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Hermite–Hadamard integral inequalities on coordinated convex functions in quantum calculus

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Abstract

At first, we recall the q-operators in the context of q-calculus and by examining these operators we will introduce new definitions of the partial q-operators. Then, we investigate some new refinements inequalities of Hermite–Hadamard ($H-H$) type on the coordinated convex functions involving the new defined partial q-operators. From our main results, we establish several specific inequalities and we point out the existing results which had already been obtained in the literature.

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1 Introduction and Preliminaries

Let Ψ be defined on an interval $J \subseteq \mathbb{R}_e$ ($\mathbb{R}_e$ is the set of real numbers), then Ψ is convex if

$$\Psi\big(\lambda u + (1-\lambda)w\big) \leq \lambda\Psi(u) + (1-\lambda)\Psi(w), \tag{1.1}$$

for all u, w in J and for any λ in $[0,1]$.

Convex functions plays a vital role in the development of many fields of mathematics and significant applications are found in a variety of applied sciences such as optimization theory, number theory, combinatorics, special means theory, approximation theory and numerical analysis, see [1–5]. The well-known Hermite–Hadamard ($H-H$) inequality has central part in this development as it gives the criterion for convex functions. The $H-H$ inequality is given as follows: Let $\Psi : [A,B] \subset \mathbb{R}_e \to \mathbb{R}_e$ be a convex function with $A < B$, then the $H-H$ inequality is given as [6]

$$\Psi\left(\frac{A+B}{2}\right) \leq \frac{1}{B-A}\int_A^B \Psi(\lambda)\,d\lambda \leq \frac{\Psi(A)+\Psi(B)}{2}. \tag{1.2}$$

The $H-H$ inequality (1.2) is converted into three equivalent integral inequalities by the help of Riemann–Liouville fractional operators; see [7–9]. Also, it has been generalized and converted into many other integral inequalities by the help of other types of fractional operators, see [10–13].

In 2001, Dragomir [14] introduced the notion of coordinated convex functions in a rectangle from of the plane $\mathbb{R}_e^2$.

Definition 1.1 ([14]) Let $\Delta := [A,B] \times [C,D]$ be a bi-dimensional interval such that $0 \le A < B < \infty$, $0 \le C < D < \infty$. Then, a function $\Psi : \Delta \to \mathbb{R}_e$ is called coordinated convex on Δ, if the partial mappings $\Psi_v : [A,B] \to \mathbb{R}_e$, $\Psi_v(x) = \Psi(x,v)$ and $\Psi_u : [C,D] \to \mathbb{R}_e$, $\Psi_u(y) = \Psi(u,y)$ are convex for each $y, v \in [C,D]$ and $x, u \in [A,B]$. Also, Ψ satisfies the inequality

$$\Psi\big(\lambda u + (1-\lambda)w, \lambda v + (1-\lambda)z\big) \le \lambda \Psi(u,v) + (1-\lambda)\Psi(w,z), \tag{1.3}$$

for all $(u,v),(w,z) \in \Delta$ and $\lambda \in [0,1]$.

By the help of above definition, Dragomir [14] established the following $H-H$ type inequalities similar to the one dimensional case.

Theorem 1.1 ([14]) *Suppose that* $\Psi : \Delta \to \mathbb{R}_e$ *is a coordinated convex function on* Δ. *Then we have*

$$\begin{aligned}
&\Psi\left(\frac{A+B}{2}, \frac{C+D}{2}\right) \\
&\le \frac{1}{2}\left[\frac{1}{B-A}\int_A^B \Psi\left(u, \frac{C+D}{2}\right) du + \frac{1}{D-C}\int_C^D \Psi\left(\frac{A+B}{2}, v\right) dv\right] \\
&\le \frac{1}{(B-A)(D-C)}\int_A^B \int_C^D \Psi(u,v)\, du\, dv \\
&\le \frac{1}{2}\left[\frac{1}{B-A}\int_A^B \big[\Psi(u,C) + \Psi(u,D)\big] du + \frac{1}{D-C}\int_C^D \big[\Psi(A,v) + \Psi(B,v)\big] dv\right] \\
&\le \frac{\Psi(A,C) + \Psi(A,D) + \Psi(B,C) + \Psi(B,D)}{4}.
\end{aligned} \tag{1.4}$$

Many relevant results have been reported in this direction with different class of convex functions; see [15–19] and the references therein.

In the early 20th century, Jackson [20, 21] worked on the classical notion of a derivative without limit allowing for easier study of number theory and ordinary calculus in his investigations. Jackson got the credit of the q-analogue of the various well-known results of calculus; see [20–24].

For a real valued function Ψ, the q-derivative is characterized by

$$\mathbf{D}_q \Psi(\varsigma) = \frac{\Psi(q\varsigma) - \Psi(\varsigma)}{q\varsigma - \varsigma}, \tag{1.5}$$

where $q \in (0,1)$. The well-known Jackson integral of a real valued function Ψ is given by the following series expansion:

$$\int_0^{\mu} \Psi(\varsigma)\, d_q\varsigma = (1-q)\mu \sum_{r=0}^{\infty} q^r \Psi\big(q^r \mu\big). \tag{1.6}$$

References [25, 26] discuss the notion of q-derivatives and q-integrals over the finite interval $[k_1, k_2]$ of real numbers and defined the q_{k_1}-derivative and q_{k_1}-integral.

Definition 1.2 ([25, 26]) For any continuous function $\Psi : [k_1, k_2] \to \mathbb{R}_e$ and $q \in (0, 1)$, the q_{k_1}-derivative of Ψ at $\varsigma \in [k_1, k_2]$ is defined by

$$_{k_1}\mathbf{D}_q\Psi(\varsigma) = \frac{\Psi(\varsigma) - \Psi(q\varsigma + (1-q)k_1)}{(1-q)(\varsigma - k_1)}, \quad \varsigma \neq k_1. \tag{1.7}$$

Definition 1.3 ([25, 26]) For a continuous function $\Psi : [k_1, k_2] \to \mathbb{R}_e$ and $q \in (0, 1)$, the q_{k_1}-integral of Ψ at $\varsigma \in [k_1, k_2]$ is defined by

$$\int_{k_1}^{k} \Psi(\varsigma)_{k_1}\, d_q\varsigma = (1-q)(k-k_1)\sum_{h=0}^{\infty} q^h \Psi\bigl(q^h k + \bigl(1-q^h\bigr)k_1\bigr), \quad k \in [k_1, k_2]. \tag{1.8}$$

If $k_1 = 0$, then

$$\int_{0}^{k} \Psi(\varsigma)_0\, d_q\varsigma = (1-q)k\sum_{h=0}^{\infty} q^h \Psi\bigl(q^h k\bigr) = \int_0^k \Psi(\varsigma)\, d_q\varsigma, \tag{1.9}$$

which is the classical q-integral (1.6).

In view of the above definitions, the authors [27] established the following inequality.

Theorem 1.2 ([27]) *Let* $\Psi : [A, B] \to \mathbb{R}_e$ *be a convex differentiable function on* $[A, B]$ *and* $q \in (0, 1)$. *Then we have*

$$\Psi\left(\frac{qA + B}{1+q}\right) \leq \frac{1}{B-A}\int_A^B \Psi(\varsigma)\,_A\, d_q\varsigma \leq \frac{q\Psi(A) + \Psi(B)}{1+q}. \tag{1.10}$$

In [28], the authors generalize the notion of q-derivatives and q-integrals by introducing the q^{k_2}-derivative and q^{k_2}-integral over the finite real interval $[k_1, k_2]$.

Definition 1.4 ([28]) For any continuous function $\Psi : [k_1, k_2] \to \mathbb{R}_e$ and $q \in (0, 1)$, the q^{k_2}-derivative of Ψ at $\varsigma \in [k_1, k_2]$ is defined by

$$^{k_2}\mathbf{D}_q\Psi(\varsigma) = \frac{\Psi(\varsigma) - \Psi(q\varsigma + (1-q)k_2)}{(1-q)(\varsigma - k_2)}, \quad \varsigma \neq k_2. \tag{1.11}$$

Definition 1.5 ([28]) For any continuous function $\Psi : [k_1, k_2] \to \mathbb{R}_e$ and $q \in (0, 1)$, the q^{k_2}-integral of Ψ at $\varsigma \in [k_1, k_2]$ is defined by

$$\int_{k}^{k_2} \Psi(\varsigma)\,^{k_2}\, d_q\varsigma = (1-q)(k_2-k)\sum_{h=0}^{\infty} q^h \Psi\bigl(q^h k + \bigl(1-q^h\bigr)k_2\bigr), \quad k \in [k_1, k_2]. \tag{1.12}$$

In [28], the authors established the following counterpart for integrals (1.10).

Theorem 1.3 ([28]) *Let* $\Psi : [A,B] \to \mathbb{R}_e$ *be a convex differentiable function on* $[A,B]$ *and* $q \in (0,1)$. *Then we have*

$$\Psi\left(\frac{A+qB}{1+q}\right) \le \frac{1}{B-A}\int_A^B \Psi(\varsigma)\,{}^B d_q\varsigma \le \frac{\Psi(A)+q\Psi(B)}{1+q}. \tag{1.13}$$

In [29], the authors extended the definition of q-derivative and q-integral to the case of two variables.

Definition 1.6 ([29]) Let $\Psi : \Delta \to \mathbb{R}_e$ be a continuous function of two variables and $0 < q_1 < 1$, $0 < q_2 < 1$. Then, the partial q_{1A}-derivative, partial q_{2C}-derivative and partial $q_{1A}q_{2C}$- derivatives are defined by

$$\frac{{}_A\partial_{q_1}\Psi(u,v)}{{}_A\partial_{q_1}u} = \frac{\Psi(u,v)-\Psi(q_1u+(1-q_1)A,v)}{(1-q_1)(u-A)}, \quad u \neq A, \tag{1.14}$$

$$\frac{{}_C\partial_{q_2}\Psi(u,v)}{{}_C\partial_{q_2}u} = \frac{\Psi(u,v)-\Psi(u,q_2v+(1-q_2)C)}{(1-q_2)(v-C)}, \quad v \neq C, \tag{1.15}$$

and

$$\begin{aligned}\frac{{}_{A,C}\partial_{q_1q_2}\Psi(u,v)}{{}_A\partial_{q_1}u\,{}_C\partial_{q_2}v} &= \frac{1}{(1-q_1)(u-A)(1-q_2)(v-C)}\big[\Psi\big(q_1u+(1-q_1)A,q_2v+(1-q_2)C\big)\\ &\quad -\Psi\big(q_1u+(1-q_1)A,v\big)-\Psi\big(u,q_2v+(1-q_2)C\big)\\ &\quad +\Psi(u,v)\big], \quad u\neq A, v\neq C.\end{aligned} \tag{1.16}$$

Definition 1.7 ([29]) Let $\Psi : \Delta \to \mathbb{R}_e$ be a continuous function of two variables and $0 < q_1 < 1$, $0 < q_2 < 1$, Then the definite $q_{1A}q_{2C}$-integral on Δ is defined by

$$\begin{aligned}&\int_A^\nu\int_C^\mu \Psi(u,v)\,{}_C d_{q_2}v\,{}_A d_{q_1}u\\ &\quad=(1-q_1)(1-q_2)(\nu-A)(\mu-C)\\ &\qquad\times\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}q_1^kq_2^h\Psi\big(q_1^k\nu+\big(1-q_1^k\big)A,q_2^h\mu+\big(1-q_2^h\big)C\big),\end{aligned} \tag{1.17}$$

for all $(\nu,\mu) \in \Delta$.

Meanwhile, the authors [29] proved the following inequality.

Theorem 1.4 ([29]) *Let* $\Psi : \Delta \to \mathbb{R}_e$ *be a coordinated convex function on* Δ, *then the following inequalities hold*:

$$\begin{aligned}&\Psi\left(\frac{A+B}{2},\frac{C+D}{2}\right)\\ &\quad\le\frac{1}{2}\left[\frac{1}{B-A}\int_A^B\Psi\left(u,\frac{C+D}{2}\right){}_A d_{q_1}u+\frac{1}{D-C}\int_C^D\Psi\left(\frac{A+B}{2},v\right){}_C d_{q_2}v\right]\\ &\quad\le\frac{1}{(B-A)(D-C)}\int_A^B\int_C^D\Psi(u,v)\,{}_C d_{q_2}v\,{}_A d_{q_1}u\end{aligned}$$

$$\begin{aligned}
&\leq \frac{q_2}{2(1+q_2)(B-A)} \int_A^B \Psi(u,C)\,{}_A d_{q_1} u \\
&\quad + \frac{1}{2(1+q_2)(B-A)} \int_A^B \Psi(u,D)\,{}_A d_{q_1} u + \frac{q_1}{2(1+q_1)(D-C)} \int_C^D \Psi(A,v)\,{}_C d_{q_2} v \\
&\quad + \frac{1}{2(1+q_1)(D-C)} \int_C^D \Psi(B,v)\,{}_C d_{q_2} v \\
&\leq \frac{q_1 q_2 \Psi(A,C) + q_1 \Psi(A,D) + q_2 \Psi(B,C) + \Psi(B,D)}{(1+q_1)(1+q_2)}. \qquad (1.18)
\end{aligned}$$

Recently, in [30], the authors disproved the inequality (1.18) by giving a counter example and proved the following correct $H-H$ inequality.

