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Difference formula defined by a new differential symmetric operator for a class of meromorphically multivalent functions

Rabha W. Ibrahim[1] and Ibtisam Aldawish[2*]

*Correspondence:
imaldawish@imamu.edu.sa
[2]Department of Mathematics and Statistics, College of Science, IMSIU (Imam Mohammad Ibn Saud Islamic University), P.O. Box 90950, Riyadh 11623, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

Symmetric operators have benefited in different fields not only in mathematics but also in other sciences. They appeared in the studies of boundary value problems and spectral theory. In this note, we present a new symmetric differential operator associated with a special class of meromorphically multivalent functions in the punctured unit disk. This study explores some of its geometric properties. We consider a new class of analytic functions employing the suggested symmetric differential operator.

MSC: 30C55; 30C45

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1 Introduction

The study of the operator is narrowly connected with problems in the theory of functions. Various operators that were studied are operators on the space of holomorphic functions. For instance, Beurling's theorem defines the invariant subspaces of bounded holomorphic functions on the open unit disk. Beurling deduced the idea as multiplication of the independent variable on the Hardy space. The realization in studying multiplication operators is seemed in Toeplitz operators, specifically in the Bergman space of holomorphic functions. The geometric function theory is likewise ironic covering a long list of operators, counting differential, integral, and convolution operators. Limited symmetric operators are studied in this field. Newly, Ibrahim and Darus (see [1] and for applications see [2–5]) offered new symmetric differential, integral, and linear symmetric operators for a class of normalized functions in the open unit disk.

In this note, we proceed to consider a differential symmetric operator (DSO) associated with a class of meromorphically multivalent functions in the punctured unit disk. Consequently, we suggest a new class of analytic functions based on DSO to study it in view of the geometric function theory. Moreover, we investigate the real case of a formula containing the DSO. We show that this operator is a solution of a type of Sturm–Liouville equation. Some examples are illustrated in the sequel.

2 Construction

In this paper, we construct a new DSO connected with the following class of multivalent meromorphic functions $\Sigma_k(\wp)$ consisting of functions φ with the power series expansion

$$\varphi(z) = z^{-\wp} + \sum_{n=k}^{\infty} \varphi_n z^{n-\wp}, \quad z \in \cup, \tag{2.1}$$

where $k \in \mathbb{N} = \{1, 2, 3, \dots\}$ and $n - \wp \in \mathbb{N}$. Recall that the functions φ of the form (2.1) are called meromorphic with a pole at $z = 0$ so that $\varphi(z) - z^{-\wp}$ is analytic in $\cup$ (see Komatu [6] or Hayman [7]). We then concentrate on a subclass of $\Sigma_k(\wp)$ formulated by a subordination and explore inclusion properties and sufficient inclusion conditions for this class and check its closure property under convolution or Hadamard product.

2.1 Differential symmetric operator (DSO)

In this place, we state a few definitions and a lemma that we shall need in the next section. First, we define a conformable differential operator for the class of meromorphic functions $\Sigma_k(\wp)$ defined by (2.1).

Definition 2.1 For functions $\varphi \in \Sigma_k(\wp)$, define the symmetric differential operator as follows:

$$\begin{aligned}
\Delta^0 \varphi(z) &= \varphi(z) = z^{-\wp} + \sum_{n=k}^{\infty} \varphi_n z^{n-\wp}, \\
\Delta^\alpha \varphi(z) &= \left(\frac{\alpha}{-\wp}\right)\left(z\varphi'(z)\right) + \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-\wp}\right)\left(z\varphi'(-z)\right) \\
&= \left(\frac{\alpha}{-\wp}\right)\left((-\wp)z^{-\wp} + \sum_{n=k}^{\infty}(n-\wp)\varphi_n z^{n-\wp}\right) + \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-\wp}\right) \\
&\quad \times \left((-\wp)(-1)^{-\wp-1} z^{-\wp} + \sum_{n=k}^{\infty}(n-\wp)\varphi_n (-1)^{n-\wp-1} z^{n-\wp}\right) \\
&= z^{-\wp} + \sum_{n=k}^{\infty}(n-\wp)\left(\frac{\alpha + (1-\alpha)(-1)^n}{-\wp}\right)\varphi_n z^{n-\wp}, \qquad (2.2) \\
\Delta^{2\alpha}\varphi(z) &= \Delta^\alpha \varphi(z)\left(\Delta^\alpha \varphi(z)\right) \\
&= z^{-\wp} + \sum_{n=k}^{\infty}(n-\wp)^2\left(\frac{\alpha + (1-\alpha)(-1)^n}{-\wp}\right)^2 \varphi_n z^{n-\wp}, \\
&\vdots \\
\Delta^{m\alpha}\varphi(z) &= \Delta^\alpha \varphi(z)\left(\Delta^{(m-1)\alpha}\varphi(z)\right) \\
&= z^{-\wp} + \sum_{n=k}^{\infty}(n-\wp)^m\left(\frac{\alpha + (1-\alpha)(-1)^n}{-\wp}\right)^m \varphi_n z^{n-\wp},
\end{aligned}$$

where $\alpha \in [0, 1]$, $\wp \in \mathbb{N}$, $m \in \mathbb{N}\cup\{0\}$, $z \in \cup$.

Clearly, $\Delta^{m\alpha}\varphi(z) \in \Sigma_k(\wp)$ as well as, for two functions φ and $\psi \in \Sigma_k(\wp)$, we have

$$\begin{aligned}
&\Delta^{\alpha}\big[A\varphi(z) + B\psi(z)\big] \\
&= \left(\frac{\alpha}{-\wp}\right)\big(z\big[A\varphi(z) + B\psi(z)\big]'\big) + \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-\wp}\right)\big(z\big[A\varphi(-z) + B\psi(-z)\big]'\big) \\
&= A\left(\left(\frac{\alpha}{-\wp}\right)\big(z\varphi'(z)\big) + \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-\wp}\right)\big(z\varphi'(-z)\big)\right) \\
&\quad + B\left(\left(\frac{\alpha}{-\wp}\right)\big(z\psi'(z)\big) + \left(\frac{(1-\alpha)(-1)^{\wp+1}}{-\wp}\right)\big(z\psi'(-z)\big)\right) \\
&= A\Delta^{\alpha}\varphi(z) + B\Delta^{\alpha}\psi(z); \quad A, B \in \mathbb{R}.
\end{aligned}$$

So, in general, we have the following proposition.

