \beta(\gamma+\lambda)\vartheta_2S_3I_3\int_0^{\kappa 2} \bar{\mathcal{H}}_2(\ell)\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right) d\ell$$
$$+ \lambda(\gamma+\lambda)L_3\int_0^{\kappa 3} \bar{\mathcal{H}}_3(\ell)\ln\left(\frac{L(t-\ell)}{L}\right) d\ell$$
$$+ \frac{b\mathcal{P}\vartheta_1S_3\mathcal{H}_6}{\varepsilon}I + \frac{b\mathcal{P}\vartheta_1S_3I_3}{\varepsilon}\int_0^{\kappa 6} \bar{\mathcal{H}}_6(\ell)\ln\left(\frac{I(t-\ell)}{I}\right) d\ell.$$

Using the steady state conditions for Đ$_3$

$$\eta = \varrho S_3 + \vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3, \qquad \mathcal{H}_1(1-\beta)(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) = (\lambda + \gamma) L_3,$$
$$\beta \mathcal{H}_2(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) + \lambda \mathcal{H}_3 L_3 = \big(a + \mu_1 C_3^I\big) I_3, \qquad I_3 = \frac{\pi_1}{\sigma_1}, \qquad V_3 = \frac{b\mathcal{H}_6}{\varepsilon} I_3,$$

we get

$$\mathcal{P}(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) = (\lambda + \gamma)\big(a + \mu_1 C_3^I\big) I_3.$$

Further, we obtain

$$\begin{aligned}
\frac{d\Phi_3}{dt} &= \mathcal{P}\left(1 - \frac{S_3}{S}\right)(\varrho S_3 - \varrho S) + \mathcal{P}(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3)\left(1 - \frac{S_3}{S}\right) + \mathcal{P}\vartheta_3 S_3 Y \\
&\quad - \lambda \mathcal{H}_3(1-\beta)\vartheta_1 S_3 V_3 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \frac{S(t-\ell)V(t-\ell)L_3}{S_3 V_3 L}\, d\ell - \lambda \mathcal{H}_3(1-\beta)\vartheta_2 S_3 I_3 \\
&\quad \times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \frac{S(t-\ell)I(t-\ell)L_3}{S_3 I_3 L}\, d\ell + \lambda \mathcal{H}_1 \mathcal{H}_3(1-\beta)(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) \\
&\quad - \beta(\gamma + \lambda)\vartheta_1 S_3 V_3 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \frac{S(t-\ell)V(t-\ell)I_3}{S_3 V_3 I}\, d\ell - \beta(\gamma + \lambda)\vartheta_2 S_3 I_3 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \\
&\quad \times \frac{S(t-\ell)I(t-\ell)}{S_3 I}\, d\ell - \lambda \mathcal{H}_1(1-\beta)(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell) \frac{L(t-\ell) I_3}{L_3 I}\, d\ell \\
&\quad + \mathcal{P}(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) - \frac{\mathcal{P}[(\delta - r\mathcal{H}_5)\psi + \delta\omega]}{\varphi\psi \mathcal{H}_4 \mathcal{H}_5} Y - \frac{\mathcal{P}\vartheta_1 S_3 V_3}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell) \\
&\quad \times \frac{I(t-\ell) V_3}{I_3 V}\, d\ell + \mathcal{P}\vartheta_1 S_3 V_3 - \frac{\mu_2 \pi_2 \mathcal{P}(\psi + \omega)}{\sigma_2 \varphi \psi \mathcal{H}_4 \mathcal{H}_5} C^Y + \lambda \mathcal{H}_3(1-\beta)\vartheta_1 S_3 V_3 \\
&\quad \times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) d\ell \\
&\quad + \lambda \mathcal{H}_3(1-\beta)\vartheta_2 S_3 I_3 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right) d\ell \\
&\quad + \beta(\gamma + \lambda)\vartheta_1 S_3 V_3 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) d\ell \\
&\quad + \beta(\gamma + \lambda)\vartheta_2 S_3 I_3 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right) d\ell \\
&\quad + \lambda \mathcal{H}_1(1-\beta)(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell) \ln\left(\frac{L(t-\ell)}{L}\right) d\ell \\
&\quad + \frac{\mathcal{P}\vartheta_1 S_3 V_3}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell) \ln\left(\frac{I(t-\ell)}{I}\right) d\ell.
\end{aligned}$$

Using the equalities given by (10) in case of $n = 3$, we get

$$\begin{aligned}
\frac{d\Phi_3}{dt} &= -\varrho \mathcal{P} \frac{(S - S_3)^2}{S} - \mathcal{P}(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3)\left[\frac{S_3}{S} - 1 - \ln\left(\frac{S_3}{S}\right)\right] \\
&\quad - \lambda \mathcal{H}_3(1-\beta)\vartheta_1 S_3 V_3 \\
&\quad \times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\left[\frac{S(t-\ell)V(t-\ell)L_3}{S_3 V_3 L} - 1 - \ln\left(\frac{S(t-\ell)V(t-\ell)L_3}{S_3 V_3 L}\right)\right] d\ell
\end{aligned}$$

$$
\begin{aligned}
&- \lambda \mathcal{H}_3(1-\beta)\vartheta_2 S_3 I_3 \\
&\times \int_0^{\kappa 1} \bar{\mathcal{H}}_1(\boldsymbol{\ell})\left[\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_3}{S_3 I_3 L} - 1 - \ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_3}{S_3 I_3 L}\right)\right] d\boldsymbol{\ell} \\
&- \beta(\gamma+\lambda)\vartheta_1 S_3 V_3 \\
&\times \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})I_3}{S_3 V_3 I} - 1 - \ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})I_3}{S_3 V_3 I}\right)\right] d\boldsymbol{\ell} \\
&- \beta(\gamma+\lambda)\vartheta_2 S_3 I_3 \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_3 I} - 1 - \ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_3 I}\right)\right] d\boldsymbol{\ell} \\
&- \lambda \mathcal{H}_1(1-\beta)(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) \\
&\times \int_0^{\kappa 3} \bar{\mathcal{H}}_3(\boldsymbol{\ell})\left[\frac{L(t-\boldsymbol{\ell})I_3}{L_3 I} - 1 - \ln\left(\frac{L(t-\boldsymbol{\ell})I_3}{L_3 I}\right)\right] d\boldsymbol{\ell} \\
&- \frac{\mathcal{P}\vartheta_1 S_3 V_3}{\mathcal{H}_6} \int_0^{\kappa 6} \bar{\mathcal{H}}_6(\boldsymbol{\ell})\left[\frac{I(t-\boldsymbol{\ell})V_3}{I_3 V} - 1 - \ln\left(\frac{I(t-\boldsymbol{\ell})V_3}{I_3 V}\right)\right] d\boldsymbol{\ell} \\
&+ \mathcal{P}\vartheta_3\left(S_3 - \frac{(\delta - r\mathcal{H}_5)\psi + \delta\omega}{\vartheta_3 \varphi \psi \mathcal{H}_4 \mathcal{H}_5}\right) Y - \frac{\mu_2 \pi_2 \mathcal{P}(\psi+\omega)}{\sigma_2 \varphi \psi \mathcal{H}_4 \mathcal{H}_5} C^Y.
\end{aligned} \tag{19}
$$

Therefore, Eq. (19) becomes

$$
\begin{aligned}
\frac{d\Phi_3}{dt} &= -\varrho \mathcal{P}\frac{(S-S_3)^2}{S} - \mathcal{P}(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) F\left(\frac{S_3}{S}\right) \\
&\quad - \lambda \mathcal{H}_3(1-\beta)\vartheta_1 S_3 V_3 \int_0^{\kappa 1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) F\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})L_3}{S_3 V_3 L}\right) d\boldsymbol{\ell} \\
&\quad - \lambda \mathcal{H}_3(1-\beta)\vartheta_2 S_3 I_3 \int_0^{\kappa 1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) F\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_3}{S_3 I_3 L}\right) d\boldsymbol{\ell} \\
&\quad - \beta(\gamma+\lambda)\vartheta_1 S_3 V_3 \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) F\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})I_3}{S_3 V_3 I}\right) d\boldsymbol{\ell} \\
&\quad - \beta(\gamma+\lambda)\vartheta_2 S_3 I_3 \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) F\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_3 I}\right) d\boldsymbol{\ell} \\
&\quad - \lambda \mathcal{H}_1(1-\beta)(\vartheta_1 S_3 V_3 + \vartheta_2 S_3 I_3) \int_0^{\kappa 3} \bar{\mathcal{H}}_3(\boldsymbol{\ell}) F\left(\frac{L(t-\boldsymbol{\ell})I_3}{L_3 I}\right) d\boldsymbol{\ell} \\
&\quad - \frac{\mathcal{P}\vartheta_1 S_3 V_3}{\mathcal{H}_6} \int_0^{\kappa 6} \bar{\mathcal{H}}_6(\boldsymbol{\ell}) F\left(\frac{I(t-\boldsymbol{\ell})V_3}{I_3 V}\right) d\boldsymbol{\ell} \\
&\quad + \mathcal{P}\vartheta_3(S_3 - S_5) Y - \frac{\mu_2 \pi_2 \mathcal{P}(\psi+\omega)}{\sigma_2 \varphi \psi \mathcal{H}_4 \mathcal{H}_5} C^Y.
\end{aligned}
$$

Hence, if $\Re_5 \leq 1$, then Đ$_5$ does not exist since $E_5 \leq 0$ and $Y_5 \leq 0$. In this case

$$
\begin{aligned}
\frac{dE(t)}{dt} &= \varphi \vartheta_3 \int_0^{\kappa 4} \bar{\mathcal{H}}_4(\boldsymbol{\ell}) S(t-\boldsymbol{\ell}) Y(t-\boldsymbol{\ell})\, d\boldsymbol{\ell} + rY(t) - (\psi+\omega)E(t) \leq 0, \\
\frac{dY(t)}{dt} &= \psi \int_0^{\kappa 5} \bar{\mathcal{H}}_5(\boldsymbol{\ell}) E(t-\boldsymbol{\ell})\, d\boldsymbol{\ell} - \delta Y(t) - \mu_2 C^Y Y \leq 0.
\end{aligned}
$$

Now we want to find the value $\bar{S}$ such that, for all $0 < S(t) \le \bar{S}$, we get $\frac{dE(t)}{dt} \le 0$ and $\frac{dY(t)}{dt} \le 0$. Let us consider

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{\mathcal{H}_4}E + \frac{\psi+\omega}{\psi\mathcal{H}_4\mathcal{H}_5}Y + \frac{\varphi\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t}S(\varkappa)Y(\varkappa)\,d\varkappa\,d\boldsymbol{\ell}\right.\\
&\quad\left.+\frac{\psi+\omega}{\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t}E(\varkappa)\,d\varkappa\,d\boldsymbol{\ell}\right]\\
&\quad=\varphi\vartheta_3SY - \frac{(\delta - r\mathcal{H}_5)\psi+\delta\omega}{\psi\mathcal{H}_4\mathcal{H}_5}Y - \frac{\mu_2(\psi+\omega)}{\psi\mathcal{H}_4\mathcal{H}_5}C^YY\\
&\quad=\varphi\vartheta_3\left(S - \frac{(\delta - r\mathcal{H}_5)\psi+\delta\omega}{\varphi\vartheta_3\psi\mathcal{H}_4\mathcal{H}_5}\right)Y - \frac{\mu_2(\psi+\omega)}{\psi\mathcal{H}_4\mathcal{H}_5}C^YY \le 0 \quad \text{for all } C^Y, Y > 0.
\end{aligned}$$

This happens when $S_3 \le \bar{S} = \frac{(\delta - r\mathcal{H}_5)\psi+\delta\omega}{\vartheta_3\varphi\psi\mathcal{H}_4\mathcal{H}_5} = S_5$. Clearly, $\frac{d\Phi_3}{dt} \le 0$ for all $S, L, I, Y, V, C^Y > 0$, where $\frac{d\Phi_3}{dt} = 0$ occurs at $S = S_3$ and $Y = C^Y = 0$. The solutions of system (5) converge to Υ_3' which includes elements satisfying $S(t) = S_3$ and $F = 0$ i.e.

$$\begin{aligned}
\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})L_3}{S_3V_3L} &= \frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_3}{S_3I_3L} = \frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})I_3}{S_3V_3I}\\
&= \frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_3I} = \frac{L(t-\boldsymbol{\ell})I_3}{L_3I} = \frac{I(t-\boldsymbol{\ell})V_3}{I_3V} = 1
\end{aligned} \tag{20}$$

for all $t \in [0, \kappa]$. If $S(t) = S_3$, then from Eq. (20) we get $L(t) = L_3$, $I(t) = I_3$, and $V(t) = V_3$ for all t. Thus, Υ_3' contains elements with $I(t) = I_3$, $V(t) = V_3$, $Y(t) = 0$, and then $\frac{dI(t)}{dt} = 0$, $\frac{dY(t)}{dt} = 0$. The third and fifth equations of system (5) become

$$\begin{aligned}
0 &= \frac{dI(t)}{dt} = \beta\mathcal{H}_2(\vartheta_1S_3V_3 + \vartheta_2S_3I_3) + \lambda\mathcal{H}_3L_3 - aI_3 - \mu_1C^I(t)I_3,\\
0 &= \frac{dY(t)}{dt} = \psi\int_0^{\kappa_5}\bar{\mathcal{H}}_5E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell},
\end{aligned}$$

which yield $C^I(t) = C_3^I$ and $E(t) = 0$ for all t. Therefore, $\Upsilon_3' = \{Đ_3\}$. Applying LLAS theorem, we get $Đ_3$ is GAS. □

Theorem 5 *If* $\Re_4 > 1$ *and* $\Re_6 \le 1$, *then* $Đ_4$ *is GAS.*

Proof Let

$$\begin{aligned}
\Phi_4 &= \mathcal{P}S_4F\left(\frac{S}{S_4}\right) + \lambda\mathcal{H}_3L + (\gamma+\lambda)I + \frac{\mathcal{P}}{\varphi\mathcal{H}_4}E_4F\left(\frac{E}{E_4}\right) + \frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y_4F\left(\frac{Y}{Y_4}\right)\\
&\quad+\frac{\mathcal{P}\vartheta_1S_4}{\varepsilon}V + \frac{\mu_1(\gamma+\lambda)}{\sigma_1}C^I + \frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C_4^YF\left(\frac{C^Y}{C_4^Y}\right)\\
&\quad+\lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t}S(\varkappa)\big[\vartheta_1V(\varkappa)+\vartheta_2I(\varkappa)\big]\,d\varkappa\,d\boldsymbol{\ell}\\
&\quad+\beta(\gamma+\lambda)\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t}S(\varkappa)\big[\vartheta_1V(\varkappa)+\vartheta_2I(\varkappa)\big]\,d\varkappa\,d\boldsymbol{\ell}\\
&\quad+\lambda(\gamma+\lambda)\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t}L(\varkappa)\,d\varkappa\,d\boldsymbol{\ell}
\end{aligned}$$

$$+\frac{\mathcal{P}\vartheta_3 S_4 Y_4}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} F\left(\frac{S(\varkappa)Y(\varkappa)}{S_4 Y_4}\right)d\varkappa\, d\boldsymbol{\ell}$$
$$+\frac{\mathcal{P}(\psi+\omega)E_4}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} F\left(\frac{E(\varkappa)}{E_4}\right)d\varkappa\, d\boldsymbol{\ell}$$
$$+\frac{b\mathcal{P}\vartheta_1 S_4}{\varepsilon}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t} I(\varkappa)\, d\varkappa\, d\boldsymbol{\ell}.$$

Calculate $\frac{d\Phi_4}{dt}$ as follows:

$$\begin{aligned}
\frac{d\Phi_4}{dt} &= \mathcal{P}\left(1-\frac{S_4}{S}\right)(\eta-\varrho S-\vartheta_1 SV-\vartheta_2 SI-\vartheta_3 SY) \\
&\quad+\lambda\mathcal{H}_3\left[(1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\left\{\vartheta_1 V(t-\boldsymbol{\ell})+\vartheta_2 I(t-\boldsymbol{\ell})\right\}d\boldsymbol{\ell}-(\lambda+\gamma)L\right] \\
&\quad+(\gamma+\lambda)\left[\beta\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\left\{\vartheta_1 V(t-\boldsymbol{\ell})+\vartheta_2 I(t-\boldsymbol{\ell})\right\}d\boldsymbol{\ell}\right. \\
&\quad\left.+\lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})L(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}-aI-\mu_1 C^I I\right] \\
&\quad+\frac{\mathcal{P}}{\varphi\mathcal{H}_4}\left(1-\frac{E_4}{E}\right)\left[\varphi\vartheta_3\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}+rY-(\psi+\omega)E\right] \\
&\quad+\frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(1-\frac{Y_4}{Y}\right)\left[\psi\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}-\delta Y-\mu_2 C^Y Y\right] \\
&\quad+\frac{\mathcal{P}\vartheta_1 S_4}{\varepsilon}\left[b\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})I(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}-\varepsilon V\right]+\frac{\mu_1(\gamma+\lambda)}{\sigma_1}\left(\sigma_1 C^I I-\pi_1 C^I\right) \\
&\quad+\frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(1-\frac{C_4^Y}{C^Y}\right)\left(\sigma_2 C^Y Y-\pi_2 C^Y\right)+\mathcal{P}(\vartheta_1 SV+\vartheta_2 SI) \\
&\quad-\lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\left[\vartheta_1 V(t-\boldsymbol{\ell})+\vartheta_2 I(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} \\
&\quad-\beta(\gamma+\lambda)\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\left[\vartheta_1 V(t-\boldsymbol{\ell})+\vartheta_2 I(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} \\
&\quad+\lambda(\gamma+\lambda)\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\left[L-L(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell} \\
&\quad+\frac{\mathcal{P}\vartheta_3 S_4 Y_4}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\left[\frac{SY}{S_4 Y_4}-\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{S_4 Y_4}\right. \\
&\quad\left.+\ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY}\right)\right]d\boldsymbol{\ell}+\frac{\mathcal{P}(\psi+\omega)E_4}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\left[\frac{E}{E_4}-\frac{E(t-\boldsymbol{\ell})}{E_4}\right. \\
&\quad\left.+\ln\left(\frac{E(t-\boldsymbol{\ell})}{E}\right)\right]d\boldsymbol{\ell}+\frac{b\mathcal{P}\vartheta_1 S_4}{\varepsilon}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\left[I-I(t-\boldsymbol{\ell})\right]d\boldsymbol{\ell}.
\end{aligned}\tag{21}$$

Summing the terms of Eq. (21), we get

$$\begin{aligned}
\frac{d\Phi_4}{dt} &= \mathcal{P}\left[\left(1-\frac{S_4}{S}\right)(\eta-\varrho S)+\vartheta_2 S_4 I+\vartheta_3 S_4 Y-\frac{a(\lambda+\gamma)}{\mathcal{P}}I+\frac{r}{\varphi\mathcal{H}_4}Y\right. \\
&\quad-\frac{\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_4}{E}\,d\boldsymbol{\ell}-\frac{r}{\varphi\mathcal{H}_4}\frac{YE_4}{E}+\frac{\psi+\omega}{\varphi\mathcal{H}_4}E_4
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y - \frac{\psi+\omega}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\frac{E(t-\boldsymbol{\ell})Y_4}{Y}\,d\boldsymbol{\ell} + \frac{\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y_4 \\
&+ \frac{\mu_2(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y Y_4 - \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1\mathcal{P}}C^I - \frac{\mu_2\pi_2(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y - \frac{\mu_2(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C_4^Y Y \\
&+ \frac{\mu_2\pi_2(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C_4^Y + \frac{\vartheta_3 S_4 Y_4}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY}\right)d\boldsymbol{\ell} \\
&+ \frac{(\psi+\omega)E_4}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\ln\left(\frac{E(t-\boldsymbol{\ell})}{E}\right)d\boldsymbol{\ell} + \frac{b\vartheta_1 S_4\mathcal{H}_6}{\varepsilon}I\Bigg].
\end{aligned}
$$