Theorem 1.5 ([30]) *Let* $\Psi : \Delta \to \mathbb{R}_e$ *be a coordinated convex function on* Δ, *then for all* $q_1, q_2 \in (0,1)$, *we have*

$$\begin{aligned}
&\Psi\left(\frac{q_1 A + B}{1+q_1}, \frac{q_2 C + D}{1+q_2}\right) \\
&\leq \frac{1}{2}\left[\frac{1}{B-A} \int_A^B \Psi\left(u, \frac{q_2 C + D}{1+q_2}\right) {}_A d_{q_1} u + \frac{1}{D-C} \int_C^D \Psi\left(\frac{q_1 A + B}{1+q_1}, v\right) {}_C d_{q_2} v\right] \\
&\leq \frac{1}{(B-A)(D-C)} \int_A^B \int_C^D \Psi(u,v)\,{}_C d_{q_2} v\,{}_A d_{q_1} u \\
&\leq \frac{q_2}{2(1+q_2)(B-A)} \int_A^B \Psi(u,C)\,{}_A d_{q_1} u \\
&\quad + \frac{1}{2(1+q_2)(B-A)} \int_A^B \Psi(u,D)\,{}_A d_{q_1} u + \frac{q_1}{2(1+q_1)(D-C)} \int_C^D \Psi(A,v)\,{}_C d_{q_2} v \\
&\quad + \frac{1}{2(1+q_1)(D-C)} \int_C^D \Psi(B,v)\,{}_C d_{q_2} v \\
&\leq \frac{q_1 q_2 \Psi(A,C) + q_1 \Psi(A,D) + q_2 \Psi(B,C) + \Psi(B,D)}{(1+q_1)(1+q_2)}. \qquad (1.19)
\end{aligned}$$

Now, by combining the two concepts of q-derivatives and q-integrals given in Definitions 1.2–1.7, we present the following mixed types of partial derivatives and integrals in the two variables along with some examples.

Definition 1.8 Let $\Psi : \Delta \to \mathbb{R}_e$ be a continuous function of two variables and $0 < q_1 < 1$, $0 < q_2 < 1$, Then the partial q_1^B-derivative, partial q_2^D-derivative and partial $q_1^B q_2^D$, $q_1^B q_{2C}$ and $q_{1A} q_2^D$ derivatives are defined, respectively, by

$$\frac{{}^B\partial_{q_1} \Psi(u,v)}{{}^B\partial_{q_1} u} = \frac{\Psi(u,v) - \Psi(q_1 u + (1-q_1)B, v)}{(1-q_1)(u-B)}, \quad u \neq B, \qquad (1.20)$$

$$\frac{{}^D\partial_{q_2} \Psi(u,v)}{{}^D\partial_{q_2} u} = \frac{\Psi(u,v) - \Psi(u, q_2 v + (1-q_2)D)}{(1-q_2)(v-D)}, \quad v \neq D, \qquad (1.21)$$

$$\begin{aligned}
&\frac{{}_{B,D}\partial_{q_1 q_2} \Psi(u,v)}{{}^B\partial_{q_1} u\, {}^D\partial_{q_2} v} \\
&= \frac{1}{(1-q_1)(u-B)(1-q_2)(v-D)} \big[\Psi\big(q_1 u + (1-q_1)B, q_2 v + (1-q_2)D\big)
\end{aligned}$$

$$- \Psi\big(q_1u + (1-q_1)B, v\big) - \Psi\big(u, q_2v + (1-q_2)D\big)$$
$$+ \Psi(u,v)\big], \quad u \neq B, v \neq D, \tag{1.22}$$

$$\frac{{}^{B}_{C}\partial_{q_1q_2}\Psi(u,v)}{{}^{B}\partial_{q_1}u\,{}_{C}\partial_{q_2}v} = \frac{1}{(1-q_1)(u-B)(1-q_2)(v-C)}\big[\Psi\big(q_1u + (1-q_1)B, q_2v + (1-q_2)C\big)$$
$$- \Psi\big(q_1u + (1-q_1)B, v\big) - \Psi\big(u, q_2v + (1-q_2)C\big)$$
$$+ \Psi(u,v)\big], \quad u \neq B, v \neq C, \tag{1.23}$$

and

$$\frac{{}^{D}_{A}\partial_{q_1q_2}\Psi(u,v)}{{}_{A}\partial_{q_1}u\,{}^{D}\partial_{q_2}v} = \frac{1}{(1-q_1)(u-A)(1-q_2)(v-D)}\big[\Psi\big(q_1u + (1-q_1)A, q_2v + (1-q_2)D\big)$$
$$- \Psi\big(q_1u + (1-q_1)A, v\big) - \Psi\big(u, q_2v + (1-q_2)D\big)$$
$$+ \Psi(u,v)\big], \quad u \neq A, v \neq D. \tag{1.24}$$

Definition 1.9 Let $\Psi : \Delta \to \mathbb{R}_e$ be a continuous function of two variables and $0 < q_1 < 1$, $0 < q_2 < 1$, Then the definite $q_1^Bq_2^D$-integral, $q_{1A}q_2^D$-integral and $q_1^Bq_{2C}$-integral on Δ are defined by

$$\int_{\nu}^{B}\int_{\mu}^{D} \Psi(u,v)\,{}^{D}d_{q_2}v\,{}^{B}d_{q_1}u = (1-q_1)(1-q_2)(B-\nu)(D-\mu)$$
$$\times \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} q_1^k q_2^h \Psi\big(q_1^k\nu + \big(1-q_1^k\big)B, q_2^h\mu + \big(1-q_2^h\big)D\big), \quad \forall(\nu,\mu) \in \Delta, \tag{1.25}$$

$$\int_{A}^{\nu}\int_{\mu}^{D} \Psi(u,v)\,{}^{D}d_{q_2}v\,{}_{A}d_{q_1}u = (1-q_1)(1-q_2)(\nu-A)(D-\mu)$$
$$\times \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} q_1^k q_2^h \Psi\big(q_1^k\nu + \big(1-q_1^k\big)A, q_2^h\mu + \big(1-q_2^h\big)D\big), \quad \forall(\nu,\mu) \in \Delta, \tag{1.26}$$

and

$$\int_{\nu}^{B}\int_{C}^{\mu} \Psi(u,v)\,{}_{C}d_{q_2}v\,{}^{B}d_{q_1}u = (1-q_1)(1-q_2)(B-\nu)(\mu-C)$$
$$\times \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} q_1^k q_2^h \Psi\big(q_1^k\nu + \big(1-q_1^k\big)B, q_2^h\mu + \big(1-q_2^h\big)C\big), \quad \forall(\nu,\mu) \in \Delta. \tag{1.27}$$

Example 1.1 All the q_1q_2-integrals are different for general functions. For instance,

$$\int_A^B \int_C^D uv \,{}^D d_{q_2}v \,{}^B d_{q_1}u = \frac{(B-A)(A+q_1B)(D-C)(C+q_2D)}{(1+q_1)(1+q_2)},$$
$$\int_A^B \int_C^D uv \,{}^D d_{q_2}v \,{}_A d_{q_1}u = \frac{(B-A)(q_1A+B)(D-C)(C+q_2D)}{(1+q_1)(1+q_2)},$$
$$\int_A^B \int_C^D uv \,{}_C d_{q_2}v \,{}^B d_{q_1}u = \frac{(B-A)(A+q_1B)(D-C)(q_2C+D)}{(1+q_1)(1+q_2)},$$
$$\int_A^B \int_C^D uv \,{}_C d_{q_2}v \,{}_A d_{q_1}u = \frac{(B-A)(q_1A+B)(D-C)(q_2C+D)}{(1+q_1)(1+q_2)}.$$

Furthermore,

$$\int_C^B \int_C^D uv\,dv\,du = \frac{(B^2-A^2)(D^2-C^2)}{4},$$

subject to the condition that both $q_1, q_2 \to 1^-$.

Now, we obtain midpoint type inequalities from the inequality (1.19).

Remark 1.1 We have four special cases for this inequality at midpoint.

- If $A=a$, $B=\frac{q_1a+b}{1+q_1}$, $C=c$, $D=\frac{q_2d+c}{1+q_2}$, then

$$\begin{aligned}
&\Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, \frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\right) \\
&\leq \frac{1+q_1}{2(b-a)} \int_a^{\frac{q_1a+b}{1+q_1}} \Psi\left(u, \frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\right) {}_a d_{q_1}u \\
&\quad + \frac{1+q_2}{2q_2(d-c)} \int_c^{\frac{q_2d+c}{1+q_2}} \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, v\right) {}_c d_{q_2}v \\
&\leq \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)} \int_a^{\frac{q_1a+b}{1+q_1}} \int_c^{\frac{q_2d+c}{1+q_2}} \Psi(u,v)\, {}_c d_{q_2}v \,{}_a d_{q_1}u \\
&\leq \frac{q_2(1+q_1)}{2(1+q_2)(b-a)} \int_a^{\frac{q_1a+b}{1+q_1}} \Psi(u,c)\, {}_a d_{q_1}u \\
&\quad + \frac{1+q_1}{2(1+q_2)(b-a)} \int_a^{\frac{q_1a+b}{1+q_1}} \Psi\left(u, \frac{q_2d+c}{1+q_2}\right) {}_a d_{q_1}u \\
&\quad + \frac{q_1(1+q_2)}{2q_2(1+q_1)(d-c)} \int_c^{\frac{q_2d+c}{1+q_2}} \Psi(a,v)\, {}_c d_{q_2}v \\
&\quad + \frac{1+q_2}{2q_2(1+q_1)(d-c)} \int_c^{\frac{q_2d+c}{1+q_2}} \Psi\left(\frac{q_1a+b}{1+q_1}, v\right) {}_c d_{q_2}v \\
&\leq \frac{q_1q_2\Psi(a,c) + q_1\Psi(a, \frac{q_2d+c}{1+q_2}) + q_2\Psi(\frac{q_1a+b}{1+q_1}, c) + \Psi(\frac{q_1a+b}{1+q_1}, \frac{q_2d+c}{1+q_2})}{(1+q_1)(1+q_2)}.
\end{aligned} \tag{1.28}$$

- If $A = a$, $B = \frac{q_1 b+a}{1+q_1}$, $C = c$, $D = \frac{q_2 d+c}{1+q_2}$, then

$$\begin{aligned}
&\Psi\left(\frac{(1+q_1+q_1^2)a+q_1 b}{(1+q_1)^2}, \frac{(1+q_2+q_2^2)c+q_2 d}{(1+q_2)^2}\right) \\
&\leq \frac{1+q_1}{2q_1(b-a)} \int_a^{\frac{q_1 b+a}{1+q_1}} \Psi\left(u, \frac{(1+q_2+q_2^2)c+q_2 d}{(1+q_2)^2}\right) {}_a d_{q_1} u \\
&\quad + \frac{1+q_2}{2q_2(d-c)} \int_c^{\frac{q_2 d+c}{1+q_2}} \Psi\left(\frac{(1+q_1+q_1^2)a+q_1 b}{(1+q_1)^2}, \nu\right) {}_c d_{q_2} \nu \\
&\leq \frac{(1+q_1)(1+q_2)}{q_1 q_2 (b-a)(d-c)} \int_a^{\frac{q_1 b+a}{1+q_1}} \int_c^{\frac{q_2 d+c}{1+q_2}} f(u,\nu) \, {}_c d_{q_2} \nu \, {}_a d_{q_1} u \\
&\leq \frac{q_2(1+q_1)}{2q_1(1+q_2)(b-a)} \int_a^{\frac{q_1 b+a}{1+q_1}} \Psi(u,c) \, {}_a d_{q_1} u \\
&\quad + \frac{1+q_1}{2q_1(1+q_2)(b-a)} \int_a^{\frac{q_1 b+a}{1+q_1}} \Psi\left(u, \frac{q_2 d+c}{1+q_2}\right) {}_a d_{q_1} u \\
&\quad + \frac{q_1(1+q_2)}{2q_2(1+q_1)(d-c)} \int_c^{\frac{q_2 d+c}{1+q_2}} \Psi(a,\nu) \, {}_c d_{q_2} \nu \\
&\quad + \frac{1+q_2}{2q_2(1+q_1)(d-c)} \int_c^{\frac{q_2 d+c}{1+q_2}} \Psi\left(\frac{q_1 a+b}{1+q_1}, \nu\right) {}_c d_{q_2} \nu \\
&\leq \frac{q_1 q_2 \Psi(a,c) + q_1 \Psi(a, \frac{q_2 d+c}{1+q_2}) + q_2 \Psi(\frac{q_1 b+a}{1+q_1}, c) + \Psi(\frac{q_1 b+a}{1+q_1}, \frac{q_2 d+c}{1+q_2})}{(1+q_1)(1+q_2)}.
\end{aligned} \tag{1.29}$$

