Ahmad et al. *Advances in Difference Equations* (2021) 2021:278
https://doi.org/10.1186/s13662-021-03440-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

A study of a nonlinear coupled system of three fractional differential equations with nonlocal coupled boundary conditions

Bashir Ahmad[1*], Soha Hamdan[1], Ahmed Alsaedi[1] and Sotiris K. Ntouyas[2,1]

*Correspondence: bashirahmad_qau@yahoo.com
[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this research we introduce and study a new coupled system of three fractional differential equations supplemented with nonlocal multi-point coupled boundary conditions. Existence and uniqueness results are established by using the Leray–Schauder alternative and Banach's contraction mapping principle. Illustrative examples are also presented.

MSC: 34A08; 34B15

Keywords: Fractional differential equations; Caputo fractional derivative; System; Existence; Fixed point theorems

1 Introduction

The methods of fractional calculus significantly improved the study of integer-order mathematical models associated with real-world problems appearing in scientific and technical disciplines. A point of central importance for choosing fractional order derivative operators is their nonlocal nature, which accounts for the history of the associated phenomena under investigation. For application details, see financial economics [1], ecology [2], immune systems [3], HIV/AIDS [4], chaotic synchronization [5, 6], etc. Inspired by the great popularity of the subject, many researchers turned to the further development of this branch of mathematical analysis. In particular, fractional order boundary value problems attracted considerable attention, and the literature on the topic was enriched with a huge number of interesting articles, for instance, see [7–12].

Fractional differential systems have also been studied by many researchers in view of their occurrence in the mathematical modeling of several physical and engineering processes [13–15]. One can find the details about the theoretical development of the topic in the articles [16–26].

We introduce and study a new class of coupled systems of mixed-order three fractional differential equations equipped with nonlocal multi-point coupled boundary conditions.

In precise terms, we consider the following fully coupled system:

$$\begin{cases} {}^{C}D^{\eta}_{a^+}u(t) = \rho(t,u(t),x(t),y(t)), \quad 1 < \eta \le 2, t \in [a,b], \\ {}^{C}D^{\xi}_{a^+}x(t) = \varphi(t,u(t),x(t),y(t)), \quad 1 < \xi \le 2, t \in [a,b], \\ {}^{C}D^{\zeta}_{a^+}y(t) = \psi(t,u(t),x(t),y(t)), \quad 2 < \zeta \le 3, t \in [a,b], \\ u(a) = u_0, \qquad u(b) = \sum_{i=1}^{m} p_i x(\alpha_i), \\ x(a) = 0, \qquad x(b) = \sum_{j=1}^{n} q_j y(\beta_j), \\ y(\xi_1) = 0, \qquad y(\xi_2) = 0, \qquad y(b) = \sum_{k=1}^{l} r_k u(\gamma_k), \\ a < \xi_1 < \xi_2 < \alpha_1 < \cdots < \alpha_m < \beta_1 < \cdots < \beta_n < \gamma_1 < \cdots < \gamma_l < b, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\chi}$ is a Caputo fractional derivative of order $\chi \in \{\eta,\xi,\zeta\}$, $\rho,\varphi,\psi : [a,b]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given functions, $p_i,q_j,r_k \in \mathbb{R}$, $i = 1,\dots,m$, $j = 1,\dots,n$, $k = 1,\dots,l$.

Here we emphasize that the novelty of the present work lies in the fact that we consider a coupled system of three fractional differential equations of different orders on an arbitrary domain equipped with coupled nonlocal multi-point boundary conditions. One can observe that the multi-point boundary conditions are of cyclic nature and involve different nonlocal positions. Moreover, it is worthwhile to mention that much of the work on coupled fractional systems involves two fractional differential equations on the fixed domain. Thus our results are more general and contribute significantly to the existing literature on the topic.

The rest of the paper is organized as follows: In Sect. 2 we recall some basic definitions from fractional calculus and present an auxiliary result, which plays a pivotal role in transforming system (1.1) into equivalent integral equations. An existence result for the problem at hand is proved via the Leray–Schauder alternative, while the existence of a unique solution is established via Banach's contraction mapping principle. These results are presented in Sect. 3. Examples are also discussed for illustration of the obtained results.

2 Preliminaries

Let us begin this section with some definitions related to our study [27].

Definition 2.1 The Riemann–Liouville fractional integral of order $\omega \in \mathbb{R}$ $(\omega > 0)$ for a locally integrable real-valued function h defined on $-\infty \le a < t < b \le +\infty$, denoted by $I^{\omega}_{a^+}h$, is defined by

$$I^{\omega}_{a^+}h(t) = \frac{1}{\Gamma(\omega)}\int_a^t (t-s)^{\omega-1}h(s)\,ds,$$

where Γ denotes the Euler gamma function.

Definition 2.2 Let $h, h^{(m)} \in L^1[a,b]$ for $-\infty \le a < t < b \le +\infty$. The Riemann–Liouville fractional derivative $D^{\omega}_{a^+}h$ of order $\omega \in (m-1,m]$, $m \in \mathbb{N}$ is defined as

$$D^{\omega}_{a^+}h(t) = \frac{1}{\Gamma(m-\omega)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-1-\omega}h(s)\,ds,$$

while the Caputo fractional derivative ${}^{C}D^{\omega}_{a^+}h$ of order $\omega \in (m-1, m]$, $m \in \mathbb{N}$ is defined as

$$ {}^{C}D^{\omega}_{a^+}h(t) = D^{\omega}_{a^+}\left[h(t) - h(a) - h'(a)\frac{(t-a)}{1!} - \cdots - h^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!}\right]. $$

Remark 2.3 The Caputo fractional derivative of order $\omega \in (m-1, m]$, $m \in \mathbb{N}$ for a continuous function $h : (0, \infty) \to \mathbb{R}$ such that $h \in C^m[a, b]$, existing almost everywhere on $[a, b]$, is defined by

$$ {}^{C}D^{\omega}h(t) = \frac{1}{\Gamma(m-\omega)}\int_a^t (t-s)^{m-\omega-1}h^{(m)}(s)\,ds. $$

Now we present an important result to analyze problem (1.1).

Lemma 2.4 *Let $\overline{\rho}, \overline{\varphi}, \overline{\psi} \in C[a, b]$ and $\Delta \neq 0$. Then the unique solution of the system*

$$ \begin{cases} {}^{C}D^{\eta}_{a^+}u(t) = \overline{\rho}(t), \quad 1 < \eta \leq 2, t \in [a, b], \\ {}^{C}D^{\xi}_{a^+}x(t) = \overline{\varphi}(t), \quad 1 < \xi \leq 2, t \in [a, b], \\ {}^{C}D^{\zeta}_{a^+}y(t) = \overline{\psi}(t), \quad 2 < \zeta \leq 3, t \in [a, b], \\ u(a) = u_0, \qquad u(b) = \sum_{i=1}^{m} p_i x(\alpha_i), \\ x(a) = 0, \qquad x(b) = \sum_{j=1}^{n} q_j y(\beta_j), \\ y(\xi_1) = 0, \qquad y(\xi_2) = 0, \qquad y(b) = \sum_{k=1}^{l} r_k u(\gamma_k), \\ a < \xi_1 < \xi_2 < \alpha_1 < \cdots < \alpha_m < \beta_1 < \cdots < \beta_n < \gamma_1 < \cdots < \gamma_l < b. \end{cases} \tag{2.1} $$

is given by the formulas

$$ \begin{aligned} u(t) = {} & \int_a^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\overline{\rho}(s)\,ds + u_0 + (t-a)\Bigg\{a_{12}\int_a^b \frac{(b-s)^{\eta-1}}{\Gamma(\eta)}\overline{\rho}(s)\,ds \\ & + \frac{\sum_{i=1}^{m} p_i}{b-a}\left(a_{13}\sum_{i=1}^{m} p_i(\alpha_i - a) + 1\right)\int_a^{\alpha_i} \frac{(\alpha_i - a)^{\xi-1}}{\Gamma(\xi)}\overline{\varphi}(s)\,ds \\ & + \frac{\sum_{i=1}^{m} p_i(\alpha_i - a)}{\Delta(b-a)}\Bigg(a_3\int_a^{\xi_1} \frac{(\xi_1 - s)^{\zeta-1}}{\Gamma(\zeta)}\overline{\psi}(s)\,ds + a_4\int_a^{\xi_2} \frac{(\xi_2 - s)^{\zeta-1}}{\Gamma(\zeta)}\overline{\psi}(s)\,ds \\ & - A_3\int_a^b \frac{(b-s)^{\zeta-1}}{\Gamma(\eta)}\overline{\psi}(s)\,ds - A_1\sum_{j=1}^{n} q_j\int_a^{\beta_j} \frac{(\beta_j - s)^{\zeta-1}}{\Gamma(\zeta)}\overline{\psi}(s)\,ds \\ & - A_1\int_a^b \frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\overline{\varphi}(s)\,ds \\ & + A_3\sum_{k=1}^{l} r_k\int_a^{\gamma_k} \frac{(\gamma_k - s)^{\eta-1}}{\Gamma(\eta)}\overline{\rho}(s)\,ds\Bigg)\Bigg\} + a_{11}(t-a), \end{aligned} \tag{2.2} $$

$$ \begin{aligned} x(t) = {} & \int_a^t \frac{(t-s)^{\xi-1}}{\Gamma(\xi)}\overline{\varphi}(s)\,ds + \frac{(t-a)}{\Delta}\Bigg\{-a_5\int_a^b \frac{(b-s)^{\eta-1}}{\Gamma(\eta)}\overline{\rho}(s)\,ds \\ & + a_5\sum_{i=1}^{m} p_i\int_a^{\alpha_i} \frac{(\alpha_i - a)^{\xi-1}}{\Gamma(\xi)}\overline{\varphi}(s)\,ds + a_3\int_a^{\xi_1} \frac{(\xi_1 - s)^{\zeta-1}}{\Gamma(\zeta)}\overline{\psi}(s)\,ds \end{aligned} $$

$$
\begin{aligned}
&+ a_4 \int_a^{\xi_2} \frac{(\xi_2 - s)^{\zeta-1}}{\Gamma(\zeta)} \overline{\psi}(s)\,ds \\
&- A_3 \int_a^{b} \frac{(b - s)^{\zeta-1}}{\Gamma(\eta)} \overline{\psi}(s)\,ds - A_1 \sum_{j=1}^{n} q_j \int_a^{\beta_j} \frac{(\beta_j - s)^{\zeta-1}}{\Gamma(\zeta)} \overline{\psi}(s)\,ds \\
&- A_1 \int_a^{b} \frac{(b - s)^{\xi-1}}{\Gamma(\xi)} \overline{\varphi}(s)\,ds \\
&+ A_3 \sum_{k=1}^{l} r_k \int_a^{\gamma_k} \frac{(\gamma_k - s)^{\eta-1}}{\Gamma(\eta)} \overline{\rho}(s)\,ds + a_6 \Bigg\}, \qquad (2.3)
\end{aligned}
$$