Proposition 2.2 (Semigroup property) *The class of DSO constructed by $\Delta^{m\alpha}$ has the semigroup property since, for φ and ψ in $\Sigma_k(\wp)$, we obtain*

$$\Delta^{m\nu}\big[A\varphi(z) + B\psi(z)\big] = A\Delta^{m\alpha}f(z) + B\Delta^{m\alpha}g(z).$$

We will need the following subordination definition for our class of meromorphic functions. For functions φ and ψ in $\Sigma_k(\wp)$, we call that φ is subordinate to ψ, denoted by $\varphi \prec \psi$, if there is a Schwarz function ϖ with $\varpi(0) = 0$ and $|\varpi(z)| \leq |z| < 1$ so that $\varphi(z) = \psi(\varpi(z))$ in $\cup$ (see [8] or [9]).

Definition 2.3 For $-1 \leq \nu < \mu \leq 1$ and $\varsigma < 0$, a function $\varphi \in \Sigma_k(\wp)$ is said to be in the class $\Sigma_k^{\alpha}(\mu, \nu, \varsigma, \wp)$ if it achieves the subordination condition

$$\begin{aligned}
&(1-\varsigma)z^{\wp}\big[\Delta^{m\alpha}\varphi(z)\big] - \left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\big[\Delta^{m\alpha}\varphi(z)\big]' \prec J_{\mu,\nu}(z) \qquad (2.3)\\
&\left(J_{\mu,\nu}(z) := \frac{1+\mu z}{1+\nu z}, z \in \cup\right).
\end{aligned}$$

The class of functions $J_{\mu,\nu}(\rho(z)) := \frac{1+\mu\rho(z)}{1+\nu\rho(z)}$ and, as a special case, the functions of the form $J_{\mu,\nu}(z) = \frac{1+\mu z}{1+\nu z}$ are of particular importance since $J_{\mu,\nu}(\rho(z))$ is the class of Caratheodory functions of order $\frac{1-\mu}{1-\nu}$, that is, $\Re\{J_{\mu,\nu}(\rho(z))\} > \frac{1-\mu}{1-\nu}$ (see Janowski [10] or Jahangiri et al. [11]).

To prove our outcomes in the next section, we need the following lemmas which are due to Miller and Mocanu [9].

Lemma 2.4 *Suppose that $f_1(z)$ is analytic in $\cup$ and $f_2(z)$ is convex univalent in $\cup$ such that $f_1(0) = f_2(0)$. If $f_1(z) + (1/\gamma)f_1'(z) \prec f_2(z)$ for a nonzero complex constant number γ with $\Re(\gamma) \geq 0$, then $f_1(z) \prec f_2(z)$.*

Lemma 2.5 *For $a \in \mathbb{C}$ and positive integer n, let $\mathbb{H}[a,n] = \{h : h(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\}$. If $c \in \mathbb{R}$, then $\Re(h(z) + czh'(z)) > 0 \Longrightarrow \Re(h(z)) > 0$. Moreover, if $c > 0$ and $h \in \mathbb{H}[1,n]$, then there are constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that the inequality*

$$h(z) + czh'(z) \prec \left[\frac{1+z}{1-z}\right]^{\lambda_1}$$

implies

$$h(z) \prec \left[\frac{1+z}{1-z}\right]^{\lambda_2}.$$

3 Main results

First we prove an inclusion theorem for the class $\Sigma_k^{\alpha}(\mu,\nu,\varsigma,\wp)$.

3.1 Inclusion properties

Theorem 3.1 *Let* $\varphi \in \Sigma_k(\wp)$. *If* $\varsigma_2 < \varsigma_1 < 0$, *then*

$$\Sigma_k^{\alpha}(\mu,\nu,\varsigma_2,\wp) \subset \Sigma_k^{\alpha}(\mu,\nu,\varsigma_1,\wp).$$

Proof Let $\varphi \in \Sigma_k^{\alpha}(\mu,\nu,\varsigma_2,\wp)$. Define a function $\phi(z) = z^{\wp}[\Delta^{m\alpha}\varphi(z)]$, which is analytic in $\cup$ with $\phi(0) = 1$. A calculation yields

$$(1-\varsigma_2)z^{\wp}\big[\Delta^{m\alpha}\varphi(z)\big] - \left(\frac{\varsigma_2}{\wp}\right)z^{1+\wp}\big[\Delta^{m\alpha}\varphi(z)\big]' = \phi(z) - \frac{\varsigma_2}{\wp}\big(z\phi'(z)\big).$$

Consequently, we get the inequality

$$\phi(z) - \frac{\varsigma_2}{\wp}\big(z\phi'(z)\big) \prec \frac{\mu z+1}{\nu z+1}.$$