Using the steady state conditions for $Đ_4$

$$
\begin{aligned}
&\eta = \varrho S_4 + \vartheta_3 S_4 Y_4, \qquad Y_4 = \frac{\pi_2}{\sigma_2}, \\
&\vartheta_3 S_4 Y_4 + \frac{rY_4}{\varphi\mathcal{H}_4} = \frac{(\psi+\omega)E_4}{\varphi\mathcal{H}_4} = \frac{\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y_4 + \frac{\mu_2(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C_4^Y Y_4,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{d\Phi_4}{dt} &= \mathcal{P}\Bigg[\left(1-\frac{S_4}{S}\right)(\varrho S_4 - \varrho S) + \vartheta_3 S_4 Y_4\left(1-\frac{S_4}{S}\right) \\
&- \frac{\vartheta_3 S_4 Y_4}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_4}{S_4 Y_4 E}\,d\boldsymbol{\ell} \\
&- \frac{rY_4}{\varphi\mathcal{H}_4}\frac{YE_4}{Y_4E} + \vartheta_3 S_4 Y_4 + \frac{rY_4}{\varphi\mathcal{H}_4} \\
&- \left(\frac{\vartheta_3 S_4 Y_4}{\mathcal{H}_5} + \frac{rY_4}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\frac{E(t-\boldsymbol{\ell})Y_4}{E_4 Y}\,d\boldsymbol{\ell} \\
&+ \vartheta_3 S_4 Y_4 + \frac{rY_4}{\varphi\mathcal{H}_4} - \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1\mathcal{P}}C^I \\
&+ \frac{\vartheta_3 S_4 Y_4}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY}\right)d\boldsymbol{\ell} \\
&+ \left(\frac{\vartheta_3 S_4 Y_4}{\mathcal{H}_5} + \frac{rY_4}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\ln\left(\frac{E(t-\boldsymbol{\ell})}{E}\right)d\boldsymbol{\ell} \\
&+ \frac{a(\lambda+\gamma)}{\mathcal{P}}\left\{\frac{\mathcal{P}S_4(\varepsilon\vartheta_2 + b\vartheta_1\mathcal{H}_6)}{a\varepsilon(\lambda+\gamma)} - 1\right\}I\Bigg].
\end{aligned}
$$

Using the equalities given by (11) in case of $m = 4$, we get

$$
\begin{aligned}
\frac{d\Phi_4}{dt} &= -\mathcal{P}\Bigg[\varrho\frac{(S-S_4)^2}{S} + \vartheta_3 S_4 Y_4\left\{\frac{S_4}{S} - 1 - \ln\left(\frac{S_4}{S}\right)\right\} + \frac{rY_4}{\varphi\mathcal{H}_4}\left\{\frac{YE_4}{Y_4E} - 1 - \ln\left(\frac{YE_4}{Y_4E}\right)\right\} \\
&+ \frac{\vartheta_3 S_4 Y_4}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\left\{\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_4}{S_4 Y_4 E} - 1 - \ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_4}{S_4 Y_4 E}\right)\right\}d\boldsymbol{\ell} \\
&+ \left(\frac{\vartheta_3 S_4 Y_4}{\mathcal{H}_5} + \frac{rY_4}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\left\{\frac{E(t-\boldsymbol{\ell})Y_4}{E_4 Y} - 1 - \ln\left(\frac{E(t-\boldsymbol{\ell})Y_4}{E_4 Y}\right)\right\}d\boldsymbol{\ell} \\
&+ \frac{a(\lambda+\gamma)}{\mathcal{P}}\left\{\frac{\mathcal{P}S_4(\varepsilon\vartheta_2 + b\vartheta_1\mathcal{H}_6)}{a\varepsilon(\lambda+\gamma)} - 1\right\}I - \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1\mathcal{P}}C^I\Bigg]. \qquad (22)
\end{aligned}
$$

Therefore, Eq. (22) becomes

$$\begin{aligned}\frac{d\Phi_4}{dt} &= -\mathcal{P}\Bigg[\varrho\frac{(S-S_4)^2}{S} + \frac{rY_4}{\varphi\mathcal{H}_4}F\left(\frac{YE_4}{Y_4E}\right) \\ &\quad + \frac{\vartheta_3 S_4 Y_4}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\ell)\left\{F\left(\frac{S(t-\ell)Y(t-\ell)E_4}{S_4Y_4E}\right) + F\left(\frac{S_4}{S}\right)\right\}d\ell \\ &\quad + \left(\frac{\vartheta_3 S_4 Y_4}{\mathcal{H}_5} + \frac{rY_4}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\ell)F\left(\frac{E(t-\ell)Y_4}{E_4Y}\right)d\ell \\ &\quad + \frac{a(\lambda+\gamma)}{\mathcal{P}}(\Re_6-1)I - \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1\mathcal{P}}C^I\Bigg].\end{aligned}$$

Hence, if $\Re_6 \le 1$, then $\frac{d\Phi_4}{dt} \le 0$ for all $S, I, E, Y, V, C^I > 0$, where $\frac{d\Phi_4}{dt} = 0$ occurs at $S = S_4$, $E = E_4$, $Y = Y_4$, and $I = C^I = 0$. The trajectories of system (5) converge to Υ_4' which includes elements with $S(t) = S_4$, $E(t) = E_4$, $Y(t) = Y_4$, and then $\frac{dS(t)}{dt} = \frac{dY(t)}{dt} = 0$. The first and fifth equations of system (5) become

$$\begin{aligned}0 &= \frac{dS(t)}{dt} = \eta - \varrho S_4 - \vartheta_1 S_4 V(t) - \vartheta_3 S_4 Y_4, \\ 0 &= \frac{dY(t)}{dt} = \psi\mathcal{H}_5 E_4 - \delta Y_4 - \mu_2 C^Y(t) Y_4,\end{aligned}$$

which imply that $V(t) = 0$ and $C^Y(t) = C_4^Y$ for all t. Moreover, we have $\frac{dI(t)}{dt} = 0$, then the third equation of system (5) becomes

$$0 = \frac{dI(t)}{dt} = \lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\ell)L(t-\ell)\,d\ell,$$

which yields $L(t) = 0$ for all t, and then $\Upsilon_4' = \{Đ_4\}$. Applying LLAS theorem, we get $Đ_4$ is GAS. □

Theorem 6 *If* $\Re_5 > 1$, $\Re_8 \le 1$, *and* $\Re_1/\Re_2 > 1$, *then* $Đ_5$ *is GAS.*

Proof Define

$$\begin{aligned}\Phi_5 &= \mathcal{P}S_5F\left(\frac{S}{S_5}\right) + \lambda\mathcal{H}_3L_5F\left(\frac{L}{L_5}\right) + (\gamma+\lambda)I_5F\left(\frac{I}{I_5}\right) + \frac{\mathcal{P}}{\varphi\mathcal{H}_4}E_5F\left(\frac{E}{E_5}\right) \\ &\quad + \frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y_5F\left(\frac{Y}{Y_5}\right) + \frac{\mathcal{P}\vartheta_1S_5}{\varepsilon}V_5F\left(\frac{V}{V_5}\right) + \frac{\mu_1(\gamma+\lambda)}{\sigma_1}C_5^IF\left(\frac{C^I}{C_5^I}\right) \\ &\quad + \frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y + \vartheta_1\lambda\mathcal{H}_3(1-\beta)S_5V_5\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)V(\varkappa)}{S_5V_5}\right)d\varkappa\,d\ell \\ &\quad + \vartheta_2\lambda\mathcal{H}_3(1-\beta)S_5I_5\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)I(\varkappa)}{S_5I_5}\right)d\varkappa\,d\ell \\ &\quad + \vartheta_1\beta(\gamma+\lambda)S_5V_5\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)V(\varkappa)}{S_5V_5}\right)d\varkappa\,d\ell \\ &\quad + \vartheta_2\beta(\gamma+\lambda)S_5I_5\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)I(\varkappa)}{S_5I_5}\right)d\varkappa\,d\ell\end{aligned}$$

$$+ \lambda(\gamma+\lambda)L_5 \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\boldsymbol{\ell}) \int_{t-\boldsymbol{\ell}}^{t} F\left(\frac{L(\varkappa)}{L_5}\right) d\varkappa\, d\boldsymbol{\ell}$$
$$+ \frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_4} \int_0^{\kappa_4} \bar{\mathcal{H}}_4(\boldsymbol{\ell}) \int_{t-\boldsymbol{\ell}}^{t} F\left(\frac{S(\varkappa)Y(\varkappa)}{S_5 Y_5}\right) d\varkappa\, d\boldsymbol{\ell}$$
$$+ \frac{\mathcal{P}(\psi+\omega)E_5}{\varphi\mathcal{H}_4\mathcal{H}_5} \int_0^{\kappa_5} \bar{\mathcal{H}}_5(\boldsymbol{\ell}) \int_{t-\boldsymbol{\ell}}^{t} F\left(\frac{E(\varkappa)}{E_5}\right) d\varkappa\, d\boldsymbol{\ell}$$
$$+ \frac{b\mathcal{P}\vartheta_1 S_5 I_5}{\varepsilon} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\boldsymbol{\ell}) \int_{t-\boldsymbol{\ell}}^{t} F\left(\frac{I(\varkappa)}{I_5}\right) d\varkappa\, d\boldsymbol{\ell}.$$

Calculate $\frac{d\Phi_5}{dt}$ as follows:

$$\begin{aligned}
\frac{d\Phi_5}{dt} &= \mathcal{P}\left(1-\frac{S_5}{S}\right)(\eta - \varrho S - \vartheta_1 SV - \vartheta_2 SI - \vartheta_3 SY) + \lambda\mathcal{H}_3\left(1-\frac{L_5}{L}\right) \\
&\quad \times \left[(1-\beta)\int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) S(t-\boldsymbol{\ell})\left\{\vartheta_1 V(t-\boldsymbol{\ell}) + \vartheta_2 I(t-\boldsymbol{\ell})\right\} d\boldsymbol{\ell} - (\lambda+\gamma)L\right] \\
&\quad + (\gamma+\lambda)\left(1-\frac{I_5}{I}\right)\left[\beta\int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) S(t-\boldsymbol{\ell})\left\{\vartheta_1 V(t-\boldsymbol{\ell}) + \vartheta_2 I(t-\boldsymbol{\ell})\right\} d\boldsymbol{\ell}\right. \\
&\quad \left. + \lambda\int_0^{\kappa_3} \bar{\mathcal{H}}_3(\boldsymbol{\ell}) L(t-\boldsymbol{\ell})\, d\boldsymbol{\ell} - aI - \mu_1 C^I I\right] \\
&\quad + \frac{\mathcal{P}}{\varphi\mathcal{H}_4}\left(1-\frac{E_5}{E}\right)\left[\varphi\vartheta_3\int_0^{\kappa_4} \bar{\mathcal{H}}_4(\boldsymbol{\ell}) S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\, d\boldsymbol{\ell} + rY - (\psi+\omega)E\right] \\
&\quad + \frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(1-\frac{Y_5}{Y}\right)\left[\psi\int_0^{\kappa_5} \bar{\mathcal{H}}_5(\boldsymbol{\ell}) E(t-\boldsymbol{\ell})\, d\boldsymbol{\ell} - \delta Y - \mu_2 C^Y Y\right] \\
&\quad + \frac{\mathcal{P}\vartheta_1 S_5}{\varepsilon}\left(1-\frac{V_5}{V}\right)\left[b\int_0^{\kappa_6} \bar{\mathcal{H}}_6(\boldsymbol{\ell}) I(t-\boldsymbol{\ell})\, d\boldsymbol{\ell} - \varepsilon V\right] \\
&\quad + \frac{\mu_1(\gamma+\lambda)}{\sigma_1}\left(1-\frac{C_5^I}{C^I}\right)\left(\sigma_1 C^I I - \pi_1 C^I\right) + \frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(\sigma_2 C^Y Y - \pi_2 C^Y\right) \\
&\quad + \vartheta_1\lambda\mathcal{H}_3(1-\beta)S_5 V_5 \\
&\quad \times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell})\left[\frac{SV}{S_5 V_5} - \frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{S_5 V_5} + \ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)\right] d\boldsymbol{\ell} \\
&\quad + \vartheta_2\lambda\mathcal{H}_3(1-\beta)S_5 I_5 \\
&\quad \times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell})\left[\frac{SI}{S_5 I_5} - \frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_5 I_5} + \ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)\right] d\boldsymbol{\ell} \\
&\quad + \vartheta_1\beta(\gamma+\lambda)S_5 V_5 \\
&\quad \times \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{SV}{S_5 V_5} - \frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{S_5 V_5} + \ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)\right] d\boldsymbol{\ell} \\
&\quad + \vartheta_2\beta(\gamma+\lambda)S_5 I_5 \\
&\quad \times \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{SI}{S_5 I_5} - \frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_5 I_5} + \ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)\right] d\boldsymbol{\ell} \\
&\quad + \lambda(\gamma+\lambda)L_5\int_0^{\kappa_3} \bar{\mathcal{H}}_3(\boldsymbol{\ell})\left[\frac{L}{L_5} - \frac{L(t-\boldsymbol{\ell})}{L_5} + \ln\left(\frac{L(t-\boldsymbol{\ell})}{L}\right)\right] d\boldsymbol{\ell}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_4} \int_0^{\kappa 4} \bar{\mathcal{H}}_4(\boldsymbol{\ell}) \left[\frac{SY}{S_5 Y_5} - \frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{S_5 Y_5} + \ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY} \right) \right] d\boldsymbol{\ell} \\
&+ \frac{\mathcal{P}(\psi+\omega)E_5}{\varphi \mathcal{H}_4 \mathcal{H}_5} \int_0^{\kappa 5} \bar{\mathcal{H}}_5(\boldsymbol{\ell}) \left[\frac{E}{E_5} - \frac{E(t-\boldsymbol{\ell})}{E_5} + \ln\left(\frac{E(t-\boldsymbol{\ell})}{E} \right) \right] d\boldsymbol{\ell} \\
&+ \frac{b\mathcal{P}\vartheta_1 S_5 I_5}{\varepsilon} \int_0^{\kappa 6} \bar{\mathcal{H}}_6(\boldsymbol{\ell}) \left[\frac{I}{I_5} - \frac{I(t-\boldsymbol{\ell})}{I_5} + \ln\left(\frac{I(t-\boldsymbol{\ell})}{I} \right) \right] d\boldsymbol{\ell}. \qquad (23)
\end{aligned}
$$

Summing the terms of Eq. (23), we get

$$
\begin{aligned}
\frac{d\Phi_5}{dt} &= \mathcal{P}\left(1 - \frac{S_5}{S}\right)(\eta - \varrho S) + \mathcal{P}\vartheta_2 S_5 I + \mathcal{P}\vartheta_3 S_5 Y - \vartheta_1 \lambda \mathcal{H}_3 (1-\beta) \int_0^{\kappa 1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) \\
&\quad \times \frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})L_5}{L} d\boldsymbol{\ell} - \vartheta_2 \lambda \mathcal{H}_3 (1-\beta) \int_0^{\kappa 1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) \frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_5}{L} d\boldsymbol{\ell} \\
&\quad + \lambda \mathcal{H}_3 (\lambda+\gamma) L_5 - a(\lambda+\gamma) I - \vartheta_1 \beta(\gamma+\lambda) \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) \frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})I_5}{I} d\boldsymbol{\ell} \\
&\quad - \vartheta_2 \beta(\gamma+\lambda) \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) \frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})I_5}{I} d\boldsymbol{\ell} \\
&\quad - \lambda(\lambda+\gamma) \int_0^{\kappa 3} \bar{\mathcal{H}}_3(\boldsymbol{\ell}) \frac{L(t-\boldsymbol{\ell})I_5}{I} d\boldsymbol{\ell} + a(\lambda+\gamma) I_5 + \mu_1 (\lambda+\gamma) C^I I_5 \\
&\quad + \frac{\mathcal{P}r}{\varphi \mathcal{H}_4} Y - \frac{\mathcal{P}\vartheta_3}{\mathcal{H}_4} \int_0^{\kappa 4} \bar{\mathcal{H}}_4(\boldsymbol{\ell}) \frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_5}{E} d\boldsymbol{\ell} \\
&\quad - \frac{\mathcal{P}r}{\varphi \mathcal{H}_4} \frac{Y E_5}{E} + \frac{\mathcal{P}(\psi+\omega)}{\varphi \mathcal{H}_4} E_5 - \frac{\mathcal{P}\delta(\psi+\omega)}{\varphi \psi \mathcal{H}_4 \mathcal{H}_5} Y \\
&\quad - \frac{\mathcal{P}(\psi+\omega)}{\varphi \mathcal{H}_4 \mathcal{H}_5} \int_0^{\kappa 5} \bar{\mathcal{H}}_5(\boldsymbol{\ell}) \frac{E(t-\boldsymbol{\ell})Y_5}{Y} d\boldsymbol{\ell} + \frac{\mathcal{P}\delta(\psi+\omega)}{\varphi \psi \mathcal{H}_4 \mathcal{H}_5} Y_5 \\
&\quad + \frac{\mu_2 \mathcal{P}(\psi+\omega)}{\varphi \psi \mathcal{H}_4 \mathcal{H}_5} C^Y Y_5 - \frac{b\mathcal{P}\vartheta_1 S_5}{\varepsilon} \int_0^{\kappa 6} \bar{\mathcal{H}}_6(\boldsymbol{\ell}) \frac{I(t-\boldsymbol{\ell})V_5}{V} d\boldsymbol{\ell} + \mathcal{P}\vartheta_1 S_5 V_5 \\
&\quad - \frac{\mu_1 \pi_1 (\gamma+\lambda)}{\sigma_1} C^I - \mu_1 (\lambda+\gamma) C_5^I I + \frac{\mu_1 \pi_1 (\lambda+\gamma)}{\sigma_1} C_5^I - \frac{\mu_2 \pi_2 \mathcal{P}(\psi+\omega)}{\sigma_2 \varphi \psi \mathcal{H}_4 \mathcal{H}_5} C^Y \\
&\quad + \lambda \mathcal{H}_3 (1-\beta) \vartheta_1 S_5 V_5 \int_0^{\kappa 1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) \ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV} \right) d\boldsymbol{\ell} \\
&\quad + \lambda \mathcal{H}_3 (1-\beta) \vartheta_2 S_5 I_5 \int_0^{\kappa 1} \bar{\mathcal{H}}_1(\boldsymbol{\ell}) \ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI} \right) d\boldsymbol{\ell} \\
&\quad + \beta(\gamma+\lambda) \vartheta_1 S_5 V_5 \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) \ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV} \right) d\boldsymbol{\ell} \\
&\quad + \beta(\gamma+\lambda) \vartheta_2 S_5 I_5 \int_0^{\kappa 2} \bar{\mathcal{H}}_2(\boldsymbol{\ell}) \ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI} \right) d\boldsymbol{\ell} \\
&\quad + \lambda(\gamma+\lambda) L_5 \int_0^{\kappa 3} \bar{\mathcal{H}}_3(\boldsymbol{\ell}) \ln\left(\frac{L(t-\boldsymbol{\ell})}{L} \right) d\boldsymbol{\ell} \\
&\quad + \frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_4} \int_0^{\kappa 4} \bar{\mathcal{H}}_4(\boldsymbol{\ell}) \ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY} \right) d\boldsymbol{\ell} \\
&\quad + \frac{\mathcal{P}(\psi+\omega)E_5}{\varphi \mathcal{H}_4 \mathcal{H}_5} \int_0^{\kappa 5} \bar{\mathcal{H}}_5(\boldsymbol{\ell}) \ln\left(\frac{E(t-\boldsymbol{\ell})}{E} \right) d\boldsymbol{\ell} \\
&\quad + \frac{b\mathcal{P}\vartheta_1 S_5 \mathcal{H}_6}{\varepsilon} I + \frac{b\mathcal{P}\vartheta_1 S_5 I_5}{\varepsilon} \int_0^{\kappa 6} \bar{\mathcal{H}}_6(\boldsymbol{\ell}) \ln\left(\frac{I(t-\boldsymbol{\ell})}{I} \right) d\boldsymbol{\ell}.
\end{aligned}
$$