$$
\begin{aligned}
y(t) = {}& \int_a^{t} \frac{(t - s)^{\zeta-1}}{\Gamma(\zeta)} \overline{\psi}(s)\,ds + b_1(t) \int_a^{\xi_1} \frac{(\xi_1 - a)^{\zeta-1}}{\Gamma(\zeta)} \overline{\psi}(s)\,ds \\
&+ b_2(t) \int_a^{\xi_2} \frac{(\xi_2 - s)^{\zeta-1}}{\Gamma(\zeta)} \overline{\psi}(s)\,ds \\
&+ b_3(t) \int_a^{b} \frac{(b - s)^{\zeta-1}}{\Gamma(\zeta)} \overline{\psi}(s)\,ds + b_4(t) \sum_{j=1}^{n} q_j \int_a^{b} \frac{(\beta_j - s)^{\zeta-1}}{\Gamma(\zeta)} \overline{\psi}(s)\,ds \\
&+ b_5(t) \int_a^{b} \frac{(b - s)^{\eta-1}}{\Gamma(\eta)} \overline{\rho}(s)\,ds \\
&- b_3(t) \sum_{k=1}^{l} r_k \int_a^{\gamma_k} \frac{(\gamma_k - s)^{\eta-1}}{\Gamma(\eta)} \overline{\rho}(s)\,ds - b_4(t) \int_a^{b} \frac{(b - s)^{\xi-1}}{\Gamma(\xi)} \overline{\varphi}(s)\,ds \\
&- b_5(t) \sum_{i=1}^{m} \alpha_i \int_a^{\alpha_i} \frac{(\alpha_i - s)^{\xi-1}}{\Gamma(\xi)} \overline{\varphi}(s)\,ds + b_6(t), \qquad (2.4)
\end{aligned}
$$

where

$$
\begin{aligned}
b_1(t) &= \frac{1}{\xi_1 - \xi_2} \left(\xi_2 - a + \frac{a_2 a_7}{\Delta} + (t-a)\left(\frac{a_1 a_7}{\Delta} - 1 \right) \right) + (t-a)^2 \frac{a_7}{\Delta}, \\
b_2(t) &= \frac{1}{\xi_1 - \xi_2} \left(-\xi_1 + a + \frac{a_2 a_8}{\Delta} + (t-a)\left(\frac{a_1 a_8}{\Delta} + 1 \right) \right) + (t-a)^2 \frac{a_8}{\Delta}, \\
b_3(t) &= \frac{b-a}{\Delta(\xi_1 - \xi_2)} \big(a_2 + a_1(t-a) + (\xi_1 - \xi_2)(t-a)^2 \big), \\
b_4(t) &= \frac{A_2}{\Delta(\xi_1 - \xi_2)} \big(a_2 + a_1(t-a) + (\xi_1 - \xi_2)(t-a)^2 \big), \\
b_5(t) &= \frac{a_9}{\Delta(\xi_1 - \xi_2)} \big(a_2 + a_1(t-a) + (\xi_1 - \xi_2)(t-a)^2 \big), \\
b_6(t) &= \frac{a_{10}}{\Delta(\xi_1 - \xi_2)} \big(a_2 + a_1(t-a) + (\xi_1 - \xi_2)(t-a)^2 \big), \\
a_1 &= (\xi_2 - a)^2 - (\xi_1 - a)^2, \qquad a_2 = (\xi_2 - a)(\xi_1 - a)(\xi_1 - \xi_2), \\
a_3 &= \frac{A_3(b - \xi_2) - A_1 \sum_{j=1}^{n} q_j(\xi_2 - \beta_j)}{\xi_1 - \xi_2}, \qquad a_4 = \frac{A_3(\xi_1 - b) - A_1 \sum_{j=1}^{n} q_j(\beta_j - \xi_1)}{\xi_1 - \xi_2}, \\
a_5 &= \frac{A_3 \sum_{k=1}^{l} r_k(\gamma_k - a)}{b-a}, \qquad a_6 = u_0 A_3 \left(\sum_{k=1}^{l} r_k - \frac{\sum_{k=1}^{l} r_k(\gamma_k - a)}{b-a} \right),
\end{aligned}
$$

$$
\begin{aligned}
a_7 &= \frac{-(b-a)(b-\xi_2) + A_2 \sum_{j=1}^{n} q_j(\xi_2 - \beta_j)}{\xi_1 - \xi_2}, \\
a_8 &= \frac{-(b-a)(\xi_1 - b) + A_2 \sum_{j=1}^{n} q_j(\beta_j - \xi_1)}{\xi_1 - \xi_2}, \\
a_9 &= \sum_{k=1}^{l} r_k(\gamma_k - a), \qquad a_{10} = -u_0(b-a)\sum_{k=1}^{l} r_k + \sum_{k=1}^{l} r_k(\gamma_k - a), \\
a_{11} &= \frac{1}{b-a}\left(\frac{a_6 \sum_{i=1}^{m} p_i(\alpha_i - a)}{\Delta} - u_0\right), \\
a_{12} &= \frac{-a_5 \sum_{i=1}^{m} p_i(\alpha_i - a)}{\Delta(b-a)} - \frac{1}{b-a}, \qquad a_{13} = \frac{a_5}{\Delta}, \\
A_1 &= \frac{a_2 + (b-a)a_1 + (b-a)^2(\xi_1 - \xi_2)}{\xi_1 - \xi_2}, \qquad A_2 = \frac{-\sum_{k=1}^{l} r_k(r_k - a)\sum_{i=1}^{m} p_i(\alpha_i - a)}{b-a}, \\
A_3 &= \frac{-\sum_{j=1}^{n} q_j}{\xi_1 - \xi_2}\bigl(a_2 + a_1(\beta_j - a) + (\beta_j - a)^2(\xi_1 - \xi_2)\bigr), \\
\Delta &= A_2A_3 - A_1(b-a).
\end{aligned} \tag{2.5}
$$

Proof The solution of system (2.1) can be written as

$$u(t) = I_{a^+}^{\eta}\overline{\rho}(t) + c_1 + c_2(t-a), \tag{2.6}$$

$$x(t) = I_{a^+}^{\xi}\overline{\varphi}(t) + c_3 + c_4(t-a), \tag{2.7}$$

$$y(t) = I_{a^+}^{\zeta}\overline{\psi}(t) + c_5 + c_6(t-a) + c_7(t-a)^2, \tag{2.8}$$

where $c_i \in \mathbb{R}$ $(i = 1,2,\dots,7)$ are unknown constants. Using the condition $u(a) = u_0$ in (2.6) gives $c_1 = u_0$ and applying the condition $x(a) = 0$ in (2.7) yields $c_3 = 0$, while making use of the conditions $y(\xi_1) = 0$ and $y(\xi_2) = 0$ leads to the equations

$$I_{a^+}^{\zeta}\overline{\psi}(\xi_1) + c_5 + c_6(\xi_1 - a) + c_7(\xi_1 - a)^2 = 0, \tag{2.9}$$

$$I_{a^+}^{\zeta}\overline{\psi}(\xi_2) + c_5 + c_6(\xi_2 - a) + c_7(\xi_2 - a)^2 = 0. \tag{2.10}$$

Using the conditions $u(b) = \sum_{i=1}^{m} p_i x(\alpha_i)$, $x(b) = \sum_{j=1}^{n} q_j y(\beta_j)$, and $y(b) = \sum_{k=1}^{l} r_k u(\gamma_k)$ with $c_1 = u_0$ and $c_3 = 0$ in (2.6)–(2.8) gives

$$I_{a^+}^{\eta}\overline{\rho}(b) + u_0 + c_2(b-a) = \sum_{i=1}^{m} p_i\bigl(I_{a^+}^{\xi}\overline{\varphi}(\alpha_i) + c_4(\alpha_i - a)\bigr), \tag{2.11}$$

$$I_{a^+}^{\xi}\overline{\varphi}(b) + c_4(b-a) = \sum_{j=1}^{n} q_j\bigl(I_{a^+}^{\zeta}\overline{\psi}(\beta_j) + c_5 + c_6(\beta_j - a) + c_7(\beta_j - a)^2\bigr), \tag{2.12}$$

$$I_{a^+}^{\zeta}\overline{\psi}(b) + c_5 + c_6(b-a) + c_7(b-a)^2 = \sum_{k=1}^{l} r_k\bigl(I_{a^+}^{\eta}\overline{\rho}(r_k) + u_0 + c_2(r_k - a)\bigr). \tag{2.13}$$

Subtracting (2.10) from (2.9), we find that

$$c_6 = \frac{1}{\xi_1 - \xi_2}\bigl(a_1c_7 - I_{a^+}^{\zeta}\overline{\psi}(\xi_1) + I_{a^+}^{\zeta}\overline{\psi}(\xi_2)\bigr), \tag{2.14}$$

where a_1 is given in (2.5). Substituting the value of c_6 into (2.9) yields

$$c_5 = \frac{1}{\xi_1 - \xi_2}\big(a_2 c_7(\xi_2 - a) I^{\zeta}_{a^+}\overline{\psi}(\xi_1) - (\xi_1 - a) I^{\zeta}_{a^+}\overline{\psi}(\xi_2)\big), \tag{2.15}$$

where a_2 is given in (2.5). From (2.11), we have

$$c_2 = \frac{1}{b-a}\left(-I^{\eta}_{a^+}\overline{\rho}(b) + \sum_{i=1}^{m} p_i I^{\xi}_{a^+}\overline{\varphi}(\alpha_i) + \sum_{i=1}^{m} p_i(\alpha_i - a)c_4 - u_0\right). \tag{2.16}$$

