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Fixed point results for a pair of fuzzy mappings and related applications in b-metric like spaces

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Abstract

This paper is devoted to finding out some realization of the concept of b-metric like space. First, we attain a fixed point for two fuzzy mappings satisfying a suitable requirement of contractiveness. Subsequently, we apply such a result to graphic contractions. Also, we attain a unique solution for a system of integral equations, and lastly we give an application to ensure that there exists a common bounded solution of a suitable functional equation in dynamic programming.

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1 Introduction and preliminaries

Fixed point theory plays a vital role in the field of functional analysis. Banach [1] proved a very useful outcome for contraction maps, called Banach contraction principle. Thanks to this achievement, many authors proved various interesting extensions of the principle (see [1–45]).

Hussain et al. [20] conceived the new idea of dislocated b-metric spaces and obtained fixed points in common for weak contractive mappings with application on integral inclusions. Afterwards, Chen et al. [15] discussed fixed point outcomes for generalized F-contraction in b-metric like spaces. After this Rasham et al. [27] obtained some multivalued fixed point achievements for pairs of F-contractive maps. Mehmood et al. [24] proved some upshot for a collection of multivalued F-contractive maps and applied this result to a system of nonlinear integral equations. We give the following definitions that we will use from now on.

Definition 1.1 ([20]) Let A be a nonempty set, and let $d_b : A \times A \to [0, +\infty)$ be a b-metric like or dislocated b-metric (or simply d_b-metric), that is, a function for which there is $b \geq 1$ such that, for $f, h, w \in A$, the following assumptions are fulfilled:

(i) If $d_b(f, h) = 0$, then $f = h$;

(ii) $d_b(f, h) = d_b(h, f)$;

(iii) $d_b(f,h) \leq b[d_b(f,w) + d_b(w,h)]$.

We call (A,d_b) a b-metric like space or, in short, $d.b - m.$ space.

Note that, if $x = y$, then $d_b(x,y)$ may not be 0. This is the dislocation that differentiates by the usual metric. An example of a dislocated b-metric space is in [20].

Definition 1.2 ([20]) Let (A,d_b) be a $d.b - m.$ space.

(i) A sequence $\{f_n\}$ in (A,d_b) is said to be Cauchy sequence if, for any $\varepsilon > 0$, we find $n_0 \in N$ so that, for $n,m \geq n_0$, then $d_b(f_m,f_n) < \varepsilon$, that is, $\lim_{n,m\to+\infty} d_b(f_n,f_m) = 0$.

(ii) A sequence $\{f_n\}$ $d.b - m$ converges (briefly d_b-converges) to f if $\lim_{n\to+\infty} d_b(f_n,f) = 0$. Such f is said to be d_b-limit of $\{f_n\}$.

(iii) (A,d_b) is said a complete $d.b - m.$ space if any Cauchy sequence in A converges to a point $f \in A$ satisfying $d_b(f,f) = 0$.

Definition 1.3 ([29]) Let C be a nonempty subset of a $d.b - m.$ space A, and let $f \in A$. A point $g_0 \in C$ is said to be the point of best approximation for f in C if

$$d_b(f,C) = d_b(f,g_0), \quad \text{where } d_b(f,C) = \inf_{g\in C} d_b(f,g).$$

We will say that C is a proximinal set if for any f in A there exists a point of best approximation in C.

Let Ψ_b, where $b \geq 1$, be the collection of all nondecreasing functions $\psi_b : [0,+\infty) \to [0,+\infty)$ for which $\sum_{k=1}^{+\infty} b^k \psi_b^k(t) < +\infty$ and $b\psi_b(t) < t$, where ψ_b^k is the kth iterated of ψ_b. Also $b^{n+1}\psi_b^{n+1}(t) = b^n b\psi_b(\psi_b^n(t)) < b^n\psi_b^n(t)$.

Let $P(A)$ be the collection of all closed proximinal subsets of A.

Definition 1.4 ([35]) The function $H_{d_b} : P(A) \times P(A) \to R^+$, defined by

$$H_{d_b}(D,E) = \max\Big\{\sup_{n\in D} d_b(n,E), \sup_{m\in E} d_b(D,m)\Big\},$$

is known as Hausdorff b-metric like on $P(A)$.

Definition 1.5 ([30]) Let $M,N : A \to P(A)$ be two closed-valued multifunctions and $\beta : A \times A \to [0,+\infty)$ be a positive real function. We say that the pair (M,N) is $\beta_\star$-admissible if, for all $f,g \in A$,

$$\beta(f,g) \geq 1 \Rightarrow \beta_\star(Mf,Ng) \geq 1, \quad \text{and} \quad \beta_\star(Nf,Mg) \geq 1,$$

where $\beta_\star(Nf,Mg) = \inf\{\beta(a,c) : a \in Nf, c \in Mg\}$. When M coincides with N, we regain the definition of α_*-admissible mapping donated in [9].

Definition 1.6 ([29]) Let (A,d_b) be a $d.b - m.$ space. Let $M : A \to P(A)$ be a multivalued mapping and $\alpha : A \times A \to [0,+\infty)$. Let $B \subseteq A$. Then we say that the M is semi α_*-dominated on B whenever $\alpha_*(r,Mr) \geq 1$ for all $r \in B$, where $\alpha_*(r,Mr) = \inf\{\alpha(r,l) : l \in Mr\}$. If $B = A$, it is said that the M is α_*-dominated. If $M : A \to A$ is a self-mapping, then M is semi α-dominated on B, whenever $\alpha(r,Mr) \geq 1$ for each $r \in B$.

Definition 1.7 ([39]) Let (A,d) be a metric space. A mapping $L : A \to A$ is called F-contraction if we can take $\tau > 0$ in a such way that, for all $j,k \in A$, $d(Lj,Lk) > 0$ implies

$$\tau + F\big(d(Lj,Lk)\big) \leq F\big(d(j,k)\big),$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a real function which fulfills the three conditions:
(F1) F is a real strictly increasing function;
(F2) For each sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers, $\lim_{n\to+\infty} a_n = 0$ if and only if $\lim_{n\to+\infty} F(a_n) = -\infty$;
(F3) We can find $q \in (0,1)$ for which $\lim_{a\to 0^+} a^q F(a) = 0$.

Lemma 1.8 ([29]) *Let (A,d_b) be a $d.b-m$. space. Let $(P(A),H_{d_b})$ be a dislocated Hausdorff b-metric space on $P(A)$. For $G,B \in P(A)$ and for all $g \in G$, let $h_g \in B$ such that $d_b(g,B) = d_b(g,h_g)$. Then $H_{d_b}(G,B) \geq d_b(g,h_g)$ holds.*

Example 1.9 ([29]) Let $A = \mathbb{R}$. We define the mapping $\alpha : A \times A \to [0,+\infty)$ by

$$\alpha(r,q) = \left\{\begin{array}{ll} 1 & \text{if } r > q \\ \frac{1}{2} & \text{otherwise} \end{array}\right\}.$$

Define $M,N : A \to P(A)$ by

$$Mr = [r-4,r-3] \quad \text{and} \quad Nq = [q-2,q-1].$$

Suppose $r = 1$ and $q = 0.5$. Since $1 > 0.5$, then $\alpha(1,0.5) \geq 1$. Now, $\alpha_*(M1,N0.5) = \inf\{\alpha(a,c) : a \in M1, c \in N0.5\} = \frac{1}{2} \ngeq 1$, which means the couple (M,N) is not α_*-admissible. Also, $\alpha_*(M1,M0.5) \ngeq 1$ and $\alpha_*(N1,N0.5) \ngeq 1$. This signifies that M and N are not α_*-admissible. Now, $\alpha_*(r,Mr) = \inf\{\alpha(r,c) : c \in Mr\} \geq 1$ for all $r \in A$. Accordingly, M is an α_*-dominated mapping. Analogously, $\alpha_*(q,Nq) = \inf\{\alpha(q,b) : b \in Nq\} \geq 1$. This means that M and N are α_*-dominated but the couple (M,N) is not α_*-admissible.