Using the steady state conditions for Đ$_5$

$$\eta = \varrho S_5 + \vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5 + \vartheta_3 S_5 Y_5, \qquad \mathcal{H}_1(1-\beta)(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5) = (\lambda + \gamma) L_5,$$

$$\beta \mathcal{H}_2(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5) + \lambda \mathcal{H}_3 L_5 = \left(a + \mu_1 C_5^I\right) I_5 \qquad I_5 = \frac{\pi_1}{\sigma_1}, \qquad V_5 = \frac{b\mathcal{H}_6}{\varepsilon} I_5,$$

$$\vartheta_3 S_5 Y_5 + \frac{r Y_5}{\varphi \mathcal{H}_4} = \frac{(\psi + \omega) E_5}{\varphi \mathcal{H}_4} = \frac{\delta(\psi + \omega) Y_5}{\varphi \psi \mathcal{H}_4 \mathcal{H}_5},$$

we obtain

$$\mathcal{P}(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5) = (\lambda + \gamma)\left(a + \mu_1 C_5^I\right) I_5.$$

Moreover, we get

$$\begin{aligned}
\frac{d\Phi_5}{dt} ={}& \mathcal{P}\left(1 - \frac{S_5}{S}\right)(\varrho S_5 - \varrho S) + \mathcal{P}(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5 + \vartheta_3 S_5 Y_5)\left(1 - \frac{S_5}{S}\right) \\
&- \lambda \mathcal{H}_3(1-\beta)\vartheta_1 S_5 V_5 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \frac{S(t-\ell)V(t-\ell)L_5}{S_5 V_5 L}\, d\ell \\
&- \lambda \mathcal{H}_3(1-\beta)\vartheta_2 S_5 I_5 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \frac{S(t-\ell)I(t-\ell)L_5}{S_5 I_5 L}\, d\ell \\
&+ \lambda \mathcal{H}_1 \mathcal{H}_3(1-\beta)(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5) \\
&- \beta(\gamma + \lambda)\vartheta_1 S_5 V_5 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \frac{S(t-\ell)V(t-\ell)I_5}{S_5 V_5 I}\, d\ell \\
&- \beta(\gamma + \lambda)\vartheta_2 S_5 I_5 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \frac{S(t-\ell)I(t-\ell)}{S_5 I}\, d\ell \\
&- \lambda \mathcal{H}_1(1-\beta)(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell) \frac{L(t-\ell)I_5}{L_5 I}\, d\ell \\
&+ \mathcal{P}(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5) - \frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_4} \int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell) \frac{S(t-\ell)Y(t-\ell)E_5}{S_5 Y_5 E}\, d\ell \\
&- \frac{\mathcal{P} r Y_5}{\varphi \mathcal{H}_4} \frac{Y E_5}{Y_5 E} + \mathcal{P}\vartheta_3 S_5 Y_5 + \frac{\mathcal{P} r Y_5}{\varphi \mathcal{H}_4} \\
&- \left(\frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_5} + \frac{\mathcal{P} r Y_5}{\varphi \mathcal{H}_4 \mathcal{H}_5}\right) \int_0^{\kappa_5} \bar{\mathcal{H}}_5(\ell) \frac{E(t-\ell)Y_5}{E_5 Y}\, d\ell + \mathcal{P}\vartheta_3 S_5 Y_5 + \frac{\mathcal{P} r Y_5}{\varphi \mathcal{H}_4} \\
&- \frac{\mathcal{P}\vartheta_1 S_5 V_5}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell) \frac{I(t-\ell)V_5}{I_5 V}\, d\ell + \mathcal{P}\vartheta_1 S_5 V_5 \\
&+ \lambda \mathcal{H}_3(1-\beta)\vartheta_1 S_5 V_5 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) d\ell \\
&+ \lambda \mathcal{H}_3(1-\beta)\vartheta_2 S_5 I_5 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right) d\ell \\
&+ \beta(\gamma + \lambda)\vartheta_1 S_5 V_5 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) d\ell \\
&+ \beta(\gamma + \lambda)\vartheta_2 S_5 I_5 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right) d\ell \\
&+ \lambda \mathcal{H}_1(1-\beta)(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell) \ln\left(\frac{L(t-\ell)}{L}\right) d\ell
\end{aligned}$$

$$+ \frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_4} \int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell) \ln\left(\frac{S(t-\ell)Y(t-\ell)}{SY}\right) d\ell$$
$$+ \left(\frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_5} + \frac{\mathcal{P}rY_5}{\varphi\mathcal{H}_4\mathcal{H}_5}\right) \int_0^{\kappa_5} \bar{\mathcal{H}}_5(\ell) \ln\left(\frac{E(t-\ell)}{E}\right) d\ell$$
$$+ \frac{\mathcal{P}\vartheta_1 S_5 V_5}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell) \ln\left(\frac{I(t-\ell)}{I}\right) d\ell + \frac{\mu_2 \mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(Y_5 - \frac{\pi_2}{\sigma_2}\right) C^Y.$$

Using the equalities given by (10) and (11) in case of $n = m = 5$, we get

$$\begin{aligned}
\frac{d\Phi_5}{dt} &= -\varrho\mathcal{P}\frac{(S-S_5)^2}{S} - \mathcal{P}(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5 + \vartheta_3 S_5 Y_5)\left[\frac{S_5}{S} - 1 - \ln\left(\frac{S_5}{S}\right)\right] \\
&\quad - \lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_5 V_5 \\
&\quad \times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\left[\frac{S(t-\ell)V(t-\ell)L_5}{S_5 V_5 L} - 1 - \ln\left(\frac{S(t-\ell)V(t-\ell)L_5}{S_5 V_5 L}\right)\right] d\ell \\
&\quad - \lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_5 I_5 \\
&\quad \times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\left[\frac{S(t-\ell)I(t-\ell)L_5}{S_5 I_5 L} - 1 - \ln\left(\frac{S(t-\ell)I(t-\ell)L_5}{S_5 I_5 L}\right)\right] d\ell \\
&\quad - \beta(\gamma+\lambda)\vartheta_1 S_5 V_5 \\
&\quad \times \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\left[\frac{S(t-\ell)V(t-\ell)I_5}{S_5 V_5 I} - 1 - \ln\left(\frac{S(t-\ell)V(t-\ell)I_5}{S_5 V_5 I}\right)\right] d\ell \\
&\quad - \beta(\gamma+\lambda)\vartheta_2 S_5 I_5 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\left[\frac{S(t-\ell)I(t-\ell)}{S_5 I} - 1 - \ln\left(\frac{S(t-\ell)I(t-\ell)}{S_5 I}\right)\right] d\ell \\
&\quad - \lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5) \\
&\quad \times \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell)\left[\frac{L(t-\ell)I_5}{L_5 I} - 1 - \ln\left(\frac{L(t-\ell)I_5}{L_5 I}\right)\right] d\ell \\
&\quad - \frac{\mathcal{P}rY_5}{\varphi\mathcal{H}_4}\left[\frac{YE_5}{Y_5 E} - 1 - \ln\left(\frac{YE_5}{Y_5 E}\right)\right] \\
&\quad - \frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_4} \int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell)\left[\frac{S(t-\ell)Y(t-\ell)E_5}{S_5 Y_5 E} - 1 - \ln\left(\frac{S(t-\ell)Y(t-\ell)E_5}{S_5 Y_5 E}\right)\right] d\ell \\
&\quad - \left(\frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_5} + \frac{\mathcal{P}rY_5}{\varphi\mathcal{H}_4\mathcal{H}_5}\right) \int_0^{\kappa_5} \bar{\mathcal{H}}_5(\ell)\left[\frac{E(t-\ell)Y_5}{E_5 Y} - 1 - \ln\left(\frac{E(t-\ell)Y_5}{E_5 Y}\right)\right] d\ell \\
&\quad - \frac{\mathcal{P}\vartheta_1 S_5 V_5}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell)\left[\frac{I(t-\ell)V_5}{I_5 V} - 1 - \ln\left(\frac{I(t-\ell)V_5}{I_5 V}\right)\right] d\ell \\
&\quad + \frac{\mu_2\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(Y_5 - \frac{\pi_2}{\sigma_2}\right) C^Y. \qquad (24)
\end{aligned}$$

Therefore, Eq. (24) becomes

$$\begin{aligned}
\frac{d\Phi_5}{dt} &= -\varrho\mathcal{P}\frac{(S-S_5)^2}{S} - \mathcal{P}(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5 + \vartheta_3 S_5 Y_5)F\left(\frac{S_5}{S}\right) \\
&\quad - \lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_5 V_5 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) F\left(\frac{S(t-\ell)V(t-\ell)L_5}{S_5 V_5 L}\right) d\ell \\
&\quad - \lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_5 I_5 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) F\left(\frac{S(t-\ell)I(t-\ell)L_5}{S_5 I_5 L}\right) d\ell
\end{aligned}$$

$$\begin{aligned}
&-\beta(\gamma+\lambda)\vartheta_1 S_5 V_5 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) F\left(\frac{S(t-\ell)V(t-\ell)I_5}{S_5 V_5 I}\right) d\ell \\
&-\beta(\gamma+\lambda)\vartheta_2 S_5 I_5 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) F\left(\frac{S(t-\ell)I(t-\ell)}{S_5 I}\right) d\ell \\
&-\lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_5 V_5+\vartheta_2 S_5 I_5) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell) F\left(\frac{L(t-\ell)I_5}{L_5 I}\right) d\ell \\
&-\frac{\mathcal{P}rY_5}{\varphi\mathcal{H}_4} F\left(\frac{YE_5}{Y_5 E}\right)-\frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_4} \int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell) F\left(\frac{S(t-\ell)Y(t-\ell)E_5}{S_5 Y_5 E}\right) d\ell \\
&-\left(\frac{\mathcal{P}\vartheta_3 S_5 Y_5}{\mathcal{H}_5}+\frac{\mathcal{P}rY_5}{\varphi\mathcal{H}_4\mathcal{H}_5}\right) \int_0^{\kappa_5} \bar{\mathcal{H}}_5(\ell) F\left(\frac{E(t-\ell)Y_5}{E_5 Y}\right) d\ell \\
&-\frac{\mathcal{P}\vartheta_1 S_5 V_5}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell) F\left(\frac{I(t-\ell)V_5}{I_5 V}\right) d\ell \\
&+\frac{\mu_2\mathcal{P}(\psi+\omega)[\pi_1\sigma_2(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)+\varepsilon\sigma_1(\pi_2\vartheta_3+\varrho\sigma_2)]}{\varphi\psi\vartheta_3\varepsilon\sigma_1\sigma_2\mathcal{H}_4\mathcal{H}_5}(\Re_8-1)C^Y.
\end{aligned}$$

Hence, if $\Re_8 \le 1$, then $\frac{d\Phi_5}{dt} \le 0$ for all $S, L, I, E, Y, V, C^Y > 0$. One can show that $\frac{d\Phi_5}{dt} = 0$ when $(S, L, I, E, Y, V, C^Y) = (S_5, L_5, I_5, E_5, Y_5, V_5, 0)$. The solutions of model (5) tend to Υ_5' which includes elements with $(S(t), L(t), I(t), V(t)) = (S_5, L_5, I_5, V_5)$, and then $\frac{dI(t)}{dt} = 0$. The third equation of system (5) becomes

$$0 = \frac{dI(t)}{dt} = \beta\mathcal{H}_2(\vartheta_1 S_5 V_5 + \vartheta_2 S_5 I_5) + \lambda\mathcal{H}_3 L_5 - aI_5 - \mu_1 C^I(t) I_5,$$

which yields $C^I(t) = C^I_5$ for all t, and hence $\Upsilon_5' = \{Đ_5\}$. Applying LLAS theorem, we get $Đ_5$ is GAS. □

Theorem 7 *If* $\Re_6 > 1$, $\Re_7 \le 1$, *and* $\Re_2/\Re_1 > 1$, *then* $Đ_6$ *is GAS.*

Proof Define

$$\begin{aligned}
\Phi_6 &= \mathcal{P}S_6 F\left(\frac{S}{S_6}\right)+\lambda\mathcal{H}_3 L_6 F\left(\frac{L}{L_6}\right)+(\gamma+\lambda)I_6 F\left(\frac{I}{I_6}\right)+\frac{\mathcal{P}}{\varphi\mathcal{H}_4}E_6 F\left(\frac{E}{E_6}\right) \\
&+\frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5} Y_6 F\left(\frac{Y}{Y_6}\right)+\frac{\mathcal{P}\vartheta_1 S_6}{\varepsilon} V_6 F\left(\frac{V}{V_6}\right)+\frac{\mu_1(\gamma+\lambda)}{\sigma_1} C^I \\
&+\frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5} C^Y_6 F\left(\frac{C^Y}{C^Y_6}\right) \\
&+\vartheta_1\lambda\mathcal{H}_3(1-\beta)S_6 V_6 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \int_{t-\ell}^{t} F\left(\frac{S(\varkappa)V(\varkappa)}{S_6 V_6}\right) d\varkappa\, d\ell \\
&+\vartheta_2\lambda\mathcal{H}_3(1-\beta)S_6 I_6 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \int_{t-\ell}^{t} F\left(\frac{S(\varkappa)I(\varkappa)}{S_6 I_6}\right) d\varkappa\, d\ell \\
&+\vartheta_1\beta(\gamma+\lambda)S_6 V_6 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \int_{t-\ell}^{t} F\left(\frac{S(\varkappa)V(\varkappa)}{S_6 V_6}\right) d\varkappa\, d\ell \\
&+\vartheta_2\beta(\gamma+\lambda)S_6 I_6 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \int_{t-\ell}^{t} F\left(\frac{S(\varkappa)I(\varkappa)}{S_6 I_6}\right) d\varkappa\, d\ell \\
&+\lambda(\gamma+\lambda)L_6 \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell) \int_{t-\ell}^{t} F\left(\frac{L(\varkappa)}{L_6}\right) d\varkappa\, d\ell
\end{aligned}$$

$$+\frac{\mathcal{P}\vartheta_3 S_6 Y_6}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\ell)\int_{t-\ell}^{t}F\left(\frac{S(\varkappa)Y(\varkappa)}{S_6 Y_6}\right)d\varkappa\, d\ell$$
$$+\frac{\mathcal{P}(\psi+\omega)E_6}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\ell)\int_{t-\ell}^{t}F\left(\frac{E(\varkappa)}{E_6}\right)d\varkappa\, d\ell$$
$$+\frac{b\mathcal{P}\vartheta_1 S_6 I_6}{\varepsilon}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\ell)\int_{t-\ell}^{t}F\left(\frac{I(\varkappa)}{I_6}\right)d\varkappa\, d\ell.$$

Calculate $\frac{d\Phi_6}{dt}$ as follows:

$$\begin{aligned}
\frac{d\Phi_6}{dt} &= \mathcal{P}\left(1-\frac{S_6}{S}\right)(\eta-\varrho S-\vartheta_1 SV-\vartheta_2 SI-\vartheta_3 SY)+\lambda\mathcal{H}_3\left(1-\frac{L_6}{L}\right)\\
&\quad\times\left[(1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)S(t-\ell)\left\{\vartheta_1 V(t-\ell)+\vartheta_2 I(t-\ell)\right\}d\ell-(\lambda+\gamma)L\right]\\
&\quad+(\gamma+\lambda)\left(1-\frac{I_6}{I}\right)\left[\beta\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)S(t-\ell)\left\{\vartheta_1 V(t-\ell)+\vartheta_2 I(t-\ell)\right\}d\ell\right.\\
&\quad\left.+\lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\ell)L(t-\ell)\,d\ell-aI-\mu_1 C^I I\right]\\
&\quad+\frac{\mathcal{P}}{\varphi\mathcal{H}_4}\left(1-\frac{E_6}{E}\right)\left[\varphi\vartheta_3\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\ell)S(t-\ell)Y(t-\ell)\,d\ell+rY-(\psi+\omega)E\right]\\
&\quad+\frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(1-\frac{Y_6}{Y}\right)\left[\psi\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\ell)E(t-\ell)\,d\ell-\delta Y-\mu_2 C^Y Y\right]\\
&\quad+\frac{\mathcal{P}\vartheta_1 S_6}{\varepsilon}\left(1-\frac{V_6}{V}\right)\left[b\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\ell)I(t-\ell)\,d\ell-\varepsilon V\right]\\
&\quad+\frac{\mu_1(\gamma+\lambda)}{\sigma_1}\left(\sigma_1 C^I I-\pi_1 C^I\right)+\frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(1-\frac{C_6^Y}{C^Y}\right)\left(\sigma_2 C^Y Y-\pi_2 C^Y\right)\\
&\quad+\vartheta_1\lambda\mathcal{H}_3(1-\beta)S_6 V_6\\
&\quad\times\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\left[\frac{SV}{S_6 V_6}-\frac{S(t-\ell)V(t-\ell)}{S_6 V_6}+\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right)\right]d\ell\\
&\quad+\vartheta_2\lambda\mathcal{H}_3(1-\beta)S_6 I_6\\
&\quad\times\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\ell)\left[\frac{SI}{S_6 I_6}-\frac{S(t-\ell)I(t-\ell)}{S_6 I_6}+\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right)\right]d\ell\\
&\quad+\vartheta_1\beta(\gamma+\lambda)S_6 V_6\\
&\quad\times\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)\left[\frac{SV}{S_6 V_6}-\frac{S(t-\ell)V(t-\ell)}{S_6 V_6}+\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right)\right]d\ell\\
&\quad+\vartheta_2\beta(\gamma+\lambda)S_6 I_6\\
&\quad\times\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\ell)\left[\frac{SI}{S_6 I_6}-\frac{S(t-\ell)I(t-\ell)}{S_6 I_6}+\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right)\right]d\ell\\
&\quad+\lambda(\gamma+\lambda)L_6\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\ell)\left[\frac{L}{L_6}-\frac{L(t-\ell)}{L_6}+\ln\left(\frac{L(t-\ell)}{L}\right)\right]d\ell\\
&\quad+\frac{\mathcal{P}\vartheta_3 S_6 Y_6}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\ell)\left[\frac{SY}{S_6 Y_6}-\frac{S(t-\ell)Y(t-\ell)}{S_6 Y_6}+\ln\left(\frac{S(t-\ell)Y(t-\ell)}{SY}\right)\right]d\ell
\end{aligned}$$

$$+\frac{\mathcal{P}(\psi+\omega)E_6}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\left[\frac{E}{E_6}-\frac{E(t-\boldsymbol{\ell})}{E_6}+\ln\left(\frac{E(t-\boldsymbol{\ell})}{E}\right)\right]d\boldsymbol{\ell}$$
$$+\frac{b\mathcal{P}\vartheta_1 S_6 I_6}{\varepsilon}\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\left[\frac{I}{I_6}-\frac{I(t-\boldsymbol{\ell})}{I_6}+\ln\left(\frac{I(t-\boldsymbol{\ell})}{I}\right)\right]d\boldsymbol{\ell}. \quad (25)$$