Substituting the values of c_2, c_5, and c_6 into (2.12) and (2.13), we get the system

$$\begin{aligned}
&A_1 c_7 + A_2 c_4 \\
&\quad = \sum_{k=1}^{l} r_k I^{\eta}_{a^+}\overline{\rho}(\gamma_k) + \sum_{k=1}^{l} r_k u_0 + \frac{\sum_{k=1}^{l} r_k(\gamma_k - a)}{b-a}\left(\sum_{i=1}^{m} p_i I^{\xi}_{a^+}\overline{\varphi}(\alpha_i) - u_0 - I^{\eta}_{a^+}\overline{\rho}(b)\right) \\
&\qquad + \frac{1}{\xi_1 - \xi_2}\big((\xi_1 - b) I^{\zeta}_{a^+}\overline{\psi}(\xi_2) + (b - \xi_2) I^{\zeta}_{a^+}\overline{\psi}(\xi_1)\big) - I^{\zeta}_{a^+}\overline{\psi}(b), \\
&A_3 c_7 + (b-a)c_4 \\
&\quad = \sum_{j=1}^{n} q_j I^{\zeta}_{a^+}\overline{\psi}(\beta_j) + \frac{\sum_{j=1}^{n} q_j}{\xi_1 - \xi_2}\big((\xi_2 - \beta_j) I^{\zeta}_{a^+}\overline{\psi}(\xi_1) + (\beta_j - \xi_1) I^{\zeta}_{a^+}\overline{\psi}(\xi_2)\big) - I^{\xi}_{a^+}\overline{\varphi}(b).
\end{aligned}$$

Solving the above system together with the notation in (2.5), we obtain

$$\begin{aligned}
c_4 &= \frac{1}{\Delta}\Bigg\{a_3 I^{\zeta}_{a^+}\overline{\psi}(\xi_1) + a_4 I^{\zeta}_{a^+}\overline{\psi}(\xi_2) - A_3 I^{\zeta}_{a^+}\overline{\psi}(b) - A_1 \sum_{j=1}^{n} q_j I^{\zeta}_{a^+}\overline{\psi}(\beta_j) - a_5 I^{\eta}_{a^+}\overline{\rho}(b) \\
&\quad + A_3 \sum_{k=1}^{l} r_k I^{\eta}_{a^+}\overline{\rho}(\gamma_k) + a_6\Bigg\}, \\
c_7 &= \frac{1}{\Delta}\Bigg\{a_7 I^{\zeta}_{a^+}\overline{\psi}(\xi_1) + a_8 I^{\zeta}_{a^+}\overline{\psi}(\xi_2) + (b-a) I^{\zeta}_{a^+}\overline{\psi}(b) \\
&\quad + A_2 \sum_{j=1}^{n} q_j I^{\zeta}_{a^+}\overline{\psi}(\beta_j) - a_9 \sum_{i=1}^{m} p_i I^{\xi}_{a^+}\overline{\varphi}(\alpha_i) \\
&\quad - A_2 I^{\xi}_{a^+}\overline{\varphi}(b) + a_9 I^{\eta}_{a^+}\overline{\rho}(b) - (b-a)\sum_{k=1}^{l} r_k I^{\eta}_{a^+}\overline{\rho}(\gamma_k) + a_{10}\Bigg\}.
\end{aligned}$$

Substituting the value of c_4 into (2.16) yields

$$\begin{aligned}
c_2 &= \left[\frac{-1}{b-a} - a_5 \frac{\sum_{i=1}^{m} p_i(\alpha_i - a)}{\Delta(b-a)}\right] I^{\eta}_{a^+}\overline{\rho}(b) + \sum_{i=1}^{m}\left[\frac{1}{b-a} + a_5 \frac{\sum_{i=1}^{m} p_i(\alpha_i - a)}{\Delta(b-a)} I^{\xi}_{a^+}\overline{\varphi}(\alpha_i)\right] \\
&\quad + \frac{\sum_{i=1}^{m} p_i(\alpha_i - a)}{\Delta(b-a)}\Bigg\{a_3 I^{\zeta}_{a^+}\overline{\psi}(\xi_1) + a_4 I^{\zeta}_{a^+}\overline{\psi}(\xi_2) - A_3 I^{\zeta}_{a^+}\overline{\psi}(b)
\end{aligned}$$

$$
\begin{aligned}
&- A_1 \sum_{j=1}^{n} q_j I_{a^+}^{\zeta} \overline{\psi}(\beta_j) - A_1 I_{a^+}^{\xi} \overline{\varphi}(b) \\
&+ A_3 \sum_{k=1}^{l} r_k I_{a^+}^{\eta} \overline{\rho}(\gamma_k) \biggr\} + a_{11}.
\end{aligned}
$$

Inserting the value of c_7 into (2.14) and (2.15), we get

$$
\begin{aligned}
c_5 = {}& \frac{1}{\xi_1 - \xi_2} \biggl\{ \biggl[\xi_2 - a + \frac{a_2 a_7}{\Delta} \biggr] I_{a^+}^{\zeta} \overline{\psi}(\xi_1) + \biggl[-\xi_1 + a + \frac{a_2 a_8}{\Delta} \biggr] I_{a^+}^{\zeta} \overline{\psi}(\xi_2) \\
&+ \frac{a_2}{\Delta} \biggl((b-a) I_{a^+}^{\zeta} \overline{\psi}(b) \\
&+ A_2 \sum_{j=1}^{n} q_j I_{a^+}^{\zeta} \overline{\psi}(\beta_j) - a_9 \sum_{i=1}^{m} p_i I_{a^+}^{\xi} \overline{\varphi}(\alpha_i) - A_2 I_{a^+}^{\xi} \overline{\varphi}(b) + a_9 I_{a^+}^{\eta} \overline{\rho}(b) \\
&- (b-a) \sum_{k=1}^{l} r_k I_{a^+}^{\eta} \overline{\rho}(\gamma_k) + a_{10} \biggr) \biggr\}, \\
c_6 = {}& \frac{1}{\xi_1 - \xi_2} \biggl\{ \biggl[\frac{a_1 a_7 - 1}{\Delta} \biggr] I_{a^+}^{\zeta} \overline{\psi}(\xi_1) + \biggl[\frac{a_1 a_8}{\Delta} + 1 \biggr] I_{a^+}^{\zeta} \overline{\psi}(\xi_2) + \frac{a_1}{\Delta} \biggl((b-a) I_{a^+}^{\zeta} \overline{\psi}(b) \\
&+ A_2 \sum_{j=1}^{n} q_j I_{a^+}^{\zeta} \overline{\psi}(\beta_j) - a_9 \sum_{i=1}^{m} p_i I_{a^+}^{\xi} \overline{\varphi}(\alpha_i) - A_2 I_{a^+}^{\xi} \overline{\varphi}(b) + a_9 I_{a^+}^{\eta} \overline{\rho}(b) \\
&- (b-a) \sum_{k=1}^{l} r_k I_{a^+}^{\eta} \overline{\rho}(\gamma_k) + a_{10} \biggr) \biggr\}.
\end{aligned}
$$

Finally, substituting the values of c_i, $i = 1, 2, \dots, 7$, into (2.6), (2.7), and (2.8), we obtain (2.2), (2.3), and (2.4). We can prove the converse of this lemma by direct computation. □

3 Main results

Let $X = C([a,b], \mathbb{R})$ be a Banach space endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [a,b]\}$. Then $(X \times X \times X, \|(u,x,y)\|_X)$ is also a Banach space equipped with the norm $\|(u,x,y)\|_X = \|u\| + \|x\| + \|y\|$, $u, x, y \in X$.

In view of Lemma 2.4, we define an operator $T : X \times X \times X \to X \times X \times X$ by

$$
T\bigl(u(t), x(t), y(t)\bigr) = \bigl(T_1\bigl(u(t), x(t), y(t)\bigr), T_2\bigl(u(t), x(t), y(t)\bigr), T_3\bigl(u(t), x(t), y(t)\bigr)\bigr),
$$

where

$$
\begin{aligned}
&T_1\bigl(u(t), x(t), y(t)\bigr) \\
&\quad = \int_a^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} \rho\bigl(s, u(s), x(s), y(s)\bigr)\, ds + u_0 \\
&\qquad + (t-a) \biggl\{ a_{12} \int_a^b \frac{(b-s)^{\eta-1}}{\Gamma(\eta)} \rho\bigl(s, u(s), x(s), y(s)\bigr)\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m} p_i}{b-a}\left(a_{13}\sum_{i=1}^{m} p_i(\alpha_i-a)+1\right)\int_a^{\alpha_i}\frac{(\alpha_i-a)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,u(s),x(s),y(s)\big)\,ds \\
&+\frac{\sum_{i=1}^{m} p_i(\alpha_i-a)}{\Delta(b-a)}\bigg(a_3\int_a^{\xi_1}\frac{(\xi_1-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds \\
&+a_4\int_a^{\xi_2}\frac{(\xi_2-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds \\
&-A_3\int_a^{b}\frac{(b-s)^{\zeta-1}}{\Gamma(\eta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds \\
&-A_1\sum_{j=1}^{n} q_j\int_a^{\beta_j}\frac{(\beta_j-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds \\
&-A_1\int_a^{b}\frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,u(s),x(s),y(s)\big)\,ds \\
&+A_3\sum_{k=1}^{l} r_k\int_a^{\gamma_k}\frac{(\gamma_k-s)^{\eta-1}}{\Gamma(\eta)}\rho\big(s,u(s),x(s),y(s)\big)\,ds+a_{12}\bigg)\Bigg\}+a_{11}(t-a),
\end{aligned}
$$