Applying Lemma 2.4 with $\gamma := -\frac{\varsigma_2}{\wp} > 0$ gives

$$\phi(z) \prec \frac{\mu z+1}{\nu z+1}, \quad z \in \cup.$$

Since $0 < \varsigma_1/\varsigma_2 < 1$ and since $J_{\mu,\nu}(z)$ is convex univalent in $\cup$, we arrive at the inequality

$$\begin{aligned}
&(1-\varsigma_1)z^{\wp}\big[\Delta^{m\alpha}\varphi(z)\big] - \left(\frac{\varsigma_1}{\wp}\right)z^{1+\wp}\big[\Delta^{m\alpha}\varphi(z)\big]' \\
&\quad = (1-\varsigma_1)\phi(z) - \left(\frac{\varsigma_1}{\wp}\right)\big(z\phi'(z) - \wp\phi(z)\big) + \left(\frac{\varsigma_1}{\varsigma_2}\phi(z) - \frac{\varsigma_1}{\varsigma_2}\phi(z)\right) \\
&\quad = \frac{\varsigma_1}{\varsigma_2}\left((1-\varsigma_2)\phi(z) - \left(\frac{\varsigma_2}{\wp}\right)\big(z\phi'(z) - \wp\phi(z)\big)\right) + \left(1 - \frac{\varsigma_1}{\varsigma_2}\right)\phi(z) \\
&\quad = \frac{\varsigma_1}{\varsigma_2}\left((1-\varsigma_2)z^{\wp}\big[\Delta^{m\alpha}\varphi(z)\big] - \left(\frac{\varsigma_2}{\wp}\right)z^{1+\wp}\big[\Delta^{m\alpha}\varphi(z)\big]'\right) + \left(1 - \frac{\varsigma_1}{\varsigma_2}\right)\phi(z) \\
&\quad \prec J_{\mu,\nu}(z).
\end{aligned}$$

Hence, by Definition 2.3, we conclude that $\varphi \in \Sigma_k^{\alpha}(\mu,\nu,\varsigma_1,\wp)$. □

3.2 Geometric properties

Next, we show a sufficient inclusion condition for the class $\Sigma_k^{\alpha}(\mu,\nu,\varsigma,\wp)$.

Theorem 3.2 *Let* $\varphi \in \Sigma_k(\wp)$ *and*

$$\Phi(z) := (1-\varsigma)z^{\wp}\big[\Delta^{m\alpha}\varphi(z)\big] - \left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\big[\Delta^{m\alpha}\varphi(z)\big]'.$$

Then $\Phi(z) \prec J_{\mu,\nu}(z)$ *if one of the following inequalities occurs*:

- $1+\varepsilon(z\Phi'(z)) \prec \sqrt{z+1}$, $\varepsilon \geq \max\{\varepsilon_0, \varepsilon_1\}$, *where*

$$\varepsilon_0 = \frac{0.452\nu + 0.452}{\mu-\nu}, \quad \nu+1 \neq 0, \mu-\nu \neq 0;$$

  and

$$\varepsilon_1 = \frac{-0.631(\nu-1)}{(\mu-\nu)}, \quad \nu-1 \neq 0, \mu-\nu \neq 0.$$

- $1+\varepsilon(z\frac{\Phi'(z)}{\Phi(z)}) \prec \sqrt{z+1}$, $\varepsilon \geq \max\{|\varepsilon_2|, |\varepsilon_3|\}$, *where*

$$\varepsilon_2 = \frac{0.6i}{2\pi n - i\log(\frac{\mu-1}{\nu-1})}, \\ \left(\log\left(\frac{\mu-1}{\nu-1}\right) + 2i\pi n \neq 0, \mu \neq 1, \nu \neq 1\right);$$

  and

$$\varepsilon_3 = \frac{0.452i}{2\pi n - i\log(\frac{\nu+1}{\mu+1})}, \\ \left(\nu+1 \neq 0, \mu+1 \neq 0, \log\left(\frac{\nu+1}{\mu+1}\right) + 2\pi ni \neq 0\right).$$

- $1+\varepsilon(z\frac{\Phi'(z)}{\Phi^2(z)}) \prec \sqrt{z+1}$, $\varepsilon \geq \max\{\varepsilon_4, \varepsilon_5\}$, *where*

$$\varepsilon_4 = \frac{0.452(\mu+1)}{(\mu-\nu)}, \quad \nu+1 \neq 0, \mu \neq \nu;$$
$$\varepsilon_5 = \frac{0.6(\nu-1)}{(\mu-\nu)}, \quad \nu-1 \neq 0\mu \neq \nu.$$

Proof Case I: $1+\varepsilon(z\Phi'(z)) \prec \sqrt{z+1}$.

Define a function $\top_\varepsilon : \cup \to \mathbb{C}$ formulating by

$$\top_\varepsilon(z) = 1 + \frac{2}{\varepsilon}\big(\sqrt{z+1} - \log(1+\sqrt{z+1}) - 1 + \log(2)\big).$$

Clearly, $\top_\varepsilon(z)$ is analytic in $\cup$ satisfying $\top_\varepsilon(0) = 1$, and it is a solution of the differential equation

$$1+\varepsilon\big(z\top'_\varepsilon(z)\big) = \sqrt{z+1}. \tag{3.1}$$

Thus, we obtain $\mathfrak{T}(z) := \varepsilon(z\top'_\varepsilon(z)) = \sqrt{z+1} - 1$ is starlike in $\cup$. So, for

$$\mathfrak{F}(z) := \mathfrak{T}(z) + 1,$$

we have

$$\Re\left(\frac{z\mathfrak{T}'(z)}{\mathfrak{T}(z)}\right) = \Re\left(\frac{z\mathfrak{F}'(z)}{\mathfrak{W}(z)}\right) > 0.$$

Thus, by Lemma 2.4, it yields

$$1+\varepsilon\big(z\Phi'(z)\big)\prec 1+\varepsilon z\top'_{\varepsilon}(z) \quad\Rightarrow\quad \Phi(z)\prec \top_{\varepsilon}(z).$$