Collecting the terms of Eq. (25), we obtain

$$\begin{aligned}
\frac{d\Phi_6}{dt} &= \mathcal{P}\left(1-\frac{S_6}{S}\right)(\eta-\varrho S)+\mathcal{P}\vartheta_2 S_6 I+\mathcal{P}\vartheta_3 S_6 Y-\vartheta_1\lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell}) \\
&\quad\times\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})L_6}{L}\,d\boldsymbol{\ell}-\vartheta_2\lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_6}{L}\,d\boldsymbol{\ell} \\
&\quad+\lambda\mathcal{H}_3(\lambda+\gamma)L_6-a(\lambda+\gamma)I-\vartheta_1\beta(\gamma+\lambda)\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})I_6}{I}\,d\boldsymbol{\ell} \\
&\quad-\vartheta_2\beta(\gamma+\lambda)\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})I_6}{I}\,d\boldsymbol{\ell} \\
&\quad-\lambda(\lambda+\gamma)\int_0^{\kappa 3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\frac{L(t-\boldsymbol{\ell})I_6}{I}\,d\boldsymbol{\ell}+a(\lambda+\gamma)I_6+\mu_1(\lambda+\gamma)C^I I_6 \\
&\quad+\frac{\mathcal{P}r}{\varphi\mathcal{H}_4}Y-\frac{\mathcal{P}\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_6}{E}\,d\boldsymbol{\ell} \\
&\quad-\frac{\mathcal{P}r}{\varphi\mathcal{H}_4}\frac{YE_6}{E}+\frac{\mathcal{P}(\psi+\omega)}{\varphi\mathcal{H}_4}E_6-\frac{\mathcal{P}\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y \\
&\quad-\frac{\mathcal{P}(\psi+\omega)}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\frac{E(t-\boldsymbol{\ell})Y_6}{Y}\,d\boldsymbol{\ell}+\frac{\mathcal{P}\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y_6+\frac{\mu_2\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y Y_6 \\
&\quad-\frac{b\mathcal{P}\vartheta_1 S_6}{\varepsilon}\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\frac{I(t-\boldsymbol{\ell})V_6}{V}\,d\boldsymbol{\ell}+\mathcal{P}\vartheta_1 S_6 V_6-\frac{\mu_1\pi_1(\lambda+\gamma)}{\sigma_1}C^I \\
&\quad-\frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y-\frac{\mu_2\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C_6^Y Y+\frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C_6^Y \\
&\quad+\lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_6 V_6\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)d\boldsymbol{\ell} \\
&\quad+\lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_6 I_6\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)d\boldsymbol{\ell} \\
&\quad+\beta(\gamma+\lambda)\vartheta_1 S_6 V_6\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)d\boldsymbol{\ell} \\
&\quad+\beta(\gamma+\lambda)\vartheta_2 S_6 I_6\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)d\boldsymbol{\ell} \\
&\quad+\lambda(\gamma+\lambda)L_6\int_0^{\kappa 3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\ln\left(\frac{L(t-\boldsymbol{\ell})}{L}\right)d\boldsymbol{\ell} \\
&\quad+\frac{\mathcal{P}\vartheta_3 S_6 Y_6}{\mathcal{H}_4}\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY}\right)d\boldsymbol{\ell} \\
&\quad+\frac{\mathcal{P}(\psi+\omega)E_6}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\ln\left(\frac{E(t-\boldsymbol{\ell})}{E}\right)d\boldsymbol{\ell} \\
&\quad+\frac{b\mathcal{P}\vartheta_1 S_6\mathcal{H}_6}{\varepsilon}I+\frac{b\mathcal{P}\vartheta_1 S_6 I_6}{\varepsilon}\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\ln\left(\frac{I(t-\boldsymbol{\ell})}{I}\right)d\boldsymbol{\ell}.
\end{aligned}$$

Using the steady state conditions for Đ$_5$

$$\eta = \varrho S_6 + \vartheta_1 S_6 V_6 + \vartheta_2 S_6 I_6 + \vartheta_3 S_6 Y_6, \qquad \mathcal{H}_1(1-\beta)(\vartheta_1 S_6 V_6 + \vartheta_2 S_6 I_6) = (\lambda+\gamma)L_6,$$
$$\beta\mathcal{H}_2(\vartheta_1 S_6 V_6 + \vartheta_2 S_6 I_6) + \lambda\mathcal{H}_3 L_6 = aI_6, \qquad Y_6 = \frac{\pi_2}{\sigma_2}, \qquad V_6 = \frac{b\mathcal{H}_6 I_6}{\varepsilon},$$
$$\vartheta_3 S_6 Y_6 + \frac{rY_6}{\varphi\mathcal{H}_4} = \frac{(\psi+\omega)E_6}{\varphi\mathcal{H}_4} = \frac{\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y_6 + \frac{\mu_2(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C_6^Y Y_6,$$

we get

$$\mathcal{P}(\vartheta_1 S_6 V_6 + \vartheta_2 S_6 I_6) = a(\lambda+\gamma)I_6.$$

Moreover, we get

$$\begin{aligned}
\frac{d\Phi_6}{dt} &= \mathcal{P}\left(1-\frac{S_6}{S}\right)(\varrho S_6 - \varrho S) + \mathcal{P}(\vartheta_1 S_6 V_6 + \vartheta_2 S_6 I_6 + \vartheta_3 S_6 Y_6)\left(1-\frac{S_6}{S}\right) \\
&\quad - \lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_6 V_6 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})L_6}{S_6 V_6 L}\,d\boldsymbol{\ell} - \lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_6 I_6 \\
&\quad \times \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_6}{S_6 I_6 L}\,d\boldsymbol{\ell} + \lambda\mathcal{H}_1\mathcal{H}_3(1-\beta)(\vartheta_1 S_6 V_6 + \vartheta_2 S_6 I_6) \\
&\quad - \beta(\gamma+\lambda)\vartheta_1 S_6 V_6 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})I_6}{S_6 V_6 I}\,d\boldsymbol{\ell} \\
&\quad - \beta(\gamma+\lambda)\vartheta_2 S_6 I_6 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_6 I}\,d\boldsymbol{\ell} \\
&\quad - \lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_6 V_6 + \vartheta_2 S_6 I_6)\int_0^{\kappa_3} \bar{\mathcal{H}}_3(\boldsymbol{\ell})\frac{L(t-\boldsymbol{\ell})I_6}{L_6 I}\,d\boldsymbol{\ell} \\
&\quad + \mathcal{P}(\vartheta_1 S_6 V_6 + \vartheta_2 S_6 I_6) - \frac{\mathcal{P}\vartheta_3 S_6 Y_6}{\mathcal{H}_4}\int_0^{\kappa_4} \bar{\mathcal{H}}_4(\boldsymbol{\ell})\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_6}{S_6 Y_6 E}\,d\boldsymbol{\ell} \\
&\quad - \frac{\mathcal{P}rY_6}{\varphi\mathcal{H}_4}\frac{YE_6}{Y_6 E} + \mathcal{P}\vartheta_3 S_6 Y_6 + \frac{\mathcal{P}rY_6}{\varphi\mathcal{H}_4} \\
&\quad - \left(\frac{\mathcal{P}\vartheta_3 S_6 Y_6}{\mathcal{H}_5} + \frac{\mathcal{P}rY_6}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5} \bar{\mathcal{H}}_5(\boldsymbol{\ell})\frac{E(t-\boldsymbol{\ell})Y_6}{E_6 Y}\,d\boldsymbol{\ell} \\
&\quad + \mathcal{P}\vartheta_3 S_6 Y_6 + \frac{\mathcal{P}rY_6}{\varphi\mathcal{H}_4} - \frac{\mathcal{P}\vartheta_1 S_6 V_6}{\mathcal{H}_6}\int_0^{\kappa_6} \bar{\mathcal{H}}_6(\boldsymbol{\ell})\frac{I(t-\boldsymbol{\ell})V_6}{I_6 V}\,d\boldsymbol{\ell} + \mathcal{P}\vartheta_1 S_6 V_6 \\
&\quad + \lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_6 V_6 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)d\boldsymbol{\ell} \\
&\quad + \lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_6 I_6 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)d\boldsymbol{\ell} \\
&\quad + \beta(\gamma+\lambda)\vartheta_1 S_6 V_6 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)d\boldsymbol{\ell} \\
&\quad + \beta(\gamma+\lambda)\vartheta_2 S_6 I_6 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)d\boldsymbol{\ell} \\
&\quad + \lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_6 V_6 + \vartheta_2 S_6 I_6)\int_0^{\kappa_3} \bar{\mathcal{H}}_3(\boldsymbol{\ell})\ln\left(\frac{L(t-\boldsymbol{\ell})}{L}\right)d\boldsymbol{\ell}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathcal{P}\vartheta_3S_6Y_6}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})}{SY}\right)d\boldsymbol{\ell}\\
&+\left(\frac{\mathcal{P}\vartheta_3S_6Y_6}{\mathcal{H}_5}+\frac{\mathcal{P}rY_6}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\ln\left(\frac{E(t-\boldsymbol{\ell})}{E}\right)d\boldsymbol{\ell}\\
&+\frac{\mathcal{P}\vartheta_1S_6V_6}{\mathcal{H}_6}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\ln\left(\frac{I(t-\boldsymbol{\ell})}{I}\right)d\boldsymbol{\ell}+\mu_1(\lambda+\gamma)\left(I_6-\frac{\pi_1}{\sigma_1}\right)C^I.
\end{aligned}
$$

Using the equalities given by (10) and (11) in case of $n = m = 6$, we get

$$
\begin{aligned}
\frac{d\Phi_6}{dt}={}&-\varrho\mathcal{P}\frac{(S-S_6)^2}{S}-\mathcal{P}(\vartheta_1S_6V_6+\vartheta_2S_6I_6+\vartheta_3S_6Y_6)\left[\frac{S_6}{S}-1-\ln\left(\frac{S_6}{S}\right)\right]\\
&-\lambda\mathcal{H}_3(1-\beta)\vartheta_1S_6V_6\\
&\times\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\left[\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})L_6}{S_6V_6L}-1-\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})L_6}{S_6V_6L}\right)\right]d\boldsymbol{\ell}\\
&-\lambda\mathcal{H}_3(1-\beta)\vartheta_2S_6I_6\\
&\times\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\left[\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_6}{S_6I_6L}-1-\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_6}{S_6I_6L}\right)\right]d\boldsymbol{\ell}\\
&-\beta(\gamma+\lambda)\vartheta_1S_6V_6\\
&\times\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})I_6}{S_6V_6I}-1-\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})I_6}{S_6V_6I}\right)\right]d\boldsymbol{\ell}\\
&-\beta(\gamma+\lambda)\vartheta_2S_6I_6\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_6I}-1-\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_6I}\right)\right]d\boldsymbol{\ell}\\
&-\lambda\mathcal{H}_1(1-\beta)(\vartheta_1S_6V_6+\vartheta_2S_6I_6)\\
&\times\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell}))\left[\frac{L(t-\boldsymbol{\ell})I_6}{L_6I}-1-\ln\left(\frac{L(t-\boldsymbol{\ell})I_6}{L_6I}\right)\right]d\boldsymbol{\ell}\\
&-\frac{\mathcal{P}\vartheta_3S_6Y_6}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\left[\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_6}{S_6Y_6E}-1-\ln\left(\frac{S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})E_6}{S_6Y_6E}\right)\right]d\boldsymbol{\ell}\\
&-\frac{\mathcal{P}rY_6}{\varphi\mathcal{H}_4}\left[\frac{YE_6}{Y_6E}-1-\ln\left(\frac{YE_6}{Y_6E}\right)\right]\\
&-\left(\frac{\mathcal{P}\vartheta_3S_6Y_6}{\mathcal{H}_5}+\frac{\mathcal{P}rY_6}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\left[\frac{E(t-\boldsymbol{\ell})Y_6}{E_6Y}-1-\ln\left(\frac{E(t-\boldsymbol{\ell})Y_6}{E_6Y}\right)\right]d\boldsymbol{\ell}\\
&-\frac{\mathcal{P}\vartheta_1S_6V_6}{\mathcal{H}_6}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\left[\frac{I(t-\boldsymbol{\ell})V_6}{I_6V}-1-\ln\left(\frac{I(t-\boldsymbol{\ell})V_6}{I_6V}\right)\right]d\boldsymbol{\ell}\\
&+\mu_1(\lambda+\gamma)\left(I_6-\frac{\pi_1}{\sigma_1}\right)C^I.
\end{aligned}
\tag{26}
$$

Therefore, Eq. (26) becomes

$$
\begin{aligned}
\frac{d\Phi_6}{dt}={}&-\varrho\mathcal{P}\frac{(S-S_6)^2}{S}-\mathcal{P}(\vartheta_1S_6V_6+\vartheta_2S_6I_6+\vartheta_3S_6Y_6)+F\left(\frac{S_6}{S}\right)\\
&-\lambda\mathcal{H}_3(1-\beta)\vartheta_1S_6V_6\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})F\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})L_6}{S_6V_6L}\right)d\boldsymbol{\ell}\\
&-\lambda\mathcal{H}_3(1-\beta)\vartheta_2S_6I_6\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})F\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})L_6}{S_6I_6L}\right)d\boldsymbol{\ell}
\end{aligned}
$$

$$
\begin{aligned}
&-\beta(\gamma+\lambda)\vartheta_1 S_6 V_6 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) F\left(\frac{S(t-\ell)V(t-\ell)I_6}{S_6 V_6 I}\right) d\ell \\
&-\beta(\gamma+\lambda)\vartheta_2 S_6 I_6 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) F\left(\frac{S(t-\ell)I(t-\ell)}{S_6 I}\right) d\ell \\
&-\lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_6 V_6+\vartheta_2 S_6 I_6) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell)) F\left(\frac{L(t-\ell)I_6}{L_6 I}\right) d\ell \\
&-\frac{\mathcal{P}\vartheta_3 S_6 Y_6}{\mathcal{H}_4} \int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell) F\left(\frac{S(t-\ell)Y(t-\ell)E_6}{S_6 Y_6 E}\right) d\ell - \frac{\mathcal{P}rY_6}{\varphi\mathcal{H}_4} F\left(\frac{YE_6}{Y_6 E}\right) \\
&-\left(\frac{\mathcal{P}\vartheta_3 S_6 Y_6}{\mathcal{H}_5}+\frac{\mathcal{P}rY_6}{\varphi\mathcal{H}_4\mathcal{H}_5}\right) \int_0^{\kappa_5} \bar{\mathcal{H}}_5(\ell) F\left(\frac{E(t-\ell)Y_6}{E_6 Y}\right) d\ell \\
&-\frac{\mathcal{P}\vartheta_1 S_6 V_6}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell) F\left(\frac{I(t-\ell)V_6}{I_6 V}\right) d\ell \\
&+\frac{\mu_1(\gamma+\lambda)[\pi_1\sigma_2(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)+\varepsilon\sigma_1(\pi_2\vartheta_3+\varrho\sigma_2)]}{\sigma_1\sigma_2(b\vartheta_1\mathcal{H}_6+\varepsilon\vartheta_2)}(\Re_7-1)C^I.
\end{aligned}
$$

Hence, if $\Re_7 \le 1$, then $\frac{d\Phi_6}{dt} \le 0$ for all $S, L, I, E, Y, V, C^I > 0$. Similar to the previous theorems, one can show that $\frac{d\Phi_6}{dt} = 0$ at $(S, L, I, E, Y, V, C^I) = (S_6, L_6, I_6, E_6, Y_6, V_6, 0)$. The solutions of system (5) reach Υ_6' which contains elements with $E(t) = E_6$, $Y(t) = Y_6$, and then $\frac{dY(t)}{dt} = 0$. The fifth equation of system (5) becomes

$$
0 = \frac{dY(t)}{dt} = \psi\mathcal{H}_5 E_6 - \delta Y_6 - \mu_2 C^Y(t) Y_6,
$$

which yields $C^Y(t) = C^Y_6$ for all t, and hence $\Upsilon_6' = \{Ð_6\}$. Applying LLAS theorem, we get $Ð_6$ is GAS. □

Theorem 8 *If* $\Re_7 > 1$ *and* $\Re_8 > 1$, *then* $Ð_7$ *is GAS.*

Proof Consider

$$
\begin{aligned}
\Phi_7 &= \mathcal{P}S_7 F\left(\frac{S}{S_7}\right) + \lambda\mathcal{H}_3 L_7 F\left(\frac{L}{L_7}\right) + (\gamma+\lambda) I_7 F\left(\frac{I}{I_7}\right) + \frac{\mathcal{P}}{\varphi\mathcal{H}_4} E_7 F\left(\frac{E}{E_7}\right) \\
&+ \frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5} Y_7 F\left(\frac{Y}{Y_7}\right) + \frac{\mathcal{P}\vartheta_1 S_7}{\varepsilon} V_7 F\left(\frac{V}{V_7}\right) + \frac{\mu_1(\gamma+\lambda)}{\sigma_1} C^I_7 F\left(\frac{C^I}{C^I_7}\right) \\
&+ \frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5} C^Y_7 F\left(\frac{C^Y}{C^Y_7}\right) \\
&+ \vartheta_1\lambda\mathcal{H}_3(1-\beta) S_7 V_7 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \int_{t-\ell}^{t} F\left(\frac{S(\varkappa)V(\varkappa)}{S_7 V_7}\right) d\varkappa\, d\ell \\
&+ \vartheta_2\lambda\mathcal{H}_3(1-\beta) S_7 I_7 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \int_{t-\ell}^{t} F\left(\frac{S(\varkappa)I(\varkappa)}{S_7 I_7}\right) d\varkappa\, d\ell \\
&+ \vartheta_1\beta(\gamma+\lambda) S_7 V_7 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \int_{t-\ell}^{t} F\left(\frac{S(\varkappa)V(\varkappa)}{S_7 V_7}\right) d\varkappa\, d\ell \\
&+ \vartheta_2\beta(\gamma+\lambda) S_7 I_7 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \int_{t-\ell}^{t} F\left(\frac{S(\varkappa)I(\varkappa)}{S_7 I_7}\right) d\varkappa\, d\ell \\
&+ \lambda(\gamma+\lambda) L_7 \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell) \int_{t-\ell}^{t} F\left(\frac{L(\varkappa)}{L_7}\right) d\varkappa\, d\ell
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_4}\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t}F\left(\frac{S(\varkappa)Y(\varkappa)}{S_7Y_7}\right)d\varkappa\,d\boldsymbol{\ell}\\
&+\frac{\mathcal{P}(\psi+\omega)E_7}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t}F\left(\frac{E(\varkappa)}{E_7}\right)d\varkappa\,d\boldsymbol{\ell}\\
&+\frac{b\mathcal{P}\vartheta_1 S_7 I_7}{\varepsilon}\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})\int_{t-\boldsymbol{\ell}}^{t}F\left(\frac{I(\varkappa)}{I_7}\right)d\varkappa\,d\boldsymbol{\ell}.
\end{aligned}
$$