$$
\begin{aligned}
&T_2\big(u(t),x(t),y(t)\big) \\
&=\int_a^{t}\frac{(t-s)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,u(s),x(s),y(s)\big)\,ds \\
&\quad+\frac{(t-a)}{\Delta}\Bigg\{-a_5\int_a^{b}\frac{(b-s)^{\eta-1}}{\Gamma(\eta)}\rho\big(s,u(s),x(s),y(s)\big)\,ds \\
&\quad+a_5\sum_{i=1}^{m} p_i\int_a^{\alpha_i}\frac{(\alpha_i-a)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,u(s),x(s),y(s)\big)\,ds \\
&\quad+a_3\int_a^{\xi_1}\frac{(\xi_1-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds \\
&\quad+a_4\int_a^{\xi_2}\frac{(\xi_2-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds \\
&\quad-A_3\int_a^{b}\frac{(b-s)^{\zeta-1}}{\Gamma(\eta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds \\
&\quad-A_1\sum_{j=1}^{n} q_j\int_a^{\beta_j}\frac{(\beta_j-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds \\
&\quad-A_1\int_a^{b}\frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\varphi\big(s,u(s),x(s),y(s)\big)\,ds \\
&\quad+A_3\sum_{k=1}^{l} r_k\int_a^{\gamma_k}\frac{(\gamma_k-s)^{\eta-1}}{\Gamma(\eta)}\rho\big(s,u(s),x(s),y(s)\big)\,ds+a_6\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
&T_3\big(u(t),x(t),y(t)\big) \\
&=\int_a^{t}\frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds \\
&\quad+b_1(t)\int_a^{\xi_1}\frac{(\xi_1-a)^{\zeta-1}}{\Gamma(\zeta)}\psi\big(s,u(s),x(s),y(s)\big)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ b_2(t)\int_a^{\xi_2}\frac{(\xi_2-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\bigl(s,u(s),x(s),y(s)\bigr)\,ds\\
&+ b_3(t)\int_a^{b}\frac{(b-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\bigl(s,u(s),x(s),y(s)\bigr)\,ds\\
&+ b_4(t)\sum_{j=1}^{n}q_j\int_a^{b}\frac{(\beta_j-s)^{\zeta-1}}{\Gamma(\zeta)}\psi\bigl(s,u(s),x(s),y(s)\bigr)\,ds\\
&+ b_5(t)\int_a^{b}\frac{(b-s)^{\eta-1}}{\Gamma(\eta)}\rho\bigl(s,u(s),x(s),y(s)\bigr)\,ds\\
&- b_3(t)\sum_{k=1}^{l}r_k\int_a^{\gamma_k}\frac{(\gamma_k-s)^{\eta-1}}{\Gamma(\eta)}\rho\bigl(s,u(s),x(s),y(s)\bigr)\,ds\\
&- b_4(t)\int_a^{b}\frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\varphi\bigl(s,u(s),x(s),y(s)\bigr)\,ds\\
&- b_5(t)\sum_{i=1}^{m}\alpha_i\int_a^{\alpha_i}\frac{(\alpha_i-s)^{\xi-1}}{\Gamma(\xi)}\varphi\bigl(s,u(s),x(s),y(s)\bigr)\,ds + b_6(t).
\end{aligned}
$$

For computational convenience, let us set

$$
\begin{aligned}
L_1 &= \frac{1}{\Gamma(\eta+1)}\left\{(b-a)^{\eta} + |a_{12}|(b-a)^{\eta+1} + \frac{|A_3|\sum_{i=1}^{m}|p_i|(\alpha_i-a)\sum_{k=1}^{l}|r_k|(\gamma_k-a)^{\eta}}{|\Delta|}\right\},\\
M_1 &= \frac{1}{\Gamma(\xi+1)}\left\{\sum_{i=1}^{m}|p_i|\left(|a_{13}|\sum_{i=1}^{m}|p_i|(\alpha_i-a)+1\right)(\alpha_i-a)^{\xi}\right.\\
&\quad\left. + \frac{|A_1|\sum_{i=1}^{m}|p_i|(\alpha_i-a)(b-a)^{\xi}}{|\Delta|}\right\},\\
N_1 &= \frac{1}{\Gamma(\zeta+1)}\left\{\frac{\sum_{i=1}^{m}|p_i|(\alpha_i-a)}{|\Delta|}\left(|a_3|(\xi_1-a)^{\zeta} + |a_4|(\xi_2-a)^{\zeta} + |A_3|(b-a)^{\zeta}\right.\right.\\
&\quad\left.\left. + |A_1|\sum_{j=1}^{n}|q_j|(\beta_j-a)^{\zeta}\right)\right\},\\
L_2 &= \frac{b-a}{|\Delta|\Gamma(\eta+1)}\left\{|a_5|(b-a)^{\eta} + |A_3|\sum_{k=1}^{l}|r_k|(\gamma_k-a)^{\eta}\right\},\\
M_2 &= \frac{1}{\Gamma(\xi+1)}\left\{(b-a)^{\xi} + \frac{b-a}{|\Delta|}\left(|a_5|\sum_{i=1}^{m}|p_i|(\alpha_i-a)^{\xi} + |A_1|(b-a)^{\xi}\right)\right\},\\
N_2 &= \frac{b-a}{|\Delta|\Gamma(\zeta+1)}\left\{|a_3|(\xi_1-a)^{\zeta} + |a_4|(\xi_2-a)^{\zeta} + |A_3|(b-a)^{\zeta} + |A_1|\sum_{j=1}^{n}|q_j|(\beta_j-a)^{\zeta}\right\},\\
L_3 &= \frac{1}{\Gamma(\eta+1)}\left\{\delta_5(b-a)^{\eta} + \delta_3\sum_{k=1}^{l}|r_k|(\gamma_k-a)^{\eta}\right\},\\
M_3 &= \frac{1}{\Gamma(\xi+1)}\left\{\delta_4(b-a)^{\xi} + \delta_5\sum_{i=1}^{m}|\alpha_i|(\alpha_i-a)^{\xi}\right\},
\end{aligned}
$$

$$
\begin{aligned}
N_3 = {} & \frac{1}{\Gamma(\zeta+1)}\Biggl\{(b-a)^{\zeta} + \delta_1(\xi_1-a)^{\zeta} + \delta_2(\xi_2-a)^{\zeta} \\
& + \delta_3(b-a)^{\zeta} + \delta_4 \sum_{j=1}^{n} |q_j|(\beta_j-a)^{\zeta}\Biggr\},
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
\delta_1 &= \frac{1}{|\xi_1-\xi_2|}\left(|\xi_2| + |a| + \frac{|a_2a_7|}{|\Delta|} + (b-a)\left|\frac{a_1a_7}{\Delta} - 1\right|\right) + (b-a)^2\frac{|a_7|}{|\Delta|}, \\
\delta_2 &= \frac{1}{|\xi_1-\xi_2|}\left(|a| + |\xi_1| + \frac{|a_2a_8|}{|\Delta|} + (b-a)\left|\frac{a_1a_8}{\Delta} + 1\right|\right) + (b-a)^2\frac{|a_8|}{|\Delta|}, \\
\delta_3 &= \frac{b-a}{|\Delta|(|\xi_1-\xi_2|)}\bigl(|a_2| + |a_1|(b-a) + \bigl(|\xi_1-\xi_2|\bigr)(b-a)^2\bigr), \\
\delta_4 &= \frac{|A_2|}{|\Delta|(|\xi_1-\xi_2|)}\bigl(|a_2| + |a_1|(b-a) + \bigl(|\xi_1-\xi_2|\bigr)(b-a)^2\bigr), \\
\delta_5 &= \frac{|a_9|}{|\Delta|(|\xi_1-\xi_2|)}\bigl(|a_2| + |a_1|(b-a) + \bigl(|\xi_1-\xi_2|\bigr)(b-a)^2\bigr), \\
\delta_6 &= \frac{|a_{10}|}{|\Delta|(|\xi_1-\xi_2|)}\bigl(|a_2| + |a_1|(b-a) + \bigl(|\xi_1-\xi_2|\bigr)(b-a)^2\bigr).
\end{aligned}
$$

In our first result, we establish the existence of solutions for system (1.1) by applying the Leray–Schauder alternative [28].

Lemma 3.1 (Leray–Schauder alternative) *Let $\mathfrak{J}:\mathcal{U}\longrightarrow\mathcal{U}$ be a completely continuous operator (i.e., a map restricted to any bounded set in $\mathcal{U}$ is compact). Let $\mathcal{Q}(\mathfrak{J}) = \{x\in\mathcal{U} : x = \eta\mathfrak{J}(x)$ for some $0<\eta<1\}$. Then either the set $\mathcal{Q}(\mathfrak{J})$ is unbounded, or $\mathfrak{J}$ has at least one fixed point.*

Theorem 3.2 *Let $\Delta\neq 0$, where Δ is defined by (2.5). In addition, we assume that:*

(H_2) *$\rho,\varphi,\psi:[a,b]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions and there exist real constants $k_i,\sigma_i,\mu_i\geq 0$ $(i=1,2,3)$ and $k_0>0$, $\sigma_0>0$, $\mu_0>0$ such that, for all $t\in[a,b]$ and $u,x,y\in\mathbb{R}$,*

$$
\begin{aligned}
\bigl|\rho(t,u,x,y)\bigr| &\leq k_0 + k_1|u| + k_2|x| + k_3|y|, \\
\bigl|\varphi(t,u,x,y)\bigr| &\leq \sigma_0 + \sigma_1|u| + \sigma_2|x| + \sigma_3|y|, \\
\bigl|\psi(t,u,x,y)\bigr| &\leq \mu_0 + \mu_1|u| + \mu_2|x| + \mu_3|y|.
\end{aligned}
$$

Then system (1.1) has at least one solution on $[a,b]$ provided that

$$
\begin{aligned}
&(L_1+L_2+L_3)k_1 + (M_1+M_2+M_3)\sigma_1 + (N_1+N_2+N_3)\mu_1 < 1, \\
&(L_1+L_2+L_3)k_2 + (M_1+M_2+M_3)\sigma_2 + (N_1+N_2+N_3)\mu_2 < 1, \\
&(L_1+L_2+L_3)k_3 + (M_1+M_2+M_3)\sigma_3 + (N_1+N_2+N_3)\mu_3 < 1,
\end{aligned}
\tag{3.2}
$$

where L_i, M_i, N_i, $i=1,2,3$, are given in (3.1).