- If $A = a$, $B = \frac{q_1 a+b}{1+q_1}$, $C = c$, $D = \frac{q_2 c+d}{1+q_2}$, then

$$\begin{aligned}
&\Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) \\
&\leq \frac{1+q_1}{2(b-a)} \int_a^{\frac{q_1 a+b}{1+q_1}} \Psi\left(u, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) {}_a d_{q_1} u \\
&\quad + \frac{1+q_2}{2(d-c)} \int_c^{\frac{q_2 c+d}{1+q_2}} \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, \nu\right) {}_c d_{q_2} \nu \\
&\leq \frac{(1+q_1)(1+q_2)}{(b-a)(d-c)} \int_a^{\frac{q_1 a+b}{1+q_1}} \int_c^{\frac{q_2 c+d}{1+q_2}} f(u,\nu) \, {}_c d_{q_2} \nu \, {}_a d_{q_1} u \\
&\leq \frac{q_2(1+q_1)}{2(1+q_2)(b-a)} \int_a^{\frac{q_1 a+b}{1+q_1}} \Psi(u,c) \, {}_a d_{q_1} u \\
&\quad + \frac{1+q_1}{2(1+q_2)(b-a)} \int_a^{\frac{q_1 a+b}{1+q_1}} \Psi\left(u, \frac{q_2 d+c}{1+q_2}\right) {}_a d_{q_1} u \\
&\quad + \frac{q_1(1+q_2)}{2(1+q_1)(d-c)} \int_c^{\frac{q_2 c+d}{1+q_2}} \Psi(a,\nu) \, {}_c d_{q_2} \nu \\
&\quad + \frac{1+q_2}{2(1+q_1)(d-c)} \int_c^{\frac{q_2 c+d}{1+q_2}} \Psi\left(\frac{q_1 a+b}{1+q_1}, \nu\right) {}_c d_{q_2} \nu
\end{aligned}$$

$$\leq \frac{q_1q_2\Psi(a,c)+q_1\Psi(a,\frac{q_2c+d}{1+q_2})+q_2\Psi(\frac{q_1a+b}{1+q_1},c)+\Psi(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2})}{(1+q_1)(1+q_2)}. \tag{1.30}$$

- If $A=a$, $B=\frac{q_1b+a}{1+q_1}$, $C=c$, $D=\frac{q_2c+d}{1+q_2}$, then

$$\begin{aligned}
&\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},\frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right)\\
&\leq \frac{1+q_1}{2q_1(b-a)}\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right)\,{}_a d_{q_1}u\\
&\quad+\frac{1+q_2}{2(d-c)}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},v\right)\,{}_c d_{q_2}v\\
&\leq \frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\int_a^{\frac{q_1b+a}{1+q_1}}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi(u,v)\,{}_c d_{q_2}v\,{}_a d_{q_1}u\\
&\leq \frac{q_2(1+q_1)}{2q_1(1+q_2)(b-a)}\int_a^{\frac{q_1b+a}{1+q_1}}\Psi(u,c)\,{}_a d_{q_1}u\\
&\quad+\frac{1+q_1}{2q_1(1+q_2)(b-a)}\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}_a d_{q_1}u\\
&\quad+\frac{q_1(1+q_2)}{2(1+q_1)(d-c)}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi(a,v)\,{}_c d_{q_2}v\\
&\quad+\frac{1+q_2}{2(1+q_1)(d-c)}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi\left(\frac{q_1b+a}{1+q_1},v\right)\,{}_c d_{q_2}v\\
&\leq \frac{q_1q_2\Psi(a,c)+q_1\Psi(a,\frac{q_2c+d}{1+q_2})+q_2\Psi(\frac{q_1b+a}{1+q_1},c)+\Psi(\frac{q_1b+a}{1+q_1},\frac{q_2c+d}{1+q_2})}{(1+q_1)(1+q_2)}.
\end{aligned} \tag{1.31}$$

In view of the above results and literatures, and following this tendency of the newly introduced q-derivatives and q-integrals, the aim of this paper is to establish some new refinements of the $H-H$ inequality in the quantum domain using coordinated convex functions. Several special cases from our main results will be given in detail and many well-known results will be recaptured. At the end, we provide a briefly conclusion as well.

2 Main results

By utilizing Theorem 1.3, we have the new result.

Theorem 2.1 *Suppose that $\Psi:\Delta\to\mathbb{R}_e$ is a coordinated convex function on Δ and $\Psi\in L_1(\Delta)$. Then we have*

$$\begin{aligned}
&\Psi\left(\frac{A+q_1B}{1+q_1},\frac{C+q_2D}{1+q_2}\right)\\
&\leq \frac{1}{2}\left[\frac{1}{B-A}\int_A^B\Psi\left(u,\frac{C+q_2D}{1+q_2}\right)\,{}^B d_{q_1}u+\frac{1}{D-C}\int_C^D\Psi\left(\frac{A+q_1B}{1+q_1},v\right)\,{}^D d_{q_2}v\right]\\
&\leq \frac{1}{(B-A)(D-C)}\int_A^B\int_C^D\Psi(u,v)\,{}^D d_{q_2}v\,{}^B d_{q_1}u
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2(1+q_2)(B-A)}\int_A^B \Psi(u,C)\,{}^B d_{q_1}u + \frac{q_2}{2(1+q_2)(B-A)}\int_A^B \Psi(u,D)\,{}^B d_{q_1}u \\
&\quad + \frac{1}{2(1+q_1)(D-C)}\int_C^D \Psi(A,v)\,{}^D d_{q_2}v + \frac{q_1}{2(1+q_1)(D-C)}\int_C^D \Psi(B,v)\,{}^D d_{q_2}v \\
&\leq \frac{\Psi(A,C)+q_2\Psi(A,D)+q_1\Psi(B,C)+q_1q_2\Psi(B,D)}{(1+q_1)(1+q_2)}. \qquad (2.1)
\end{aligned}$$

Proof Due to the coordinated convexity of $\Psi : \Delta \to \mathbb{R}_e$, the partial mapping $\Psi_u : [C,D] \to \mathbb{R}_e$ defined by $\Psi_u(y) = \Psi(u,y)$ for all $u \in [A,B]$ will be convex on $[C,D]$. Analogously, $\Psi_v : [A,B] \to \mathbb{R}_e$ defined by $\Psi_v(x) = \Psi(x,v)$ for all $v \in [C,D]$ is convex on $[A,B]$. Then, by the Theorem 1.3, we get

$$\Psi_u\left(\frac{C+q_2D}{1+q_2}\right) \leq \frac{1}{D-C}\int_C^D \Psi_u(v)\,{}^D d_{q_2}v \leq \frac{\Psi_u(C)+q_2\Psi_u(D)}{1+q_2},$$

or

$$\Psi\left(u,\frac{C+q_2D}{1+q_2}\right) \leq \frac{1}{D-C}\int_C^D \Psi(u,v)\,{}^D d_{q_2}v \leq \frac{\Psi(u,C)+q_2\Psi(u,D)}{1+q_2}. \qquad (2.2)$$

Integrating the inequality over $[A,B]$, we have

$$\begin{aligned}
\frac{1}{B-A}\int_A^B \Psi\left(u,\frac{C+q_2D}{1+q_2}\right)\,{}^B d_{q_1}u &\leq \frac{1}{(B-A)(D-C)}\int_A^B\int_C^D \Psi(u,v)^D d_{q_2}v\,{}^B d_{q_1}u \\
&\leq \frac{1}{(1+q_2)(B-A)}\int_A^B \Psi(u,C)^B d_{q_1}u \\
&\quad + \frac{q_2}{(1+q_2)(B-A)}\int_A^B \Psi(u,D)\,{}^B d_{q_1}u. \qquad (2.3)
\end{aligned}$$

Now, by the convexity of Ψ_v, by the Theorem 1.3, we have

$$\Psi_v\left(\frac{A+q_1B}{1+q_1}\right) \leq \frac{1}{B-A}\int_A^B \Psi_v(u)\,{}^B d_{q_1}u \leq \frac{\Psi_v(A)+q_1\Psi_v(B)}{1+q_1}, \qquad (2.4)$$

or

$$\Psi\left(\frac{A+q_1B}{1+q_1},v\right) \leq \frac{1}{B-A}\int_A^B \Psi(u,v)\,{}^B d_{q_1}u \leq \frac{\Psi(A,v)+q_1\Psi(B,v)}{1+q_1}. \qquad (2.5)$$

Evaluating the average integral over $[C,D]$, we have

$$\begin{aligned}
&\frac{1}{D-C}\int_C^D \Psi\left(\frac{A+q_1B}{1+q_2},v\right)\,{}^D d_{q_2}v \\
&\leq \frac{1}{(B-A)(D-C)}\int_A^B\int_C^D \Psi(u,v)\,{}^D d_{q_2}v\,{}^B d_{q_1}u \\
&\leq \frac{1}{(1+q_1)(D-C)}\int_C^D \Psi(A,v)\,{}^D d_{q_2}v + \frac{q_1}{(1+q_1)(D-C)}\int_C^D \Psi(B,v)\,{}^D d_{q_2}v. \qquad (2.6)
\end{aligned}$$

Adding (2.5) and (2.6), we get

$$\begin{aligned}
&\frac{1}{B-A}\int_A^B \Psi\left(u,\frac{C+q_2D}{1+q_2}\right)\,{}^B d_{q_1}u+\frac{1}{D-C}\int_C^D \Psi\left(\frac{A+q_1B}{1+q_2},v\right)\,{}^D d_{q_2}v\\
&\quad\le \frac{2}{(B-A)(D-C)}\int_A^B\int_C^D \Psi(u,v)\,{}^D d_{q_2}v\,{}^B d_{q_1}u\\
&\quad\le \frac{1}{(1+q_2)(B-A)}\int_A^B \Psi(u,C)\,{}^B d_{q_1}u+\frac{q_2}{(1+q_2)(B-A)}\int_A^B \Psi(u,D)\,{}^B d_{q_1}u\\
&\qquad+\frac{1}{(1+q_1)(D-C)}\int_C^D \Psi(A,v)\,{}^D d_{q_2}v+\frac{q_1}{(1+q_1)(D-C)}\int_C^D \Psi(B,v)\,{}^D d_{q_2}v.
\end{aligned}\tag{2.7}$$

Again, by the convexity and applying the first inequality of Theorem 1.3, we have

$$\Psi\left(\frac{A+q_1B}{1+q_2},\frac{C+q_2D}{1+q_2}\right)\le\frac{1}{D-C}\int_C^D\Psi\left(\frac{A+q_1B}{1+q_2},v\right)\,{}^D d_{q_2}v\tag{2.8}$$

and

$$\Psi\left(\frac{A+q_1B}{1+q_2},\frac{C+q_2D}{1+q_2}\right)\le\frac{1}{B-A}\int_A^B\Psi\left(u,\frac{C+q_2D}{1+q_2}\right)\,{}^B d_{q_1}u.\tag{2.9}$$

Similarly, applying the second inequality of Theorem 1.3 and convexity of Ψ_u and Ψ_v, we have

$$\begin{aligned}
&\frac{1}{(1+q_2)(B-A)}\int_A^B \Psi(u,C)\,{}^B d_{q_1}u+\frac{q_2}{(1+q_2)(B-A)}\int_A^B \Psi(u,D)\,{}^B d_{q_1}u\\
&\qquad+\frac{1}{(1+q_1)(D-C)}\int_C^D \Psi(A,v)\,{}^D d_{q_2}v+\frac{q_1}{(1+q_1)(D-C)}\int_C^D \Psi(B,v)\,{}^D d_{q_2}v\\
&\quad\le\frac{2}{(1+q_1)(1+q_2)}\big[\Psi(A,C)+q_2\Psi(A,D)+q_1\Psi(B,C)+q_1q_2\Psi(B,D)\big].
\end{aligned}\tag{2.10}$$

Combining inequalities (2.7), (2.8), (2.9) and (2.10), we directly obtain our desired inequality. □

Remark 2.1 From Theorem 2.1, we can deduce the following midpoint special cases.