2 Application to fuzzy maps

The notion of fuzzy set was introduced and its related information was discussed by Zadeh in [41]. In fixed point theory, Weiss [40] and Butnariu [14] presented the content of fuzzy maps and obtained many related results. Heilpern [17] established a fixed point theorem for fuzzy maps that can be considered an analogue of Nadler's multivalued result [25] in metric spaces. Motivated by the Heilpern's results, the fixed point theory for fuzzy contraction using the Hausdorff metric spaces has become more mature in different directions by various authors [32–34].

In the present paper we prove fixed point outcomes for F-contractions generalized in two directions: one is a more extended class of semi-dominated fuzzy maps (in place of admissible mappings) and the other is a wide class of mappings F in place of the mappings F used by Wardowski [39]. The existence of a fuzzy common fixed point for two fuzzy graphic contractions defined on a closed set is given.

Recently, Rasham et al. [29] achieved fixed point outcomes for two families of fuzzy A-dominated maps defined on a closed ball in a complete $d.b-m$. space. Example and usages are given to illustrate the wideness of the results. In this paper, moreover, we achieve our results in a more general setting of $d.b - m.$ space.

Definition 2.1 ([32]) A fuzzy set B is a function from A in $[0,1]$, $F(A)$ is a family of all fuzzy sets in A. Whenever B is a fuzzy set and $f \in A$, the value $B(f)$ is said to be the grade of membership of f in B. The η-level set of fuzzy set B is denoted by $[B]_\eta$ and defined as follows:

$$[B]_\eta = \{f : B(f) \geq \eta\} \quad \text{where } 0 < \eta \leq 1,$$
$$[B]_0 = \overline{\{f : B(f) > 0\}}.$$

Now we select, by the family $F(A)$ of all fuzzy sets, a subfamily with stronger properties, i.e., the subfamily of the approximate quantities, denoted by $W(A)$ and defined by the following.

Definition 2.2 ([17]) A fuzzy subset B of A is an approximate quantity if its η-level set is a compact convex subset of A for each $\eta \in [0,1]$ and $\sup_{f\in A} B(f) = 1$.

Definition 2.3 ([17]) Let A be a set, and let Y be a metric linear space. We call a fuzzy map any map from A to $W(Y)$.

Note that we can see a fuzzy mapping $T : A \to W(Y)$ as a fuzzy subset of $A \times Y$, $T : A \times Y \to [0,1]$ in the sense that $T(f,g) = T(f)(g)$.

Definition 2.4 ([32]) A point $f \in A$ is called a fuzzy fixed point of a fuzzy mapping $T : A \to W(A)$ if there exists $0 < \eta \leq 1$ such that $f \in [Tf]_\eta$.

2.1 Main results

Let (A,d_b) be a $d.b - m.$ space, $f_0 \in A$, and $M,N : A \to W(A)$ be fuzzy mappings on A. Moreover, let $\eta,\upsilon : A \to [0,1]$ be two real-valued functions. Let $f_1 \in [Mf_0]_{\eta(f_0)}$ be an element such that $d_b(f_0,[Mf_0]_{\eta(f_0)}) = d_b(f_0,f_1)$. Let $f_2 \in [Nf_1]_{\upsilon(f_1)}$ be such that $d_b(f_1,[Nf_1]_{\upsilon(f_1)}) = d_b(f_1,f_2)$. Let $f_3 \in [Mf_2]_{\eta(f_2)}$ be such that $d_b(f_2,[Mf_2]_{\eta(f_2)}) = d_b(f_2,f_3)$. Doing so, we obtain a sequence $\{f_n\}$ in A which satisfies $f_{2n+1} \in [Mf_{2n}]_{\eta(f_{2n})}$ and $f_{2n+2} \in [Nf_{2n+1}]_{\upsilon(f_{2n+1})}$ for $n = 0,1,2,\ldots$. Besides $d_b(f_{2n},[Mf_{2n}]_{\eta(f_{2n})}) = d_b(f_{2n},f_{2n+1})$, $d_b(f_{2n+1},[Nf_{2n+1}]_{\upsilon(f_{2n+1})}) = d_b(f_{2n+1}, f_{2n+2})$. We indicate the iterative sequence by $\{NM(f_n)\}$. We will say that $\{NM(f_n)\}$ is a sequence in A generated by f_0. For $f,g \in A$ and $a > 0$, we define $D_b(f,g)$ as

$$D_b(f,g) = \max\left\{\begin{array}{c} d_b(f,g), \\ \frac{d_b(f,[Mf]_{\eta(f)}).d_b(g,[Ng]_{\upsilon(g)})}{a+d_b(f,g)}, \\ d_b(f,[Mf]_{\eta(f)}), d_b(g,[Ng]_{\upsilon(g)}) \end{array}\right\}.$$

Theorem 2.5 *Let (A,d_b) be a complete $d.b - m.$ space. Let $\alpha : A \times A \to [0,+\infty)$. Let $r > 0$, $f_0 \in \overline{B_{d_b}(f_0,r)}$, F be a strictly increasing function, and $M,N : A \to W(A)$ be two α_*-dominated fuzzy mappings on $\overline{B_{d_b}(f_0,r)}$. Hypothesize that with $\psi_b \in \Psi_b$ and $\eta(f),\upsilon(g) \in (0,1]$ the following holds*:

$$\tau + F\big(H_{d_b}\big([Mf]_{\eta(f)},[Ng]_{\upsilon(g)}\big)\big) \leq F\big(\psi_b\big(D_b(f,g)\big)\big) \tag{2.1}$$

for each $f,g \in \overline{B_{d_b}(f_0,r)} \cap \{NM(f_n)\}$, $\alpha(f,g) \geq 1$ *and* $H_{d_b}([Mf]_{\eta(f)},[Ng]_{\upsilon(g)}) > 0$. *Furthermore, suppose that*

$$\sum_{m=0}^{n} b^{m+1}\{\psi_b^m(d_b(f_0,[Mf_0]_{\eta(f_0)}))\} \leq r \tag{2.2}$$

for each $n \in \mathbb{N} \cup \{0\}$ *and* $b \geq 1$. *Then* $\{NM(f_n)\}$ *is a sequence in* $\overline{B_{d_b}(f_0,r)}$, $\alpha(f_n,f_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\{NM(f_n)\} \to f^* \in \overline{B_{d_b}(f_0,r)}$. *Again if inequality* (2.1) *holds for* f^*, $\overline{B_{d_b}(f_0,r)}$ *is a closed set and either* $\alpha(f_n,f^*) \geq 1$ *or* $\alpha(f^*,f_n) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$, *then* M *and* N *have a common fuzzy fixed point* f^* *in* $\overline{B_{d_b}(f_0,r)}$.