Proof Observe that the continuity of the operator $T : X \times X \times X \to X \times X \times X$ follows from that of the functions ρ, φ, and ψ. Next, let $\Omega \subset X \times X \times X$ be bounded such that

$$\begin{aligned} &\bigl|\rho\bigl(t,u(t),x(t),y(t)\bigr)\bigr| \le K_1, \\ &\bigl|\varphi\bigl(t,u(t),x(t),y(t)\bigr)\bigr| \le K_2, \\ &\bigl|\psi\bigl(t,u(t),x(t),y(t)\bigr)\bigr| \le K_3, \quad \forall (u,x,y) \in \Omega, \end{aligned}$$

for positive constants K_1, K_2, and K_3. Then, for any $(u,x,y) \in \Omega$, we have

$$\begin{aligned} &\bigl|T_1\bigl(u(t),x(t),y(t)\bigr)\bigr| \\ &\quad \le \int_a^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)}\bigl|\rho\bigl(s,u(s),x(s),y(s)\bigr)\bigr|\,ds + |u_0| \\ &\qquad + (b-a)\Biggl\{|a_{12}|\int_a^b \frac{(b-s)^{\eta-1}}{\Gamma(\eta)}\bigl|\rho\bigl(s,u(s),x(s),y(s)\bigr)\bigr|\,ds \\ &\qquad + \frac{\sum_{i=1}^m |p_i|}{b-a}\Biggl(|a_{13}|\sum_{i=1}^m |p_i|(\alpha_i-a)+1\Biggr)\int_a^{\alpha_i} \frac{(\alpha_i-a)^{\xi-1}}{\Gamma(\xi)}\bigl|\varphi\bigl(s,u(s),x(s),y(s)\bigr)\bigr|\,ds \\ &\qquad + \frac{\sum_{i=1}^m |p_i|(\alpha_i-a)}{|\Delta|(b-a)}\biggl(|a_3|\int_a^{\xi_1} \frac{(\xi_1-s)^{\zeta-1}}{\Gamma(\zeta)}\bigl|\psi\bigl(s,u(s),x(s),y(s)\bigr)\bigr|\,ds \\ &\qquad + |a_4|\int_a^{\xi_2} \frac{(\xi_2-s)^{\zeta-1}}{\Gamma(\zeta)}\bigl|\psi\bigl(s,u(s),x(s),y(s)\bigr)\bigr|\,ds \\ &\qquad + |A_3|\int_a^b \frac{(b-s)^{\zeta-1}}{\Gamma(\eta)}\bigl|\psi\bigl(s,u(s),x(s),y(s)\bigr)\bigr|\,ds \\ &\qquad + |A_1|\sum_{j=1}^n |q_j|\int_a^{\beta_j} \frac{(\beta_j-s)^{\zeta-1}}{\Gamma(\zeta)}\bigl|\psi\bigl(s,u(s),x(s),y(s)\bigr)\bigr|\,ds \\ &\qquad + |A_1|\int_a^b \frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\bigl|\varphi\bigl(s,u(s),x(s),y(s)\bigr)\bigr|\,ds \\ &\qquad + |A_3|\sum_{k=1}^l |r_k|\int_a^{\gamma_k} \frac{(\gamma_k-s)^{\eta-1}}{\Gamma(\eta)}\bigl|\rho\bigl(s,u(s),x(s),y(s)\bigr)\bigr|\,ds + |a_{12}|\biggr)\Biggr\} + |a_{11}|(b-a) \\ &\quad \le |u_0| + |a_{11}|(b-a) + \Biggl\{\frac{1}{\Gamma(\eta+1)}\Biggl\{(b-a)^\eta + |a_{12}|(b-a)^{\eta+1} \\ &\qquad + \frac{|A_3|\sum_{i=1}^m |p_i|(\alpha_i-a)\sum_{k=1}^l |r_k|(\gamma_k-a)^\eta}{|\Delta|}\Biggr\}\Biggr\}\|\rho\| \\ &\qquad + \Biggl\{\frac{1}{\Gamma(\xi+1)}\Biggl\{\sum_{i=1}^m |p_i|\Biggl(\sum_{i=1}^m |p_i|(\alpha_i-a)+1\Biggr)(\alpha_i-a)^\xi \\ &\qquad + \frac{|A_1|\sum_{i=1}^m |p_i|(\alpha_i-a)(b-a)^\xi}{|\Delta|}\Biggr\}\Biggr\}\|\varphi\| \\ &\qquad + \Biggl\{\frac{1}{\Gamma(\zeta+1)}\Biggl\{\frac{\sum_{i=1}^m |p_i(\alpha_i-a)|}{|\Delta|}\Biggl(|a_3|(\xi_1-a)^\zeta + |a_4|(\xi_2-a)^\zeta \end{aligned}$$

$$
\begin{aligned}
&+ |A_3|(b-a)^{\zeta} + |A_1| \sum_{j=1}^{n} |q_j|(\beta_j - a)^{\zeta} \Bigg) \Bigg\} \Bigg\} \|\psi\| \\
&\leq |u_0| + |a_{11}|(b-a) + L_1K_1 + M_1K_2 + N_1K_3,
\end{aligned}
$$

which implies that

$$
\big\| T_1(u,x,y) \big\|_X \leq |u_0| + |a_{11}|(b-a) + L_1K_1 + M_1K_2 + N_1K_3.
$$

In a similar way, we can find that

$$
\big\| T_2(u,x,y) \big\|_X \leq \frac{|a_6|(b-a)}{|\Delta|} + L_2K_1 + M_2K_2 + N_2K_3
$$

and

$$
\begin{aligned}
\big| T_3(u,x,y)(t) \big| \leq{}& \int_a^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} \big| \psi\big(s,u(s),x(s),y(s)\big) \big| \, ds \\
&+ \max_{t\in[a,b]} \big\{ b_1(t) \big\} \int_a^{\xi_1} \frac{(\xi_1 - a)^{\zeta-1}}{\Gamma(\zeta)} \big| \psi\big(s,u(s),x(s),y(s)\big) \big| \, ds \\
&+ \max_{t\in[a,b]} \big\{ b_2(t) \big\} \int_a^{\xi_2} \frac{(\xi_2 - s)^{\zeta-1}}{\Gamma(\zeta)} \big| \psi\big(s,u(s),x(s),y(s)\big) \big| \, ds \\
&+ \max_{t\in[a,b]} \big\{ b_3(t) \big\} \int_a^{b} \frac{(b - s)^{\zeta-1}}{\Gamma(\zeta)} \big| \psi\big(s,u(s),x(s),y(s)\big) \big| \, ds \\
&+ \max_{t\in[a,b]} \big\{ b_4(t) \big\} \sum_{j=1}^{n} |q_j| \int_a^{b} \frac{(\beta_j - s)^{\zeta-1}}{\Gamma(\zeta)} \big| \psi\big(s,u(s),x(s),y(s)\big) \big| \, ds \\
&+ \max_{t\in[a,b]} \big\{ b_5(t) \big\} \int_a^{b} \frac{(b - s)^{\eta-1}}{\Gamma(\eta)} \big| \rho\big(s,u(s),x(s),y(s)\big) \big| \, ds \\
&+ \max_{t\in[a,b]} \big\{ b_3(t) \big\} \sum_{k=1}^{l} |r_k| \int_a^{\gamma_k} \frac{(\gamma_k - s)^{\eta-1}}{\Gamma(\eta)} \big| \rho\big(s,u(s),x(s),y(s)\big) \big| \, ds \\
&+ \max_{t\in[a,b]} \big\{ b_4(t) \big\} \int_a^{b} \frac{(b - s)^{\xi-1}}{\Gamma(\xi)} \big| \varphi\big(s,u(s),x(s),y(s)\big) \big| \, ds \\
&+ \max_{t\in[a,b]} \big\{ b_5(t) \big\} \sum_{i=1}^{m} \bigg| p_i \int_a^{\alpha_i} \frac{(\alpha_i - s)^{\xi-1}}{\Gamma(\xi)} \bigg| \varphi \big| \big(s,u(s),x(s),y(s)\big) \, ds \\
&+ \max_{t\in[a,b]} \big\{ b_6(t) \big\} \\
\leq{}& \delta_6 + L_3K_1 + M_3K_2 + N_3K_3,
\end{aligned}
$$

which implies that

$$
\big\| T_3(u,x,y) \big\|_X \leq \delta_6 + L_3K_1 + M_3K_2 + N_3K_3.
$$

From the above argument, we deduce that the operator T is uniformly bounded, that is,

$$
\begin{aligned}
\big\| T(u,x,y) \big\|_X \leq{}& |u_0| + |a_{11}|(b-a) + \frac{|a_6(b-a)|}{|\Delta|} + \delta_6 \\
&+ (L_1 + L_2 + L_3)K_1 + (M_1 + M_2 + M_3)K_2 + (N_1 + N_2 + N_3)K_3.
\end{aligned}
$$

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\bigl|T_1\bigl(u(t_2),x(t_2),y(t_2)\bigr) - T_1\bigl(u(t_1),x(t_1),y(t_1)\bigr)\bigr| \\
&\quad\le \frac{K_1}{\Gamma(\eta+1)}\bigl[2(t_2-t_1)^\eta + \bigl|t_2^\eta - t_1^\eta\bigr|\bigr] + (t_2-t_1)|a_{11}| + \biggl\{\frac{t_2-t_1}{\Gamma(\eta+1)}\biggl\{(|a_{12}|(b-a)^{\eta+1} \\
&\qquad + \frac{|A_3|\sum_{i=1}^m |p_i|(\alpha_i-a)\sum_{k=1}^l |r_k|(\gamma_k-a)^\eta}{|\Delta|(b-a)}\biggr\}\biggr\}K_1 \\
&\qquad + \biggl\{\frac{t_2-t_1}{\Gamma(\xi+1)}\biggl\{\frac{\sum_{i=1}^m |p_i|(\sum_{i=1}^m |p_i|(\alpha_i-a)+1)(\alpha_i-a)^\xi}{|\Delta|(b-a)} \\
&\qquad + \frac{|A_1|\sum_{i=1}^m |p_i|(\alpha_i-a)(b-a)^\xi}{|\Delta|(b-a)}\biggr\}\biggr\}K_2 \\
&\qquad + \biggl\{\frac{t_2-t_1}{\Gamma(\zeta+1)}\biggl\{\frac{\sum_{i=1}^m |p_i|(\alpha_i-a)}{|\Delta|(b-a)}\biggl(|a_3|(\xi_1-a)^\zeta + |a_4|(\xi_2-a)^\zeta + |A_3|(b-a)^\zeta \\
&\qquad + |A_1|\sum_{j=1}^n |q_j|(\beta_j-a)^\zeta\biggr)\biggr\}\biggr\}K_3.
\end{aligned}$$