To complete this argument, we must prove that $\top_{\varepsilon}(z)\prec J_{\mu,\nu}(z)$. Evidently, the function $\top_{\varepsilon}(z)$ is increasing in the interval $(-1,1)$ that satisfies the inequality

$$\top_{\varepsilon}(-1)\le \top_{\varepsilon}(1).$$

Since

$$\frac{1-\mu}{1-\nu}\le \top_{\varepsilon}(-1)\le \top_{\varepsilon}(1)\le \frac{1+\mu}{1+\nu},$$

where $\varepsilon\ge \max\{\varepsilon_0,\varepsilon_1\}$,

$$\varepsilon_0=\frac{0.452\nu+0.452}{\mu-\nu},\quad \nu+1\neq 0,\mu-\nu\neq 0$$

and

$$\varepsilon_1=\frac{-0.631(\nu-1)}{(\mu-\nu)},\quad \nu-1\neq 0,\mu-\nu\neq 0,$$

then we get the conclusion

$$\Phi(z)\prec \top_{\varepsilon}(z)\prec J_{\mu,\nu}(z) \quad\Rightarrow\quad \Phi(z)\prec J_{\mu,\nu}(z).$$

Case II: $1+\varepsilon(\frac{z\Phi'(z)}{\Phi(z)})\prec\sqrt{z+1}$.
Define a function $\Omega_{\varepsilon}:\cup\to\mathbb{C}$ formulating by the structure

$$\Omega_{\varepsilon}(z)=\exp\left(\frac{2}{\varepsilon}\big(\sqrt{z+1}-\log(1+\sqrt{z+1})-1+\log(2)\big)\right).$$

Obviously, $\Omega_{\varepsilon}(z)$ is analytic in $\cup$ having $\Omega_{\varepsilon}(0)=1$, and it is a solution of the differential equation

$$1+\varepsilon\left(\frac{z\Omega'_{\varepsilon}(z)}{\Omega_{\varepsilon}(z)}\right)=\sqrt{z+1},\quad z\in\cup. \tag{3.2}$$

By assuming $\mathfrak{T}(z)=\sqrt{z+1}-1$, which is starlike in $\cup$ and $\mathfrak{F}(z)=\mathfrak{T}(z)+1$, we obtain

$$\Re\left(\frac{z\mathfrak{T}'(z)}{\mathfrak{T}(z)}\right)=\Re\left(\frac{z\mathfrak{F}'(z)}{\mathfrak{T}(z)}\right)>0.$$

Then again, by virtue of Lemma 2.4, we have

$$1+\varepsilon\left(\frac{z\Phi'(z)}{\Phi(z)}\right)\prec 1+\varepsilon\left(\frac{z\Omega'_{\varepsilon}(z)}{\Omega_{\varepsilon}(z)}\right) \quad\Rightarrow\quad \Phi(z)\prec\Omega_{\varepsilon}(z).$$

Consequently,

$$\frac{1-\mu}{1-\nu} \le \Omega_\varepsilon(-1) \le \Omega_\varepsilon(1) \le \frac{1+\mu}{1+\nu}$$

whenever $\varepsilon \ge \max\{|\varepsilon_2|, |\varepsilon_3|\}$, where

$$\varepsilon_2 = \frac{0.6i}{2\pi n - i\log(\frac{\mu-1}{\nu-1})}$$
$$\left(\log\left(\frac{\mu-1}{\nu-1}\right) + 2i\pi n \neq 0, \mu \neq 1, \nu \neq 1\right)$$

and

$$\varepsilon_3 = \frac{0.452i}{2\pi n - i\log(\frac{\nu+1}{\mu+1})}.$$

This introduces the subordination conclusions

$$\Phi(z) \prec \Omega_\varepsilon(z) \prec J_{\mu,\nu}(z) \quad \Rightarrow \quad \Phi(z) \prec J_{\mu,\nu}(z).$$

Case III: $1 + \varepsilon(\frac{z\Phi'(z)}{\Phi^2(z)}) \prec \sqrt{z+1}$.
Define a function $\eth_\varepsilon : \cup \to \mathbb{C}$ by the formula

$$\eth_\varepsilon(z) = \frac{1}{(1 - \frac{2}{\varepsilon}(\sqrt{z+1} - \log(1+\sqrt{z+1}) - 1 + \log(2)))}.$$

Clearly, $\eth_\varepsilon(z)$ is analytic in U achieving $\eth_\varepsilon(0) = 1$, and it is the result of the differential equation

$$1 + \varepsilon\left(\frac{z\eth'_\varepsilon(z)}{\eth_\varepsilon(z)}\right) = \sqrt{z+1}. \tag{3.3}$$

By employing the function $\mathfrak{T}(z) = \sqrt{z+1} - 1$, which is starlike in $\cup$ and $\mathfrak{F}(z) = \mathfrak{T}(z) + 1$, we obtain

$$\Re\left(\frac{z\mathfrak{T}'(z)}{\mathfrak{T}(z)}\right) = \Re\left(\frac{z\mathfrak{F}'(z)}{\mathfrak{T}(z)}\right) > 0.$$

Hence, Lemma 2.4 implies

$$1 + \varepsilon\left(\frac{z\Phi'(z)}{\Phi^2(z)}\right) \prec 1 + \varepsilon\left(\frac{z\eth'_\varepsilon(z)}{\eth^2_\varepsilon(z)}\right) \quad \Rightarrow \quad \Phi(z) \prec \eth_\varepsilon(z).$$

Accordingly, we have

$$\frac{1-\mu}{1-\nu} \le \eth_\varepsilon(-1) \le \eth_\varepsilon(1) \le \frac{1+\mu}{1+\nu}$$

whenever $\varepsilon \geq \max\{\varepsilon_4, \varepsilon_5\}$, where

$$\varepsilon_4 = \frac{0.452(\mu + 1)}{(\mu - \nu)}, \quad \nu + 1 \neq 0, \mu \neq \nu;$$
$$\varepsilon_5 = \frac{0.6(\nu - 1)}{(\mu - \nu)}, \quad \nu - 1 \neq 0 \mu \neq \nu.$$

This implies the subordination

$$\Phi(z) \prec \eth_\varepsilon(z) \prec J_{\mu,\nu}(z) \quad \Rightarrow \quad \Phi(z) \prec J_{\mu,\nu}(z).$$