Calculate $\frac{d\Phi_7}{dt}$ as follows:

$$
\begin{aligned}
\frac{d\Phi_7}{dt}={}&\mathcal{P}\left(1-\frac{S_7}{S}\right)(\eta-\varrho S-\vartheta_1 SV-\vartheta_2 SI-\vartheta_3 SY)+\lambda\mathcal{H}_3\left(1-\frac{L_7}{L}\right)\\
&\times\left[(1-\beta)\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\bigl\{\vartheta_1 V(t-\boldsymbol{\ell})+\vartheta_2 I(t-\boldsymbol{\ell})\bigr\}\,d\boldsymbol{\ell}-(\lambda+\gamma)L\right]\\
&+(\gamma+\lambda)\left(1-\frac{I_7}{I}\right)\left[\beta\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})S(t-\boldsymbol{\ell})\bigl\{\vartheta_1 V(t-\boldsymbol{\ell})+\vartheta_2 I(t-\boldsymbol{\ell})\bigr\}\,d\boldsymbol{\ell}\right.\\
&\left.+\lambda\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})L(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}-aI-\mu_1 C^I I\right]\\
&+\frac{\mathcal{P}}{\varphi\mathcal{H}_4}\left(1-\frac{E_7}{E}\right)\left[\varphi\vartheta_3\int_0^{\kappa_4}\bar{\mathcal{H}}_4(\boldsymbol{\ell})S(t-\boldsymbol{\ell})Y(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}+rY-(\psi+\omega)E\right]\\
&+\frac{\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(1-\frac{Y_7}{Y}\right)\left[\psi\int_0^{\kappa_5}\bar{\mathcal{H}}_5(\boldsymbol{\ell})E(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}-\delta Y-\mu_2 C^Y Y\right]\\
&+\frac{\mathcal{P}\vartheta_1 S_7}{\varepsilon}\left(1-\frac{V_7}{V}\right)\left[b\int_0^{\kappa_6}\bar{\mathcal{H}}_6(\boldsymbol{\ell})I(t-\boldsymbol{\ell})\,d\boldsymbol{\ell}-\varepsilon V\right]\\
&+\frac{\mu_1(\gamma+\lambda)}{\sigma_1}\left(1-\frac{C^I_7}{C^I}\right)\left(\sigma_1 C^I I-\pi_1 C^I\right)\\
&+\frac{\mu_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}\left(1-\frac{C^Y_7}{C^Y}\right)\left(\sigma_2 C^Y Y-\pi_2 C^Y\right)\\
&+\vartheta_1\lambda\mathcal{H}_3(1-\beta)S_7V_7\\
&\times\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\left[\frac{SV}{S_7V_7}-\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{S_7V_7}+\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)\right]d\boldsymbol{\ell}\\
&+\vartheta_2\lambda\mathcal{H}_3(1-\beta)S_7I_7\\
&\times\int_0^{\kappa_1}\bar{\mathcal{H}}_1(\boldsymbol{\ell})\left[\frac{SI}{S_7I_7}-\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_7I_7}+\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)\right]d\boldsymbol{\ell}\\
&+\vartheta_1\beta(\gamma+\lambda)S_7V_7\\
&\times\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{SV}{S_7V_7}-\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{S_7V_7}+\ln\left(\frac{S(t-\boldsymbol{\ell})V(t-\boldsymbol{\ell})}{SV}\right)\right]d\boldsymbol{\ell}\\
&+\vartheta_2\beta(\gamma+\lambda)S_7I_7\\
&\times\int_0^{\kappa_2}\bar{\mathcal{H}}_2(\boldsymbol{\ell})\left[\frac{SI}{S_7I_7}-\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{S_7I_7}+\ln\left(\frac{S(t-\boldsymbol{\ell})I(t-\boldsymbol{\ell})}{SI}\right)\right]d\boldsymbol{\ell}\\
&+\lambda(\gamma+\lambda)L_7\int_0^{\kappa_3}\bar{\mathcal{H}}_3(\boldsymbol{\ell})\left[\frac{L}{L_7}-\frac{L(t-\boldsymbol{\ell})}{L_7}+\ln\left(\frac{L(t-\boldsymbol{\ell})}{L}\right)\right]d\boldsymbol{\ell}
\end{aligned}
$$

$$+\frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_4}\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\ell)\left[\frac{SY}{S_7Y_7}-\frac{S(t-\ell)Y(t-\ell)}{S_7Y_7}+\ln\left(\frac{S(t-\ell)Y(t-\ell)}{SY}\right)\right]d\ell$$
$$+\frac{\mathcal{P}(\psi+\omega)E_7}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\ell)\left[\frac{E}{E_7}-\frac{E(t-\ell)}{E_7}+\ln\left(\frac{E(t-\ell)}{E}\right)\right]d\ell$$
$$+\frac{b\mathcal{P}\vartheta_1 S_7 I_7}{\varepsilon}\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\ell)\left[\frac{I}{I_7}-\frac{I(t-\ell)}{I_7}+\ln\left(\frac{I(t-\ell)}{I}\right)\right]d\ell. \quad (27)$$

Summing the terms of Eq. (27), we get

$$\frac{d\Phi_7}{dt}=\mathcal{P}\left(1-\frac{S_7}{S}\right)(\eta-\varrho S)+\mathcal{P}\vartheta_2 S_7 I+\mathcal{P}\vartheta_3 S_7 Y-\vartheta_1\lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\ell)$$
$$\times\frac{S(t-\ell)V(t-\ell)L_7}{L}d\ell-\vartheta_2\lambda\mathcal{H}_3(1-\beta)\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\ell)\frac{S(t-\ell)I(t-\ell)L_7}{L}d\ell$$
$$+\lambda\mathcal{H}_3(\lambda+\gamma)L_7-a(\lambda+\gamma)I-\vartheta_1\beta(\gamma+\lambda)\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\ell)\frac{S(t-\ell)V(t-\ell)I_7}{I}d\ell$$
$$-\vartheta_2\beta(\gamma+\lambda)\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\ell)\frac{S(t-\ell)I(t-\ell)I_7}{I}d\ell$$
$$-\lambda(\lambda+\gamma)\int_0^{\kappa 3}\bar{\mathcal{H}}_3(\ell)\frac{L(t-\ell)I_7}{I}d\ell+a(\lambda+\gamma)I_7+\mu_1(\lambda+\gamma)C^I I_7$$
$$+\frac{\mathcal{P}r}{\varphi\mathcal{H}_4}Y-\frac{\mathcal{P}\vartheta_3}{\mathcal{H}_4}\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\ell)\frac{S(t-\ell)Y(t-\ell)E_7}{E}d\ell-\frac{\mathcal{P}r}{\varphi\mathcal{H}_4}\frac{YE_7}{E}$$
$$+\frac{\mathcal{P}(\psi+\omega)}{\varphi\mathcal{H}_4}E_7-\frac{\mathcal{P}\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y-\frac{\mathcal{P}(\psi+\omega)}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\ell)\frac{E(t-\ell)Y_7}{Y}d\ell$$
$$+\frac{\mathcal{P}\delta(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}Y_7+\frac{\mu_2\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y Y_7-\frac{b\mathcal{P}\vartheta_1 S_7}{\varepsilon}\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\ell)\frac{I(t-\ell)V_7}{V}d\ell$$
$$+\mathcal{P}\vartheta_1 S_7 V_7-\frac{\mu_1\pi_1(\lambda+\gamma)}{\sigma_1}C^I-\mu_1(\lambda+\gamma)C_7^I I+\frac{\mu_1\pi_1(\lambda+\gamma)}{\sigma_1}C_7^I$$
$$-\frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C^Y-\frac{\mu_2\mathcal{P}(\psi+\omega)}{\varphi\psi\mathcal{H}_4\mathcal{H}_5}C_7^Y Y+\frac{\mu_2\pi_2\mathcal{P}(\psi+\omega)}{\sigma_2\varphi\psi\mathcal{H}_4\mathcal{H}_5}C_7^Y$$
$$+\lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_7 V_7\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\ell)\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right)d\ell$$
$$+\lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_7 I_7\int_0^{\kappa 1}\bar{\mathcal{H}}_1(\ell)\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right)d\ell$$
$$+\beta(\gamma+\lambda)\vartheta_1 S_7 V_7\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\ell)\ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right)d\ell$$
$$+\beta(\gamma+\lambda)\vartheta_2 S_7 I_7\int_0^{\kappa 2}\bar{\mathcal{H}}_2(\ell)\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right)d\ell$$
$$+\lambda(\gamma+\lambda)L_7\int_0^{\kappa 3}\bar{\mathcal{H}}_3(\ell)\ln\left(\frac{L(t-\ell)}{L}\right)d\ell$$
$$+\frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_4}\int_0^{\kappa 4}\bar{\mathcal{H}}_4(\ell)\ln\left(\frac{S(t-\ell)Y(t-\ell)}{SY}\right)d\ell$$
$$+\frac{\mathcal{P}(\psi+\omega)E_7}{\varphi\mathcal{H}_4\mathcal{H}_5}\int_0^{\kappa 5}\bar{\mathcal{H}}_5(\ell)\ln\left(\frac{E(t-\ell)}{E}\right)d\ell+\frac{b\mathcal{P}\vartheta_1 S_7\mathcal{H}_6}{\varepsilon}I$$
$$+\frac{b\mathcal{P}\vartheta_1 S_7 I_7}{\varepsilon}\int_0^{\kappa 6}\bar{\mathcal{H}}_6(\ell)\ln\left(\frac{I(t-\ell)}{I}\right)d\ell.$$

Using the steady state conditions for $Đ_7$

$$\eta = \varrho S_7 + \vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7 + \vartheta_3 S_7 Y_7,$$
$$\mathcal{H}_1(1-\beta)(\vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7) = (\lambda + \gamma) L_7,$$
$$\beta \mathcal{H}_2(\vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7) + \lambda \mathcal{H}_3 L_7 = \left(a + \mu_1 C_7^I\right) I_7,$$
$$I_7 = \frac{\pi_1}{\sigma_1}, \qquad Y_7 = \frac{\pi_2}{\sigma_2}, \qquad V_7 = \frac{b\mathcal{H}_6 I_7}{\varepsilon},$$
$$\vartheta_3 S_7 Y_7 + \frac{rY_7}{\varphi \mathcal{H}_4} = \frac{(\psi + \omega)E_7}{\varphi \mathcal{H}_4} = \frac{\delta(\psi + \omega)}{\varphi \psi \mathcal{H}_4 \mathcal{H}_5} Y_7 + \frac{\mu_2(\psi + \omega)}{\varphi \psi \mathcal{H}_4 \mathcal{H}_5} C_7^Y Y_7,$$

we get

$$\mathcal{P}(\vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7) = (\lambda + \gamma)\left(a + \mu_1 C_7^I\right) I_7.$$

Moreover, we get

$$\begin{aligned}
\frac{d\Phi_7}{dt} &= \mathcal{P}\left(1 - \frac{S_7}{S}\right)(\varrho S_7 - \varrho S) + \mathcal{P}(\vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7 + \vartheta_3 S_7 Y_7)\left(1 - \frac{S_7}{S}\right) \\
&\quad - \lambda \mathcal{H}_3(1-\beta)\vartheta_1 S_7 V_7 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \frac{S(t-\ell)V(t-\ell)L_7}{S_7 V_7 L}\, d\ell \\
&\quad - \lambda \mathcal{H}_3(1-\beta)\vartheta_2 S_7 I_7 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \frac{S(t-\ell)I(t-\ell)L_7}{S_7 I_7 L}\, d\ell \\
&\quad + \lambda \mathcal{H}_1 \mathcal{H}_3(1-\beta)(\vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7) \\
&\quad - \beta(\gamma + \lambda)\vartheta_1 S_7 V_7 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \frac{S(t-\ell)V(t-\ell)I_7}{S_7 V_7 I}\, d\ell \\
&\quad - \beta(\gamma + \lambda)\vartheta_2 S_7 I_7 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \frac{S(t-\ell)I(t-\ell)}{S_7 I}\, d\ell \\
&\quad - \lambda \mathcal{H}_1(1-\beta)(\vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell) \frac{L(t-\ell)I_7}{L_7 I}\, d\ell \\
&\quad + \mathcal{P}(\vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7) - \frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_4} \int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell) \frac{S(t-\ell)Y(t-\ell)E_7}{S_7 Y_7 E}\, d\ell \\
&\quad - \frac{\mathcal{P}rY_7}{\varphi \mathcal{H}_4} \frac{YE_7}{Y_7 E} + \mathcal{P}\vartheta_3 S_7 Y_7 + \frac{\mathcal{P}rY_7}{\varphi \mathcal{H}_4} \\
&\quad - \left(\frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_5} + \frac{\mathcal{P}rY_7}{\varphi \mathcal{H}_4 \mathcal{H}_5}\right) \int_0^{\kappa_5} \bar{\mathcal{H}}_5(\ell) \frac{E(t-\ell)Y_7}{E_7 Y}\, d\ell \\
&\quad + \mathcal{P}\vartheta_3 S_7 Y_7 + \frac{\mathcal{P}rY_7}{\varphi \mathcal{H}_4} Y_7 - \frac{\mathcal{P}\vartheta_1 S_7 V_7}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell) \frac{I(t-\ell)V_7}{I_7 V}\, d\ell \\
&\quad + \mathcal{P}\vartheta_1 S_7 V_7 + \lambda \mathcal{H}_3(1-\beta)\vartheta_1 S_7 V_7 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) d\ell \\
&\quad + \lambda \mathcal{H}_3(1-\beta)\vartheta_2 S_7 I_7 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) \ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right) d\ell \\
&\quad + \beta(\gamma + \lambda)\vartheta_1 S_7 V_7 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) \ln\left(\frac{S(t-\ell)V(t-\ell)}{SV}\right) d\ell
\end{aligned}$$

$$
\begin{aligned}
&+\beta(\gamma+\lambda)\vartheta_2 S_7 I_7 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\ln\left(\frac{S(t-\ell)I(t-\ell)}{SI}\right)d\ell\\
&+\lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_7 V_7+\vartheta_2 S_7 I_7)\int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell)\ln\left(\frac{L(t-\ell)}{L}\right)d\ell\\
&+\frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_4}\int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell)\ln\left(\frac{S(t-\ell)Y(t-\ell)}{SY}\right)d\ell\\
&+\left(\frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_5}+\frac{\mathcal{P}rY_7}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5} \bar{\mathcal{H}}_5(\ell)\ln\left(\frac{E(t-\ell)}{E}\right)d\ell\\
&+\frac{\mathcal{P}\vartheta_1 S_7 V_7}{\mathcal{H}_6}\int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell)\ln\left(\frac{I(t-\ell)}{I}\right)d\ell.
\end{aligned}
$$

Using the equalities given by (10) and (11) in case of $n = m = 7$, we get

$$
\begin{aligned}
\frac{d\Phi_7}{dt} ={}& -\varrho\mathcal{P}\frac{(S-S_7)^2}{S}-\mathcal{P}(\vartheta_1 S_7 V_7+\vartheta_2 S_7 I_7+\vartheta_3 S_7 Y_7)\left[\frac{S_7}{S}-1-\ln\left(\frac{S_7}{S}\right)\right]\\
&-\lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_7 V_7\\
&\times\int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\left[\frac{S(t-\ell)V(t-\ell)L_7}{S_7 V_7 L}-1-\ln\left(\frac{S(t-\ell)V(t-\ell)L_7}{S_7 V_7 L}\right)\right]d\ell\\
&-\lambda\mathcal{H}_3(1-\beta)\vartheta_2 S_7 I_7\\
&\times\int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)\left[\frac{S(t-\ell)I(t-\ell)L_7}{S_7 I_7 L}-1-\ln\left(\frac{S(t-\ell)I(t-\ell)L_7}{S_7 I_7 L}\right)\right]d\ell\\
&-\beta(\gamma+\lambda)\vartheta_1 S_7 V_7\\
&\times\int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\left[\frac{S(t-\ell)V(t-\ell)I_7}{S_7 V_7 I}-1-\ln\left(\frac{S(t-\ell)V(t-\ell)I_7}{S_7 V_7 I}\right)\right]d\ell\\
&-\beta(\gamma+\lambda)\vartheta_2 S_7 I_7\int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell)\left[\frac{S(t-\ell)I(t-\ell)}{S_7 I}-1-\ln\left(\frac{S(t-\ell)I(t-\ell)}{S_7 I}\right)\right]d\ell\\
&-\lambda\mathcal{H}_1(1-\beta)(\vartheta_1 S_7 V_7+\vartheta_2 S_7 I_7)\\
&\times\int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell))\left[\frac{L(t-\ell)I_7}{L_7 I}-1-\ln\left(\frac{L(t-\ell)I_7}{L_7 I}\right)\right]d\ell\\
&-\frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_4}\int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell)\left[\frac{S(t-\ell)Y(t-\ell)E_7}{S_7 Y_7 E}-1-\ln\left(\frac{S(t-\ell)Y(t-\ell)E_7}{S_7 Y_7 E}\right)\right]d\ell\\
&-\frac{\mathcal{P}rY_7}{\varphi\mathcal{H}_4}\left[\frac{YE_7}{Y_7 E}-1-\ln\left(\frac{YE_7}{Y_7 E}\right)\right]\\
&-\left(\frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_5}+\frac{\mathcal{P}rY_7}{\varphi\mathcal{H}_4\mathcal{H}_5}\right)\int_0^{\kappa_5} \bar{\mathcal{H}}_5(\ell)\left[\frac{E(t-\ell)Y_7}{E_7 Y}-1-\ln\left(\frac{E(t-\ell)Y_7}{E_7 Y}\right)\right]d\ell\\
&-\frac{\mathcal{P}\vartheta_1 S_7 V_7}{\mathcal{H}_6}\int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell)\left[\frac{I(t-\ell)V_7}{I_7 V}-1-\ln\left(\frac{I(t-\ell)V_7}{I_7 V}\right)\right]d\ell. \qquad (28)
\end{aligned}
$$

Therefore, Eq. (28) becomes

$$
\begin{aligned}
\frac{d\Phi_7}{dt} ={}& -\varrho\mathcal{P}\frac{(S-S_7)^2}{S}-\mathcal{P}(\vartheta_1 S_7 V_7+\vartheta_2 S_7 I_7+\vartheta_3 S_7 Y_7)F\left(\frac{S_7}{S}\right)\\
&-\lambda\mathcal{H}_3(1-\beta)\vartheta_1 S_7 V_7\int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell)F\left(\frac{S(t-\ell)V(t-\ell)L_7}{S_7 V_7 L}\right)d\ell
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda \mathcal{H}_3(1-\beta)\vartheta_2 S_7 I_7 \int_0^{\kappa_1} \bar{\mathcal{H}}_1(\ell) F\left(\frac{S(t-\ell)I(t-\ell)L_7}{S_7 I_7 L}\right) d\ell \\
&- \beta(\gamma+\lambda)\vartheta_1 S_7 V_7 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) F\left(\frac{S(t-\ell)V(t-\ell)I_7}{S_7 V_7 I}\right) d\ell \\
&- \beta(\gamma+\lambda)\vartheta_2 S_7 I_7 \int_0^{\kappa_2} \bar{\mathcal{H}}_2(\ell) F\left(\frac{S(t-\ell)I(t-\ell)}{S_7 I}\right) d\ell \\
&- \lambda \mathcal{H}_1(1-\beta)(\vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7) \int_0^{\kappa_3} \bar{\mathcal{H}}_3(\ell)) F\left(\frac{L(t-\ell)I_7}{L_7 I}\right) d\ell \\
&- \frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_4} \int_0^{\kappa_4} \bar{\mathcal{H}}_4(\ell) F\left(\frac{S(t-\ell)Y(t-\ell)E_7}{S_7 Y_7 E}\right) d\ell - \frac{\mathcal{P} r Y_7}{\varphi \mathcal{H}_4} F\left(\frac{YE_7}{Y_7 E}\right) \\
&- \left(\frac{\mathcal{P}\vartheta_3 S_7 Y_7}{\mathcal{H}_5} + \frac{\mathcal{P} r Y_7}{\varphi \mathcal{H}_4 \mathcal{H}_5}\right) \int_0^{\kappa_5} \bar{\mathcal{H}}_5(\ell) F\left(\frac{E(t-\ell)Y_7}{E_7 Y}\right) d\ell \\
&- \frac{\mathcal{P}\vartheta_1 S_7 V_7}{\mathcal{H}_6} \int_0^{\kappa_6} \bar{\mathcal{H}}_6(\ell) F\left(\frac{I(t-\ell)V_7}{I_7 V}\right) d\ell.
\end{aligned}
$$