Analogously,we can obtain

$$\begin{aligned}
&\bigl|T_2\bigl(u(t_2),x(t_2),y(t_2)\bigr) - T_2\bigl(u(t_1),x(t_1),y(t_1)\bigr)\bigr| \\
&\quad\le \frac{K_2}{\Gamma(\xi+1)}\bigl[2(t_2-t_1)^\xi + \bigl|t_2^\xi - t_1^\xi\bigr|\bigr] + (t_2-t_1)\frac{|a_6|}{|\Delta|} + \frac{t_2-t_1}{\Delta}\biggl\{\frac{1}{\Gamma(\eta+1)}\biggl\{|a_5|(b-a)^\eta \\
&\qquad + |A_3|\sum_{k=1}^l |r_k|(\gamma_k-a)^\eta\biggr\}\biggr\}K_1 \\
&\qquad + \frac{t_2-t_1}{\Delta\Gamma(\xi+1)}\biggl\{|a_5|\sum_{i=1}^m |p_i|(\alpha_i-a)^\xi + |A_1|(b-a)^\xi\biggr\}K_2 \\
&\qquad + \frac{t_2-t_1}{|\Delta|\Gamma(\zeta+1)}\biggl\{|a_3|(\xi_1-a)^\zeta + |a_4|(\xi_2-a)^\zeta + |A_3|(b-a)^\zeta \\
&\qquad + |A_1|\sum_{j=1}^n |q_j|(\beta_j-a)^\zeta\biggr\}K_3,
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|T_3(t_2),x(t_2),y(t_2)) - T_3u(t_1),x(t_1),y(t_1))\bigr| \\
&\quad\le \frac{K_3}{\Gamma(\zeta+1)}\bigl[2(t_2-t_1)^\zeta + \bigl|t_2^\zeta - t_1^\zeta\bigr|\bigr] \\
&\qquad + (t_2-t_1)\frac{|a_1a_{10}|}{|\Delta|(\xi_1-\xi_2)} \\
&\qquad + \frac{(t_2-t_1)|a_1|}{|\Delta|(|\xi_1-\xi_2|)\Gamma(\eta+1)}\biggl\{|a_9|(b-a)^\eta + (b-a)\sum_{k=1}^l |r_k|(\gamma_k-a)^\eta\biggr\}K_1
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(t_2-t_1)|a_1|}{|\Delta|(|\xi_1-\xi_2|)\Gamma(\xi+1)}\left\{|A_2|(b-a)^{\xi}+|a_9|\sum_{i=1}^{m}|p_i|(\alpha_i-a)^{\xi}\right\}K_2 \\
&+ \frac{(t_2-t_1}{(|\xi_1-\xi_2|)\Gamma(\zeta+1)}\left\{\left|\frac{a_1a_7}{\Delta}-1\right|(\xi_1-a)^{\zeta}+\left|\frac{a_1a_8}{\Delta}+1\right|(\xi_2-a)^{\zeta}\right. \\
&\left.+ \frac{|a_1|}{|\Delta|}(b-a)^{\zeta}+\frac{|A_2a_1|}{|\Delta|}\sum_{j=1}^{n}q_j(\beta_j-a)^{\zeta}\right\}K_3+\left\{\left(t_2^2-t_1^2\right)-2a(t_2-t_1)\right\}\left\{\frac{|a_{10}|}{|\Delta|}\right. \\
&+ \left\{\frac{1}{|\Delta|\Gamma(\eta+1)}\left(|a_9|(b-a)^{\eta}+(b-a)\sum_{k=1}^{l}|r_k|(\gamma_k-a)^{\eta}\right)\right\}K_1 \\
&+ \left\{\frac{1}{|\Delta|\Gamma(\xi+1)}\left(|A_2|(b-a)^{\xi}+|a_9|\sum_{i=1}^{m}p_i(\alpha_i-a)^{\xi}\right)\right\}K_2 \\
&+ \left\{\frac{1}{|\Delta|\Gamma(\zeta+1)}\left(|a_7|(\xi_1-a)^{\zeta}+|a_8|(\xi_2-a)^{\zeta}+(b-a)^{\zeta+1}\right.\right. \\
&\left.\left.\left.+ |A_2|\sum_{j=1}^{n}q_j(\beta_j-a)^{\zeta}\right)\right\}K_3\right\}.
\end{aligned}
$$

The above inequalities are independent of u, x, y and tend to zero as $t_1 \to t_2$. This shows that the operator $T(u,x,y)$ is equicontinuous. In consequence, we deduce that the operator $T(u,x,y)$ is completely continuous.

Finally, we consider the set $\mathcal{P} = \{(u,x,y) \in X \times X \times X : (u,x,y) = \nu T(u,x,y), 0 \le \nu \le 1\}$ and show that it is bounded.

Let $(u,x,y) \in \mathcal{P}$ with $(u,x,y) = \nu T(u,x,y)$. For any $t \in [a,b]$, we have $u(t) = \nu T_1(u,x,y)(t)$, $x(t) = \nu T_2(u,x,y)(t)$, $y(t) = \nu T_3(u,x,y)(t)$. Then, by (H_2), we have

$$
\begin{aligned}
\left|u(t)\right| &\le |u_0| + |a_{11}|(b-a) + L_1\left(k_0 + k_1|u| + k_2|x| + k_3|y|\right) \\
&\quad + M_1\left(\sigma_0 + \sigma_1|u| + \sigma_2|x| + \sigma_3|y|\right) \\
&\quad + N_1\left(\mu_0 + \mu_1|u| + \mu_2|x| + \mu_3|y|\right) \\
&= |u_0| + |a_{11}|(b-a) + L_1k_0 + M_1\sigma_0 + N_1\mu_0 + (L_1k_1 + M_1\sigma_1 + N_1\mu_1)|u|_2 \\
&\quad + (L_1k_2 + M_1\sigma + N_1\mu_2)|x| + (L_1k_3 + M_1\sigma_3 + N_1\mu_3)|y|, \\
\left|x(t)\right| &\le \frac{|a_6|(b-a)}{|\Delta|} + L_2k_0 + M_2\sigma_0 + N_2\mu_0 + (L_2k_1 + M_2\sigma_1 + N_2\mu_1)|u| \\
&\quad + (L_2k_2 + M_2\sigma_2 + N_2\mu_2)|x| \\
&\quad + (L_2k_3 + M_2\sigma_3 + N_2\mu_3)|y|,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|y(t)\right| &\le \delta_6 + L_3k_0 + M_3\sigma_0 + N_3\mu_0 + (L_3k_1 + M_3\sigma_1 + N_3\mu_1)|u| \\
&\quad + (L_3k_2 + M_3\sigma_2 + N_3\mu_2)|x| \\
&\quad + (L_3k_3 + M_3\sigma_3 + N_3\mu_3)|y|.
\end{aligned}
$$

It follows from the foregoing arguments that

$$\begin{aligned}
\|u\| &\le |u_0| + |a_{11}|(b-a) + L_1k_0 + M_1\sigma_0 + N_1\mu_0 + (L_1k_1 + M_1\sigma_1 + N_1\mu_1)\|u\| \\
&\quad + (L_1k_2 + M_1\sigma_2 + N_1\mu_2)\|x\| + (L_1k_3 + M_1\sigma_3 + N_1\mu_3)\|y\|, \\
\|x\| &\le \frac{|a_6|(b-a)}{|\Delta|} + L_2k_0 + M_2\sigma_0 + N_2\mu_0 + (L_2k_1 + M_2\sigma_1 + N_2\mu_1)\|u\| \\
&\quad + (L_2k_2 + M_2\sigma_2 + N_2\mu_2)\|x\| + (L_2k_3 + M_2\sigma_3 + N_2\mu_3)\|y\|, \\
\|y\| &\le \delta_6 + L_3k_0 + M_3\sigma_0 + N_3\mu_0 + (L_3k_1 + M_3\sigma_1 + N_3\mu_1)\|u\| \\
&\quad + (L_3k_2 + M_3\sigma_2 + N_3\mu_2)\|x\| \\
&\quad + (L_3k_3 + M_3\sigma_3 + N_3\mu_3)\|y\|.
\end{aligned}$$

Adding the above three inequalities, we have

$$\begin{aligned}
&\|u\| + \|x\| + \|y\| \\
&\quad \le |u_0| + |a_{11}|(b-a) + \frac{|a_6|(b-a)}{|\Delta|} + \delta_6 + (L_1 + L_2 + L_3)k_0 + (M_1 + M_2 + M_3)\sigma_0 \\
&\qquad + (N_1 + N_2 + N_3)\mu_0 \\
&\qquad + \big[(L_1 + L_2 + L_3)k_1 + (M_1 + M_2 + M_3)\sigma_1 + (N_1 + N_2 + N_3)\mu_1\big]\|u\| \\
&\qquad + \big[(L_1 + L_2 + L_3)k_2 + (M_1 + M_2 + M_3)\sigma_2 + (N_1 + N_2 + N_3)\mu_2\big]\|x\| \\
&\qquad + \big[(L_1 + L_2 + L_3)k_3 + (M_1 + M_2 + M_3)\sigma_3 + (N_1 + N_2 + N_3)\mu_3\big]\|y\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\big\|(u,x,y)\big\|_X \\
&\quad \le \frac{1}{M_0}\Bigg[|u_0| + |a_{11}|(b-a) + \frac{|a_6|(b-a)}{|\Delta|} \\
&\qquad + \delta_6 + (L_1 + L_2 + L_3)k_0 + (M_1 + M_2 + M_3)\sigma_0 \\
&\qquad + (N_1 + N_2 + N_3)\mu_0\Bigg],
\end{aligned}$$

where $M_0 = \min\{1 - [(L_1 + L_2 + L_3)k_i + (M_1 + M_2 + M_3)\sigma_i + (N_1 + N_2 + N_3)\mu_i], i = 1,2,3\}$. Hence the set $\mathcal{P}$ is bounded. Thus, by the Leray–Schauder alternative, we deduce that the operator T has at least one fixed point, which implies that problem (1.1) has at least one solution on $[a,b]$. This completes the proof. □

Our next existence and uniqueness result is based on the contraction mapping principle due to Banach.

Theorem 3.3 *Let $\Delta \neq 0$, where Δ is defined by* (2.5). *In addition, we assume that*:

(H_2) *$\rho, \varphi, \psi : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exist positive constants l_1, l_2, and l_3 such that, for all $t \in [a,b]$ and $u_i, x_i, y_i \in \mathbb{R}$, $i = 1,2,3$, we* *have*

$$\begin{aligned}
&\bigl|\rho(t,x_1,x_2,x_3)-\rho(t,y_1,y_2,y_3)\bigr|\le l_1\bigl(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\bigr),\\
&\bigl|\varphi(t,x_1,x_2,x_3)-\varphi(t,y_1,y_2,y_3)\bigr|\le l_2\bigl(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\bigr),\\
&\bigl|\psi(t,x_1,x_2,x_3)-\psi(t,y_1,y_2,y_3)\bigr|\le l_3\bigl(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\bigr).
\end{aligned}$$

If

$$(L_1+L_2+L_3)l_1+(M_1+M_2+M_3)l_2+(N_1+N_2+N_3)l_3<1, \tag{3.3}$$

where L_i, M_i, N_i are given in (3.1), *then system* (1.1) *has a unique solution on* $[a,b]$.

Proof Define $\sup_{t\in[a,b]}\rho(t,0,0,0)=Q_1<\infty$, $\sup_{t\in[a,b]}\varphi(t,0,0,0)=Q_2<\infty$, $\sup_{t\in[a,b]}\psi(t,0,0,0)=Q_3<\infty$, and $r>0$ such that

$$r>\frac{|u_0|+(b-a)|a_{11}|+\frac{|a_6|(b-a)}{|\Delta|}+\delta_6+E}{1-(L_1+L_2+L_3)l_1-(M_1+M_2+M_3)l_2-(N_1+N_2+N_3)l_3},$$

where $E=(L_1+L_2+L_3)Q_1+(M_1+M_2+M_3)Q_2+(N_1+N_2+N_3)Q_3$.