- If $A=\frac{q_1a+b}{1+q_1}$, $B=b$, $C=\frac{q_2d+c}{1+q_2}$, $D=d$, then

$$\begin{aligned}
&\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\\
&\quad\le\frac{1+q_1}{2q_1(b-a)}\int_{\frac{q_1a+b}{1+q_1}}^b\Psi\left(u,\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}^b d_{q_1}u\\
&\qquad+\frac{1+q_2}{2(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^d\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},v\right)\,{}^d d_{q_2}v\\
&\quad\le\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\int_{\frac{q_1a+b}{1+q_1}}^b\int_{\frac{q_2d+c}{1+q_2}}^d\Psi(u,v)\,{}^d d_{q_2}v\,{}^b d_{q_1}u\\
&\quad\le\frac{1+q_1}{2q_1(1+q_2)(b-a)}\int_{\frac{q_1a+b}{1+q_1}}^b\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}^b d_{q_1}u
\end{aligned}$$

$$
\begin{aligned}
&+\frac{q_2(1+q_1)}{2q_1(1+q_2)(b-a)}\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi(u,d)\,{}^{b}d_{q_1}u\\
&+\frac{1+q_2}{2(1+q_1)(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{q_1a+b}{1+q_1},v\right){}^{d}d_{q_2}v\\
&+\frac{q_1(1+q_2)}{2(1+q_1)(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(b,v)\,{}^{d}d_{q_2}v\\
&\le\frac{\Psi(\frac{q_1a+b}{1+q_1},\frac{q_2d+c}{1+q_2})+q_2\Psi(\frac{q_1a+b}{1+q_1},d)+q_1\Psi(b,\frac{q_2d+c}{1+q_2})+q_1q_2\Psi(b,d)}{(1+q_1)(1+q_2)}.
\end{aligned}
\tag{2.11}
$$

- If $A=\frac{q_1b+a}{1+q_1}$, $B=b$, $C=\frac{q_2d+c}{1+q_2}$, $D=d$, then

$$
\begin{aligned}
&\Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\\
&\le\frac{1+q_1}{2(b-a)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\Psi\left(u,\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right){}^{b}d_{q_1}u\\
&+\frac{1+q_2}{2(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},v\right){}^{d}d_{q_2}v\\
&\le\frac{(1+q_1)(1+q_2)}{(b-a)(d-c)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(u,v)\,{}^{d}d_{q_2}v\,{}^{b}d_{q_1}u\\
&\le\frac{1+q_1}{2(1+q_2)(b-a)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right){}^{b}d_{q_1}u\\
&+\frac{q_2(1+q_1)}{2(1+q_2)(b-a)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\Psi(u,d)\,{}^{b}d_{q_1}u\\
&+\frac{1+q_2}{2(1+q_1)(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{q_1b+a}{1+q_1},v\right){}^{d}d_{q_2}v\\
&+\frac{q_1(1+q_2)}{2(1+q_1)(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(b,v)\,{}^{d}d_{q_2}v\\
&\le\frac{\Psi(\frac{q_1b+a}{1+q_1},\frac{q_2d+c}{1+q_2})+q_2\Psi(\frac{q_1b+a}{1+q_1},d)+q_1\Psi(b,\frac{q_2d+c}{1+q_2})+q_1q_2\Psi(b,d)}{(1+q_1)(1+q_2)}.
\end{aligned}
\tag{2.12}
$$

- If $A=\frac{q_1a+b}{1+q_1}$, $B=b$, $C=\frac{q_2c+d}{1+q_2}$, $D=d$, then

$$
\begin{aligned}
&\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\\
&\le\frac{1+q_1}{2q_1(b-a)}\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi\left(u,\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right){}^{b}d_{q_1}u\\
&+\frac{1+q_2}{2q_2(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},v\right){}^{d}d_{q_2}v\\
&\le\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\int_{\frac{q_1a+b}{1+q_1}}^{b}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi(u,v)\,{}^{d}d_{q_2}v\,{}^{b}d_{q_1}u
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1+q_1}{2q_1(1+q_2)(b-a)}\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}^{b}d_{q_1}u \\
&\quad+\frac{q_2(1+q_1)}{2q_1(1+q_2)(b-a)}\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi(u,d)\,{}^{b}d_{q_1}u \\
&\quad+\frac{1+q_2}{2q_2(1+q_1)(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi\left(\frac{q_1a+b}{1+q_1},v\right)\,{}^{d}d_{q_2}v \\
&\quad+\frac{q_1(1+q_2)}{2q_2(1+q_1)(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi(b,v)\,{}^{d}d_{q_2}v \\
&\leq \frac{\Psi(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2})+q_2\Psi(\frac{q_1a+b}{1+q_1},d)+q_1\Psi(b,\frac{q_2c+d}{1+q_2})+q_1q_2\Psi(b,d)}{(1+q_1)(1+q_2)}.
\end{aligned}
\tag{2.13}
$$

- If $A=\frac{q_1b+a}{1+q_1}$, $B=b$, $C=\frac{q_2c+d}{1+q_2}$, $D=d$, then

$$
\begin{aligned}
&\Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right) \\
&\quad\leq \frac{1+q_1}{2(b-a)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\Psi\left(u,\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}^{b}d_{q_1}u \\
&\quad\quad+\frac{1+q_2}{2q_2(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},v\right)\,{}^{d}d_{q_2}v \\
&\quad\leq \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi(u,v)\,{}^{d}d_{q_2}v\,{}^{b}d_{q_1}u \\
&\quad\leq \frac{1+q_1}{2(1+q_2)(b-a)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}^{b}d_{q_1}u \\
&\quad\quad+\frac{q_2(1+q_1)}{2(1+q_2)(b-a)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\Psi(u,d)\,{}^{b}d_{q_1}u \\
&\quad\quad+\frac{1+q_2}{2q_2(1+q_1)(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi\left(\frac{q_1b+a}{1+q_1},v\right)\,{}^{d}d_{q_2}v \\
&\quad\quad+\frac{q_1(1+q_2)}{2q_2(1+q_1)(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi(b,v)\,{}^{d}d_{q_2}v \\
&\quad\leq \frac{\Psi(\frac{q_1b+a}{1+q_1},\frac{q_2c+d}{1+q_2})+q_2\Psi(\frac{q_1b+a}{1+q_1},d)+q_1\Psi(b,\frac{q_2c+d}{1+q_2})+q_1q_2\Psi(b,d)}{(1+q_1)(1+q_2)}.
\end{aligned}
\tag{2.14}
$$

The application of the Theorems 1.2 and 1.3 leads to the following result.

Theorem 2.2 *Suppose that $\Psi:\Delta\to\mathbb{R}_e$ is a coordinated convex function on Δ and $\Psi\in L_1(\Delta)$. Then we have*

$$
\begin{aligned}
&\Psi\left(\frac{A+q_1B}{1+q_1},\frac{q_2C+D}{1+q_2}\right) \\
&\quad\leq \frac{1}{2}\left[\frac{1}{B-A}\int_A^B\Psi\left(u,\frac{q_2C+D}{1+q_2}\right)\,{}^{B}d_{q_1}u+\frac{1}{D-C}\int_C^D\Psi\left(\frac{A+q_1B}{1+q_1},v\right)\,{}_{C}d_{q_2}v\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{(B-A)(D-C)}\int_A^B\int_C^D \Psi(u,v)\,{}_C d_{q_2}v\,{}^B d_{q_1}u \\
&\le \biggl[\frac{q_2}{2(1+q_2)(B-A)}\int_A^B \Psi(u,C)\,{}^B d_{q_1}u + \frac{1}{2(1+q_2)(B-A)}\int_A^B \Psi(u,D)\,{}^B d_{q_1}u \\
&\quad + \frac{1}{2(1+q_1)(D-C)}\int_C^D \Psi(A,v)\,{}_C d_{q_2}v + \frac{q_1}{2(1+q_1)(D-C)}\int_C^D \Psi(B,v)\,{}_C d_{q_2}v\biggr] \\
&\le \frac{q_2\Psi(A,C)+\Psi(A,D)+q_1q_2\Psi(B,C)+q_1\Psi(B,D)}{(1+q_1)(1+q_2)}.
\end{aligned}
\tag{2.15}
$$

Proof The proof is omitted. □

Remark 2.2 From Theorem 2.2, we can deduce the following midpoint special cases.

- If $A=\frac{q_1a+b}{1+q_1}$, $B=b$, $C=c$, $D=\frac{q_2d+c}{1+q_2}$, then

$$
\begin{aligned}
&\Psi\biggl(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},\frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\biggr) \\
&\quad\le \frac{1+q_1}{2q_1(b-a)}\int_{\frac{q_1a+b}{1+q_1}}^b \Psi\biggl(u,\frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\biggr)\,{}^b d_{q_1}u \\
&\qquad + \frac{1+q_2}{2q_2(d-c)}\int_c^{\frac{q_2d+c}{1+q_2}} \Psi\biggl(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},v\biggr)\,{}_c d_{q_2}v \\
&\quad\le \frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\int_{\frac{q_1a+b}{1+q_1}}^b\int_c^{\frac{q_2d+c}{1+q_2}} \Psi(u,v)\,{}_c d_{q_2}v\,{}^b d_{q_1}u \\
&\quad\le \frac{q_2(1+q_1)}{2q_1(1+q_2)(b-a)}\int_{\frac{q_1a+b}{1+q_1}}^b \Psi(u,c)\,{}^b d_{q_1}u \\
&\qquad + \frac{1+q_1}{2q_1(1+q_2)(b-a)}\int_{\frac{q_1a+b}{1+q_1}}^b \Psi\biggl(u,\frac{q_2d+c}{1+q_2}\biggr)\,{}^b d_{q_1}u \\
&\qquad + \frac{1+q_2}{2q_2(1+q_1)(d-c)}\int_c^{\frac{q_2d+c}{1+q_2}} \Psi\biggl(\frac{q_1a+b}{1+q_1},v\biggr)\,{}_c d_{q_2}v \\
&\qquad + \frac{q_1(1+q_2)}{2q_2(1+q_1)(d-c)}\int_c^{\frac{q_2d+c}{1+q_2}} \Psi(b,v)\,{}_c d_{q_2}v \\
&\quad\le \frac{q_2\Psi(\frac{q_1a+b}{1+q_1},c)+\Psi(\frac{q_1a+b}{1+q_1},\frac{q_2d+c}{1+q_2})+q_1q_2\Psi(b,c)+q_1\Psi(b,\frac{q_2d+c}{1+q_2})}{(1+q_1)(1+q_2)}.
\end{aligned}
\tag{2.16}
$$

- If $A=\frac{q_1b+a}{1+q_1}$, $B=b$, $C=c$, $D=\frac{q_2d+c}{1+q_2}$, then

$$
\begin{aligned}
&\Psi\biggl(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},\frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\biggr) \\
&\quad\le \frac{1+q_1}{2(b-a)}\int_{\frac{q_1b+a}{1+q_1}}^b \Psi\biggl(u,\frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\biggr)\,{}^b d_{q_1}u \\
&\qquad + \frac{1+q_2}{2q_2(d-c)}\int_c^{\frac{q_2d+c}{1+q_2}} \Psi\biggl(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},v\biggr)\,{}_c d_{q_2}v
\end{aligned}
$$

$$\leq \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)} \int_{\frac{q_1 b+a}{1+q_1}}^{b} \int_{c}^{\frac{q_2 d+c}{1+q_2}} \Psi(u,v)\, {}_c d_{q_2} v\, {}^b d_{q_1} u$$

$$\leq \frac{q_2(1+q_1)}{2(1+q_2)(b-a)} \int_{\frac{q_1 b+a}{1+q_1}}^{b} \Psi(u,c)\, {}^b d_{q_1} u$$

$$+ \frac{1+q_1}{2(1+q_2)(b-a)} \int_{\frac{q_1 b+a}{1+q_1}}^{b} \Psi\left(u, \frac{q_2 d+c}{1+q_2}\right) {}^b d_{q_1} u$$

$$+ \frac{1+q_2}{2q_2(1+q_1)(d-c)} \int_{c}^{\frac{q_2 d+c}{1+q_2}} \Psi\left(\frac{q_1 b+a}{1+q_1}, v\right) {}_c d_{q_2} v$$

$$+ \frac{q_1(1+q_2)}{2q_2(1+q_1)(d-c)} \int_{c}^{\frac{q_2 d+c}{1+q_2}} \Psi(b,v)\, {}_c d_{q_2} v$$

$$\leq \frac{q_2 \Psi(\frac{q_1 b+a}{1+q_1}, c) + \Psi(\frac{q_1 b+a}{1+q_1}, \frac{q_2 d+c}{1+q_2}) + q_1 q_2 \Psi(b,c) + q_1 \Psi(b, \frac{q_2 d+c}{1+q_2})}{(1+q_1)(1+q_2)}. \tag{2.17}$$