As a conclusion, we have

$$(1-\varsigma)z^{\wp}\big[\Delta^{m\alpha}\varphi(z)\big] - \left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\big[\Delta^{m\alpha}\varphi(z)\big]' \prec J_{\mu,\nu}(z)$$

for all $\varsigma < 0$ and $\wp \in \mathbb{N}$. Consequently, $\varphi \in \Sigma_k^\alpha(\mu, \nu, \varsigma, \wp)$. □

Theorem 3.3 *Let*

$$\Phi(z) = (1-\varsigma)z^{\wp}\big[\Delta^{m\alpha}\varphi(z)\big] - \left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\big[\Delta^{m\alpha}\varphi(z)\big]'.$$

Then

$$\begin{aligned}&\ell_1(1+\wp)z^{\wp}\Delta^{m\alpha}\varphi(z) + \big[\ell_1 - \ell_2(1+\wp) - \ell_2\big]z^{1+\wp}\big(\Delta^{m\alpha}\varphi(z)\big)' - \ell_2 z^{2+\wp}\big(\Delta^{m\alpha}\varphi(z)\big)'' \\ &\quad \prec \left(\frac{1+z}{1-z}\right)^{\lambda_1} \quad \Rightarrow \quad \Phi(z) \prec \left(\frac{1+z}{1-z}\right)^{\lambda_2} \\ &\left(\lambda_1 > 0, \lambda_2 > 0, \ell_1 = 1-\varsigma, \ell_2 = \frac{\varsigma}{\wp}, \wp < 0\right).\end{aligned}$$

Proof A calculation implies that

$$\begin{aligned}\Phi(z) + z\Phi'(z) &= (1-\varsigma)z^{\wp}\big[\Delta^{m\alpha}\varphi(z)\big] - \left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\big[\Delta^{m\alpha}\varphi(z)\big]' \\ &\quad + z\left((1-\varsigma)z^{\wp}\big[\Delta^{m\alpha}\varphi(z)\big] - \left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\big[\Delta^{m\alpha}\varphi(z)\big]'\right)' \\ &= \ell_1(1+\wp)z^{\wp}\Delta^{m\alpha}\varphi(z) + \big[\ell_1 - \ell_2(1+\wp) - \ell_2\big]z^{1+\wp}\big(\Delta^{m\alpha}\varphi(z)\big)' \\ &\quad - \ell_2 z^{2+\wp}\big(\Delta^{m\alpha}\varphi(z)\big)'' \\ &\prec \left(\frac{1+z}{1-z}\right)^{\lambda_1}.\end{aligned}$$

Then, in view of Lemma 2.5 with $c = 1$, we obtain $\Phi(z) \prec (\frac{1+z}{1-z})^{\lambda_2}$. □

Note that when $\lambda_1 = \lambda_2 = 1$, then we have the following result.

Corollary 3.4 *For $\Phi(z)$ in Theorem 3.3, if the subordination*

$$\begin{aligned}&\ell_1(1+\wp)z^{\wp}\Delta^{m\alpha}\varphi(z)+\left[\ell_1-\ell_2(1+\wp)-\ell_2\right]z^{1+\wp}\left(\Delta^{m\alpha}\varphi(z)\right)'-\ell_2 z^{2+\wp}\left(\Delta^{m\alpha}\varphi(z)\right)''\\&\quad\prec\left(\frac{1+z}{1-z}\right),\\&\left(\ell_1=1-\varsigma,\ell_2=\frac{\varsigma}{\wp},\wp<0\right)\end{aligned}$$

holds, then $\varphi\in\Sigma_k^{\alpha}(1,-1,\varsigma,\wp)$.

Proof Let $\lambda_1=\lambda_2=1$ in Theorem 3.3, then this implies that $\Phi(z)\prec(\frac{1+z}{1-z})$; consequently, we have $\varphi\in\Sigma_k^{\alpha}(1,-1,\varsigma,\wp)$. □

Finally, we prove a convolution condition for the class $\Sigma_k^{\nu}(\mu,\nu,\varsigma,\wp)$.

Definition 3.5 The Hadamard product or convolution of two power series

$$\varphi(z)=z^{-\wp}+\sum_{n=k}^{\infty}\varphi_n z^{n-\wp}$$

and

$$\psi(z)=z^{-\wp}+\sum_{n=k}^{\infty}\psi_n z^{n-\wp}$$

in $\Sigma_k(\wp)$ is denoted by

$$\begin{aligned}(\varphi*\psi)(z)&=\varphi(z)*\psi(z)\\&=z^{-\wp}+\sum_{n=k}^{\infty}\varphi_n\psi_n z^{n-\wp}.\end{aligned}$$

Theorem 3.6 *Let $\varphi\in\Sigma_k^{\alpha}(\mu,\nu,\varsigma,\wp)$ and $f\in\Sigma_k(\wp)$. Then $\varphi*f\in\Sigma_k^{\alpha}(\mu,\nu,\varsigma,\wp)$ if*

$$\Re\left(z^{\wp}\Delta^{m\alpha}f(z)\right)>\frac{1}{2}.\tag{3.4}$$

Proof By the properties of the Hadamard product, we indicate that

$$\begin{aligned}&(1-\varsigma)z^{\wp}\left[\Delta^{m\alpha}(\varphi*f)(z)\right]-\left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\left[\Delta^{m\alpha}(\varphi*f)(z)\right]'\\&\quad=(1-\varsigma)\left(z^{\wp}\left[\Delta^{m\alpha}\varphi(z)\right]*z^{\wp}\left[\Delta^{m\alpha}f(z)\right]\right)\\&\qquad-\left(\frac{\varsigma}{\wp}\right)\left(z^{1+\wp}\left[\Delta^{m\alpha}f(z)\right]'*\left(z^{\wp}\left[\Delta^{m\alpha}f(z)\right]\right)\right)\\&\quad=\left((1-\varsigma)z^{\wp}\left[\Delta^{m\alpha}\varphi(z)\right]-\left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\left[\Delta^{m\alpha}f(z)\right]'\right)*\left(z^{\wp}\Delta^{m\alpha}f(z)\right)\\&\quad=\Phi(z)*\left(z^{\wp}\Delta^{m\alpha}f(z)\right),\end{aligned}$$

