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A novel proof of two partial fraction decompositions

Jun-Ming Zhu[1] and Qiu-Ming Luo[2*]

*Correspondence: luomath2007@163.com
[2]Department of Mathematics, Chongqing Normal University, Chongqing Higher Education Mega Center, Huxi Campus, Chongqing 401331, People's Republic of China
Full list of author information is available at the end of the article

Abstract

In this paper, by constructing contour integral and using Cauchy's residue theorem, we provide a novel proof of Chu's two partial fraction decompositions.

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Keywords: Contour integral; Cauchy's residue theorem; Rational function; Partial fraction decomposition; Bell polynomial

1 Introduction

The generalized harmonic numbers are defined by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} \quad \text{for } n, r = 1, 2, \ldots;$$

when $r = 1$, they reduce to the classical harmonic numbers $H_n = H_n^{(1)}$.

For $z \in \mathbb{C}$, the shifted factorial is defined by

$$(z)_0 = 1 \quad \text{and} \quad (z)_n = z(z+1)\cdots(z+n-1) \quad \text{for } n = 1, 2, \ldots.$$

The complete Bell polynomials $\mathbf{B}_n(x_1, x_2, \ldots, x_n)$ are defined by [10, p. 134]

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k!}\right) = \sum_{n=0}^{\infty} \mathbf{B}_n(x_1, x_2, \ldots, x_n) \frac{z^n}{n!}, \qquad \mathbf{B}_0 := 1, \tag{1}$$

with explicit expression

$$\mathbf{B}_n(x_1, x_2, \ldots, x_n) = \sum_{\pi(n)} \frac{n!}{k_1! k_2! \cdots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots \left(\frac{x_n}{n!}\right)^{k_n}, \tag{2}$$

where $\pi(n)$ denotes all partitions of n into nonnegative parts, that is, all nonnegative integer solutions of the equation

$$m_1 + 2m_2 + \cdots + nm_n = n.$$

The way of the partial fraction decomposition plays an important role in studying the combinatorial identities and related questions (see, e.g., [1–9, 11, 12, 14, 16–22, 24] and the references therein).

In 2005, Chu [4] established the partial fraction decompositions of two rational functions $\frac{1}{(x)_{n+1}^{\lambda}}$ and $\frac{x^{\theta}}{(x)_{n+1}^{\lambda}}$ based on the induction principle and famous the Faà di Bruno formula and obtained several striking harmonic number identities from two partial fraction decompositions. He constructed the generalized Hermite–Padé approximates to the logarithm and therefore resolved completely the open problem of Driver et al. [13].

We now rewrite two main results of Chu.

Theorem A ([4, Theorem 2]) *Let λ, θ, and n be three natural numbers such that $0 \le \theta < \lambda(n+1)$. Then we have the algebraic identity*

$$\frac{(n!)^{\lambda}}{(x)_{n+1}^{\lambda}} = \sum_{k=0}^{n}(-1)^{k\lambda}\binom{n}{k}^{\lambda}\sum_{j=0}^{\lambda-1}\frac{\Omega_j(\lambda,-k)}{j!(x+k)^{\lambda-j}}, \tag{3}$$

where the Ω-coefficients are defined as

$$\Omega_{\ell}(\lambda,x) = (-1)^{\ell}\ell!\sum_{\|\tilde{m}\|=\ell}\frac{\lambda^{|\tilde{m}|}}{\tilde{m}!}\prod_{i=1}^{\ell}\frac{\mathcal{H}_i^{m_i}(x)}{i^{m_i}},$$

$$\Omega_{\ell}(\lambda,-k) = \ell!\sum_{\|\tilde{m}\|=\ell}\frac{\lambda^{|\tilde{m}|}}{\tilde{m}!}\prod_{i=1}^{\ell}\frac{\{\theta-\lambda k^i[H_k^{(i)}+(-1)^i H_{n-k}^{(i)}]\}^{m_i}}{i^{m_i}}.$$