Proof Take the sequence $\{NM(f_n)\}$ generated by f_0. Then, by (2.2),

$$d_b(f_0,f_1) \leq \sum_{m=0}^{n} b^{m+1}\{\psi_b^m(d_b(f_0,[Mf_0]_{\eta(f_0)}))\} \leq r.$$

This means that

$$f_1 \in \overline{B_{d_b}(f_0,r)}.$$

For induction, suppose $f_2,\ldots,f_j \in \overline{B_{d_b}(f_0,r)}$ for some $j \in \mathbb{N}$. Suppose first $j = 2m+1$, where $m = 1,2,\ldots,\frac{j-1}{2}$. Since $M,N : A \to W(A)$ are two α_*-dominated fuzzy mappings on $\overline{B_{d_b}(f_0,r)}$, so $\alpha_*(f_{2m},[Mf_{2m}]_{\eta(f_{2m})}) \geq 1$ and $\alpha_*(f_{2m+1},[Nf_{2m+1}]_{\upsilon(f_{2m+1})}) \geq 1$. As $\alpha_*(f_{2m},[Mf_{2m}]_{\eta(f_{2m})}) \geq 1$, it follows $\inf\{\alpha(f_{2m},b) : b \in [Mf_{2m}]_{\eta(f_{2m})}\} \geq 1$. Also $f_{2m+1} \in [Mf_{2m}]_{\eta(f_{2m})}$, so $\alpha(f_{2m},f_{2m+1}) \geq 1$. Now, by means of Lemma 1.8, we obtain

$$\begin{aligned}
&\tau + F(d_b(f_{2m+1},f_{2m+2})) \\
&\quad \leq \tau + F(H_{d_b}([Mf_{2m}]_{\eta(f_{2m})},[Nf_{2m+1}]_{\upsilon(f_{2m+1})})) \\
&\quad \leq F(\psi_b(D_b(f_{2m},f_{2m+1}))) \\
&\quad \leq F\left(\psi_b\left(\max\left\{\begin{array}{c} d_b(f_{2m},f_{2m+1}), \\ \frac{d_b(f_{2m},f_{2m+1}).d_b(f_{2m+1},f_{2m+2})}{a+d_b(f_{2m},f_{2m+1})}, \\ d_b(f_{2m},f_{2m+1}),d_b(f_{2m+1},f_{2m+2}) \end{array}\right\}\right)\right) \\
&\quad \leq F(\psi_b(\max\{d_b(f_{2m},f_{2m+1}),d_b(f_{2m+1},f_{2m+2})\}).
\end{aligned}$$

If

$$\max\{d_b(f_{2m},f_{2m+1}),d_b(f_{2m+1},f_{2m+2})\} = d_b(f_{2m+1},f_{2m+2}),$$

then

$$\tau + F(d_b(f_{2m+1},f_{2m+2})) \leq F(\psi_b(d_b(f_{2m+1},f_{2m+2}))).$$

By the strict increasing of F we obtain

$$d_b(f_{2m+1},f_{2m+2}) < b\psi_b(d_b(f_{2m+1},f_{2m+2})).$$

This contradicts the assumption $b\psi_b(t) < t$ whenever $t > 0$. So

$$\max\{d_b(f_{2m},f_{2m+1}),d_b(f_{2m+1},f_{2m+2})\} = d_b(f_{2m},f_{2m+1}).$$

Hence, we obtain

$$d_b(f_{2m+1},f_{2m+2}) < \psi_b\bigl(d_b(f_{2m},f_{2m+1})\bigr). \tag{2.3}$$

As $\alpha_*(f_{2m-1},[Nf_{2m-1}]_{\upsilon(f_{2m-1})}) \geq 1$ and $f_{2m} \in [Nf_{2m-1}]_{\upsilon(f_{2m-1})}$, so $\alpha(f_{2m-1},f_{2m}) \geq 1$. Now, by using Lemma 1.8, we have

$$\begin{aligned}
&\tau + F\bigl(d_b(f_{2m},f_{2m+1})\bigr) \\
&\quad\leq \tau + F\bigl(H_{d_b}\bigl([Nf_{2m-1}]_{\upsilon(f_{2m-1})},[Mf_{2m}]_{\eta(f_{2m})}\bigr)\bigr) \\
&\quad\leq F\bigl(\psi_b\bigl(D_b(f_{2m},f_{2m-1})\bigr)\bigr) \\
&\quad\leq F\left(\psi_b\left(\max\left\{\begin{array}{c} d_b(f_{2m},f_{2m-1}), \\ \frac{d_b(f_{2m},f_{2m+1}).d_b(f_{2m-1},f_{2m})}{a+d_b(f_{2m},f_{2m-1})}, \\ d_b(f_{2m},f_{2m+1}),d_b(f_{2m-1},f_{2m}) \end{array}\right\}\right)\right) \\
&\quad\leq F\bigl(\psi_b\bigl(\max\{d_b(f_{2m},f_{2m-1}),d_b(f_{2m},f_{2m+1})\}\bigr)\bigr).
\end{aligned}$$

Since F is a strictly increasing function, we have

$$d_b(f_{2m},f_{2m+1}) < \psi_b\bigl(\max\{d_b(f_{2m},f_{2m-1}),d_b(f_{2m},f_{2m+1})\}\bigr).$$

If $\max\{d_b(f_{2m},f_{2m-1}),d_b(f_{2m},f_{2m+1})\} = d_b(f_{2m},f_{2m+1})$, then

$$d_b(f_{2m},f_{2m+1}) < \psi_b\bigl(d_b(f_{2m},f_{2m+1})\bigr) < b\psi_b\bigl(d_b(f_{2m},f_{2m+1})\bigr).$$

This contradicts the assumption $b\psi_b(t) < t$ for positive t. Therefore, we get

$$d_b(f_{2m},f_{2m+1}) < \psi_b\bigl(d_b(f_{2m-1},f_{2m})\bigr). \tag{2.4}$$

So, the nondecreasing of ψ_b yields

$$\psi_b\bigl(d_b(f_{2m},f_{2m+1})\bigr) < \psi_b\bigl(\psi_b\bigl(d_b(f_{2m-1},f_{2m})\bigr)\bigr).$$

This last inequality, together with (2.3), gives

$$d_b(f_{2m+1},f_{2m+2}) < \psi_b^2\bigl(d_b(f_{2m-1},f_{2m})\bigr).$$

Iterating this reasoning, we obtain

$$d_b(f_{2m+1},f_{2m+2}) < \psi_b^{2m+1}\bigl(d_b(f_0,f_1)\bigr). \tag{2.5}$$

Instead, if $j = 2m$, where $m = 1, 2, \dots, \frac{j}{2}$, by using (2.4) and similar procedure as above, we have

$$d_b(f_{2m}, f_{2m+1}) < \psi_b^{2m}\big(d_b(f_0, f_1)\big). \tag{2.6}$$

Now, by combining (2.5) and (2.6),

$$d_b(f_j, f_{j+1}) < \psi_b^{j}\big(d_b(f_0, f_1)\big) \quad \text{for all } j \in \mathbb{N}. \tag{2.7}$$

Now, making use of (2.4) and reasoning similarly to the odd case, we get

$$\begin{aligned} d_b(f_0, f_{j+1}) &\leq b d_b(f_0, f_1) + b^2 d_b(f_1, f_2) + \dots + b^{j+1} d_b(f_j, f_{j+1}) \\ &< b d_b(f_0, f_1) + b^2 \psi_b\big(d_b(f_0, f_1)\big) + \dots + b^{j+1} \psi_b^{j}\big(d_b(f_0, f_1)\big) \\ &< \sum_{m=0}^{j} b^{m+1}\big\{\psi_b^{m}\big(d_b(f_0, f_1)\big)\big\} < r. \end{aligned}$$