where $\Phi(z) \prec J_{\mu,\nu}(z)$. Given condition (3.4) yields that $(z^{\wp}\Delta^{m\alpha}f(z))$ has the Herglotz integral formula (e.g. see [12])

$$\left(z^{\wp}\Delta^{m\alpha}f(z)\right) = \int_{|\chi|=1} \frac{d\sigma(\chi)}{1-\chi z},$$

where $d\sigma$ presents the probability measure on the unit circle $|\chi| = 1$ and

$$\int_{|\chi|=1} d\sigma(\chi) = 1.$$

Since $J_{\mu,\nu}(z)$ is convex in $\cup$, we have

$$\begin{aligned}
&(1-\varsigma)z^{\wp}\left[\Delta^{m\alpha}(\varphi * f)(z)\right] - \left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\left[\Delta^{m\alpha}(\varphi * f)(z)\right]' \\
&\quad = \Phi(z) * \left(z^{\wp}\Delta^{m\alpha}f(z)\right) \\
&\quad = \int_{|\chi|=1} \Phi(\chi z)\, d\sigma(\chi) \\
&\quad \prec J_{\mu,\nu}(z).
\end{aligned}$$

Hence, $\varphi * f \in \Sigma_k^{\alpha}(\mu,\nu,\varsigma,\wp)$. □

We have the following geometric results.

Theorem 3.7 *For the function $\varphi \in \Sigma_k(\wp)$, define a functional*

$$\begin{aligned}
\Phi(z) &= (1-\varsigma)z^{\wp}\left[\Delta^{m\alpha}\varphi(z)\right] - \left(\frac{\varsigma}{\wp}\right)z^{1+\wp}\left[\Delta^{m\alpha}\varphi(z)\right]', \quad \varsigma < 0 \\
&= 1 + \sum_{n=1}^{\infty} \phi_n z^n, \quad z \in \cup.
\end{aligned}$$

Then

$$\Re\left(\Phi(z)\right) > 0 \quad \Rightarrow \quad |\phi_n| \le 2\int_0^{2\pi} \left|e^{-in\theta}\right| d\upsilon(\theta),$$

where $d\upsilon$ is a probability measure. Moreover,

$$\Re\left(e^{i\varpi}\Phi(z)\right) > 0 \quad \Rightarrow \quad \Phi(z) \in \mathcal{C},$$

where $\mathcal{C}$ is the class of analytic convex in $\cup$.

Proof For the first part of the theorem, we suppose that

$$\Re\left(\Phi(z)\right) = \Re\left(1 + \sum_{n=1}^{\infty}\phi_n z^n\right) > 0.$$

Then, by the Carathéodory positivist theorem for holomorphic functions, we have

$$|\phi_n| \leq 2\int_0^{2\pi} \left|e^{-in\theta}\right| d\upsilon(\theta),$$

where $d\upsilon$ is a probability measure. Lastly, if

$$\Re\big(e^{i\varrho}\Phi(z)\big) > 0, \quad z \in \cup, \varrho \in \mathbb{R},$$

then in view of [13]-Theorem 1.6(P22) and for some real numbers ϱ, we get

$$\Phi(z) \approx \frac{\mu z+1}{\nu z+1}, \quad z \in \cup.$$

But $\frac{\mu z+1}{\nu z+1}$ is convex in $\cup$, then by the majority concept, we obtain that $\Phi(z) \in \mathcal{C}$. □

Theorem 3.7 implies the sufficient conditions to a function $\varphi \in \Sigma_k(\wp)$ to be in $\Sigma_k^\alpha(\mu,\nu,\varsigma,\wp)$.

Theorem 3.8 *For the function* $\varphi \in \Sigma_k(\wp)$, *define a functional* $\flat(z) := z^{\wp+1}\Delta^{m\alpha}\varphi(z)$, $z \in \cup$. *If the subordination*

$$\flat(z) \prec \frac{z}{(1+z)^2}$$

holds, then $\flat(z) \in \mathbb{S}^*$ (*the class of starlike analytic functions*) *and*

$$\left(\int_0^z \frac{\sqrt{\flat(\zeta)}}{\zeta}\,d\zeta\right)^2 \prec \big(2\tan^{-1}\sqrt{z}\big)^2$$

such that

$$-\frac{\pi}{2} < -2\tan^{-1}\sqrt{r} \leq \Re\left(\int_0^z \frac{\sqrt{\flat(\zeta)}}{\zeta}\,d\zeta\right) < 2\tan^{-1}\sqrt{r} \leq \frac{\pi}{2}.$$

Proof Let $\flat(z) = z^{\wp+1}\Delta^{m\alpha}\varphi(z)$, $z \in \cup$. Then

$$\flat(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \cup$$

is analytic in the open unit disk. Obviously,

$$\begin{aligned}\mathcal{B}(z) &:= \big(2\tan^{-1}\sqrt{z}\big)^2 \\ &= 4z - 8\frac{z^2}{3} + \frac{92z^3}{45} + O\big(z^4\big).\end{aligned}$$

Since the function (see [9]-P177)

$$\begin{aligned}p(z) &= \frac{z}{(1+z)^2} \\ &= z - 2z^2 + 3z^3 - 4z^4 + 5z^5 + O\big(z^6\big) \in \mathbb{S}^*, \quad z \in \cup,\end{aligned}$$

then by the majority concept, we have $\flat(z) \in \mathbb{S}^*$. The second and third assertions are verified by [9]-Corollary 3.6a.1. □

Similarly, we have the next result.