Theorem B ([4, Theorem 5]) *Let λ, θ, and n be three natural numbers such that $0 \le \theta < \lambda(n+1)$. Then we have the algebraic identity*

$$\frac{(n!)^{\lambda}x^{\theta}}{(x)_{n+1}^{\lambda}} = \sum_{k=0}^{n}(-1)^{k\lambda}\binom{n}{k}^{\lambda}\sum_{j=0}^{\lambda-1}\frac{\Omega_j(\lambda,\theta,-k)}{j!(x+k)^{\lambda-j}}, \tag{4}$$

where Ω-coefficients are defined as

$$\Omega_{\ell}(\lambda,\theta,x) = x^{\theta-\ell}\sum_{\|\tilde{m}\|=\ell}(-1)^{\ell+|\tilde{m}|}\frac{\ell!}{\tilde{m}!}\prod_{i=1}^{\ell}\frac{\{\theta-\lambda x^i\mathcal{H}_i(x)\}^{m_i}}{i^{m_i}},$$

$$\Omega_{\ell}(\lambda,\theta,-k) = k^{\theta-\ell}\sum_{\|\tilde{m}\|=\ell}(-1)^{\theta+|\tilde{m}|}\frac{\ell!}{\tilde{m}!}\prod_{i=1}^{\ell}\frac{\{\theta-\lambda k^i[H_k^{(i)}+(-1)^i H_{n-k}^{(i)}]\}^{m_i}}{i^{m_i}}.$$

For definitions of $\tilde{m}!$, $|\tilde{m}|$, $\|\tilde{m}\|$, and $\mathcal{H}_i(x)$, see [4, p. 43, and p. 44, (1.4a)].

Comparing $\Omega_{\ell}(\lambda,-k)$, and $\Omega_{\ell}(\lambda,\theta,-k)$ with expression (2) of complete Bell polynomials, it is not difficult to reformulate Theorem A and Theorem B as follows (let $\theta \longmapsto M$).

Theorem 1 *Suppose that λ and n are positive integers and $x \in \mathbb{C} \setminus \{0,-1,\dots,-n\}$. Let $N = \lambda(n+1)$. Then we have the partial fraction decomposition*

$$\frac{1}{(x)_{n+1}^{\lambda}} = \sum_{k=0}^{n}\frac{(-1)^{k\lambda}}{(n!)^{\lambda}}\binom{n}{k}^{\lambda}\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(x_1,x_2,\dots,x_j)}{j!(x+k)^{\lambda-j}}, \tag{5}$$

where

$$x_i = \lambda(i-1)!\big(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\big), \quad i = 1,2,\dots,\lambda-1.$$

Theorem 2 *Let λ, M, and n be three natural numbers such that $\lambda \le M < \lambda(n+1)$. Then we have the partial fraction decomposition*

$$\frac{x^M}{(x)_{n+1}^{\lambda}} = \sum_{k=0}^{n} \frac{(-1)^{\lambda k+M}}{(n!)^{\lambda}} \binom{n}{k}^{\lambda} k^M \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(x_1,x_2,\dots,x_j)}{j!(x+k)^{\lambda-j}},$$

where

$$x_i = (i-1)!\left[\lambda\big(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\big) - \frac{M}{k^i}\right], \quad i = 1,2,\dots,\lambda-1.$$

In the present paper, we give a novel proof of Theorem 1 and Theorem 2 by constructing an appropriate contour integral.

We also use the following lemma in Sects. 2 and 3.

Lemma 3 ([15]) *Let $P(z)$ and $Q(z)$ be polynomials (in the complex variable z) of degrees m and n, respectively, given by*

$$P(z) = a_0z^m + a_1z^{m-1} + \cdots + a_m \quad \text{and} \quad Q(z) = b_0z^n + b_1z^{n-1} + \cdots + b_n.$$

Suppose that $P(z)$ and $Q(z)$ have no common zeros. If $\mathcal{C}$ is a simple closed path containing the poles of $P(z)/Q(z)$ in its interior, then

$$\oint_{\mathcal{C}} \frac{P(z)}{Q(z)}\,\mathrm{d}z = \begin{cases} \frac{2\pi i a_0}{b_0}, & n-m=1, \\ 0, & n-m \ge 2. \end{cases} \tag{6}$$

2 The proof of Theorem 1

In this section, we give a novel proof of Theorem 1 using a contour integral and Cauchy's residue theorem. We need two lemmas.