In the first step, we show that $TB_r\subset B_r$, where $B_r=\{(u,x,y)\in X\times X\times X:\|(u,x,y)\|\le r\}$. By assumption (H_2), for $(u,x,y)\in B_r$, $t\in[a,b]$, we have

$$\begin{aligned}
\bigl|\rho\bigl(t,u(t),x(t),y(t)\bigr)\bigr|&\le\bigl|\rho\bigl(t,u(t),x(t),y(t)\bigr)-\rho(t,0,0,0)\bigr|\\
&\le l_1\bigl(|u|+\bigl|x(t)\bigr|+\bigl|y(t)\bigr|\bigr)+Q_1\\
&\le l_1\bigl(\|u\|+\|x\|+\|y\|\bigr)+Q_1\le l_1r+Q_1.
\end{aligned}\tag{3.4}$$

Similarly, we can get

$$\begin{aligned}
\bigl|\varphi\bigl(t,u(t),x(t),y(t)\bigr)\bigr|&\le\bigl|\varphi\bigl(t,u(t),x(t),y(t)\bigr)-\varphi(t,0,0,0)\bigr|\\
&\le l_2\bigl(|u|+\bigl|x(t)\bigr|+\bigl|y(t)\bigr|\bigr)+Q_2\\
&\le l_2\bigl(\|u\|+\|x\|+\|y\|\bigr)+Q_2\le l_2r+Q_2,
\end{aligned}\tag{3.5}$$

and

$$\bigl|\psi\bigl(t,u(t),x(t),y(t)\bigr)\bigr|\le l_3\bigl(\|u\|+\|x\|+\|y\|\bigr)+Q_3\le l_3r+Q_3. \tag{3.6}$$

Using (3.4), (3.5), and (3.6), we obtain

$$\begin{aligned}
\bigl|T_1(u,x,y)(t))\bigr|\le{}&|u_0|+|a_{11}|(b-a)+\biggl\{\frac{1}{\Gamma(\eta+1)}\biggl\{(b-a)^{\eta}+|a_{12}|(b-a)^{\eta+1}\\
&+\frac{|A_3|\sum_{i=1}^{m}|p_i|(\alpha_i-a)\sum_{k=1}^{l}|r_k|(\gamma_k-a)^{\eta}}{|\Delta|}\biggr\}\biggr\}\|\rho\|\\
&+\biggl\{\frac{1}{\Gamma(\xi+1)}\biggl\{\sum_{i=1}^{m}|p_i|\sum_{i=1}^{m}|p_i|(\alpha_i-a)+1)(\alpha_i-a)^{\xi}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|A_1| \sum_{i=1}^{m} |p_i|(\alpha_i - a)(b-a)^{\xi}}{|\Delta|} \Bigg\}\Bigg\} \|\varphi\| \\
&+ \Bigg\{ \frac{1}{\Gamma(\zeta+1)} \Bigg\{ \frac{\sum_{i=1}^{m} |p_i(\alpha_i - a)|}{|\Delta|} \Bigg(|a_3|(\xi_1 - a)^{\zeta} + |a_4|(\xi_2 - a)^{\zeta} \\
&+ |A_3|(b-a)^{\zeta} + |A_1| \sum_{j=1}^{n} |q_j|(\beta_j - a)^{\zeta} \Bigg) \Bigg\}\Bigg\} \|\psi\| \\
\leq{}& |u_0| + |a_{11}|(b-a) + L_1(l_1 r + Q_1) + M_1(l_2 r + Q_2) + N_1(l_3 r + Q_3) \\
\leq{}& |u_0| + |a_{11}|(b-a) + (L_1 l_1 + M_1 l_2 + N_1 l_3) r + L_1 Q_1 + M_1 Q_2 + N_1 Q_3,
\end{aligned}
$$

which, on taking the norm for $t \in [a,b]$, yields

$$
\|T_1(u,x,y)\|_X \leq |u_0| + |a_{11}|(b-a) + (L_1 l_1 + M_1 l_2 + N_1 l_3) r + L_1 Q_1 + M_1 Q_2 + N_1 Q_3.
$$

Likewise, we can find that

$$
\|T_2(u,x,y)\|_X \leq \frac{|a_6|(b-a)}{|\Delta|} + (L_2 l_1 + M_2 l_2 + N_2 l_3) r + L_2 Q_1 + M_2 Q_2 + N_2 Q_3
$$

and

$$
\|T_3(u,x,y)\|_X \leq \delta_6 + (L_3 l_1 + M_3 l_2 + N_3 l_3) r + L_3 Q_1 + M_3 Q_2 + N_3 Q_3.
$$

Consequently,

$$
\begin{aligned}
&\|T(u,x,y)\|_X \\
&\quad \leq |u_0| + |a_{11}|(b-a) + \frac{|a_6|(b-a)}{|\Delta|} + \delta_6 + \big[(L_1 + L_2 + L_3) l_1 + (M_1 + M_2 + M_3) l_2 \\
&\qquad + (N_1 + N_2 + N_3) l_3\big] r + (L_1 + L_2 + L_3) Q_1 + (M_1 + M_2 + M_3) Q_2 \\
&\qquad + (N_1 + N_2 + N_3) Q_3 \leq r.
\end{aligned}
$$

Now, for $(u_1, x_1, y_1), (u_2, x_2, y_2) \in X \times X \times X$ and for any $t \in [a,b]$, we get

$$
\begin{aligned}
&\big|T_1(u_2,x_2,y_2)(t) - T_1(u_1,x_1,y_1)(t)\big| \\
&\quad \leq \Bigg\{ \frac{1}{\Gamma(\eta+1)} \Bigg\{ (b-a)^{\eta} + |a_{12}|(b-a)^{\eta+1} \\
&\qquad + \frac{|A_3| \sum_{i=1}^{m} |p_i|(\alpha_i - a) \sum_{k=1}^{l} |r_k|(\gamma_k - a)^{\eta}}{|\Delta|} \Bigg\}\Bigg\} \\
&\qquad \times l_1\big(\|u_2 - u_1\| + \|x_2 - x_1\| + \|y_2 - y_1\|\big) \\
&\qquad + \Bigg\{ \frac{1}{\Gamma(\xi+1)} \Bigg\{ \sum_{i=1}^{m} |p_i| \Bigg(\sum_{i=1}^{m} |p_i|(\alpha_i - a) + 1 \Bigg) (\alpha_i - a)^{\xi} \\
&\qquad + \frac{|A_1| \sum_{i=1}^{m} |p_i|(\alpha_i - a)(b-a)^{\xi}}{|\Delta|} \Bigg\}\Bigg\} l_2\big(\|u_2 - u_1\| + \|x_2 - x_1\| + \|y_2 - y_1\|\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\frac{1}{\Gamma(\zeta+1)}\left\{\frac{\sum_{i=1}^{m}|p_i|(\alpha_i-a)}{|\Delta|}\left(|a_3|(\xi_1-a)^{\zeta}+|a_4|(\xi_2-a)^{\zeta}\right.\right.\right.\\
&\left.\left.\left.+|A_3|(b-a)^{\zeta}+|A_1|\sum_{j=1}^{n}|q_j|(\beta_j-a)^{\zeta}\right)\right\}\right\}l_3\big(\|u_2-u_1\|+\|x_2-x_1\|+\|y_2-y_1\|\big)\\
\le{}&(L_1l_1+M_1l_2+N_1l_3)\big(\|u_2-u_1\|+\|x_2-x_1\|+\|y_2-y_1\|\big),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\big\|T_1(u_2,x_2,y_2)-T_1(u_1,x_1,y_1)\big\|_X\\
&\quad\le(L_1l_1+M_1l_2+N_1l_3)\big(\|u_2-u_1+\|\|x_2-x_1\|+\|y_2-y_1\|\big).
\end{aligned}
\tag{3.7}
$$

Similarly, we find that

$$
\begin{aligned}
&\big\|T_2(u_2,x_2,y_2)-T_2(u_1,x_1,y_1)\big\|_X\\
&\quad\le(L_2l_1+M_2l_2+N_2l_3)\big(\|u_2-u_1\|+\|x_2-x_1\|+\|y_2-y_1\|\big)
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&\big\|T_3(u_2,x_2,y_2)-T_3(u_1,x_1,y_1)\big\|_X\\
&\quad\le(L_3l_1+M_3l_2+N_3l_3)\big(\|u_2-u_1\|+\|x_2-x_1\|+\|y_2-y_1\|\big).
\end{aligned}
\tag{3.9}
$$

It follows from (3.7), (3.8), and (3.9) that

$$
\begin{aligned}
&\big\|T(u_2,x_2,y_2)-T(u_1,x_1,y_1)\big\|_X\\
&\quad\le\big[(L_1+L_2+L_3)l_1+(M_1+M_2+M_3)l_2+(N_1+N_2+N_3)l_3\big]\\
&\qquad\times\big(\|u_2-u_1\|+\|x_2-x_1\|+\|y_2-y_1\|\big).
\end{aligned}
$$

The above inequality together with (3.3) implies that T is a contraction. Hence it follows by Banach's fixed point theorem that there exists a unique fixed point for the operator T, which corresponds to a unique solution of problem (1.1) on $[a,b]$. The proof is completed. □

4 Examples

Let us consider the following mixed-type coupled fractional differential system:

$$
\begin{cases}
D_{a^+}^{\frac{3}{2}}u(t)=\rho(t,u(t),x(t),y(t)), & t\in[1,2],\\
D_{a^+}^{\frac{7}{4}}x(t)=\varphi(t,u(t),x(t),y(t)), & t\in[1,2],\\
D_{a^+}^{\frac{5}{2}}x(t)=\psi(t,u(t),x(t),y(t)), & t\in[1,2],\\
u(1)=1/400, \qquad u(2)=\frac{1}{20}x(\frac{13}{10}),\\
x(1)=0, \qquad x(2)=\sum_{j=1}^{2}q_jy(\beta-j),\\
y(\frac{11}{10})=0, \qquad y(\frac{6}{5})=0, \qquad y(b)=\sum_{k=1}^{2}r_ku(\gamma_k).
\end{cases}
\tag{4.1}
$$

Here, $\eta = 3/4$, $\xi = 7/4$, $\zeta = 5/2$, $a = 1$, $b = 2$, $u_0 = 1/400$, $m = 1$, $n = 2$, $l = 2$, $p_1 = 1/20$, $q_1 = 1/100$, $q_2 = 1/50$, $r_1 = 1/1000$, $r_2 = 1/500$, $\alpha_1 = 13/10$, $\beta_1 = 7/5$, $\beta_2 = 3/2$, $\gamma_1 = 8/5$, $\gamma_2 = 17/10$. With the given data, it is found that $L_1 \simeq 0.75223$, $L_2 \simeq eq1397 \times 10^{-5}$, $L_3 \simeq 5.0146 \times 10^{-3}$, $M_1 \simeq 4.0414 \times 10^{-2}$, $M_2 \simeq 1.2435$, $M_3 \simeq 4.8058 \times 10^{-5}$, $N_1 \simeq 1.8944 \times 10^{-3}$, $N_2 \simeq 3.0567 \times 10^{-3}$, $N_3 \simeq 1.0942$.