Theorem 3.9 *Assume that* $\varphi \in \Sigma_k(\wp)$ *and a functional* $\flat(z) = z^{\wp+1}\Delta^{m\alpha}\varphi(z)$, $z \in \cup$. *If the subordination*

$$\flat(z) \prec \frac{z}{(1+z)^2}$$

holds, then $\flat(z) \in \mathbb{S}^*$ (*the class of starlike analytic functions*) *and*

$$\left(\int_0^z \frac{\sqrt{\flat(\zeta)}}{\zeta}\,d\zeta\right)^2 \prec \left(2\cot^{-1}\sqrt{1/z}\right)^2$$

such that

$$-\frac{\pi}{2} < -2\cot^{-1}\sqrt{1/r} \le \Re\left(\int_0^z \frac{\sqrt{\flat(\zeta)}}{\zeta}\,d\zeta\right) < 2\cot^{-1}\sqrt{1/r} \le \frac{\pi}{2}.$$

3.3 Real cases

From the proof of Theorem 3.3, we indicate the real construction as follows:

$$\begin{aligned}\Re\big(\Phi(z) + z\Phi'(z)\big) &= \Re\big(\ell_1(1+\wp)z^{\wp}\Delta^{m\alpha}\varphi(z) + [\ell_1 - \ell_2(1+\wp) - \ell_2]z^{1+\wp}\big(\Delta^{m\alpha}\varphi(z)\big)' \\ &\quad - \ell_2 z^{2+\wp}\big(\Delta^{m\alpha}\varphi(z)\big)''\big) \\ &= \ell_1(1+\wp)xy + \left(\frac{(1+\wp)(2\ell_1-1)-1}{\wp}\right)x^{1-\wp}y' - \left(\frac{1-\ell_1}{\wp}\right)x^{1-2\wp}y'',\end{aligned}$$

where $\Re(z^{\wp}) := x$, $\ell_1 = 1 - \varsigma > 0$, $\ell_2 = (1-\ell_1)/\wp$ and $\Re(\Delta^{m\alpha}\varphi(z)) := y(x)$. By approximate $\ell_1 \to 2$, we have

$$\Re\big(\Phi(z) + z\Phi'(z)\big) = 2(1+\wp)xy + \left(\frac{3(1+\wp)-1}{\wp}\right)x^{1-\wp}y' + \left(\frac{1}{\wp}\right)x^{1-2\wp}y'',$$

then the real solution of $\Re(\Phi(z) + z\Phi'(z)) = 0$ is equivalent to the solution of

$$2(1+\wp)xy + \left(\frac{3(1+\wp)-1}{\wp}\right)x^{1-\wp}y' + \left(\frac{1}{\wp}\right)x^{1-2\wp}y'' = 0. \tag{3.5}$$

The exact and the approximate solutions of Eq. (3.5) are formulated in the next result.

Theorem 3.10 *Consider Eq.* (3.5). *Then the exact solution is formulated as a linear combination of a confluent hypergeometric function with the Laguerre polynomials*

$$\begin{aligned}y(x) &= 2^{\wp/(2\wp+2)}e^{-2x^{\wp+1}}\big(x^{\wp+1}\big)^{\wp/(2\wp+2)}x^{-\wp/2} \\ &\quad\times\left\{c_1 U\left(\frac{2\wp}{\wp+2}, \frac{\wp}{\wp+1}, \frac{\wp+2x^{\wp+1}}{\wp+1}\right) + c_2 L^{(-1/(\wp+1))}_{(-2\wp/(\wp+2))}\left(\frac{(\wp+2)x^{\wp+1}}{(\wp+1)}\right)\right\}\end{aligned} \tag{3.6}$$

and an approximate solution

$$y(x) \approx 2^{\wp/(2\wp+2)}(2.718)^{-2x^{1+\wp}}\left(x^{\wp+1}\right)^{\wp/(2\wp+2)}x^{-\wp/2} \times \left\{c_1 U\left(\frac{2\wp}{\wp+2}, \frac{\wp}{\wp+1}, \frac{\wp+2x^{\wp+1}}{\wp+1}\right) + c_2 L_{(-2\wp/(\wp+2))}^{(-1/(\wp+1))}\left(\frac{(\wp+2)x^{\wp+1}}{(\wp+1)}\right)\right\}, \tag{3.7}$$

where U is the confluent hypergeometric function of the second type and L is the Laguerre polynomial.

Proof Equation (3.5) indicates the structure of the Sturm–Liouville equation. Thus we obtain the conclusion

$$\frac{d}{dx}\left(e^{((2+3\wp)x^{1+\wp})/(1+\wp))}y'(x)\right) + 2e^{((2+3\wp)x^{1+\wp})/(1+\wp))}\wp(1+\wp)x^{2\wp}y(x) = 0 \tag{3.8}$$

with the exact and the approximated solutions in (3.6) and (3.7) respectively. □

Example 3.11 Let $\wp = 1$, then Eq. (3.6) becomes the Sturm–Liouville equation

$$\frac{d}{dx}\left(e^{(5x^2)/2}y'(x)\right) + 4e^{(5x^2)/2}x^2 y(x) = 0, \tag{3.9}$$

with the solution (see Fig. 1)

$$y(x) = e^{-2x^2}\left\{c_1 H_{-4/3}\left(\sqrt{\frac{3}{2}}x\right) + c_2\,{}_1F_1\left(\frac{2}{3}; \frac{1}{2}; \frac{3x^2}{2}\right)\right\},$$

where $H_n(\chi)$ is the Hermite polynomial and ${}_1F_1$ is the hypergeometric function. It is clear that solution (3.9) is defined at the boundary of $\cup$ (see Fig. 1-left column). That is, the functional $\Re(\Delta^{m\alpha}\varphi(z)) \approx y(x)$, $x \to 1$. Now, by letting $y(0) = 1$, this implies the solution (see Fig. 1-right column)

$$y(x) = \frac{e^{-2x^2}}{4\Gamma(\frac{7}{6})}\left\{4c_1\Gamma\left(\frac{7}{6}\right)H_{-4/3}\left(\sqrt{\frac{3}{2}}x\right) - \left(2^{2/3}\sqrt{\pi}c_1 - 4\Gamma\left(\frac{7}{6}\right)\right){}_1F_1\left(\frac{2}{3}; \frac{1}{2}; \frac{3x^2}{2}\right)\right\}.$$