Lemma 4 *Suppose that λ and n are positive integers and $x \in \mathbb{C} \setminus \{0,-1,\dots,-n\}$. Let $N = \lambda(n+1)$. Then we have the algebraic identity*

$$\frac{(n!)^{\lambda}}{(x)_{n+1}^{\lambda}} = \sum_{k=0}^{n} \frac{(-1)^{\lambda k}}{(n!)^{\lambda}(x+k)} \binom{n}{k}^{\lambda} \frac{\mathbf{B}_{\lambda-1}(y_1,\dots,y_{\lambda-1})}{(\lambda-1)!}, \tag{7}$$

where

$$y_i = (i-1)!\left[\lambda\big(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\big) + \frac{1}{(x+k)^i}\right]. \tag{8}$$

Proof We first consider two polynomials $P(z)$ and $Q(z)$ of degrees 0 and $N+1$, respectively, given by

$$P(z) = 1 \quad \text{and} \quad Q(z) = (z-x)\prod_{j=0}^{n}(z+j)^{\lambda}.$$

We next construct the following contour integrals for the rational functions $1/Q(z)$:

$\oint_\Gamma \frac{1}{Q(z)}\,\mathrm{d}z$, where Γ is a simple closed contour, which only surrounds the single pole x of $1/Q(z)$;

$\oint_{\Gamma_1} \frac{1}{Q(z)}\,\mathrm{d}z$, where Γ_1 is a simple closed contour, which surrounds the poles $0, -1, \dots, -n$ of $1/Q(z)$.

Applying Cauchy's residue theorem, we compute the contour integral $\oint_\Gamma \frac{1}{Q(z)}\,\mathrm{d}z$:

$$\begin{aligned}\oint_\Gamma \frac{1}{Q(z)}\,\mathrm{d}z &= 2\pi i \operatorname*{Res}_{z=x} \frac{1}{(z-x)\prod_{j=0}^{n}(z+j)^\lambda} = 2\pi i \lim_{z\to x} \frac{1}{\prod_{j=0}^{n}(z+j)^\lambda}\\ &= \frac{2\pi i}{\prod_{j=0}^{n}(x+j)^\lambda} = \frac{2\pi i}{(x)_{n+1}^\lambda}.\end{aligned}$$

We compute the contour integral $\oint_{\Gamma_1} \frac{1}{Q(z)}\,\mathrm{d}z$. By utilizing Cauchy's residue theorem, the power series expansion of logarithmic function

$$\log(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n} \quad \bigl(|z|<1\bigr),$$

and the definition of complete Bell polynomials, we obtain

$$\begin{aligned}\oint_{\Gamma_1} \frac{1}{Q(z)}\,\mathrm{d}z &= 2\pi i \sum_{k=0}^{n} \operatorname*{Res}_{z=-k} \frac{1}{(z-x)\prod_{j=0}^{n}(z+j)^\lambda}\\ &= 2\pi i \sum_{k=0}^{n} \bigl[(z+k)^{\lambda-1}\bigr] \frac{1}{(z-x)\prod_{\substack{j=0\\ j\neq k}}^{n}(z+j)^\lambda}\\ &= 2\pi i \sum_{k=0}^{n} \bigl[z^{\lambda-1}\bigr] \frac{1}{(z-x-k)\prod_{\substack{j=0\\ j\neq k}}^{n}(z-k+j)^\lambda}\\ &= -2\pi i \sum_{k=0}^{n} \Biggl\{ \frac{1}{(x+k)\prod_{\substack{j=0\\ j\neq k}}^{n}(j-k)^\lambda}\\ &\quad \times \bigl[z^{\lambda-1}\bigr] \exp\Biggl[-\log\biggl(1-\frac{z}{x+k}\biggr) - \lambda \sum_{j=0, j\neq k}^{n} \log\biggl(1+\frac{z}{j-k}\biggr)\Biggr]\Biggr\}\\ &= -2\pi i \sum_{k=0}^{n} \Biggl\{ \frac{(-1)^{\lambda k}}{(n!)^\lambda (x+k)} \binom{n}{k}^\lambda\\ &\quad \times \bigl[z^{\lambda-1}\bigr] \exp\Biggl[\sum_{i=1}^{\infty}(i-1)!\biggl(\lambda\bigl(H_k^{(i)} + (-1)^i H_{n-k}^{(i)}\bigr) + \frac{1}{(x+k)^i}\biggr)\Biggr]\frac{z^i}{i!}\Biggr\}\\ &= -2\pi i \sum_{k=0}^{n} \frac{(-1)^{\lambda k}}{(n!)^\lambda (x+k)} \binom{n}{k}^\lambda \frac{\mathbf{B}_{\lambda-1}(y_1, \dots, y_{\lambda-1})}{(\lambda-1)!}.\end{aligned}$$