- If $A = \frac{q_1 a+b}{1+q_1}$, $B = b$, $C = c$, $D = \frac{q_2 c+d}{1+q_2}$, then

$$\Psi\left(\frac{q_1 a + (1+q_1+q_1^2)b}{(1+q_1)^2}, \frac{c(2q_2+q_2^2)+d}{(1+q_2)^2}\right)$$

$$\leq \frac{1+q_1}{2q_1(b-a)} \int_{\frac{q_1 a+b}{1+q_1}}^{b} \Psi\left(u, \frac{c(2q_2+q_2^2)+d}{(1+q_2)^2}\right) {}^b d_{q_1} u$$

$$+ \frac{1+q_2}{2(d-c)} \int_{c}^{\frac{q_2 c+d}{1+q_2}} \Psi\left(\frac{q_1 a + (1+q_1+q_1^2)b}{(1+q_1)^2}, v\right) {}_c d_{q_2} v$$

$$\leq \frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)} \int_{\frac{q_1 a+b}{1+q_1}}^{b} \int_{c}^{\frac{q_2 c+d}{1+q_2}} \Psi(u,v)\, {}_c d_{q_2} v\, {}^b d_{q_1} u$$

$$\leq \frac{q_2(1+q_1)}{2q_1(1+q_2)(b-a)} \int_{\frac{q_1 a+b}{1+q_1}}^{b} \Psi(u,c)\, {}^b d_{q_1} u$$

$$+ \frac{1+q_1}{2q_1(1+q_2)(b-a)} \int_{\frac{q_1 a+b}{1+q_1}}^{b} \Psi\left(u, \frac{q_2 d+c}{1+q_2}\right) {}^b d_{q_1} u$$

$$+ \frac{1+q_2}{2(1+q_1)(d-c)} \int_{c}^{\frac{q_2 c+d}{1+q_2}} \Psi\left(\frac{q_1 a+b}{1+q_1}, v\right) {}_c d_{q_2} v$$

$$+ \frac{q_1(1+q_2)}{2(1+q_1)(d-c)} \int_{c}^{\frac{q_2 c+d}{1+q_2}} \Psi(b,v)\, {}_c d_{q_2} v$$

$$\leq \frac{q_2 \Psi(\frac{q_1 a+b}{1+q_1}, c) + \Psi(\frac{q_1 a+b}{1+q_1}, \frac{q_2 c+d}{1+q_2}) + q_1 q_2 \Psi(b,c) + q_1 \Psi(b, \frac{q_2 c+d}{1+q_2})}{(1+q_1)(1+q_2)}. \tag{2.18}$$

- If $A = \frac{q_1 b+a}{1+q_1}$, $B = b$, $C = c$, $D = \frac{q_2 c+d}{1+q_2}$, then

$$\Psi\left(\frac{a + (2q_1+q_1^2)b}{(1+q_1)^2}, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right)$$

$$\leq \frac{1+q_1}{2(b-a)} \int_{\frac{q_1 b+a}{1+q_1}}^{b} \Psi\left(u, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) {}^b d_{q_1} u$$

$$
\begin{aligned}
&+\frac{1+q_2}{2(d-c)}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},\nu\right)\,_c d_{q_2}\nu\\
&\leq\frac{(1+q_1)(1+q_2)}{(b-a)(d-c)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi(u,\nu)\,_c d_{q_2}\nu\,^b d_{q_1}u\\
&\leq\frac{q_2(1+q_1)}{2(1+q_2)(b-a)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\Psi(u,c)\,^b d_{q_1}u\\
&+\frac{1+q_1}{2(1+q_2)(b-a)}\int_{\frac{q_1b+a}{1+q_1}}^{b}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,^b d_{q_1}u\\
&+\frac{1+q_2}{2(1+q_1)(d-c)}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi\left(\frac{q_1b+a}{1+q_1},\nu\right)\,_c d_{q_2}\nu\\
&+\frac{q_1(1+q_2)}{2(1+q_1)(d-c)}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi(b,\nu)\,_c d_{q_2}\nu\\
&\leq\frac{q_2\Psi(\frac{q_1b+a}{1+q_1},c)+\Psi(\frac{q_1b+a}{1+q_1},\frac{q_2c+d}{1+q_2})+q_1q_2\Psi(b,c)+q_1\Psi(b,\frac{q_2c+d}{1+q_2})}{(1+q_1)(1+q_2)}.
\end{aligned}
\tag{2.19}
$$

Finally, we have the following inequalities by the utilizing Theorems 1.2 and 1.3.

Theorem 2.3 *Let $\Psi:\Delta\to\mathbb{R}_e$ be a coordinated convex function on Δ and $q_1,q_2\in(0,1)$. Then one has*

$$
\begin{aligned}
&\Psi\left(\frac{q_1A+B}{1+q_1},\frac{C+q_2D}{1+q_2}\right)\\
&\leq\frac{1}{2}\left[\frac{1}{B-A}\int_A^B\Psi\left(u,\frac{C+q_2D}{1+q_2}\right)\,_A d_{q_1}u\right.\\
&\left.+\frac{1}{D-C}\int_C^D\Psi\left(\frac{q_1A+B}{1+q_1},\nu\right)\,^D d_{q_2}\nu\right]\\
&\leq\frac{1}{(B-A)(D-C)}\int_A^B\int_C^D\Psi(u,\nu)\,^D d_{q_2}\nu\,_A d_{q_1}u\\
&\leq\left[\frac{1}{2(1+q_2)(B-A)}\int_A^B\Psi(u,C)\,_A d_{q_1}u\right.\\
&+\frac{q_2}{2(1+q_2)(B-A)}\int_A^B\Psi(u,D)\,_A d_{q_1}u\\
&+\frac{q_1}{2(1+q_1)(D-C)}\int_C^D\Psi(A,\nu)\,^D d_{q_2}\nu\\
&\left.+\frac{1}{2(1+q_1)(D-C)}\int_C^D\Psi(B,\nu)\,^D d_{q_2}\nu\right]\\
&\leq\frac{q_1\Psi(A,C)+q_1q_2\Psi(A,D)+\Psi(B,C)+q_2\Psi(B,D)}{(1+q_1)(1+q_2)}.
\end{aligned}
\tag{2.20}
$$

Proof This proof is similar to our proof of Theorem 2.1, so we omit it. □

Remark 2.3 From Theorem 2.3, we can deduce the following midpoint special cases.

- If $A=a$, $B=\frac{q_1a+b}{1+q_1}$, $C=\frac{q_2d+c}{1+q_2}$, $D=d$, then

$$\begin{aligned}
&\Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2},\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\\
&\quad\le\frac{1+q_1}{2(b-a)}\int_a^{\frac{q_1a+b}{1+q_1}}\Psi\left(u,\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}_ad_{q_1}u\\
&\qquad+\frac{1+q_2}{2(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2},v\right)\,{}^dd_{q_2}v\\
&\quad\le\frac{(1+q_1)(1+q_2)}{(b-a)(d-c)}\int_a^{\frac{q_1a+b}{1+q_1}}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(u,v)\,{}^dd_{q_2}v\,{}_ad_{q_1}u\\
&\quad\le\frac{1+q_1}{2(1+q_2)(b-a)}\int_a^{\frac{q_1a+b}{1+q_1}}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}_ad_{q_1}u\\
&\qquad+\frac{q_2(1+q_1)}{2(1+q_2)(b-a)}\int_a^{\frac{q_1a+b}{1+q_1}}\Psi(u,d)\,{}_ad_{q_1}u\\
&\qquad+\frac{q_1(1+q_2)}{2(1+q_1)(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(a,v)\,{}^dd_{q_2}v\\
&\qquad+\frac{1+q_2}{2(1+q_1)(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{q_1a+b}{1+q_1},v\right)\,{}^dd_{q_2}v\\
&\quad\le\frac{q_1\Psi(a,\frac{q_2d+c}{1+q_2})+q_1q_2\Psi(a,d)+\Psi(\frac{q_1a+b}{1+q_1},\frac{q_2d+c}{1+q_2})+q_2\Psi(\frac{q_1a+b}{1+q_1},d)}{(1+q_1)(1+q_2)}.
\end{aligned}\tag{2.21}$$

- If $A=a$, $B=\frac{q_1b+a}{1+q_1}$, $C=\frac{q_2d+c}{1+q_2}$, $D=d$, then

$$\begin{aligned}
&\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\\
&\quad\le\frac{1+q_1}{2q_1(b-a)}\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}_ad_{q_1}u\\
&\qquad+\frac{1+q_2}{2(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},v\right)\,{}^dd_{q_2}v\\
&\quad\le\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\int_a^{\frac{q_1b+a}{1+q_1}}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(u,v)\,{}^dd_{q_2}v\,{}_ad_{q_1}u\\
&\quad\le\frac{1+q_1}{2q_1(1+q_2)(b-a)}\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}_ad_{q_1}u\\
&\qquad+\frac{q_2(1+q_1)}{2q_1(1+q_2)(b-a)}\int_a^{\frac{q_1b+a}{1+q_1}}\Psi(u,d)\,{}_ad_{q_1}u\\
&\qquad+\frac{q_1(1+q_2)}{2(1+q_1)(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(a,v)\,{}^dd_{q_2}v\\
&\qquad+\frac{1+q_2}{2(1+q_1)(d-c)}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{q_1b+a}{1+q_1},v\right)\,{}^dd_{q_2}v
\end{aligned}$$

$$\leq \frac{q_1\Psi(a,\frac{q_2d+c}{1+q_2}) + q_1q_2\Psi(a,d) + \Psi(\frac{q_1b+a}{1+q_1},\frac{q_2d+c}{1+q_2}) + q_2\Psi(\frac{q_1b+a}{1+q_1},d)}{(1+q_1)(1+q_2)}. \quad (2.22)$$

- If $A = a$, $B = \frac{q_1a+b}{1+q_1}$, $C = \frac{q_2c+d}{1+q_2}$, $D = d$, then

$$\begin{aligned}
&\Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2},\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\\
&\leq \frac{1+q_1}{2(b-a)}\int_a^{\frac{q_1a+b}{1+q_1}}\Psi\left(u,\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}_ad_{q_1}u\\
&\quad+\frac{1+q_2}{2q_2(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^d\Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2},v\right)\,{}^dd_{q_2}v\\
&\leq \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\int_a^{\frac{q_1a+b}{1+q_1}}\int_{\frac{q_2c+d}{1+q_2}}^d\Psi(u,v)\,{}^dd_{q_2}v\,{}_ad_{q_1}u\\
&\leq \frac{1+q_1}{2(1+q_2)(b-a)}\int_a^{\frac{q_1a+b}{1+q_1}}\Psi\left(u,\frac{q_2c+d}{1+q_2}\right)\,{}_ad_{q_1}u\\
&\quad+\frac{q_2(1+q_1)}{2(1+q_2)(b-a)}\int_a^{\frac{q_1a+b}{1+q_1}}\Psi(u,d)\,{}_ad_{q_1}u\\
&\quad+\frac{q_1(1+q_2)}{2q_2(1+q_1)(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^d\Psi(a,v)\,{}^dd_{q_2}v\\
&\quad+\frac{1+q_2}{2q_2(1+q_1)(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^d\Psi\left(\frac{q_1a+b}{1+q_1},v\right)\,{}^dd_{q_2}v\\
&\leq \frac{q_1\Psi(a,\frac{q_2c+d}{1+q_2}) + q_1q_2\Psi(a,d) + \Psi(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2}) + q_2\Psi(\frac{q_1a+b}{1+q_1},d)}{(1+q_1)(1+q_2)}. \quad (2.23)
\end{aligned}$$