This means $f_{j+1} \in \overline{B_{d_b}(f_0, r)}$. Hence f_n belongs to $\overline{B_{d_b}(f_0, r)}$ for each $n \in \mathbb{N}$, therefore the entire sequence $\{NM(f_n)\}$ is in $\overline{B_{d_b}(f_0, r)}$. As the mappings M, N are α_*-dominated on $\overline{B_{d_b}(f_0, r)}$, this implies that $\alpha_*(f_{2n}, [Mf_{2n}]_{\eta(f_{2n})}) \geq 1$ and $\alpha_*(f_{2n+1}, [Nf_{2n+1}]_{\upsilon(f_{2n+1})}) \geq 1$. This implies $\alpha(f_n, f_{n+1}) \geq 1$. Also inequality (2.7) can be written as

$$d_b(f_n, f_{n+1}) < \psi_b^{n}\big(d_b(f_0, f_1)\big) \quad \text{for } n \in \mathbb{N}. \tag{2.8}$$

Thanks to the hypothesis $\sum_{k=1}^{+\infty} b^k \psi_b^k(t) < +\infty$ for all t, if we take a natural number p, we have the convergence of the series $\sum_{k=1}^{+\infty} b^k \psi_b^k(\psi_b^{p-1}(d_b(f_0, f_1)))$. Moreover, thanks to the assumption $b\psi_b(t) < t$, because one has that, for any natural number n,

$$b^{n+1}\psi_b^{n+1}\big(\psi_b^{p-1}\big(d_b(f_0, f_1)\big)\big) < b^n \psi_b^n\big(\psi_b^{p-1}\big(d_b(f_0, f_1)\big)\big).$$

Fix $\varepsilon > 0$, from the convergence of the previous series, it follows that there is $p(\varepsilon) \in \mathbb{N}$, for which

$$b\psi_b\big(\psi_b^{p(\varepsilon)-1}\big(d_b(f_0, f_1)\big)\big) + b^2\psi_b^2\big(\psi_b^{p(\varepsilon)-1}\big(d_b(f_0, f_1)\big)\big) + \dots < \varepsilon.$$

Take $n, m \in \mathbb{N}$ with $m > n > p(\varepsilon)$, then we have

$$\begin{aligned} d_b(f_n, f_m) &\leq b d_b(f_n, f_{n+1}) + b^2 d_b(f_{n+1}, f_{n+2}) \\ &\quad + \dots + b^{m-n} d_b(f_{m-1}, f_m) \\ &< b\psi_b^n\big(d_b(f_0, f_1)\big) + b^2\psi_b^{n+1}\big(d_b(f_0, f_1)\big) \\ &\quad + \dots + b^{m-n}\psi_b^{m-1}\big(d_b(f_0, f_1)\big) \\ &< b\psi_b\big(\psi_b^{p(\varepsilon)-1}\big(d_b(f_0, f_1)\big)\big) \\ &\quad + b^2\psi_b^2\big(\psi_b^{p(\varepsilon)-1}\big(d_b(f_0, f_1)\big)\big) + \dots \\ &< \varepsilon. \end{aligned}$$

So we get that $\{NM(f_n)\}$ is a Cauchy sequence in $(\overline{B_{d_b}(f_0,r)},d_b)$. From the completeness of the $d.b-m.$ space and from the closure of a closed ball, it follows that the sequence converges to a point in the closed ball, i.e., there is $f^* \in \overline{B_{d_b}(f_0,r)}$ such that $\{NM(f_n)\} \to f^*$, that is,

$$\lim_{n\to+\infty} d_b(f_n,f^*) = 0. \tag{2.9}$$

Now,

$$\begin{aligned} &d_b\bigl(f^*,\bigl[Mf^*\bigr]_{\eta(f^*)}\bigr) \\ &\quad \le bd_b\bigl(f^*,f_{2n+2}\bigr) + bd_b\bigl(f_{2n+2},\bigl[Mf^*\bigr]_{\eta(f^*)}\bigr) \\ &\quad \le bd_b\bigl(f^*,f_{2n+2}\bigr) + bH_{d_b}\bigl([Nf_{2n+1}]_{\upsilon(f_{2n+1})},\bigl[Mf^*\bigr]_{\eta(f^*)}\bigr). \end{aligned}$$

By assumption, $\alpha(f_n,f^*) \ge 1$. For contradiction, suppose that $d_b(f^*,[Mf^*]_{\eta(f^*)}) > 0$. Then there is a sufficiently big k for which $d_b(f_n,[Mf^*]_{\eta(f^*)}) > 0$ for n greater than k. For such n, one has

$$\begin{aligned} d_b\bigl(f^*,\bigl[Mf^*\bigr]_{\eta(f^*)}\bigr) < {}& bd_b\bigl(f^*,f_{2n+2}\bigr) \\ & + b\psi_b\Biggl(\max\biggl\{d_b\bigl(f^*,f_{2n+1}\bigr),d_b\bigl(f^*,\bigl[Mf^*\bigr]_{\eta(f^*)}\bigr), \\ & \frac{d_b(f^*,[Mf^*]_{\eta(f^*)}).d_b(f_{2n+1},f_{2n+2})}{a+d_b(f^*,f_{2n+1})}, \\ & d_b(f_{2n+1},f_{2n+2})\biggr\}\Biggr). \end{aligned}$$

For $n \to +\infty$, thanks to (2.9) we have $d_b(f^*,[Mf^*]_{\eta(f^*)}) < b\psi_b(d_b(f^*,[Mf^*]_{\eta(f^*)})) < d_b(f^*,[Mf^*]_{\eta(f^*)})$, which is a counter-sense. This means that our guess is not acceptable. Hence $d_b(f^*,[Mf^*]_{\eta(f^*)}) = 0$ or $f^* \in [Mf^*]_{\eta(f^*)}$. Analogously, from Lemma 1.8 and inequality (2.9), following the same reasoning, one obtains that $f^* \in [Nf^*]_{\upsilon(f^*)}$. So M and N detain a fuzzy fixed point in common f^* in $\overline{B_{d_l}(c_0,r)}$. Now

$$d_l\bigl(f^*,f^*\bigr) \le bd_b\bigl(f^*,\bigl[Nf^*\bigr]_{\upsilon(f^*)}\bigr) + bd_b\bigl(\bigl[Nf^*\bigr]_{\upsilon(f^*)},f^*\bigr).$$

This implies that $d_b(f^*,f^*) = 0$. □

The next theorem is an immediate corollary of the previous main result in the case of only a mapping defined on the entire space, not on a ball.

Theorem 2.6 *Let (A,d_b) be a complete $d.b-m.$ space. Assume that $M: A \to W(A)$ is a fuzzy mapping and F is a strictly increasing function. Suppose that, for suitable $\psi_b \in \Psi_b$ and $\eta(f),\upsilon(g) \in (0,1]$, the following holds*:

$$\tau + F\bigl(H_{d_b}\bigl([Mf]_{\eta(f)},[Mg]_{\upsilon(g)}\bigr)\bigr) \le F\bigl(\psi_b\bigl(D_b(f,g)\bigr)\bigr)$$

for all $f,g \in \{MM(f_n)\}$. Then $\{MM(f_n)\} \to f^* \in A$ and M has a fuzzy fixed point f^* in A and $d_b(f^*,f^*) = 0$.