Let $\mathcal{C} = \Gamma + \Gamma_1$. Applying the second result of Lemma 3, we have that $\oint_{\Gamma+\Gamma_1} \frac{1}{Q(z)}\,\mathrm{d}z = 0$ or, equivalently, that $\oint_\Gamma \frac{1}{Q(z)}\,\mathrm{d}z = -\oint_{\Gamma_1} \frac{1}{Q(z)}\,\mathrm{d}z$. Therefore we directly obtain the algebraic identity (7). □

Lemma 5 *The complete Bell polynomials have the following recursive relations*:

$$\frac{\mathbf{B}_{\lambda-1}(y_1,\dots,y_{\lambda-1})}{(\lambda-1)!}=\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(x_1,\dots,x_j)}{j!(x+k)^{\lambda-j-1}}. \tag{9}$$

Proof Let

$$x_i=(i-1)!\bigl[\lambda\bigl(H_k^{(i)}+(-1)^iH_{n-k}^{(i)}\bigr)\bigr].$$

Write $y_i=x_i+\frac{(i-1)!}{(x+k)^i}$ in (8). By using the definition of complete Bell polynomials, power series expansion of the logarithmic function

$$\log(1+z)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n},$$

and the geometric series

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,$$

we get

$$\begin{aligned}\frac{\mathbf{B}_{\lambda-1}(y_1,\dots,y_{\lambda-1})}{(\lambda-1)!}&=\bigl[t^{\lambda-1}\bigr]\exp\left(\sum_{n=1}^{\infty}y_n\frac{t^n}{n!}\right)\\&=\bigl[t^{\lambda-1}\bigr]\exp\left\{\sum_{n=1}^{\infty}\left(x_n+\frac{(n-1)!}{(x+k)^n}\right)\frac{t^n}{n!}\right\}\\&=\sum_{j=0}^{\lambda-1}\bigl[t^j\bigr]\exp\left\{\sum_{n=1}^{\infty}x_n\frac{t^n}{n!}\right\}\bigl[t^{\lambda-1-j}\bigr]\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{t}{x+k}\right)^n\right\}\\&=\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(x_1,\dots,x_j)}{j!}\bigl[t^{\lambda-1-j}\bigr]\exp\left\{-\log\left(1-\frac{t}{x+k}\right)\right\}\\&=\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(x_1,\dots,x_j)}{j!}\bigl[t^{\lambda-1-j}\bigr]\sum_{n=0}^{\infty}\left(\frac{t}{x+k}\right)^n\\&=\sum_{j=0}^{\lambda-1}\frac{\mathbf{B}_j(x_1,\dots,x_j)}{j!(x+k)^{\lambda-j-1}}.\end{aligned}$$

The proof is complete. □

Proof of Theorem 1 Replacing (9) of Lemma 5 by (7) of Lemma 4, we immediately obtain Theorem 1. □

3 The proof of Theorem 2

In this section, we give a different proof of Theorem 2 in the same line. We first state the following result.

Theorem 6 *Let M be a nonnegative integer, let λ and n be positive integers, and let $x \in \mathbb{C} \setminus \{0, -1, \dots, -n\}$. Let $N = \lambda n$ for $M < \lambda n$. Then we have the partial fraction decomposition*

$$\frac{x^M}{(x+1)_n^\lambda} = \sum_{k=1}^{n} \frac{(-1)^{\lambda(k+1)+M}}{(n!)^\lambda} \binom{n}{k}^\lambda k^{\lambda+M} \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(y_1, y_2, \dots, y_j)}{j!(x+k)^{\lambda-j}}, \tag{10}$$

where

$$y_i = (i-1)! \left[\lambda \left(H_k^{(i)} + (-1)^i H_{n-k}^{(i)} \right) - \frac{\lambda + M}{k^i} \right], \quad i = 1, 2, \dots, \lambda - 1. \tag{11}$$

Proof The proof is similar to that of Theorem 1, and thus we only present the important steps and omit many details.