- If $A = a$, $B = \frac{q_1b+a}{1+q_1}$, $C = \frac{q_2c+d}{1+q_2}$, $D = d$, then

$$\begin{aligned}
&\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\\
&\leq \frac{1+q_1}{2q_1(b-a)}\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}_ad_{q_1}u\\
&\quad+\frac{1+q_2}{2q_2(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^d\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},v\right)\,{}^dd_{q_2}v\\
&\leq \frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\int_a^{\frac{q_1b+a}{1+q_1}}\int_{\frac{q_2c+d}{1+q_2}}^d\Psi(u,v)\,{}^dd_{q_2}v\,{}_ad_{q_1}u\\
&\leq \frac{1+q_1}{2q_1(1+q_2)(b-a)}\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{q_2c+d}{1+q_2}\right)\,{}_ad_{q_1}u\\
&\quad+\frac{q_2(1+q_1)}{2q_1(1+q_2)(b-a)}\int_a^{\frac{q_1b+a}{1+q_1}}\Psi(u,d)\,{}_ad_{q_1}u\\
&\quad+\frac{q_1(1+q_2)}{2q_2(1+q_1)(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^d\Psi(a,v)\,{}^dd_{q_2}v
\end{aligned}$$

$$+\frac{1+q_2}{2q_2(1+q_1)(d-c)}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi\left(\frac{q_1b+a}{1+q_1},\nu\right)\,{}^{d}d_{q_2}\nu$$

$$\leq\frac{q_1\Psi(a,\frac{q_2c+d}{1+q_2})+q_1q_2\Psi(a,d)+\Psi(\frac{q_1b+a}{1+q_1},\frac{q_2c+d}{1+q_2})+q_2\Psi(\frac{q_1b+a}{1+q_1},d)}{(1+q_1)(1+q_2)}. \tag{2.24}$$

By combining Theorems 1.5, 2.1, 2.2 and Theorem 2.3, we can deduce similar bounds to the inequalities in Theorem 1.1.

Theorem 2.4 *Suppose that $\Psi:\Delta\to\mathbb{R}_e$ is a coordinated convex function on Δ and $\Psi\in L_1(\Delta)$. Then we have*

$$\begin{aligned}
&2\Psi\left(\frac{A+B}{2},\frac{C+D}{2}\right)\\
&\leq\frac{1}{2(B-A)}\left[\int_A^B\left(\Psi\left(u,\frac{C+q_2D}{1+q_2}\right)+\Psi\left(u,\frac{q_2C+D}{1+q_2}\right)\right)\,{}_{A}d_{q_1}u\right.\\
&\quad\left.+\int_A^B\left(\Psi\left(u,\frac{C+q_2D}{1+q_2}\right)+\Psi\left(u,\frac{q_2C+D}{1+q_2}\right)\right)\,{}^{B}d_{q_1}u\right]\\
&\quad+\frac{1}{2(D-C)}\left[\int_C^D\left(\Psi\left(\frac{A+q_1B}{1+q_1},\nu\right)+\Psi\left(\frac{q_1A+B}{1+q_1},\nu\right)\right)\,{}_{C}d_{q_2}\nu\right.\\
&\quad\left.+\int_C^D\left(\Psi\left(\frac{A+q_1B}{1+q_1},\nu\right)+\Psi\left(\frac{q_1A+B}{1+q_1},\nu\right)\right)\,{}^{D}d_{q_2}\nu\right]\\
&\leq\frac{1}{(B-A)(D-C)}\left[\int_A^B\int_C^D\Psi(u,\nu)\,{}^{D}d_{q_2}\nu\,{}_{A}d_{q_1}u+\int_A^B\int_C^D\Psi(u,\nu)\,{}^{D}d_{q_2}\nu\,{}^{B}d_{q_1}u\right.\\
&\quad\left.+\int_A^B\int_C^D\Psi(u,\nu)\,{}_{C}d_{q_2}\nu\,{}^{B}d_{q_1}u+\int_A^B\int_C^D\Psi(u,\nu)\,{}_{C}d_{q_2}\nu\,{}_{A}d_{q_1}u\right]\\
&\leq\frac{1}{2(B-A)}\int_A^B\big(\Psi(u,C)+\Psi(u,D)\big)\,{}_{A}d_{q_1}u\\
&\quad+\frac{1}{2(B-A)}\int_A^B\big(\Psi(u,C)+\Psi(u,D)\big)\,{}^{B}d_{q_1}u\\
&\quad+\frac{1}{2(D-C)}\int_A^B\big(\Psi(A,\nu)+\Psi(B,\nu)\big)\,{}_{C}d_{q_2}\nu\\
&\quad+\frac{1}{2(D-C)}\int_A^B\big(\Psi(A,\nu)+\Psi(B,\nu)\big)\,{}^{D}d_{q_2}\nu\bigg]\\
&\leq\Psi(A,C)+\Psi(A,D)+\Psi(B,C)+\Psi(B,D).
\end{aligned}\tag{2.25}$$

Proof It suffices to see that we have, due to the coordinated convexity of Ψ,

$$\begin{aligned}
4\Psi\left(\frac{A+B}{2},\frac{C+D}{2}\right)&=4\Psi\left(\frac{q_1A+B+A+q_1B}{2(1+q_1)},\frac{q_2C+D+C+q_2D}{2(1+q_2)}\right)\\
&\leq\Psi\left(\frac{q_1A+B}{1+q_1},\frac{q_2C+D}{1+q_2}\right)+\Psi\left(\frac{q_1A+B}{1+q_1},\frac{C+q_2D}{1+q_2}\right)\\
&\quad+\Psi\left(\frac{A+q_1B}{1+q_1},\frac{q_2C+D}{1+q_2}\right)+\Psi\left(\frac{A+q_1B}{1+q_1},\frac{C+q_2D}{1+q_2}\right).
\end{aligned}$$

Then, we can obtain the desired inequality by utilizing Theorems 1.5, 2.1, 2.2 and Theorem 2.3. □

Remark 2.4 If $q_1, q_2 \to 1^-$ in Theorem 2.4, then Theorem 2.4 reduces to Theorem 1.1.

Finally, by utilizing Remarks 1.1, 2.1, 2.2 and Remark 2.3, we obtain the following refinements for previous results.

Theorem 2.5 *Let $\Psi : \Delta \to \mathbb{R}_e$ be a coordinated convex function on Δ and $q_1, q_2 \in (0,1)$, then we have*

$$
\begin{aligned}
&4\Psi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\leq \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) + \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, \frac{(2q_2+q_2^2)d+c}{(1+q_2)^2}\right) \\
&\quad + \Psi\left(\frac{(2q_1+q_1^2)b+a}{(1+q_1)^2}, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) + \Psi\left(\frac{(2q_1+q_1^2)b+a}{(1+q_1)^2}, \frac{(2q_2+q_2^2)d+c}{(1+q_2)^2}\right) \\
&\leq \frac{1+q_1}{2(b-a)}\left[\int_a^{\frac{q_1a+b}{1+q_1}} \Psi\left(u, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) {}_a d_{q_1}u\right. \\
&\quad + \int_{\frac{q_1b+a}{1+q_1}}^b \Psi\left(u, \frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right) {}^b d_{q_1}u \\
&\quad \left. + \int_{\frac{q_1b+a}{1+q_1}}^b \Psi\left(u, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) {}^b d_{q_1}u + \int_a^{\frac{q_1a+b}{1+q_1}} \Psi\left(u, \frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right) {}_a d_{q_1}u\right] \\
&\quad + \frac{1+q_2}{2(d-c)}\left[\int_c^{\frac{q_2c+d}{1+q_2}} \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, v\right) {}_c d_{q_2}v\right. \\
&\quad + \int_{\frac{q_2d+c}{1+q_2}}^d \Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2}, v\right) {}^d d_{q_2}v \\
&\quad \left. + \int_c^{\frac{q_2c+d}{1+q_2}} \Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2}, v\right) {}_c d_{q_2}v + \int_{\frac{q_2d+c}{1+q_2}}^d \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, v\right) {}^d d_{q_2}v\right] \\
&\leq \frac{(1+q_1)(1+q_2)}{(b-a)(d-c)}\left[\int_a^{\frac{q_1a+b}{1+q_1}} \int_c^{\frac{q_2c+d}{1+q_2}} \Psi(u,v) \, {}_c d_{q_2}v \, {}_a d_{q_1}u\right. \\
&\quad + \int_{\frac{q_1b+a}{1+q_1}}^b \int_{\frac{q_2d+c}{1+q_2}}^d \Psi(u,v) \, {}^d d_{q_2}v \, {}^b d_{q_1}u \\
&\quad \left. + \int_{\frac{q_1b+a}{1+q_1}}^b \int_c^{\frac{q_2c+d}{1+q_2}} \Psi(u,v) \, {}_c d_{q_2}v \, {}^b d_{q_1}u + \int_a^{\frac{q_1a+b}{1+q_1}} \int_{\frac{q_2d+c}{1+q_2}}^d \Psi(u,v) \, {}^d d_{q_2}v \, {}_a d_{q_1}u\right] \\
&\leq \frac{1+q_1}{2(1+q_2)(b-a)}\left[q_2 \int_a^{\frac{q_1a+b}{1+q_1}} \Psi(u,c) \, {}_a d_{q_1}u + \int_a^{\frac{q_1a+b}{1+q_1}} \Psi\left(u, \frac{q_2c+d}{1+q_2}\right) {}_a d_{q_1}u\right. \\
&\quad + \int_{\frac{q_1b+a}{1+q_1}}^b \Psi\left(u, \frac{q_2d+c}{1+q_2}\right) {}^b d_{q_1}u + q_2 \int_{\frac{q_1b+a}{1+q_1}}^b \Psi(u,d) \, {}^b d_{q_1}u
\end{aligned}
$$

$$
\begin{aligned}
&+ q_2 \int_{\frac{q_1 b+a}{1+q_1}}^{b} \Psi(u,c)\, {}^{b}d_{q_1}u + \int_{\frac{q_1 b+a}{1+q_1}}^{b} \Psi\left(u, \frac{q_2 c+d}{1+q_2}\right) {}^{b}d_{q_1}u \\
&+ q_2 \int_{a}^{\frac{q_1 a+b}{1+q_1}} \Psi(u,d)\, {}_{a}d_{q_1}u + \int_{a}^{\frac{q_1 a+b}{1+q_1}} \Psi\left(u, \frac{q_2 d+c}{1+q_2}\right) {}_{a}d_{q_1}u\Bigg] \\
&+ \frac{1+q_2}{2(1+q_1)(d-c)}\Bigg[q_1 \int_{c}^{\frac{q_2 c+d}{1+q_2}} \Psi(a,v)\, {}_{c}d_{q_2}v \\
&+ \int_{c}^{\frac{q_2 c+d}{1+q_2}} \Psi\left(\frac{q_1 a+b}{1+q_1}, v\right) {}_{c}d_{q_2}v + \int_{\frac{q_2 d+c}{1+q_2}}^{d} \Psi\left(\frac{q_1 b+a}{1+q_1}, v\right) {}^{d}d_{q_2}v \\
&+ q_1 \int_{\frac{q_2 d+c}{1+q_2}}^{d} \Psi(b,v)\, {}^{d}d_{q_2}v + \int_{c}^{\frac{q_2 c+d}{1+q_2}} \Psi\left(\frac{q_1 b+a}{1+q_1}, v\right) {}_{c}d_{q_2}v \\
&+ q_1 \int_{c}^{\frac{q_2 c+d}{1+q_2}} \Psi(b,v)\, {}_{c}d_{q_2}v + \int_{\frac{q_2 d+c}{1+q_2}}^{d} \Psi\left(\frac{q_1 a+b}{1+q_1}, v\right) {}^{d}d_{q_2}v \\
&+ q_1 \int_{\frac{q_2 d+c}{1+q_2}}^{d} \Psi(a,v)\, {}^{d}d_{q_2}v\Bigg] \\
&\le \frac{q_1 q_2[\Psi(a,c)+\Psi(a,d)+\Psi(b,d)+\Psi(b,c)]}{(1+q_1)(1+q_2)} \\
&+ \frac{q_1[\Psi(a,\frac{q_2 c+d}{1+q_2})+\Psi(a,\frac{q_2 d+c}{1+q_2})+\Psi(b,\frac{q_2 c+d}{1+q_2})+\Psi(b,\frac{q_2 d+c}{1+q_2})]}{(1+q_1)(1+q_2)} \\
&+ \frac{q_2[\Psi(\frac{q_1 a+b}{1+q_1},c)+\Psi(\frac{q_1 b+a}{1+q_1},d)+\Psi(\frac{q_1 b+a}{1+q_1},c)+\Psi(\frac{q_1 a+b}{1+q_1},d)]}{(1+q_1)(1+q_2)} \\
&+ \frac{\Psi(\frac{q_1 a+b}{1+q_1},\frac{q_2 c+d}{1+q_2})+\Psi(\frac{q_1 a+b}{1+q_1},\frac{q_2 d+c}{1+q_2})+\Psi(\frac{q_1 b+a}{1+q_1},\frac{q_2 c+d}{1+q_2})+\Psi(\frac{q_1 b+a}{1+q_1},\frac{q_2 d+c}{1+q_2})}{(1+q_1)(1+q_2)} \\
&\le \Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d). \qquad (2.26)
\end{aligned}
$$

Proof By summing up the inequalities (1.30), (2.12), (2.19) and (2.21), we can obtain the second, third, fourth and fifth inequalities of the desired inequality (2.26). The last inequality is the consequence of the coordinated convexity of the function Ψ.