Definition 2.7 ([5]) Let A be a nonempty set, $\preceq$ be a partial order on A and $K \subseteq A$. We say that $g \preceq P$ whenever, for all $p \in P$, we have $g \preceq p$. A mapping $M : A \to W(A)$ is called prevalent on K if $g \preceq Mg$ for each $g \in K \subseteq A$. If $K = A$, then $M : A \to W(A)$ is called totally prevalent.

We have the following result for fuzzy prevalent mappings on $\overline{B_{d_b}(f_0, r)}$ in an ordered complete $d.b - m.$ space.

Theorem 2.8 *Let $(A, \preceq, d_b)$ be an ordered complete $d.b - m.$ space. Let $r > 0, f_0 \in \overline{B_{d_b}(f_0, r)}$, F be a strictly increasing function, and $M, N : A \to W(A)$ be two fuzzy prevalent mappings on $\overline{B_{d_b}(f_0, r)}$. Suppose that, for some $\psi_b \in \Psi_b$ and $\eta(f), \upsilon(g) \in (0,1]$ the following holds:*

$$\tau + F\bigl(H_{d_b}\bigl([Mf]_{\eta(f)}, [Ng]_{\upsilon(g)}\bigr)\bigr) \le F\bigl(\psi_b\bigl(D_b(f,g)\bigr)\bigr) \tag{2.10}$$

for all $f, g \in \overline{B_{d_b}(f_0, r)} \cap \{NM(f_n)\}, f \preceq g$ and $H_{d_b}([Mf]_{\eta(f)}, [Mg]_{\upsilon(g)}) > 0$. Furthermore, suppose that

$$\sum_{m=0}^{n} b^{m+1}\bigl\{\psi_b^m\bigl(d_b(f_0,f_1)\bigr)\bigr\} \le r \tag{2.11}$$

for all natural numbers $n = 0, 1, 2, \dots$ Then the sequence $\{NM(f_n)\}$ is in $\overline{B_{d_b}(f_0, r)}$ and $\{NM(f_n)\} \to f^* \in \overline{B_{d_b}(f_0, r)}$. Also, if (2.10) holds for f^* and either $f_n \preceq f^*$ or $f^* \preceq f_n$ for each $n \in \mathbb{N} \cup \{0\}$, then the mappings M and N possess a common fuzzy fixed point f^* in $\overline{B_{d_b}(f_0, r)}$ and $d_b(f^*, f^*) = 0$.

Proof Let $\alpha : A \times A \to [0, +\infty)$ be a mapping fixed by $\alpha(f,g) = 1$ for all $f \in \overline{B_{d_b}(f_0, r)}, f \preceq g$, and $\alpha(f,g) = 0$ and $g \in A$. From the fact that M and N are fuzzy prevalent mappings on $\overline{B_{d_b}(f_0, r)}$, it follows $f \preceq [Mf]_{\eta(f)}$ and $f \preceq [Nf]_{\upsilon(f)}$ for all $f \in \overline{B_{d_b}(f_0, r)}$. From this it follows that $f \preceq s$ for all $s \in [Mf]_{\eta(f)}$ and $f \preceq d$ for all $d \in [Nf]_{\upsilon(f)}$. So, $\alpha(f,s) = 1$ for all $s \in [Mf]_{\eta(f)}$ and $\alpha(f,c) = 1$ for all $c \in [Nf]_{\upsilon(f)}$. Therefore $\inf\{\alpha(f,g) : g \in [Mf]_{\eta(f)}\} = 1$ and $\inf\{\alpha(f,g) : g \in [Nf]_{\upsilon(f)}\} = 1$. So, $\alpha_*(f, [Mf]_{\eta(f)}) = 1$, $\alpha_*(f, [Nf]_{\upsilon(f)}) = 1$ for all $f \in \overline{B_{d_b}(f_0, r)}$. So, $M, N : A \to W(A)$ are the α_*-dominated mappings on $\overline{B_{d_b}(f_0, r)}$. Recall that inequality (2.10) can be rewritten as

$$\tau + F\bigl(H_{d_b}\bigl([Mf]_{\eta(f)}, [Ng]_{\upsilon(g)}\bigr)\bigr) \le F\bigl(\psi_b\bigl(D_b(f,g)\bigr)\bigr)$$

for f, g in $\overline{B_{d_b}(f_0, r)} \cap \{NM(f_n)\}$, $\alpha(f,g) \ge 1$. Also, inequality (2.11) holds. So, by Theorem 2.5, we deduce that the sequence $\{NM(f_n)\}$ is in $\overline{B_{d_b}(f_0, r)}$ and $\{NM(f_n)\} \to f^* \in \overline{B_{d_b}(f_0, r)}$. Now, $f_n, f^* \in \overline{B_{d_b}(f_0, r)}$ and both $f_n \preceq f^*$ and $f^* \preceq f_n$ imply that either $\alpha(f_n, f^*) \ge 1$ or $\alpha(f^*, f_n) \ge 1$. So, the assumptions of Theorem 2.5 are fulfilled, and so M and N have a common fuzzy fixed point f^* in $\overline{B_{d_b}(f_0, r)}$ and $d_b(f^*, f^*) = 0$. □

An immediate corollary of the above result in the case of a fuzzy prevalent mapping defined on the entire space is the following.

Theorem 2.9 *Let $(A, \preceq, d_b)$ be an ordered complete $d.b - m.$ space. Let $M : A \to W(A)$ be a fuzzy prevalent mapping on A and F be a strictly increasing function. Assume that, for* *some* $\psi_b \in \Psi_b$ *and* $\eta(f), \upsilon(g) \in (0,1]$, *the following holds*:

$$\tau + F\bigl(H_{d_b}\bigl([Mf]_{\eta(f)}, [Mg]_{\upsilon(g)}\bigr)\bigr) \leq F\bigl(\psi_b\bigl(D_b(f,g)\bigr)\bigr) \tag{2.12}$$

for each $f, g \in \{MM(f_n)\}$ *with* $f \preceq g$. *Then* $\{MM(f_n)\} \to f^* \in A$. *Moreover, if inequality* (2.12) *holds for* f^* *and either* $f_n \preceq f^*$ *or* $f^* \preceq f_n$ *for all* $n \in \mathbb{N} \cup \{0\}$, *then* M *has a fuzzy fixed point* f^* *and* $d_b(f^*, f^*) = 0$.

Example 2.10 Take $A = [0, +\infty)$ and take $d_b : A \times A \to A$ defined by

$$d_b(l,r) = (l+r)^2 \quad \text{for all } l, r \in A$$

with constant $b = 2$. Now, for $f, h \in A$, $\gamma, \beta \in [0,1]$, define $M, N : A \to W(A)$ by

$$(Mf)(t) = \begin{cases} \gamma & \text{if } 0 \leq t < \frac{f}{2}, \\ \frac{\gamma}{2} & \text{if } \frac{f}{2} \leq t \leq \frac{3f}{4}, \\ \frac{\gamma}{4} & \text{if } \frac{3f}{4} < t \leq f, \\ 0 & \text{if } f < t < \infty \end{cases}$$

and

$$(Nf)(t) = \begin{cases} \beta & \text{if } 0 \leq t < \frac{f}{2}, \\ \frac{\beta}{4} & \text{if } \frac{f}{2} \leq t \leq \frac{2f}{3}, \\ \frac{\beta}{6} & \text{if } \frac{2f}{3} < t \leq f, \\ 0 & \text{if } f < t < \infty. \end{cases}$$

Now, we consider

$$[Mf]_{\frac{\gamma}{2}} = \left[\frac{f}{2}, \frac{3f}{4}\right] \quad \text{and} \quad [Nf]_{\frac{\beta}{4}} = \left[\frac{f}{3}, \frac{2f}{3}\right].$$