Hence, $\frac{d\Phi_7}{dt} \le 0$ for all $S, L, I, E, Y, V > 0$. Similar to the previous theorems, one can show that $\frac{d\Phi_7}{dt} = 0$ when $(S, L, I, E, Y, V) = (S_7, L_7, I_7, E_7, Y_7, V_7)$. The solutions of system (5) converge to Υ_7' which includes elements with $(S, L, I, E, Y, V)(t) = (S_7, L_7, I_7, E_7, Y_7, V_7)$. Then $\frac{dI(t)}{dt} = \frac{dY(t)}{dt} = 0$. The third and fifth equations of system (5) become

$$
\begin{aligned}
0 &= \frac{dI(t)}{dt} = \beta \mathcal{H}_2(\vartheta_1 S_7 V_7 + \vartheta_2 S_7 I_7) + \lambda \mathcal{H}_3 L_7 - a I_7 - \mu_1 C^I(t) I_7, \\
0 &= \frac{dY(t)}{dt} = \psi \mathcal{H}_5 E_7 - \delta Y_7 - \mu_2 C^Y(t) Y_7,
\end{aligned}
$$

which yield $C^I(t) = C_7^I$ and $C^Y(t) = C_7^Y$ for all t, and hence $\Upsilon_7' = \{Đ_7\}$. Applying LLAS theorem, we get $Đ_7$ is GAS. □

6 Numerical simulations

In this section, we perform numerical simulations to illustrate the results of Theorems 1–8. Moreover, we study the influence of time delays on the dynamical behavior of the system. Let us choose a Dirac delta function $D(\cdot)$ as a special form of the kernel $\Lambda_i(\cdot)$ as follows:

$$
\Lambda_i(x) = D(x - \ell_i), \quad \ell_i \in [0, \kappa_i], i = 1, 2, \dots, 6.
$$

Let $\kappa_i \to \infty$, then we get

$$
\begin{aligned}
&\int_0^\infty \Lambda_j(\varsigma)\, d\varsigma = 1, \\
&\mathcal{H}_j = \int_0^\infty D(\varsigma - \ell_j) e^{-\hbar_j \varsigma}\, d\varsigma = e^{-\hbar_j \ell_j}, \quad j = 1, 2, \dots, 6.
\end{aligned}
$$

Thus, model (4) reduces to

$$\begin{cases} \frac{dS(t)}{dt} = \eta - \varrho S(t) - \vartheta_1 S(t)V(t) - \vartheta_2 S(t)I(t) - \vartheta_3 S(t)Y(t), \\ \frac{dL(t)}{dt} = (1-\beta)e^{-\hbar_1 \ell_1} S(t-\ell_1)[\vartheta_1 V(t-\ell_1) + \vartheta_2 I(t-\ell_1)] - (\lambda+\gamma)L(t), \\ \frac{dI(t)}{dt} = \beta e^{-\hbar_2 \ell_2} S(t-\ell_2)[\vartheta_1 V(t-\ell_2) + \vartheta_2 I(t-\ell_2)] \\ \qquad + \lambda e^{-\hbar_3 \ell_3} L(t-\ell_3) - aI(t) - \mu_1 C^I(t)I(t), \\ \frac{dE(t)}{dt} = \varphi \vartheta_3 e^{-\hbar_4 \ell_4} S(t-\ell_4)Y(t-\ell_4) + \mathcal{K} r^* Y(t) - (\psi+\omega)E(t), \\ \frac{dY(t)}{dt} = \psi e^{-\hbar_5 \ell_5} E(t-\ell_5) + (1-\mathcal{K}) r^* Y(t) - \delta^* Y(t) - \mu_2 C^Y(t)Y(t), \\ \frac{dV(t)}{dt} = b e^{-\hbar_6 \ell_6} I(t-\ell_6) - \varepsilon V(t), \\ \frac{dC^I(t)}{dt} = \sigma_1 C^I(t)I(t) - \pi_1 C^I(t), \\ \frac{dC^Y(t)}{dt} = \sigma_2 C^Y(t)Y(t) - \pi_2 C^Y(t). \end{cases} \tag{29}$$

For model (29), the threshold parameters are given by

$$\begin{aligned}
\Re_1 &= \frac{\mathcal{P} S_0 (b\vartheta_1 e^{-\hbar_6 \ell_6} + \varepsilon \vartheta_2)}{a\varepsilon(\gamma+\lambda)}, \qquad \Re_2 = \frac{\varphi \vartheta_3 \psi e^{-(\hbar_4 \ell_4 + \hbar_5 \ell_5)} S_0}{(\delta - r e^{-\hbar_5 \ell_5})\psi + \delta\omega}, \\
\Re_3 &= \frac{\sigma_1 \eta \mathcal{P}(b\vartheta_1 e^{-\hbar_6 \ell_6} + \varepsilon\vartheta_2)}{a(\gamma+\lambda)[\pi_1(b\vartheta_1 e^{-\hbar_6 \ell_6} + \varepsilon\vartheta_2) + \varrho\varepsilon\sigma_1]}, \\
\Re_4 &= \frac{\sigma_2 \eta \varphi \vartheta_3 \psi e^{-(\hbar_4 \ell_4 + \hbar_5 \ell_5)}}{(\pi_2 \vartheta_3 + \varrho\sigma_2)[(\delta - r e^{-\hbar_5 \ell_5})\psi + \delta\omega]}, \\
\Re_5 &= \frac{\eta \varphi \varepsilon \vartheta_3 \sigma_1 \psi e^{-(\hbar_4 \ell_4 + \hbar_5 \ell_5)}}{[(\delta - r e^{-\hbar_5 \ell_5})\psi + \delta\omega][\pi_1(b\vartheta_1 e^{-\hbar_6 \ell_6} + \varepsilon\vartheta_2) + \varrho\varepsilon\sigma_1]}, \\
\Re_6 &= \frac{\eta \sigma_2 \mathcal{P}(b\vartheta_1 e^{-\hbar_6 \ell_6} + \varepsilon\vartheta_2)}{a\varepsilon(\gamma+\lambda)(\pi_2\vartheta_3 + \varrho\sigma_2)}, \\
\Re_7 &= \frac{\sigma_1 \sigma_2 \eta \mathcal{P}(b\vartheta_1 e^{-\hbar_6 \ell_6} + \varepsilon\vartheta_2)}{a(\gamma+\lambda)[\pi_1\sigma_2(b\vartheta_1 e^{-\hbar_6 \ell_6} + \varepsilon\vartheta_2) + \varepsilon\sigma_1(\pi_2\vartheta_3 + \varrho\sigma_2)]}, \\
\Re_8 &= \frac{\varepsilon \vartheta_3 \eta \sigma_1 \sigma_2 \varphi \psi e^{-(\hbar_4 \ell_4 + \hbar_5 \ell_5)}}{[\pi_1\sigma_2(b\vartheta_1 e^{-\hbar_6 \ell_6} + \varepsilon\vartheta_2) + \varepsilon\sigma_1(\pi_2\vartheta_3 + \varrho\sigma_2)][(\delta - r e^{-\hbar_5 \ell_5})\psi + \delta\omega]},
\end{aligned} \tag{30}$$

where

$$\mathcal{P} = \lambda e^{-(\hbar_1 \ell_1 + \hbar_3 \ell_3)}(1-\beta) + \beta e^{-\hbar_2 \ell_2}(\gamma+\lambda). \tag{31}$$

To solve system (29) numerically, we fix the values of some parameters (see Table 1) and the others will be varied.

Table 1 The data of model (29)

Parameter	Value	Parameter	Value	Parameter	Value
η	10	b	5	ω	0.03
ϱ	0.01	π_1	0.1	ψ	0.003
$\vartheta_i, i = 1,2,3$	Varied	π_2	0.1	$\hbar_1$	0.2
β	0.7	μ_1	0.2	$\hbar_2$	0.3
a	0.5	μ_2	0.2	$\hbar_3$	0.4
φ	0.2	ε	2	$\hbar_4$	0.5
$\mathcal{K}$	0.9	γ	0.1	$\hbar_5$	0.6
r^*	0.008	$\sigma_i, i = 1,2$	Varied	$\hbar_6$	0.9
δ^*	0.05	λ	0.2	$\xi_i, i = 1,2,\dots,6$	Varied

6.1 Stability of the steady states

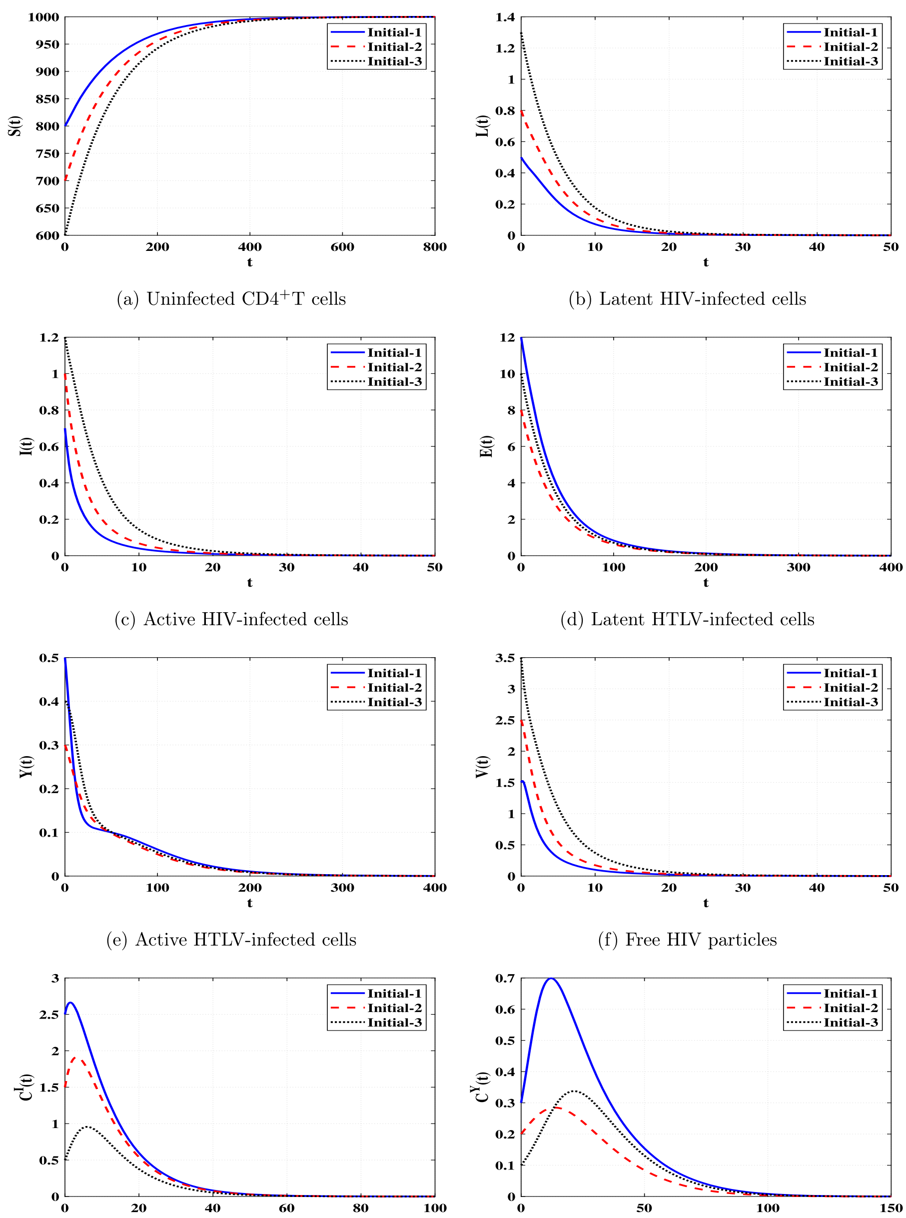
In this subsection, we select the delay parameters as $\ell_1 = 1$, $\ell_2 = 0.8$, $\ell_3 = 0.6$, $\ell_4 = 0.4$, $\ell_5 = 0.2$, $\ell_6 = 0.1$. Besides, we choose the following three different initial conditions for system (29):

Initial-1: $(S(\ell), L(\ell), I(\ell), E(\ell), Y(\ell), V(\ell), C^I(\ell), C^Y(\ell)) = (800, 0.5, 0.7, 12, 0.51.52.5, 0.3)$,

Initial-2: $(S(\ell), L(\ell), I(\ell), E(\ell), Y(\ell), V(\ell), C^I(\ell), C^Y(\ell)) = (700, 0.8, 1, 8, 0.3, 2.5, 1.5, 0.2)$,

Initial-3: $(S(\ell), L(\ell), I(\ell), E(\ell), Y(\ell), V(\ell), C^I(\ell), C^Y(\ell)) = (600, 1.31.2, 10, 0.4, 3.5, 0.5, 0.1)$,

where $\ell \in [-1, 0]$.



(a) Uninfected CD4$^+$T cells (b) Latent HIV-infected cells

(c) Active HIV-infected cells (d) Latent HTLV-infected cells

(e) Active HTLV-infected cells (f) Free HIV particles

(g) HIV-specific CTLs (h) HTLV-specific CTLs

Figure 1 Solutions of system (29) when $\Re_1 \leq 1$ and $\Re_2 \leq 1$

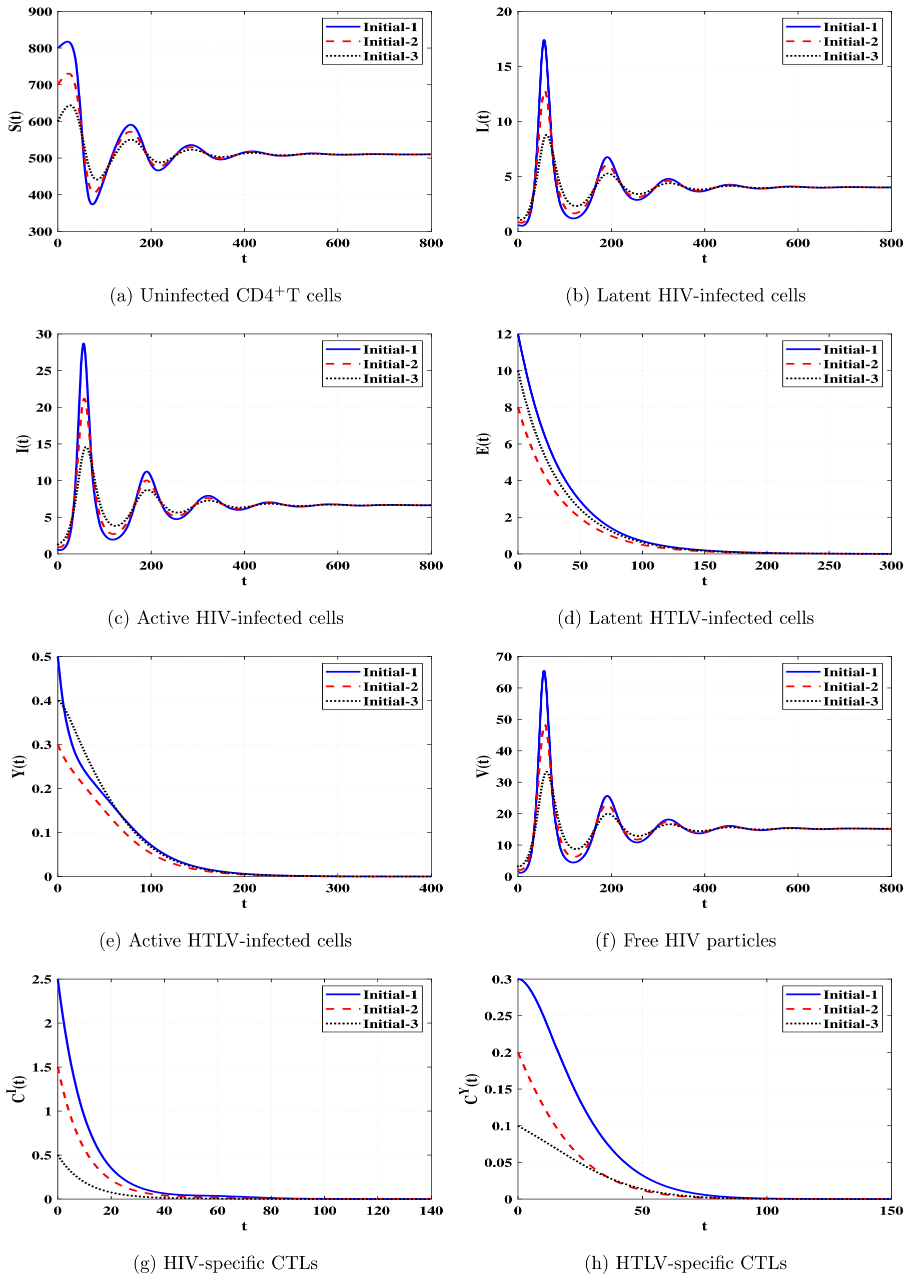
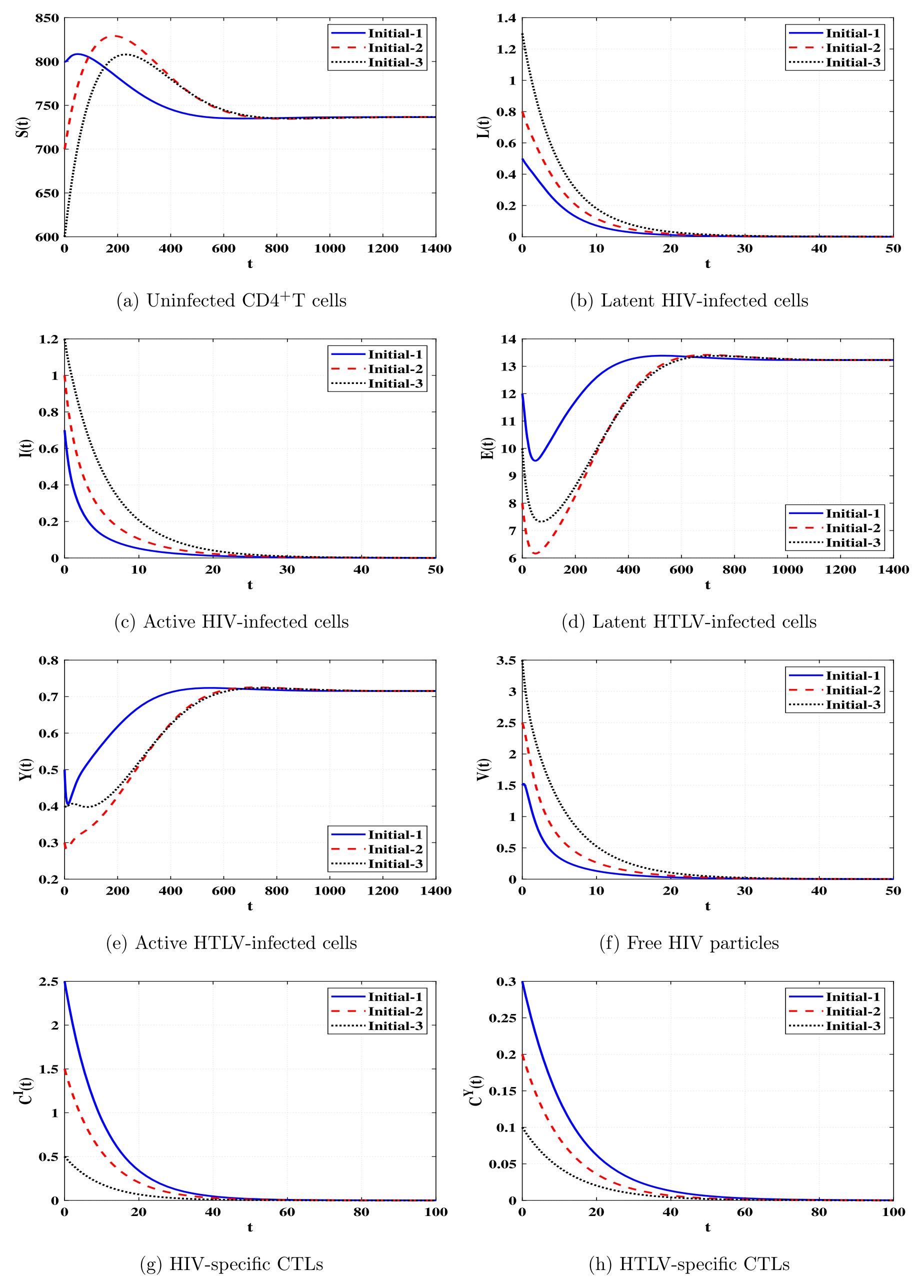


Figure 2 Solutions of system (29) when $\Re_1 > 1$, $\Re_2/\Re_1 \leq 1$, and $\Re_3 \leq 1$

Choosing different values of ϑ_1, ϑ_2, ϑ_3, ϑ_4, σ_1, and σ_2 under the above initial conditions leads to the following sets:

Set 1 (Stability of Đ_0*)*: $\vartheta_1 = 0.0002$, $\vartheta_2 = 0.0001$, $\vartheta_3 = 0.001$, $\sigma_1 = 0.3$, and $\sigma_2 = 0.5$. For this set of parameters, we have $\Re_1 = 0.76 < 1$ and $\Re_2 = 0.27 < 1$. Figure 1 illustrates that the trajectories starting different initials converge to the steady state $\text{Đ}_0 = (1000, 0, 0, 0, 0, 0, 0, 0)$. This supports the global stability result of Theorem 1. Here, a healthy state will be reached where both viruses are absent.