For the first inequality, we note that

$$
\begin{aligned}
&4\Psi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&= 4\Psi\left(\frac{(2q_1+q_1^2)a+b+a+(2q_1+q_1^2)b}{2(1+q_1^2)}, \frac{(2q_2+q_2^2)c+d+c+(2q_2+q_2^2)d}{2(1+q_1^2)}\right) \\
&\le \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) + \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2}, \frac{(2q_2+q_2^2)d+c}{(1+q_2)^2}\right) \\
&+ \Psi\left(\frac{(2q_1+q_1^2)b+a}{(1+q_1)^2}, \frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) \\
&+ \Psi\left(\frac{(2q_1+q_1^2)b+a}{(1+q_1)^2}, \frac{(2q_2+q_2^2)d+c}{(1+q_2)^2}\right), \qquad (2.27)
\end{aligned}
$$

which completes our proof. □

In a similar way, by using inequalities (1.29), (2.13), (2.16) and (2.24), we can deduce the following theorem.

Theorem 2.6 *Let $\Psi:\Delta\to\mathbb{R}_e$ be a coordinated convex function on Δ and $q_1,q_2\in(0,1)$, then we have*

$$
\begin{aligned}
&4\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leq \Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},\frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\right)\\
&\quad+\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\\
&\quad+\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},\frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\right)\\
&\quad+\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\\
&\leq \frac{1+q_1}{2q_1(b-a)}\left[\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\right)\,{}_ad_{q_1}u\right.\\
&\quad+\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi\left(u,\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}^bd_{q_1}u\\
&\quad+\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi\left(u,\frac{(1+q_2+q_2^2)c+q_2d}{(1+q_2)^2}\right)\,{}^bd_{q_1}u\\
&\quad\left.+\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{q_2c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}_ad_{q_1}u\right]\\
&\quad+\frac{1+q_2}{2q_2(d-c)}\left[\int_c^{\frac{q_2d+c}{1+q_2}}\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},v\right)\,{}_cd_{q_2}v\right.\\
&\quad+\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},v\right)\,{}^dd_{q_2}v\\
&\quad+\int_c^{\frac{q_2d+c}{1+q_2}}\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},v\right)\,{}_cd_{q_2}v\\
&\quad\left.+\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},v\right)\,{}^dd_{q_2}v\right]\\
&\leq \frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\left[\int_a^{\frac{q_1b+a}{1+q_1}}\int_c^{\frac{q_2d+c}{1+q_2}}\Psi(u,v)\,{}_cd_{q_2}v\,{}_ad_{q_1}u\right.\\
&\quad+\int_{\frac{q_1a+b}{1+q_1}}^{b}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi(u,v)\,{}^dd_{q_2}v\,{}^bd_{q_1}u\\
&\quad\left.+\int_{\frac{q_1a+b}{1+q_1}}^{b}\int_c^{\frac{q_2d+c}{1+q_2}}\Psi(u,v)\,{}_cd_{q_2}v\,{}^bd_{q_1}u+\int_a^{\frac{q_1b+a}{1+q_1}}\int_{\frac{q_2c+d}{1+q_2}}^{d}\Psi(u,v)\,{}^dd_{q_2}v\,{}_ad_{q_1}u\right]\\
&\leq \frac{1+q_1}{2q_1(1+q_2)(b-a)}\left[q_2\int_a^{\frac{q_1b+a}{1+q_1}}\Psi(u,c)\,{}_ad_{q_1}u\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_a^{\frac{q_1b+a}{1+q_1}} \Psi\left(u,\frac{q_2d+c}{1+q_2}\right) {}_a d_{q_1}u \\
&+\int_{\frac{q_1a+b}{1+q_1}}^b \Psi\left(u,\frac{q_2c+d}{1+q_2}\right) {}^b d_{q_1}u + q_2\int_{\frac{q_1a+b}{1+q_1}}^b \Psi(u,d) {}^b d_{q_1}u \\
&+q_2\int_{\frac{q_1a+b}{1+q_1}}^b \Psi(u,c) {}^b d_{q_1}u + \int_{\frac{q_1a+b}{1+q_1}}^b \Psi\left(u,\frac{q_2d+c}{1+q_2}\right) {}^b d_{q_1}u \\
&+q_2\int_a^{\frac{q_1b+a}{1+q_1}} \Psi(u,d) {}_a d_{q_1}u + \int_a^{\frac{q_1b+a}{1+q_1}} \Psi\left(u,\frac{q_2c+d}{1+q_2}\right) {}_a d_{q_1}u\Bigg] \\
&+\frac{1+q_2}{2q_2(1+q_1)(d-c)}\Bigg[q_1\int_c^{\frac{q_2d+c}{1+q_2}} \Psi(a,v) {}_c d_{q_2}v \\
&+\int_c^{\frac{q_2d+c}{1+q_2}} \Psi\left(\frac{q_1b+a}{1+q_1},v\right) {}_c d_{q_2}v \\
&+\int_{\frac{q_2c+d}{1+q_2}}^d \Psi\left(\frac{q_1a+b}{1+q_1},v\right) {}^d d_{q_2}v + q_1\int_{\frac{q_2c+d}{1+q_2}}^d \Psi(b,v) {}^d d_{q_2}v \\
&+\int_c^{\frac{q_2d+c}{1+q_2}} \Psi\left(\frac{q_1a+b}{1+q_1},v\right) {}_c d_{q_2}v + q_1\int_c^{\frac{q_2d+c}{1+q_2}} \Psi(b,v) {}_c d_{q_2}v \\
&+\int_{\frac{q_2c+d}{1+q_2}}^d \Psi\left(\frac{q_1b+a}{1+q_1},v\right) {}^d d_{q_2}v + q_1\int_{\frac{q_2c+d}{1+q_2}}^d \Psi(a,v) {}^d d_{q_2}v\Bigg] \\
\le{}& \frac{q_1q_2[\Psi(a,c)+\Psi(a,d)+\Psi(b,d)+\Psi(b,c)]}{(1+q_1)(1+q_2)} \\
&+\frac{q_1[\Psi(a,\frac{q_2d+c}{1+q_2})+\Psi(a,\frac{q_2c+d}{1+q_2})+\Psi(b,\frac{q_2c+d}{1+q_2})+\Psi(b,\frac{q_2d+c}{1+q_2})]}{(1+q_1)(1+q_2)} \\
&+\frac{q_2[\Psi(\frac{q_1a+b}{1+q_1},c)+\Psi(\frac{q_1b+a}{1+q_1},d)+\Psi(\frac{q_1b+a}{1+q_1},c)+\Psi(\frac{q_1a+b}{1+q_1},d)]}{(1+q_1)(1+q_2)} \\
&+\frac{\Psi(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2})+\Psi(\frac{q_1a+b}{1+q_1},\frac{q_2d+c}{1+q_2})+\Psi(\frac{q_1b+a}{1+q_1},\frac{q_2c+d}{1+q_2})+\Psi(\frac{q_1b+a}{1+q_1},\frac{q_2d+c}{1+q_2})}{(1+q_1)(1+q_2)} \\
\le{}& \Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d). \qquad (2.28)
\end{aligned}
$$

In a similar way, by using inequalities (1.31), (2.11), (2.18) and (2.22), we can deduce the following theorem.

Theorem 2.7 *Let $\Psi:\Delta\to\mathbb{R}_e$ be Δ a coordinated convex function on Δ and $q_1,q_2\in(0,1)$, then one has*

$$
\begin{aligned}
&4\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\le \Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},\frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right) \\
&+\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},\frac{c(2q_2+q_2^2)+d}{(1+q_2)^2}\right)\\
&+\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\\
\le{}&\frac{1+q_1}{2q_1(b-a)}\left[\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{(2q_2+q_2^2)c+d}{(1+q_2)^2}\right)\,{}_a d_{q_1}u\right.\\
&+\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi\left(u,\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}^b d_{q_1}u\\
&\left.+\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi\left(u,\frac{c(2q_2+q_2^2)+d}{(1+q_2)^2}\right)\,{}^b d_{q_1}u+\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{c+(2q_2+q_2^2)d}{(1+q_2)^2}\right)\,{}_a d_{q_1}u\right]\\
&+\frac{1+q_2}{2(d-c)}\left[\int_c^{\frac{q_2c+d}{1+q_2}}\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},v\right)\,{}_c d_{q_2}v\right.\\
&+\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},v\right)\,{}^d d_{q_2}v\\
&+\int_c^{\frac{q_2c+d}{1+q_2}}\Psi\left(\frac{q_1a+(1+q_1+q_1^2)b}{(1+q_1)^2},v\right)\,{}_c d_{q_2}v\\
&\left.+\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{(1+q_1+q_1^2)a+q_1b}{(1+q_1)^2},v\right)\,{}^d d_{q_2}v\right]\\
\le{}&\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\left[\int_a^{\frac{q_1b+a}{1+q_1}}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi(u,v)\,{}_c d_{q_2}v\,{}_a d_{q_1}u\right.\\
&+\int_{\frac{q_1a+b}{1+q_1}}^{b}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(u,v)\,{}^d d_{q_2}v\,{}^b d_{q_1}u\\
&\left.+\int_{\frac{q_1a+b}{1+q_1}}^{b}\int_c^{\frac{q_2c+d}{1+q_2}}\Psi(u,v)\,{}_c d_{q_2}v\,{}^b d_{q_1}u+\int_a^{\frac{q_1b+a}{1+q_1}}\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(u,v)\,{}^d d_{q_2}v\,{}_a d_{q_1}u\right]\\
\le{}&\frac{(1+q_1)}{2q_1(1+q_2)(b-a)}\left[q_2\int_a^{\frac{q_1b+a}{1+q_1}}\Psi(u,c)\,{}_a d_{q_1}u+\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}_a d_{q_1}u\right.\\
&+\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}^b d_{q_1}u+q_2\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi(u,d)\,{}^b d_{q_1}u\\
&+q_2\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi(u,c)\,{}^b d_{q_1}u+\int_{\frac{q_1a+b}{1+q_1}}^{b}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}^b d_{q_1}u\\
&\left.+\int_a^{\frac{q_1b+a}{1+q_1}}\Psi\left(u,\frac{q_2d+c}{1+q_2}\right)\,{}_a d_{q_1}u+q_2\int_a^{\frac{q_1b+a}{1+q_1}}\Psi(u,d)\,{}_a d_{q_1}u\right]\\
&+\frac{(1+q_2)}{2(1+q_1)(d-c)}\left[q_1\int_c^{\frac{q_2c+d}{1+q_2}}\Psi(a,v)\,{}_c d_{q_2}v+\int_c^{\frac{q_2c+d}{1+q_2}}\Psi\left(\frac{q_1b+a}{1+q_1},v\right)\,{}_c d_{q_2}v\right.\\
&+\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi\left(\frac{q_1a+b}{1+q_1},v\right)\,{}^d d_{q_2}v+q_1\int_{\frac{q_2d+c}{1+q_2}}^{d}\Psi(b,v)\,{}^d d_{q_2}v\\
&+\int_c^{\frac{q_2c+d}{1+q_2}}\Psi\left(\frac{q_1a+b}{1+q_1},v\right)\,{}_c d_{q_2}v+q_1\int_c^{\frac{q_2c+d}{1+q_2}}\Psi(b,v)\,{}_c d_{q_2}v
\end{aligned}
$$

$$
\begin{aligned}
&+ q_1 \int_{\frac{q_2 d+c}{1+q_2}}^{d} \Psi(a,\nu)\, {}^{d}d_{q_2}\nu + \int_{\frac{q_2 d+c}{1+q_2}}^{d} \Psi\left(\frac{q_1 b + a}{1+q_1},\nu\right) {}^{d}d_{q_2}\nu\Bigg] \\
&\le \frac{q_1 q_2[\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)]}{(1+q_1)(1+q_2)} \\
&\quad + \frac{q_1[\Psi(a,\frac{q_2 c+d}{1+q_2})+\Psi(b,\frac{q_2 c+d}{1+q_2})+\Psi(a,\frac{q_2 d+c}{1+q_2})+\Psi(b,\frac{q_2 d+c}{1+q_2})]}{(1+q_1)(1+q_2)} \\
&\quad + \frac{q_2[\Psi(\frac{q_1 a+b}{1+q_1},c)+\Psi(\frac{q_1 b+a}{1+q_1},d)+\Psi(\frac{q_1 b+a}{1+q_1},c)+\Psi(\frac{q_1 a+b}{1+q_1},d)]}{(1+q_1)(1+q_2)} \\
&\quad + \frac{\Psi(\frac{q_1 a+b}{1+q_1},\frac{q_2 d+c}{1+q_2})+\Psi(\frac{q_1 a+b}{1+q_1},\frac{q_2 c+d}{1+q_2})+\Psi(\frac{q_1 b+a}{1+q_1},\frac{q_2 c+d}{1+q_2})+\Psi(\frac{q_1 b+a}{1+q_1},\frac{q_2 d+c}{1+q_2})}{(1+q_1)(1+q_2)} \\
&\le \Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d).
\end{aligned}
$$

Also, by using inequalities (1.28), (2.14), (2.17) and (2.23), we can deduce the following theorem.