Let $f_0 = \frac{1}{2}$, $r = 36$. Then $\overline{B_{d_b}(f_0, r)} = [0, \frac{11}{2}]$. Now, we have $d_b(f_0, [Mf_0]_{\frac{\gamma}{2}}) = d_b(\frac{1}{2}, [M\frac{1}{2}]_{\frac{\gamma}{2}}) = d_b(\frac{1}{2}, \frac{1}{8}) = \frac{25}{64}$. So we obtain a sequence $\{NM(f_n)\} = \{\frac{1}{2}, \frac{1}{8}, \frac{1}{24}, \frac{1}{96}, \ldots\}$ in A generated by f_0. Let $\psi(k) = \frac{9k}{10}$ and $a = \frac{1}{2}$. Define

$$\alpha(f,h) = \begin{cases} 1 & \text{if } f > h, \\ \frac{8}{9} & \text{otherwise.} \end{cases}$$

Now, for $f, h \in \overline{B_{d_b}(f_0, r)} \cap \{NM(f_n)\}$ with $\alpha(f,h) \geq 1$, we have

$$\begin{aligned} H_{d_b}\bigl([Mf]_{\frac{\gamma}{2}}, [Nh]_{\frac{\beta}{4}}\bigr) &= \max\Bigl\{\sup_{a \in [Mc]_{\frac{\gamma}{2}}} d_b\bigl(a, [Nh]_{\frac{\beta}{4}}\bigr), \sup_{b \in [Nh]_{\frac{\beta}{4}}} d_b\bigl([Mf]_{\frac{\gamma}{2}}, b\bigr)\Bigr\} \\ &= \max\left\{d_b\left(\frac{3f}{4}, \left[\frac{h}{3}, \frac{2h}{3}\right]\right), d_b\left(\left[\frac{f}{2}, \frac{3f}{4}\right], \frac{2h}{3}\right)\right\} \\ &= \max\left\{d_b\left(\frac{3f}{4}, \frac{h}{3}\right), d_b\left(\frac{f}{2}, \frac{2h}{3}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{\left(\frac{3f}{4}+\frac{h}{3}\right)^2, \left(\frac{f}{2}+\frac{2h}{3}\right)^2\right\} \\
&\leq \psi\left(\max\left\{\begin{array}{c}(f+h)^2, \frac{50f^2h^2}{9(1+2(f+h)^2)}, \\ (\frac{4h}{3})^2, (\frac{5f}{4})^2\end{array}\right\}\right) \\
&\leq \psi\left(D_b(f,h)\right)
\end{aligned}
$$

in such a way that, for any $\tau \in (0, \frac{12}{95}]$ and for the strict increasing of the logarithm function $F(t) = \ln(t)$, we obtain

$$\tau + F\left(H_{d_b}\left([Mf]_{\frac{\gamma}{2}}, [Nh]_{\frac{\beta}{4}}\right)\right) \leq F\left(\psi_b\left(D_b(f,h)\right)\right).$$

Now take $5, 6 \in A$, then $\alpha(5,6) \geq 1$. But we have

$$\tau + F\left(H_{d_b}\left([M5]_{\frac{\gamma}{2}}, [N6]_{\frac{\beta}{4}}\right)\right) > F\left(\psi_b\left(D_b(f,h)\right)\right).$$

So assumption (2.1) is not satisfied on A. Moreover, for each $n \in \mathbb{N} \cup \{0\}$,

$$\sum_{m=0}^{n} b^{m+1}\left\{\psi_b^m\left(d_b(f_0,f_1)\right)\right\} = \frac{25}{64} \times 2\sum_{m=0}^{n}\left(\frac{9}{10}\right)^m < 36 = r.$$

This means that the mappings M and N satisfy all the hypotheses of Theorem 2.5 for f, $h \in \overline{B_{d_b}(f_0,r)} \cap \{NM(f_n)\}$ with $\alpha(f,g) \geq 1$. So, M and N possess a common fuzzy fixed point.

3 A realization of Theorem 2.5 for graphic contractions

Here we present an application of Theorem 2.5 in graph theory. Jachymski [22] proved an analogous result in the special occurrence of contraction maps defined on a metric space with a graph. Hussain et al. [19] gave fixed point results for graphic contraction with a realization to integral equations. For the sake of completeness, recall here that a graph G is a connected graph when there exists a path that connects any two different vertices (see for details [12]).

Definition 3.1 Let A be a nonempty set and $G = (V(G), E(G))$ be a graph such that $V(G) = A$, $H \subseteq A$. A mapping $M : A \to W(A)$ is said to be fuzzy graph dominated on H when, for each f belonging to H and for each g belonging to Mf, it results that (f,g) is an edge belonging to $E(G)$.

Theorem 3.2 *Let (A, d_b) be a complete d.b – m. space endowed with a graph G. Following the notations of Theorem 2.5, let r be a positive real number, $f_0 \in \overline{B_{d_b}(f_0,r)}$ and $M, N : A \to W(A)$. Suppose that, for some $\psi_b \in \Psi_b$ and $\eta(f), \upsilon(g) \in (0,1]$, the following three conditions are satisfied:*

(i) *M and N are fuzzy graphs dominated on $\overline{B_{d_b}(f_0,r)} \cap \{NM(f_n)\}$.*

(ii) *There are $\tau > 0$ and a strictly increasing mapping F that satisfy the contractivity condition*

$$\tau + F\left(H_{d_b}\left([Mf]_{\eta(f)}, [Ng]_{\upsilon(g)}\right)\right) \leq F\left(\psi_b\left(D_b(f,g)\right)\right), \tag{3.1}$$

whenever $f, g \in \overline{B_{d_b}(f_0,r)} \cap \{NM(f_n)\}$, $(f,g) \in E(G)$ and $H_{d_b}([Mf]_{\eta(f)}, [Ng]_{\upsilon(g)}) > 0$.

(iii) $\sum_{i=0}^{+\infty} b^{i+1}\{\psi_b^i(d_b(f_0,f_1))\} \le r$.
Then $\{NM(f_n)\}$ *is a sequence in* $\overline{B_{d_b}(f_0,r)}$, $(f_n,f_{n+1}) \in E(G)$ *and* $\{NM(f_n)\} \to m^*$. *Furthermore, suppose that inequality* (3.1) *is satisfied for* m^* *and* $(f_n,m^*) \in E(G)$ *or* $(m^*,f_n) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$. *Then both the mappings* M *and* N *have a fuzzy fixed point in common* m^* *in* $\overline{B_{d_b}(f_0,r)}$.