(a) Uninfected CD4$^+$T cells
(b) Latent HIV-infected cells
(c) Active HIV-infected cells
(d) Latent HTLV-infected cells
(e) Active HTLV-infected cells
(f) Free HIV particles
(g) HIV-specific CTLs
(h) HTLV-specific CTLs

Figure 3 Solutions of system (29) when $\Re_2 > 1$, $\Re_1/\Re_2 \leq 1$, and $\Re_4 \leq 1$

Set 2 (Stability of Đ$_1$*)*: $\vartheta_1 = 0.0005$, $\vartheta_2 = 0.0003$, $\vartheta_3 = 0.0007$, $\sigma_1 = 0.003$, and $\sigma_2 = 0.2$. With such a choice we get $\Re_2 = 0.19 < 1 < 1.96 = \Re_1$, $\Re_3 = 0.34 < 1$, and hence $\Re_2/\Re_1 = 0.1 < 1$. The steady state Đ$_1$ exists with Đ$_1 = (510.18, 4.01, 6.66, 0, 0, 15.21, 0, 0)$. The stability of Đ$_1$ given in Theorem 2 is shown in Fig. 2. This leads to the case where HIV monoinfection is chronic but with an ineffective CTL immunity.

Set 3 (Stability of Đ$_2$*)*: $\vartheta_1 = 0.0001$, $\vartheta_2 = 0.0003$, $\vartheta_3 = 0.005$, $\sigma_1 = 0.001$, and $\sigma_2 = 0.05$. Then we calculate $\Re_1 = 0.72 < 1 < 1.34 = \Re_2$, $\Re_4 = 0.67 < 1$, and then $\Re_1/\Re_2 = 0.54 < 1$ and Đ$_2 = (736.51, 0, 0, 13.23, 0.72, 0, 0, 0)$. We can see from Fig. 3 that the system's solutions tend

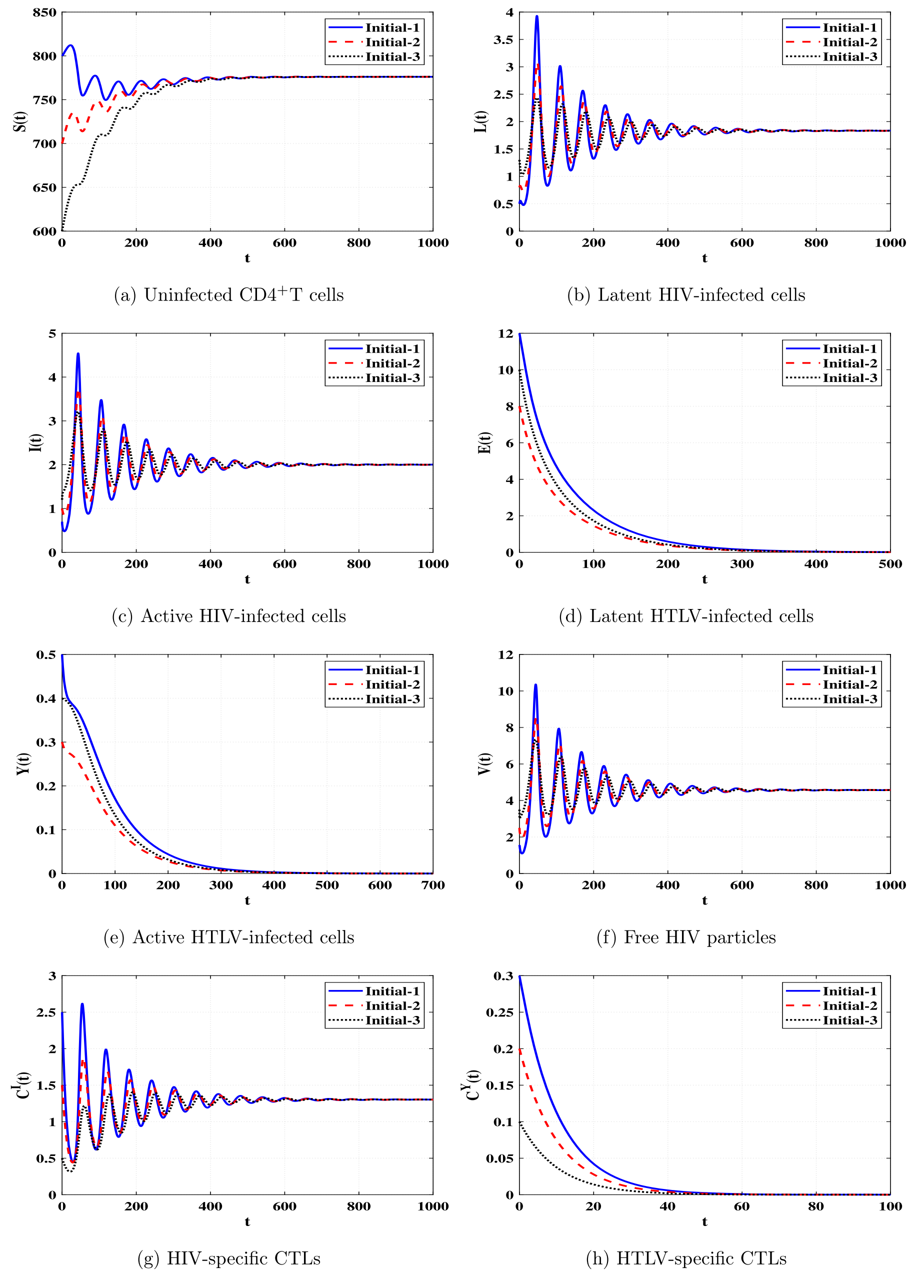
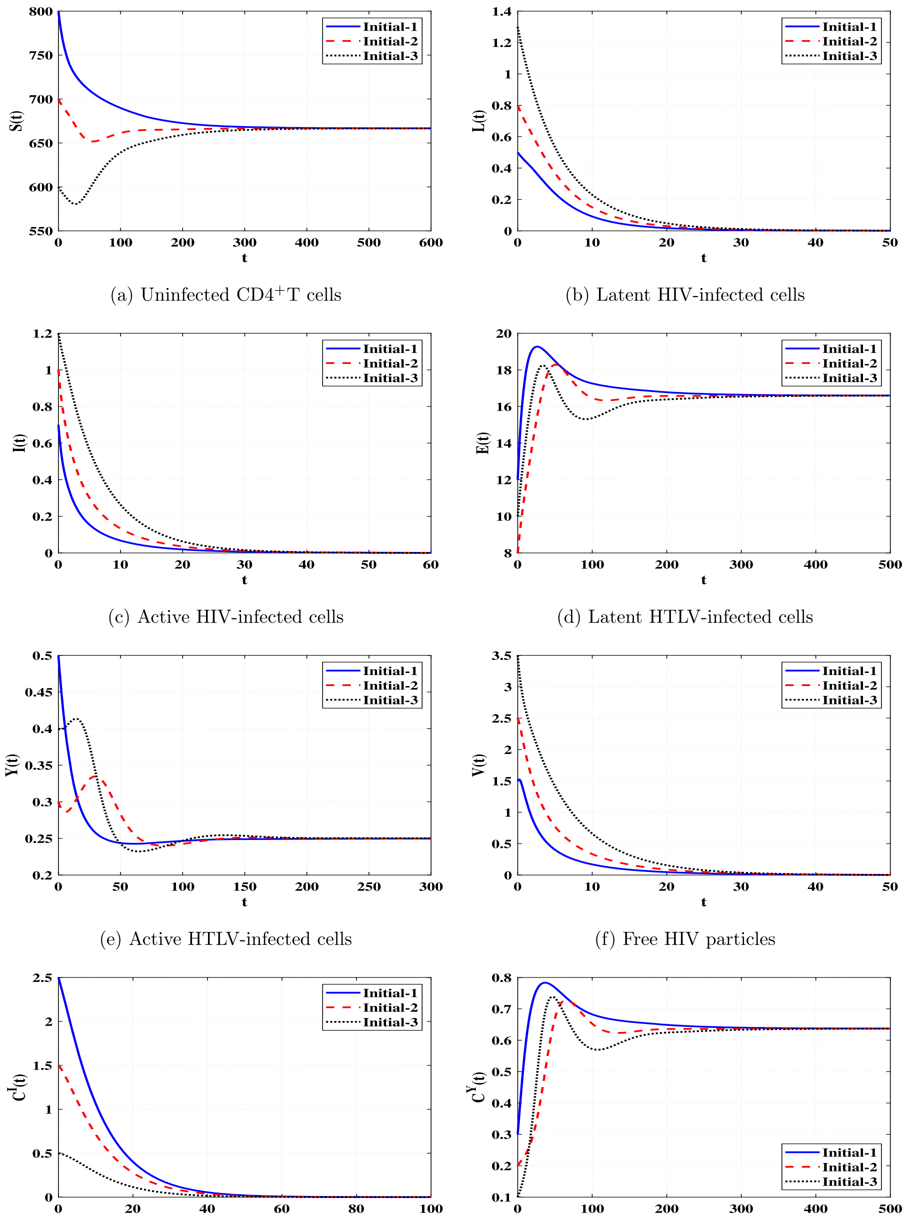


Figure 4 Solutions of system (29) when $\Re_3 > 1$ and $\Re_5 \leq 1$

to $Ð_2$, which is compatible with Theorem 3. This case means that an HTLV monoinfection is chronic with an ineffective CTL immunity.

Set 4 (Stability of $Ð_3$*)*: $\vartheta_1 = 0.0005$, $\vartheta_2 = 0.0003$, $\vartheta_3 = 0.002$, $\sigma_1 = 0.05$, and $\sigma_2 = 0.005$. Then we calculate $\Re_3 = 1.52 > 1$ and $\Re_5 = 0.42 < 1$. Figure 4 shows that the trajectories starting with different states tend to $Ð_3 = (776.11, 1.83, 2, 0, 0, 4.57, 1.30, 0)$. Therefore, $Ð_3$ is GAS, and this supports Theorem 4. Hence, an HIV monoinfection is chronic with effective HIV-specific CTL immunity.



(a) Uninfected CD4$^+$T cells (b) Latent HIV-infected cells

(c) Active HIV-infected cells (d) Latent HTLV-infected cells

(e) Active HTLV-infected cells (f) Free HIV particles

(g) HIV-specific CTLs (h) HTLV-specific CTLs

Figure 5 Solutions of system (29) when $\Re_4 > 1$ and $\Re_6 \le 1$

Set 5 (Stability of $Ð_4$*)*: $\vartheta_1 = 0.0002$, $\vartheta_2 = 0.0002$, $\vartheta_3 = 0.02$, $\sigma_1 = 0.07$, and $\sigma_2 = 0.4$. Then we calculate $\Re_4 = 3.57 > 1$ and $\Re_6 = 0.60 < 1$, and $Ð_4$ exists with $Ð_4 = (666.67, 0, 0, 16.59, 0.25, 0, 0, 0.64)$. We observe from Fig. 5 that the system's trajectories tend to $Ð_4$ and it is GAS. Here, an HTLV monoinfection is chronic with effective HTLV-specific CTL immunity.

Set 6 (Stability of $Ð_5$*)*: $\vartheta_1 = 0.001$, $\vartheta_2 = 0.0001$, $\vartheta_3 = 0.005$, $\sigma_1 = 0.15$, and $\sigma_2 = 0.01$. Then we calculate $\Re_5 = 1.15 > 1$, $\Re_8 = 0.22 < 1$, and $\Re_1/\Re_2 = 2.42 > 1$. The numerical solutions of the system drawn in Fig. 6 confirm that $Ð_5 = (736.51, 0.96, 0.67, 7.35, 0.40, 1.52, 3.47, 0)$

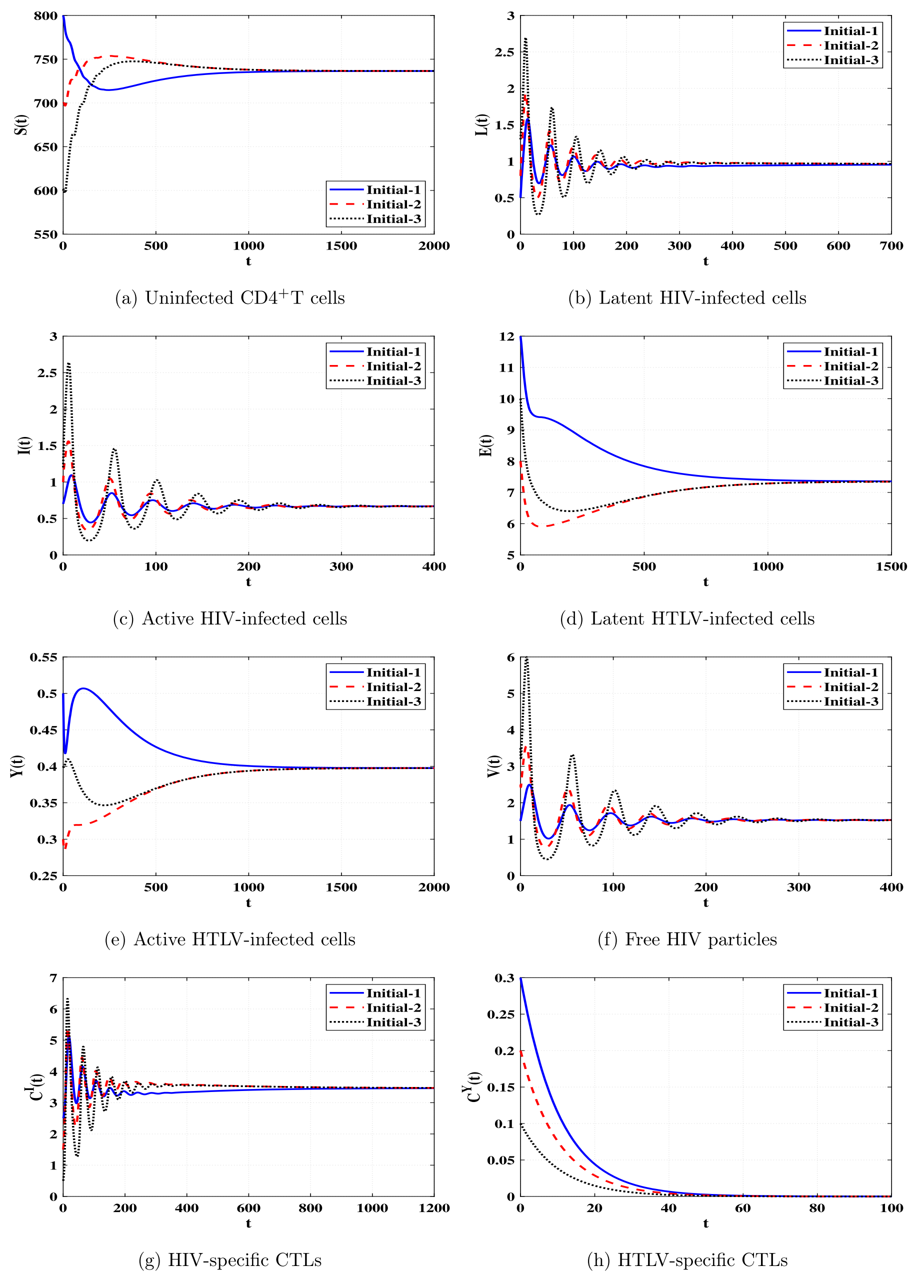
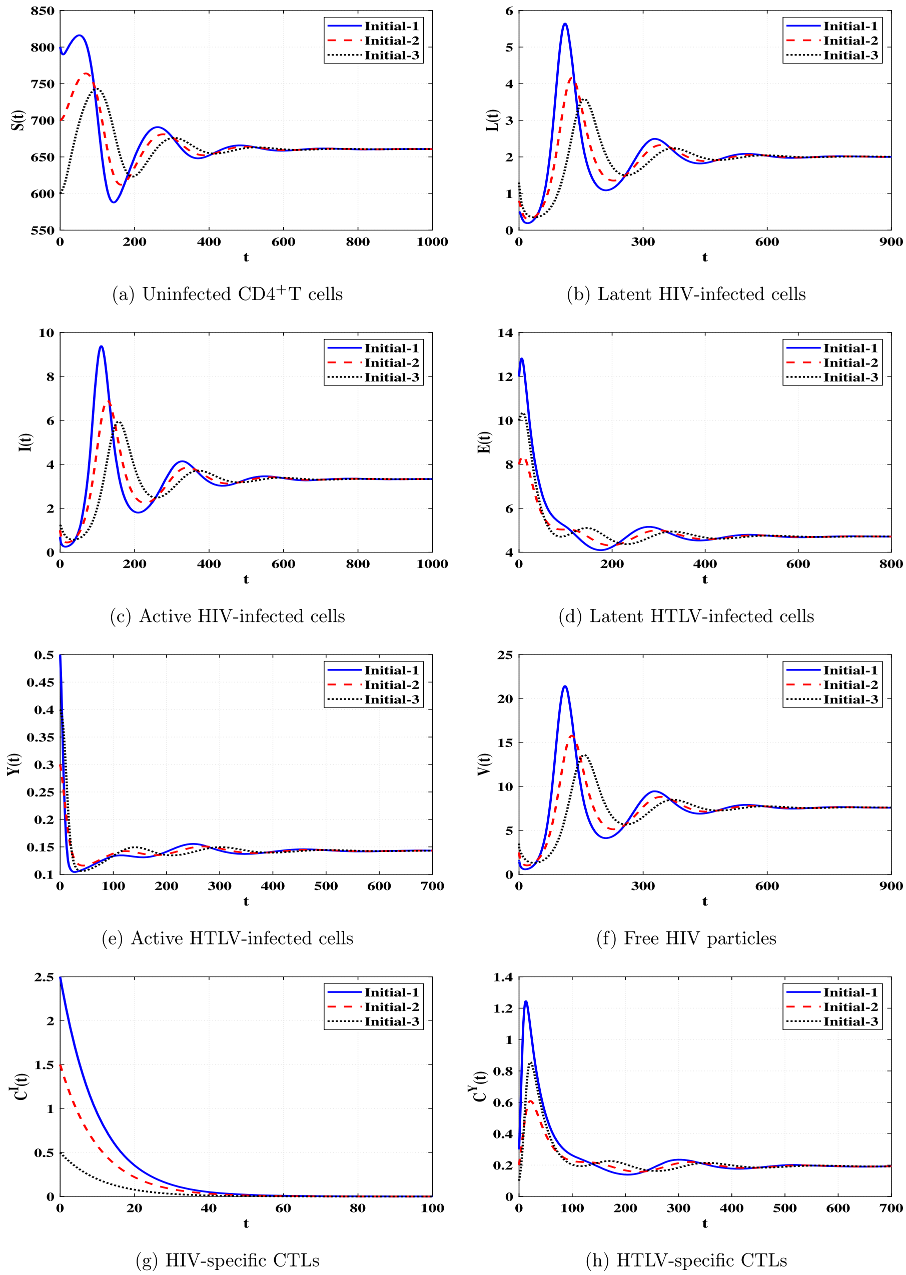


Figure 6 Solutions of system (29) when $\Re_5 > 1$, $\Re_8 \leq 1$, and $\Re_1/\Re_2 > 1$

exists and is GAS. This case leads to a chronic coinfection with HTLV and HIV where the HIV-specific CTL immunity is effective while the HTLV-specific CTL immunity is ineffective.