(1) In order to illustrate Theorem 3.2, we take

$$\begin{aligned} \rho(t,u,x,y) &= e^{-2t} + \frac{1}{8}u\cos x + \frac{e^{-t}}{3}x\sin y + \frac{e^{-t}}{4}y\cos u, \\ \varphi(t,u,x,y) &= t\sqrt{t^2+3} + \frac{e^{-t}}{3\Pi}u\tan^{-1}x + \frac{1}{\sqrt{48+t^2}}x + \frac{1}{4}y\sin u, \\ \psi(t,u,x,y) &= \frac{e^{-t}}{10} + \frac{e^{-t}}{3}u + \frac{1}{4+t}x + \frac{e^{-t}}{4}y\cos x. \end{aligned} \tag{4.2}$$

It is easy to check that condition (H_1) is satisfied with $k_0 = 1/e^2$, $k_1 = 1/8$, $k_2 = 1/(3e)$, $k_3 = 1/(4e)$, $\sigma_0 = 2\sqrt{7}$, $\sigma_1 = 1/(6e)$, $\sigma_2 = 1/7$, $\sigma_3 = 1/4$, $\mu_0 = 1/(10e)$, $\mu_1 = 1/(3e)$, $\mu_2 = 1/5$, $\mu_3 = 1/(4e)$. Furthermore,

$$\begin{aligned} &(L_1+L_2+L_3)k_1 + (M_1+M_2+M_3)\sigma_1 + (N_1+N_2+N_3)\mu_1 \simeq 0.30801 < 1, \\ &(L_1+L_2+L_3)k_2 + (M_1+M_2+M_3)\sigma_2 + (N_1+N_2+N_3)\mu_2 \simeq 0.49596 < 1, \\ &(L_1+L_2+L_3)k_3 + (M_1+M_2+M_3)\sigma_3 + (N_1+N_2+N_3)\mu_3 \simeq 0.49161 < 1. \end{aligned}$$

Clearly, the hypotheses of Theorem 3.2 are satisfied, and hence the conclusion of Theorem 3.2 applies to problem (4.1) with ρ, φ, ψ given by (4.2).

(2) In order to illustrate Theorem 3.3, we take

$$\begin{aligned} \rho(t,u,x,y) &= \frac{e^{-t}}{\sqrt{3+t^2}}\cos u + \cos t, \\ \varphi(t,u,x,y) &= \frac{1}{5+t^4}\bigl(\sin u + |x|\bigr) + e^{-t}, \\ \psi(t,u,x,u) &= \frac{e^{-t}}{3}\sin y + \tan^{-1} t, \end{aligned} \tag{4.3}$$

which clearly satisfies condition (H_2) with $l_1 = 1/(2e)$, $l_2 = 1/6$, and $l_3 = 1/(3e)$. Moreover, $(L_1+L_2+L_3)l_1 + (M_1+M_2+M_3)l_2 + (N_1+N_2+N_3)l_3 \simeq 0.49596 < 1$. Thus the hypothesis of Theorem 3.3 holds true, and consequently there exists a unique solution for problem (4.1) on $[1,2]$ with ρ, φ, ψ given by (4.3).

5 Conclusions

This paper studies a tripled system of nonlinear fractional differential equations of different orders on an arbitrary domain complemented with the multi-point boundary conditions of cyclic nature involving different nonlocal positions. Applying the standard fixed point theorems, we have proved the existence and uniqueness results for the given problem, which are well illustrated with the aid of examples. By taking all $p_i = 0$, $i = 1,\dots,m$, $q_j = 0$, $j = 1,\dots,$ and $r_k = 0$, $k = 1,\dots,l$, we obtain the new results for the given tripled system of nonlinear fractional differential equations equipped with the conditions: $u(a) = u_0$,

$u(b) = 0$, $x(a) = 0$, $x(b) = 0$, $y(\xi_1) = 0$, $y(\xi_2) = 0$, $y(b) = 0$. To the best of our knowledge, it is the first paper dealing with a nonlocal multi-point boundary value problem involving a tripled system of nonlinear fractional differential equations of different orders on an arbitrary domain.

Acknowledgements
This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PhD-41-130-41). The authors, therefore, acknowledge with thanks DSR technical and financial support. The authors also thank the reviewers for their constructive remarks on our work.

Funding
This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PhD-41-130-41).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, SH, AA, and SKN, contributed equally to each part of this work. All authors read and approved the final manuscript.

Author details
[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia. [2]Department of Mathematics, University of Ioannina, 451 10, Ioannina, Greece.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 17 February 2021 Accepted: 27 May 2021 Published online: 06 June 2021

References

1. Fallahgoul, H.A., Focardi, S.M., Fabozzi, F.J.: Fractional Calculus and Fractional Processes with Applications to Financial Economics, Theory and Application. Elsevier, London (2017)
2. Javidi, M., Ahmad, B.: Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton–zooplankton system. Ecol. Model. **318**, 8–18 (2015)
3. Ding, Y., Wang, Z., Ye, H.: Optimal control of a fractional-order HIV-immune system with memory. IEEE Trans. Control Syst. Technol. **20**, 763–769 (2012)
4. Carvalho, A., Pinto, C.M.A.: A delay fractional order model for the co-infection of malaria and HIV/AIDS. Int. J. Dyn. Control **5**, 168–186 (2017)
5. Xu, Y., Li, W.: Finite-time synchronization of fractional-order complex-valued coupled systems. Physica A **549**, 123903 (2020)
6. Zhang, F., Chen, G., Li, C., Kurths, J.: Chaos synchronization in fractional differential systems. Philos. Trans. R. Soc. A **371**, 201201553 (2013)
7. Ming, Z., Zhang, G., Li, H.: Positive solutions of a derivative dependent second-order problem subject to Stieltjes integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2019**, 98 (2019)
8. Wang, Y., Liang, S., Wang, Q.: Existence results for fractional differential equations with integral and multi-point boundary conditions. Bound. Value Probl. **2018**, 4 (2018)
9. Cen, Z., Liu, L.-B., Huang, J.: A posteriori error estimation in maximum norm for a two-point boundary value problem with a Riemann–Liouville fractional derivative. Appl. Math. Lett. **102**, 106086 (2020)
10. Iskenderoglu, G., Kaya, D.: Symmetry analysis of initial and boundary value problems for fractional differential equations in Caputo sense. Chaos Solitons Fractals **134**, 109684 (2020)
11. Goodrich, C.S.: Coercive nonlocal elements in fractional differential equations. Positivity **21**, 377–394 (2017)
12. Ahmad, B., Alruwaily, Y., Alsaedi, A., Nieto, J.J.: Fractional integro-differential equations with dual anti-periodic boundary conditions. Differ. Integral Equ. **33**, 181–206 (2020)
13. Ben-Avraham, D., Havlin, S.: Diffusion and Reactions in Fractals and Disordered Systems. Cambridge University Press, Cambridge (2000)
14. Deshpande, A.S., Daftardar-Gejji, V.: On disappearance of chaos in fractional systems. Chaos Solitons Fractals **102**, 119–126 (2017)
15. Wang, S., Xu, M.: Axial Couette flow of two kinds of fractional viscoelastic fluids in an annulus. Nonlinear Anal., Real World Appl. **10**, 1087–1096 (2009)

16. Balachandran, K., Kokila, J.: Controllability of non-linear implicit fractional dynamical systems. IMA J. Appl. Math. **79**, 562–570 (2014)
17. Wang, J.R., Zhang, Y.: Analysis of fractional order differential coupled systems. Math. Methods Appl. Sci. **38**, 3322–3338 (2015)
18. Ahmad, B., Alghamdi, N., Alsaedi, A., Ntouyas, S.K.: A system of coupled multi-term fractional differential equations with three-point coupled boundary conditions. Fract. Calc. Appl. Anal. **22**, 601–618 (2019)
19. Ahmad, B., Alsaedi, A., Ntouyas, S.K.: Nonlinear coupled fractional order systems with integro-multistrip-multipoint boundary conditions. Int. J. Anal. Appl. **17**, 940–957 (2019)
20. Ahmad, B., Alsaedi, A., Ntouyas, S.K.: Fractional order nonlinear mixed coupled systems with coupled integro-differential boundary conditions. J. Appl. Anal. Comput. **10**, 892–903 (2020)
21. Ahmad, B., Alsaedi, A., Alruwaily, Y., Ntouyas, S.K.: Nonlinear multi-term fractional differential equations with Riemann-Stieltjes integro-multipoint boundary conditions. AIMS Math. **5**, 1446–1461 (2020)
22. Henderson, J., Luca, R., Tudorache, A.: On a system of fractional differential equations with coupled integral boundary conditions. Fract. Calc. Appl. Anal. **18**, 361–386 (2015)
23. Ntouyas, S.K., Al-Sulami, H.H.: A study of coupled systems of mixed order fractional differential equations and inclusions with coupled integral fractional boundary conditions. Adv. Differ. Equ. **2020**, 73 (2020)
24. Ahmad, B., Alghanmi, M., Alsaedi, A.: Existence results for a nonlinear coupled system involving both Caputo and Riemann–Liouville generalized fractional derivatives and coupled integral boundary conditions. Rocky Mt. J. Math. **50**, 1901–1922 (2020)
25. Ahmad, B., Alghanmi, M., Alsaedi, A., Nieto, J.J.: Existence and uniqueness results for a nonlinear coupled system involving Caputo fractional derivatives with a new kind of coupled boundary conditions. Appl. Math. Lett. **116**, 107018 (2021)
26. Matar, M.M., Abo Amra, I., Alzabut, J.: Existence of solutions for tripled system of fractional differential equations involving cyclic permutation boundary conditions. Bound. Value Probl. **2020**, 140 (2020)
27. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
28. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2005)