Proof Define $\alpha : A \times A \to [0,+\infty)$ by

$$\alpha(f,h) = \begin{cases} 1, & \text{if } f \in \overline{B_{d_l}(f_0,r)}, (f,h) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Now, assumption (ii) ensures that M and N are graphs dominated on $\overline{B_{d_b}(f_0,r)}$, then for $f \in \overline{B_{d_b}(f_0,r)}$, $(f,h) \in E(G)$ for all $h \in [Mf]_{\eta(f)}$ and $(f,h) \in E(G)$ for all $h \in [Nf]_{\upsilon(f)}$. So, $\alpha(f,h) = 1$ for all $h \in [Mf]_{\eta(f)}$ and $\alpha(f,h) = 1$ for all $h \in [Nf]_{\upsilon(f)}$. This means that $\inf\{\alpha(f,h) : h \in [Mf]_{\eta(f)}\} = 1$ and $\inf\{\alpha(f,h) : h \in [Nf]_{\upsilon(f)}\} = 1$. Hence $\alpha_*(f,[Mf]_{\eta(f)}) = 1$, $\alpha_*(f,[Nf]_{\upsilon(f)}) = 1$ for all $f \in \overline{B_{d_b}(f_0,r)}$. So, $M,N : A \to W(A)$ are semi α_*-dominated fuzzy mappings on $\overline{B_{d_b}(f_0,r)}$. Also, we can rewrite inequality (3.1) as follows:

$$\tau + F\bigl(H_{d_b}\bigl([Mf]_{\eta(f)},[Nh]_{\upsilon(h)}\bigr)\bigr) \le F\bigl(\psi_b\bigl(D_b(f,h)\bigr)\bigr),$$

whenever $f,h \in \overline{B_{d_b}(f_0,r)} \cap \{NM(f_n)\}$, $\alpha(f,h) \ge 1$ and $H_{d_b}([Mf]_{\eta(f)},[Nh]_{\upsilon(h)}) > 0$. Furthermore, assumption (iii) permits Theorem 2.5 to guarantee that $\{NM(f_n)\}$ is a sequence in $\overline{B_{d_b}(f_0,r)}$ and $\{NM(f_n)\} \to m^* \in \overline{B_{d_b}(f_0,r)}$. Lastly, $f_n,m^* \in \overline{B_{d_b}(f_0,r)}$ and either $(f_n,m^*) \in E(G)$ or $(m^*,f_n) \in E(G)$ implies that either $\alpha(f_n,m^*) \ge 1$ or $\alpha(m^*,f_n) \ge 1$. Thus, all requirements of Theorem 2.5 are satisfied. Hence, by Theorem 2.5, M and N have a common fuzzy fixed point m^* in $\overline{B_{d_b}(f_0,r)}$ and $d_b(m^*,m^*) = 0$. □

4 A realization to integral equations

Theorem 4.1 *Let* (A,d_b) *be a complete* $d.b - m$. *space with constant* $b \ge 1$. *Let* $u \in A$ *and* $M,N : A \to A$. *Assume that there are* $\tau > 0$ *and a strictly increasing mapping* F *for which, for a suitable function* $\psi_b \in \Psi_b$, *the following contractiveness condition holds*:

$$\tau + F\bigl(d_b(Mf,Ng)\bigr) \le F\bigl(\psi_b\bigl(D_b(f,g)\bigr)\bigr), \tag{4.1}$$

whenever $f,g \in \{NM(f_n)\}$ *and* $d_b(Mf,Ng) > 0$. *Then* $\{NM(f_n)\} \to q \in A$. *Further, if the contractiveness condition is fulfilled for* q, *then the mappings* M *and* N *have a unique common fixed point* q *in* A.

Proof The proof of existence is very suchlike to that of Theorem 2.5, and so we omit it. It remains only to prove the uniqueness. For this, let p be another common fixed point of M and N. For contradiction, hypothesize $d_b(Mq,Np) > 0$. So it follows

$$\tau + F\bigl(d_b(Mq,Np)\bigr) \le F\bigl(\psi_b\bigl(D_b(q,p)\bigr)\bigr).$$

This implies that

$$d_b(q,p) < \psi_b(d_b(q,p) < d_b(q,p),$$

which is not true. So $d_b(Mq, Np) = 0$. Hence $q = p$.

Now, we give a realization of this last theorem to a Volterra integral system.

$$m(p) = \int_0^p K_1\bigl(p, t, m(t)\bigr)\,dt, \tag{4.2}$$

$$n(p) = \int_0^p K_2\bigl(p, t, n(t)\bigr)\,dt \tag{4.3}$$

for all $p \in [0,1]$. We find the solution of (4.2) and (4.3). Let $A = \hat{C}([0,1], \mathbb{R}_+)$ be the set of all nonnegative real-valued continuous functions provided with the complete $d.b - m.$ defined below. First, for $m \in \hat{C}([0,1], \mathbb{R}_+)$, define a supremum norm as follows: $\|m\|_\tau = \sup_{p\in[0,1]}\{m(p)f^{-\tau p}\}$, where $\tau, f > 0$ is taken arbitrarily. Then define

$$\begin{aligned} d_\tau(m,n) &= \Bigl[\sup_{p\in[0,1]}\bigl\{\bigl[m(p)+n(p)\bigr]f^{-\tau p}\bigr\}\Bigr]^2 \\ &= \|m+n\|_\tau^2 \end{aligned}$$

for all $m, n \in \hat{C}([0,1], \mathbb{R}_+)$, with these settings $(\hat{C}([0,1], \mathbb{R}_+), d_\tau)$ becomes a complete $d.b - m.$ space. □

Now we are ready to prove the theorem to find the solution of integral equations.

Theorem 4.2 *Hypothesize that conditions* (*i*) *and* (*ii*) *are satisfied*:

(i) $K_1, K_2 : [0,1] \times [0,1] \times \hat{C}([0,1], \mathbb{R}_+) \to \mathbb{R}$;

(ii) *Define* $M, N : \hat{C}([0,1], \mathbb{R}_+) \to \hat{C}([0,1], \mathbb{R}_+)$ *by*

$$\begin{aligned} (Mm)(p) &= \int_0^p K_1\bigl(p, t, m(t)\bigr)\,dt, \\ (Nn)(p) &= \int_0^p K_2\bigl(p, t, n(t)\bigr)\,dt. \end{aligned}$$

Take $\tau > 0$ *in such a way that*

$$\bigl|K_1(p,t,m) + K_2(p,t,n)\bigr| \le \frac{\tau Z(m,n)}{\tau Z(m,n)+1}$$

for all $p, t \in [0,1]$ *and* $m, n \in \hat{C}([0,1], \mathbb{R})$, *where*

$$Z(m,n) = \sup\left\{\psi_b\begin{pmatrix} [|m(p)+n(p)|]^2, \\ \frac{[|m(p)+(Mm)(p)|]^2.[|n(p)+(Nn)(p)|]^2}{1+[|u(k)+c(k)|]^4}, \\ [|m(p)+(Mm)(p)|]^2, \\ [|m(p)+(Nn)(p)|]^2 \end{pmatrix}\right\}.$$

Then integral equations (4.2) *and* (4.3) *have a unique solution.*

Proof By definition (ii)

$$\begin{aligned}
&\left|(Mm)(p) + (Nn)(p)\right| \\
&\quad = \int_0^p \left|K_1\big(p,h,m(t) + K_2\big(p,h,n(t)\big)\big)\right| dt, \\
&\quad \le \int_0^p \frac{\tau}{\tau Z(m,n) + 1}\big(\big[Z(m,n)\big]f^{-\tau t}\big)f^{\tau t}\,dt \\
&\quad \le \int_0^p \frac{\tau}{\tau Z(m,n) + 1} Z(m,n) f^{\tau t}\,dt \\
&\quad \le \frac{\tau Z(m,n)}{\tau Z(m,n) + 1}\int_0^p f^{\tau t}\,dt \\
&\quad \le \frac{Z(m,n)}{\tau Z(m,n) + 1} f^{\tau p}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\left|(Mm)(p) + (Nn)(p)\right| f^{-\tau p} \le \frac{Z(m,n)}{\tau Z(m,n) + 1}, \\
&\left\|(Mu)(p) + (Nc)(p)\right\|_\tau \le \frac{Z(m,n)}{\tau Z(m,n) + 1}, \\
&\frac{\tau Z(m,n) + 1}{Z(m,n)} \le \frac{1}{\|(Mm)(p) + (Nn)(p)\|_\tau}, \\
&\tau + \frac{1}{Z(m,n)} \le \frac{1}{\|(Mm)(p) + (Nn)(p)\|_\tau},
\end{aligned}$$

that is,

$$\tau - \frac{1}{\|(Mm)(p) + (Nn)(p)\|_\tau} \le \frac{-1}{Z(m,n)}.$$

So, all the requirements of Theorem 4.1 are fulfilled for $F(n) = \frac{-1}{\sqrt{n}}; n > 0$ and $d_\tau(m,n) = \|m+n\|_\tau^2$. Hence equations (4.2) and (4.3) have a unique common solution. □