Set 7 (Stability of Đ_6*)*: $\vartheta_1 = 0.0004$, $\vartheta_2 = 0.0002$, $\vartheta_3 = 0.01$, $\sigma_1 = 0.007$, and $\sigma_2 = 0.7$. We compute $\Re_6 = 1.32 > 1$, $\Re_7 = 0.55 < 1$, and $\Re_2/\Re_1 = 1.77 > 1$. According to these values, we obtain that $\text{Đ}_6 = (660.63, 2.01, 3.33, 4.71, 0.14, 7.61, 0, 0.19)$ exists. The numerical solutions of our system plotted in Fig. 7 show that Đ_6 is GAS (Theorem 7). This case leads to a chronic coinfection with HTLV and HIV where the HTLV-specific CTL immunity is effective and the HIV-specific CTL immunity is not working.



(a) Uninfected CD4$^+$T cells (b) Latent HIV-infected cells

(c) Active HIV-infected cells (d) Latent HTLV-infected cells

(e) Active HTLV-infected cells (f) Free HIV particles

(g) HIV-specific CTLs (h) HTLV-specific CTLs

Figure 7 Solutions of system (29) when $\Re_6 > 1$, $\Re_7 \leq 1$, and $\Re_2/\Re_1 > 1$

Set 8 (Stability of $Đ_7$*)*: $\vartheta_1 = 0.0005$, $\vartheta_2 = 0.0003$, $\vartheta_3 = 0.01$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.3$. These data give $\Re_7 = 1.33 > 1$ and $\Re_8 = 1.81 > 1$. According to these values, the steady state $Đ_7 = (676.78, 0.80, 1, 11.27, 0.33, 2.28, 0.82, 0.20)$ exists. Figure 8 illustrates that the solutions of the system initiating with three different states tend to $Đ_7$. In this case, a chronic coinfection with HTLV and HIV is reached where both immune responses are well working.

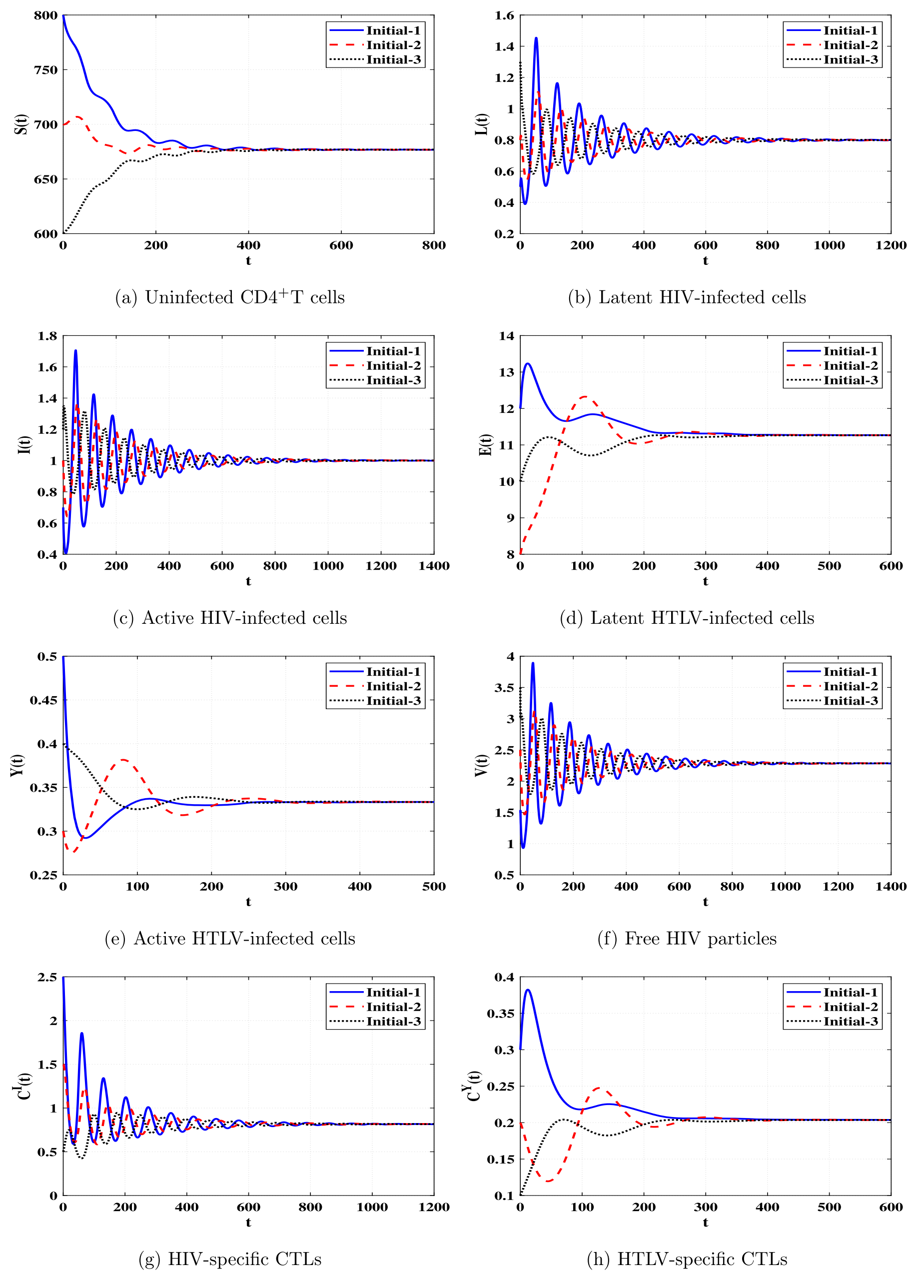


Figure 8 Solutions of system (29) when $\Re_7 > 1$ and $\Re_8 > 1$

6.2 Effect of time delays on the HTLV-HIV dynamics

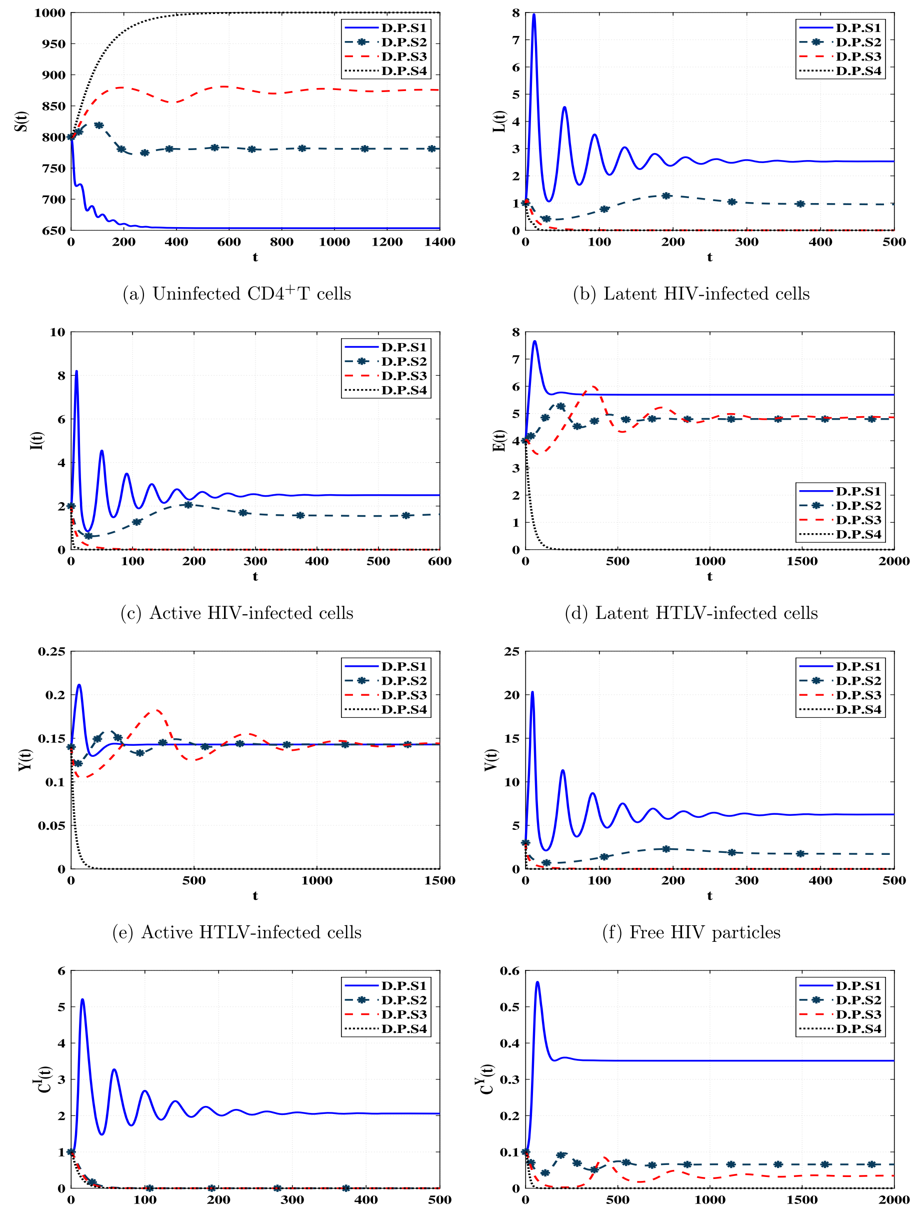
In this part we vary the delay parameters ℓ_i, $i = 1,2,\dots,6$, and fix the parameters $\vartheta_1 = 0.0005$, $\vartheta_2 = 0.0003$, $\vartheta_3 = 0.01$, $\sigma_1 = 0.04$, and $\sigma_2 = 0.7$. Since $\Re_1$ and $\Re_2$ given by Eqs. (30) and (31) depend on ℓ_i, $i = 1,2,\dots,6$, then changing the parameters ℓ_i will change the stability of steady states. Let us consider the following situations:

(D.P.S1) $\ell_1 = \ell_2 = \ell_3 = \ell_4 = \ell_5 = \ell_6 = 0$,

(D.P.S2) $\ell_1 = 0.4$, $\ell_2 = 0.5$, $\ell_3 = 0.6$, $\ell_4 = 0.7$, $\ell_5 = 0.8$, and $\ell_6 = 0.9$,

(D.P.S3) $\ell_1 = 0.6$, $\ell_2 = 0.7$, $\ell_3 = 0.8$, $\ell_4 = 0.9$, $\ell_5 = 1$, and $\ell_6 = 1.2$,

(D.P.S4) $\ell_1 = 10$, $\ell_2 = 11$, $\ell_3 = 12$, $\ell_4 = 13$, $\ell_5 = 14$, and $\ell_6 = 15$.



(a) Uninfected CD4$^+$T cells
(b) Latent HIV-infected cells
(c) Active HIV-infected cells
(d) Latent HTLV-infected cells
(e) Active HTLV-infected cells
(f) Free HIV particles
(g) HIV-specific CTLs
(h) HTLV-specific CTLs

Figure 9 Impact of delay parameters ℓ_i, $i = 1, 2, \ldots, 6$, on the behavior of solution trajectories of system (29)

With these values we solve system (29) under the following initial condition:

Initial-4: $(S(\ell), L(\ell), I(\ell), E(\ell), Y(\ell), V(\ell), C^I(\ell), C^Y(\ell)) = (800, 1, 2, 4, 0.14, 3, 1, 0.1)$, where $\ell \in [-\max \ell_i, 0]$, $i = 1, 2, \ldots, 6$.

From Fig. 9 we observe that the presence of time delays can increase the number of uninfected CD4$^+$ T cells and decrease the number of other compartments. Table 2 presents the values $\Re_1$ and $\Re_2$ for selected values of ℓ_i, $i = 1, 2, \ldots, 6$. It is clear that $\Re_1$ and $\Re_2$ are decreased when ℓ_i are increased, and thus the stability of $Đ_0$ can be changed. Let us calculate the critical value of the time delay that changes the stability of $Đ_0$. Without loss of generality, we let the parameters $\ell = \ell_1 = \ell_2 = \ell_3$ and fix ℓ_j, $j = 5, 6$, and write $\Re_1$ and $\Re_2$ as

Table 2 The variation of $\Re_1$ and $\Re_2$ with respect to the delay parameters

Delay parameters	$\Re_1$	$\Re_2$
$\ell_1 = \ell_2 = \ell_3 = \ell_4 = \ell_5 = \ell_6 = 0$	2.790	3.690
$\ell_1 = 0.3$, $\ell_2 = 0.4$, $\ell_3 = 0.5$, $\ell_4 = 0.6$, $\ell_5 = 0.7$, and $\ell_6 = 0.8$	1.408	1.787
$\ell_1 = 0.4$, $\ell_2 = 0.5$, $\ell_3 = 0.6$, $\ell_4 = 0.7$, $\ell_5 = 0.8$, and $\ell_6 = 0.9$	1.280	1.600
$\ell_1 = 0.6$, $\ell_2 = 0.7$, $\ell_3 = 0.8$, $\ell_4 = 0.9$, $\ell_5 = 1$, and $\ell_6 = 1.2$	1.009	1.283
$\ell_1 = 1$, $\ell_2 = 1.5$, $\ell_3 = 2$, $\ell_4 = 2.5$, $\ell_5 = 3$, and $\ell_6 = 3.5$	0.368	0.173
$\ell_1 = 2$, $\ell_2 = 3$, $\ell_3 = 4$, $\ell_4 = 5$, $\ell_5 = 6$, and $\ell_6 = 7$	0.188	0.008
$\ell_1 = 3$, $\ell_2 = 4$, $\ell_3 = 5$, $\ell_4 = 6$, $\ell_5 = 7$, and $\ell_6 = 8$	0.136	0.003
$\ell_1 = 4$, $\ell_2 = 6$, $\ell_3 = 8$, $\ell_4 = 9$, $\ell_5 = 10$, and $\ell_6 = 11$	0.072	0.1×10^{-3}
$\ell_1 = 6$, $\ell_2 = 7$, $\ell_3 = 9$, $\ell_4 = 10$, $\ell_5 = 11$, and $\ell_6 = 12$	0.052	0.3×10^{-4}
$\ell_1 = 10$, $\ell_2 = 11$, $\ell_3 = 12$, $\ell_4 = 13$, $\ell_5 = 14$, and $\ell_6 = 15$	0.016	1.2×10^{-6}

functions of $\boldsymbol{\ell}$ and $\boldsymbol{\ell}_4$, respectively, as follows:

$$\Re_1(\boldsymbol{\ell}) = \frac{[\lambda e^{-\boldsymbol{\ell}(\hbar_1+\hbar_3)}(1-\beta) + \beta e^{-\hbar_2\boldsymbol{\ell}}(\gamma+\lambda)]S_0(b\vartheta_1 e^{-\hbar_6\boldsymbol{\ell}_6} + \varepsilon\vartheta_2)}{a\varepsilon(\gamma+\lambda)},$$
$$\Re_2(\boldsymbol{\ell}_4) = \frac{\varphi\vartheta_3\psi e^{-(\hbar_4\boldsymbol{\ell}_4+\hbar_5\boldsymbol{\ell}_5)}S_0}{(\delta - re^{-\hbar_5\boldsymbol{\ell}_5})\psi + \delta\omega}.$$

To force the threshold parameters $\Re_1$ and $\Re_2$ to satisfy $\Re_1(\boldsymbol{\ell}) \leq 1$ and $\Re_2(\boldsymbol{\ell}_4) \leq 1$, we choose $\boldsymbol{\ell} \geq \boldsymbol{\ell}^{\min}$, where $\boldsymbol{\ell}^{\min}$ is the solution of

$$\frac{[\lambda e^{-\boldsymbol{\ell}^{\min}(\hbar_1+\hbar_3)}(1-\beta) + \beta e^{-\hbar_2\boldsymbol{\ell}^{\min}}(\gamma+\lambda)]S_0(b\vartheta_1 e^{-\hbar_6\boldsymbol{\ell}_6} + \varepsilon\vartheta_2)}{a\varepsilon(\gamma+\lambda)} = 1,$$

and

$$\boldsymbol{\ell}_4 \geq \boldsymbol{\ell}_4^{\min}, \quad \text{where } \boldsymbol{\ell}_4^{\min} = \max\left\{0, \frac{1}{\hbar_4}\ln\frac{\varphi\vartheta_3\psi e^{-\hbar_5\boldsymbol{\ell}_5}S_0}{(\delta - re^{-\hbar_5\boldsymbol{\ell}_5})\psi + \delta\omega}\right\}.$$

Therefore, if $\boldsymbol{\ell} \geq \boldsymbol{\ell}^{\min}$ and $\boldsymbol{\ell}_4 \geq \boldsymbol{\ell}_4^{\min}$, then $Đ_0$ is GAS. Let us choose the value $\boldsymbol{\ell}_5 = 0.2$ and $\boldsymbol{\ell}_6 = 0.1$ and compute $\boldsymbol{\ell}^{\min}$, $\boldsymbol{\ell}_4^{\min}$ as $\boldsymbol{\ell}^{\min} = 2.73728$, $\boldsymbol{\ell}_4^{\min} = 2.36794$. It follows that:

(i) If $\boldsymbol{\ell} \geq 2.73728$ and $\boldsymbol{\ell}_4 \geq 2.36794$, then $\Re_1(\boldsymbol{\ell}) \leq 1$, $\Re_2(\boldsymbol{\ell}_4) \leq 1$ and $Đ_0$ is GAS.

(ii) If $\boldsymbol{\ell} < 2.73728$ or $\boldsymbol{\ell}_4 < 2.36794$, then $\Re_1(\boldsymbol{\ell}) > 1$ or $\Re_2(\boldsymbol{\ell}_4) > 1$ and $Đ_0$ will lose its stability.

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