We construct two polynomials $P(z)$ and $Q(z)$ of degrees M and $N+1$, respectively, given by

$$P(z) = z^M \quad \text{and} \quad Q(z) = (z - x) \prod_{j=0}^{n-1} (z + j + 1)^\lambda.$$

We consider two contour integrals for the rational functions $P(z)/Q(z)$:
$\oint_\Gamma \frac{P(z)}{Q(z)}\,\mathrm{d}z$, where Γ is a simple closed contour, which only surrounds the single pole x of $P(z)/Q(z)$;
$\oint_{\Gamma_1} \frac{P(z)}{Q(z)}\,\mathrm{d}z$, let Γ_1 be a simple closed contour which surrounds the poles $-1, -2, \dots, -n$ of $P(z)/Q(z)$.
We obtain contour integrals $\oint_\Gamma \frac{P(z)}{Q(z)}\,\mathrm{d}z$ and $\oint_{\Gamma_1} \frac{P(z)}{Q(z)}\,\mathrm{d}z$:

$$\oint_\Gamma \frac{P(z)}{Q(z)}\,\mathrm{d}z = 2\pi i \frac{x^M}{(x+1)_n^\lambda},$$
$$\oint_{\Gamma_1} \frac{P(z)}{Q(z)}\,\mathrm{d}z = -2\pi i \sum_{k=1}^{n} \frac{(-1)^{\lambda k}(-k)^{\lambda+M}}{(n!)^\lambda (x+k)} \binom{n}{k}^\lambda \frac{\mathbf{B}_{\lambda-1}(y_1, \dots, y_{\lambda-1})}{(\lambda-1)!}.$$

Applying the second result of Lemma 3, we have the following algebraic identity:

$$\frac{x^M}{(x+1)_n^\lambda} = \sum_{k=1}^{n} \frac{(-1)^{\lambda k}(-k)^{\lambda+M}}{(n!)^\lambda (x+k)} \binom{n}{k}^\lambda \frac{\mathbf{B}_{\lambda-1}(w_1, \dots, w_{\lambda-1})}{(\lambda-1)!}, \tag{12}$$

where

$$w_i = (i-1)! \left[\lambda \left(H_k^{(i)} + (-1)^i H_{n-k}^{(i)} \right) - \frac{\lambda + M}{k^i} + \frac{1}{(x+k)^i} \right], \quad i = 1, 2, \dots, \lambda - 1.$$

We can also obtain that

$$\frac{\mathbf{B}_{\lambda-1}(w_1, \dots, w_{\lambda-1})}{(\lambda-1)!} = \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(y_1, \dots, y_j)}{j!(x+k)^{\lambda-j-1}}. \tag{13}$$

Substituting (13) into (12), we complete the proof of Theorem 6. □

Proof of Theorem 2 We obviously have

$$\frac{x^M}{(x)_{n+1}^{\lambda}} \equiv \frac{x^{M-\lambda}}{(x+1)_n^{\lambda}}.$$

Letting $M \longmapsto M - \lambda$ in Theorem 6, we immediately obtain Theorem 2. □

4 Conclusion

The basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-)hypergeometric polynomials, are known to have widespread applications, particularly, in several areas of number theory and combinatorial analysis such as the theory of partitions.

Recently, Srivastava [23] published a survey-cum-expository paper on the q-calculus and fractional q-calculus in geometric function theory of complex analysis. Remarkably, a considerably large group of authors have made use of the so-called (p,q)-analysis by introducing a seemingly redundant parameter p in the already known results dealing with the classical q-analysis. On page 340, Professor Srivastava pointed out an important demonstrated observation that any (p,q)-variations of the proposed q-results would be trivially inconsequential, because the additional parameter p is obviously redundant.

In this concluding section, we also suggest the corresponding basic (or q-) extensions of the results of this paper to the interested reader.

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Authors' contributions
There was an equal amount of contributions of the authors. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Luoyang Normal University, Luoyang City 471934, Henan Province, People's Republic of China. [2]Department of Mathematics, Chongqing Normal University, Chongqing Higher Education Mega Center, Huxi Campus, Chongqing 401331, People's Republic of China.

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