Theorem 2.8 *Let $\Psi : \Delta \to \mathbb{R}_e$ be a coordinated convex function on Δ and $q_1, q_2 \in (0,1)$, then one has*

$$
\begin{aligned}
&4\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\le \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2},\frac{(1+q_2+q_2^2)c+q_2 d}{(1+q_2)^2}\right) \\
&\quad + \Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},\frac{q_2 c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right) \\
&\quad + \Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},\frac{(1+q_2+q_2^2)c+q_2 d}{(1+q_2)^2}\right) \\
&\quad + \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2},\frac{q_2 c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right) \\
&\le \frac{1+q_1}{2(b-a)}\Bigg[\int_a^{\frac{q_1 a+b}{1+q_1}} \Psi\left(u,\frac{(1+q_2+q_2^2)c+q_2 d}{(1+q_2)^2}\right) {}_a d_{q_1}u \\
&\quad + \int_{\frac{q_1 b+a}{1+q_1}}^{b} \Psi\left(u,\frac{q_2 c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right) {}^{b}d_{q_1}u \\
&\quad + \int_{\frac{q_1 b+a}{1+q_1}}^{b} \Psi\left(u,\frac{(1+q_2+q_2^2)c+q_2 d}{(1+q_2)^2}\right) {}^{b}d_{q_1}u \\
&\quad + \int_a^{\frac{q_1 a+b}{1+q_1}} f\left(u,\frac{q_2 c+(1+q_2+q_2^2)d}{(1+q_2)^2}\right) {}_a d_{q_1}u\Bigg] \\
&\quad + \frac{1+q_2}{2q_2(d-c)}\Bigg[\int_c^{\frac{q_2 d+c}{1+q_2}} \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2},\nu\right) {}_c d_{q_2}\nu \\
&\quad + \int_{\frac{q_2 c+d}{1+q_2}}^{d} \Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},\nu\right) {}^{d}d_{q_2}\nu
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{c}^{\frac{q_2d+c}{1+q_2}} \Psi\left(\frac{a+(2q_1+q_1^2)b}{(1+q_1)^2},\nu\right) {}_{c}d_{q_2}\nu \\
&+\int_{\frac{q_2c+d}{1+q_2}}^{d} \Psi\left(\frac{(2q_1+q_1^2)a+b}{(1+q_1)^2},\nu\right) {}^{d}d_{q_2}\nu\Bigg] \\
&\leq \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\Bigg[\int_{a}^{\frac{q_1a+b}{1+q_1}}\int_{c}^{\frac{q_2d+c}{1+q_2}} \Psi(u,\nu)\,{}_{c}d_{q_2}\nu\,{}_{a}d_{q_1}u \\
&+\int_{\frac{q_1b+a}{1+q_1}}^{b}\int_{\frac{q_2c+d}{1+q_2}}^{d} \Psi(u,\nu)\,{}^{d}d_{q_2}\nu\,{}^{b}d_{q_1}u \\
&+\int_{\frac{q_1b+a}{1+q_1}}^{b}\int_{c}^{\frac{q_2d+c}{1+q_2}} \Psi(u,\nu)\,{}_{c}d_{q_2}\nu\,{}^{b}d_{q_1}u+\int_{a}^{\frac{q_1a+b}{1+q_1}}\int_{\frac{q_2c+d}{1+q_2}}^{d} \Psi(u,\nu)\,{}^{d}d_{q_2}\nu\,{}_{a}d_{q_1}u\Bigg] \\
&\leq \frac{(1+q_1)}{2(1+q_2)(b-a)}\Bigg[q_2\int_{a}^{\frac{q_1a+b}{1+q_1}} \Psi(u,c)\,{}_{a}d_{q_1}u+\int_{a}^{\frac{q_1a+b}{1+q_1}} \Psi\left(u,\frac{q_2d+c}{1+q_2}\right){}_{a}d_{q_1}u \\
&+\int_{\frac{q_1b+a}{1+q_1}}^{b} \Psi\left(u,\frac{q_2d+c}{1+q_2}\right){}^{b}d_{q_1}u+q_2\int_{\frac{q_1b+a}{1+q_1}}^{b} \Psi(u,d)\,{}^{b}d_{q_1}u \\
&+q_2\int_{\frac{q_1b+a}{1+q_1}}^{b} \Psi(u,c)\,{}^{b}d_{q_1}u+\int_{\frac{q_1b+a}{1+q_1}}^{b} \Psi\left(u,\frac{q_2d+c}{1+q_2}\right){}^{b}d_{q_1}u \\
&+\int_{a}^{\frac{q_1a+b}{1+q_1}} \Psi\left(u,\frac{q_2c+d}{1+q_2}\right){}_{a}d_{q_1}u+q_2\int_{a}^{\frac{q_1a+b}{1+q_1}} \Psi(u,d)\,{}_{a}d_{q_1}u\Bigg] \\
&+\frac{(1+q_2)}{2q_2(1+q_1)(d-c)}\Bigg[q_1\int_{c}^{\frac{q_2d+c}{1+q_2}} \Psi(a,\nu)\,{}_{c}d_{q_2}\nu \\
&+\int_{c}^{\frac{q_2d+c}{1+q_2}} \Psi\left(\frac{q_1a+b}{1+q_1},\nu\right){}_{c}d_{q_2}\nu \\
&+\int_{\frac{q_2c+d}{1+q_2}}^{d} \Psi\left(\frac{q_1b+a}{1+q_1},\nu\right){}^{d}d_{q_2}\nu+q_1\int_{\frac{q_2c+d}{1+q_2}}^{d} \Psi(b,\nu)\,{}^{d}d_{q_2}\nu \\
&+\int_{c}^{\frac{q_2d+c}{1+q_2}} \Psi\left(\frac{q_1b+a}{1+q_1},\nu\right){}_{c}d_{q_2}\nu+q_1\int_{c}^{\frac{q_2d+c}{1+q_2}} \Psi(b,\nu)\,{}_{c}d_{q_2}\nu \\
&+q_1\int_{\frac{q_2c+d}{1+q_2}}^{d} \Psi(a,\nu)\,{}^{d}d_{q_2}\nu+\int_{\frac{q_2c+d}{1+q_2}}^{d} \Psi\left(\frac{q_1a+b}{1+q_1},\nu\right){}^{d}d_{q_2}\nu\Bigg] \\
&\leq \frac{q_1q_2[\Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d)]}{(1+q_1)(1+q_2)} \\
&+\frac{q_1[\Psi(b,\frac{q_2c+d}{1+q_2})+\Psi(a,\frac{q_2d+c}{1+q_2})+\Psi(a,\frac{q_2c+d}{1+q_2})+\Psi(b,\frac{q_2d+c}{1+q_2})]}{(1+q_1)(1+q_2)} \\
&+\frac{q_2[\Psi(\frac{q_1b+a}{1+q_1},d)+\Psi(\frac{q_1b+a}{1+q_1},c)+\Psi(\frac{q_1a+b}{1+q_1},c)+\Psi(\frac{q_1a+b}{1+q_1},d)]}{(1+q_1)(1+q_2)} \\
&+\frac{\Psi(\frac{q_1b+a}{1+q_1},\frac{q_2c+d}{1+q_2})+\Psi(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2})+\Psi(\frac{q_1a+b}{1+q_1},\frac{q_2d+c}{1+q_2})+\Psi(\frac{q_1b+a}{1+q_1},\frac{q_2d+c}{1+q_2})}{(1+q_1)(1+q_2)} \\
&\leq \Psi(a,c)+\Psi(a,d)+\Psi(b,c)+\Psi(b,d).
\end{aligned}
$$

Remark 2.5 If $q_1, q_2 \to 1^-$, then Theorems 2.5–2.8 give to the following inequalities, which is the refinement for the classical inequalities (1.1):

$$\begin{aligned}
&4\Psi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad \leq \Psi\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right) + \Psi\left(\frac{3a+b}{4}, \frac{3d+c}{4}\right) \\
&\qquad + \Psi\left(\frac{a+3b}{4}, \frac{3c+d}{4}\right) + \Psi\left(\frac{a+3b}{4}, \frac{c+3d}{4}\right) \\
&\quad \leq \frac{1}{b-a}\left[\int_a^b \Psi\left(u, \frac{3c+d}{4}\right) du + \int_a^b \Psi\left(u, \frac{c+3d}{4}\right) du\right] \\
&\qquad + \frac{1}{d-c}\left[\int_c^d \Psi\left(\frac{3a+b}{4}, v\right) dv + \int_c^d \Psi\left(\frac{a+3b}{4}, v\right) dv\right] \\
&\leq \frac{4}{(b-a)(d-c)} \int_a^b \int_c^d \Psi(u,v)\, dv\, du \\
&\leq \frac{1}{2(b-a)}\left[\int_a^b \Psi(u,c)\, du + 2\int_a^b \Psi\left(u, \frac{c+d}{2}\right) du + \int_a^b \Psi(u,d)\, du\right] \\
&\qquad + \frac{1}{2(d-c)}\left[\int_c^d \Psi(a,v)\, dv + 2\int_c^d \Psi\left(\frac{a+b}{2}, v\right) dv + \int_c^d \Psi(b,v)\, dv\right] \\
&\quad \leq \frac{[\Psi(a,c) + \Psi(a,d) + \Psi(b,c) + \Psi(b,d)]}{4} + \frac{2\Psi(a, \frac{c+d}{2}) + 2\Psi(b, \frac{c+d}{2})}{4} \\
&\qquad + \frac{2\Psi(\frac{a+b}{2}, c) + 2\Psi(\frac{a+b}{2}, d)}{4} + \frac{4\Psi(\frac{a+b}{2}, \frac{c+d}{2})}{4} \\
&\quad \leq \Psi(a,c) + \Psi(a,d) + \Psi(b,c) + \Psi(b,d).
\end{aligned} \tag{2.29}$$

3 Conclusion

In this study, we have extended the definition of q-derivatives and q-integrals over the interval $[A,B]$ of the real lines. We have considered new $H-H$ inequalities in the context of q-calculus. For the desired results, we have developed an inequality with the same lower and upper estimates as in classical Theorem 1.1. Also, we have established new midpoint $H-H$ type inequalities, which confirm the refinements to the previously known inequalities.

Our results suggest that two different partitions exist for midpoint type inequalities in the q-analogues. Indeed, for the interval $[A,B]$, we have $A \leq \frac{qA+B}{1+q} \leq B$ and $A \leq \frac{A+qB}{1+q} \leq B$ with $q \in (0,1)$. Similarly for $[C,D]$, we have $C \leq \frac{qC+D}{1+q} \leq D$ and $C \leq \frac{C+qD}{1+q} \leq D$ with $q \in (0,1)$. The last four inequalities use all partitions for the desired results. We believe that the results of this paper can be extended to establish new inequalities via different kinds of convex functions in the premises of q-calculus.

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