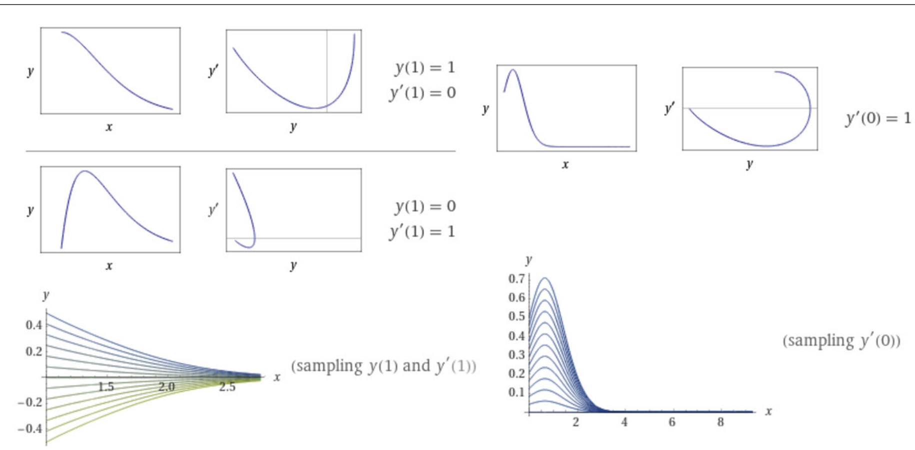

Figure 1 The solution of (3.9) for $\wp = 1$

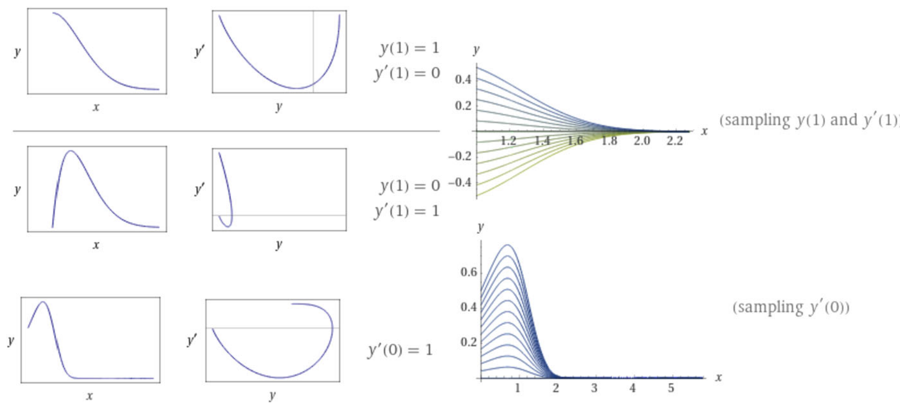

Figure 2 The solution of (3.10) for $\wp = 2$

Example 3.12 Let $\wp = 2$, then Eq. (3.6) becomes the Sturm–Liouville equation

$$\frac{d}{dx}\left(e^{(8x^3)/3}y'(x)\right) + 12e^{(8x^3)/3}x^4y(x) = 0, \tag{3.10}$$

with the solution approximating the boundary of $\cup$ (see Fig. 2-first row)

$$y(x) = c_1e^{-(2x^3)/3}x + \frac{2^{2/3}c_2e^{(-(2x^3)/3}(x^3)^{1/3}\Gamma(\frac{-1}{3}, \frac{4x^3}{3})}{3^{1/3}}.$$

Moreover, the solution, when $y(0) = 1$, is given by the formula (see Fig. 2-second row)

$$y(x) = \frac{1}{9}e^{-(2x^3)/3}\left(c_1x + 6^{2/3}\left(x^3\right)^{1/3}\Gamma\left(\frac{-1}{3}, \frac{4x^3}{3}\right)\right).$$

Proposition 3.13 *If*

$$\Re\left(\Phi(z) + z\Phi'(z)\right) > 0, \quad z \in U, \tag{3.11}$$

then the equation

$$2(1+\wp)xy + \left(\frac{3(1+\wp)-1}{\wp}\right)x^{1-\wp}y' + \left(\frac{1}{\wp}\right)x^{1-2\wp}y'' = \Bbbk, \quad \Bbbk > 0 \tag{3.12}$$

admits a positive solution.

Proof By condition (3.11) and Lemma 2.5 (the first part), we obtain that $\Re(\Phi) > 0$. This leads to

$$\Re\left(\Delta^{m\alpha}\varphi(z)\right) = y(x), \quad \varsigma \to 0.$$

Hence, Eq. (3.12) has a positive solution. □

4 Conclusion

From what has been presented above, it is apparent that we formulated a new differential symmetric operator (DSO) associated with a class of meromorphically multivalent functions. We presented some outcomes covering the geometric studies of the suggested operator joining the Janowski function in the open unit disk. Our consequences indicated, under some conditions, that the proposed operator converges to the Janowski function. Moreover, we discussed the functional $\Phi(z) + z\Phi'(z)$ and the solution for real cases when $\wp = 1$ and $\wp = 2$

$$\Re\big(\Phi(z) + z\Phi'(z)\big) = 0.$$

We discovered that the real cases are converging to the Sturm–Liouville equation, and the solutions are found to be a combination of special functions. We presented the condition that gives (Theorem 3.3)

$$\Phi(z) \prec \left(\frac{1+z}{1-z}\right)^{\lambda_2}$$

for $\lambda_2 > 0$.

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Author details
[1]IEEE, 94086547, Kuala Lumpur, 59200, Malaysia. [2]Department of Mathematics and Statistics, College of Science, IMSIU (Imam Mohammad Ibn Saud Islamic University), P.O. Box 90950, Riyadh 11623, Saudi Arabia.

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