5 Application to functional equations

Here, we derive an application for the solution of a functional equation arising in dynamic programming. Consider U and V to be two Banach spaces, $Z \subseteq U$, $H \subseteq V$, and

$$\begin{aligned}
&\tilde{u} : Z \times H \to Z, \\
&g, u : Z \times H \to \mathbb{R}, \\
&M, N : Z \times H \times \mathbb{R} \to \mathbb{R}.
\end{aligned}$$

For further results on dynamic programming, we refer to [7, 10, 11, 26]. We can assume that Z and H represent the states and decisions spaces, respectively. The problem related to dynamic programming is brought back to solve the following functional equations:

$$p(\gamma) = \sup_{\alpha \in H}\big\{g(\gamma,\alpha) + M\big(\gamma,\alpha,p\big(\tilde{u}(\gamma,\alpha)\big)\big)\big\}, \tag{5.1}$$

$$q(\gamma) = \sup_{\alpha\in H}\big\{u(\gamma,\alpha) + N\big(\gamma,\alpha,q\big(\tilde{u}(\gamma,\alpha)\big)\big)\big\} \tag{5.2}$$

for $\gamma \in Z$. We ensure the existence and uniqueness of a common and bounded solution of Eqs. (5.1) and (5.2). Suppose that $B(Z)$ is the set of all bounded real-valued functions on Z. Consider

$$d_b(h,k) = \|h-k\|_\infty^2 = \sup_{\gamma\in\tilde{N}}\big|h(\gamma)-k(\gamma)\big|^2 \tag{5.3}$$

for all $h,k \in B(Z)$. In such a way, $(B(Z),d_b)$ becomes a dislocated b-metric space. Assume that

$(C1)$: M, N, g, and u are bounded.

$(C2)$: For $\gamma \in Z$, $h \in B(Z)$, $S,T : B(Z) \to B(Z)$, take

$$Sh(\gamma) = \sup_{\alpha\in H}\big\{g(\gamma,\alpha) + M\big(\gamma,\alpha,h\big(\tilde{u}(\gamma,\alpha)\big)\big)\big\}. \tag{5.4}$$

$$Th(\gamma) = \sup_{\alpha\in H}\big\{u(\gamma,\alpha) + N\big(\gamma,\alpha,h\big(\tilde{u}(\gamma,\alpha)\big)\big)\big\}. \tag{5.5}$$

Moreover, assume also that there exist $\tau, f > 0$ such that, for every $(\gamma,\alpha) \in Z \times H$, $h,k \in B(Z)$, $t \in Z$,

$$|M\big(\gamma,\alpha,h(t)\big) - N(\gamma,\alpha,k(t)| \le D(h,k)f^{-\tau}, \tag{5.6}$$

where

$$D(h,k) = \sup\left\{\psi_b\begin{pmatrix} [|h(t)-k(t)|]^2, \\ \frac{[|h(t)-Sh(t)|]^2.[|k(t)-Tk(t)|]^2}{1+[|h(t)-k(t)|]^4}, \\ [|h(t)-Sh(t)|]^2, \\ [|h(t)-Tk(t)|]^2 \end{pmatrix}\right\}.$$

Theorem 5.1 *Assume that conditions* $(C1)$, $(C2)$ *and* (5.6) *hold. Then Eqs.* (5.1) *and* (5.2) *have a common and bounded solution in* $B(Z)$.

Proof Take any $\lambda > 0$. From (5.4) and (5.5), there exist $h_1,h_2 \in B(Z)$ and $\alpha_1,\alpha_2 \in H$ such that

$$(Sh_1) < g(\gamma,\alpha_1) + M\big(\gamma,\alpha_1,h_1\big(\tilde{u}(\gamma,\alpha_1)\big)\big) + \lambda, \tag{5.7}$$

$$(Th_2) < g(\gamma,\alpha_2) + N\big(\gamma,\alpha_2,h_2\big(\tilde{u}(\gamma,\alpha_2)\big)\big) + \lambda. \tag{5.8}$$

Again using the definition of supremum, we have

$$(Sh_1) \ge g(\gamma,\alpha_2) + M\big(\gamma,\alpha_2,h_1\big(\tilde{u}(\gamma,\alpha_2)\big)\big), \tag{5.9}$$

$$(Th_2) \ge g(\gamma,\alpha_1) + N\big(\gamma,\alpha_1,h_2\big(\tilde{u}(\gamma,\alpha_1)\big)\big). \tag{5.10}$$

Then, from inequalities (5.6), (5.7), and (5.10), we have

$$\begin{aligned}
&(Sh_1)(\gamma) - (Th_2)(\gamma) \\
&\quad \le M\bigl(\gamma, \alpha_1, h_1\bigl(\tilde{u}(\gamma, \alpha_1)\bigr)\bigr) - N\bigl(\gamma, \alpha_1, h_2\bigl(\tilde{u}(\gamma, \alpha_1)\bigr)\bigr) + \lambda \\
&\quad \le \bigl|M\bigl(\gamma, \alpha_1, h_1\bigl(\tilde{u}(\gamma, \alpha_1)\bigr)\bigr) - N\bigl(\gamma, \alpha_1, h_2\bigl(\tilde{u}(\gamma, \alpha_1)\bigr)\bigr)\bigr| + \lambda \\
&\quad \le D(h,k) f^{-\tau} + \lambda.
\end{aligned}$$

Since $\lambda > 0$ is arbitrary, we get

$$\begin{aligned}
&\bigl|Sh_1(\gamma) - Th_2(\gamma)\bigr| \le D(h,k) f^{-\tau}, \\
&f^{\tau}\bigl|Sh_1(\gamma) - Th_2(\gamma)\bigr| \le D(h,k).
\end{aligned}$$

This further implies that

$$\tau + \ln\bigl|Sh_1(\gamma) - Th_2(\gamma)\bigr| \le \ln(D(h,k).$$

Therefore, all the requirements of Theorem 4.1 hold for $F(g) = \ln g$; $g > 0$ and $d_\tau(h,k) = \|h-k\|_\tau^2$. Thus, we obtain a common fixed point $h^* \in B(W)$ of M and T, that is, $h^*(\gamma)$ is a common solution of Eqs. (5.1) and (5.2). □

6 Conclusion

In this work we have discussed the notion of b-metric like space, and we have given several applications. We attained fixed point achievements for general rational type F-contraction for a pair of semi α_*-dominated fuzzy mappings. The notion of fuzzy graph dominated mappings on a closed set has been introduced. Applications of two different types of Volterra type nonlinear integral inclusions and dynamic process are presented. Our results generalized and extended many recent fixed point results of Rasham et al. [24, 27, 28], Wardowski's result [39], Ameer et al. [7], and many classical results in the current literature (see [15, 21, 23, 31, 32]).

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