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On interpolative F-contractions with shrink map

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Abstract

In this article, we introduce two notions of interpolative F-contractions with shrink map and F-contractions with shrink map. We also study the existence of E-fixed points by using these notations on a metric space endowed with a binary relation. As an application and consequence of the main results, we also get some other interesting results like a common fixed point result, an E-fixed point result on a metric space equipped with graph, and an existence theorem for a solution of integral equations.

MSC: 47H10; 54H25

Keywords: E-fixed points; Interpolative Kannan contraction; Interpolative F-contraction with shrink map

1 Introduction and preliminaries

The concept of interpolative Kannan contraction mapping was derived by Karapınar [1] to redefine the famous Kannan contraction mapping in the following way: A mapping $V:(K,d_K)\to(K,d_K)$ is an interpolative Kannan contraction [1] if

$$d_K(Vk,Vl)\le\gamma\bigl[d_K(k,Vk)\bigr]^{\tau_1}\bigl[d_K(l,Vl)\bigr]^{1-\tau_1}$$

for all $k,l\in K$ with $k\neq Vk$, where $\gamma\in[0,1)$ and $\tau_1\in(0,1)$.

After that, Karapınar, Agarwal, and Aydi [2] refined the above inequality as

$$d_K(Vk,Vl)\le\gamma\bigl[d_K(k,Vk)\bigr]^{\tau_1}\bigl[d_K(l,Vl)\bigr]^{1-\tau_1}$$

for all $k,l\in K\setminus\operatorname{Fix}(V)$, where $\gamma\in[0,1)$, $\tau_1\in(0,1)$ and $\operatorname{Fix}(V)=\{k\in K:Vk=k\}$.

This concept of interpolative contraction provides a new dimension of study whether existing contraction inequalities can be redefined in this way or not. This concept of Karapınar [1] provoked research in this field, and within a short duration we have seen many new studies related to this topic. For example, Gaba and Karapınar [3] modified the interpolative Kannan contraction by using different exponential values instead of $1-\tau_1$, which were independent of τ_1. Karapınar *et al.* [4] redefined the Hardy–Rogers type contraction by the interpolative Hardy–Rogers type contraction. Reich–Rus–Ćirić type contractions were extended to interpolative Reich–Rus–Ćirić type contractions by Karapınar *et* *al.* [2] and Aydi *et al.* [5]. Karapınar [6] also studied interpolative contractions based on simulation functions. Mohammadi *et al.* [7] studied interpolative Ćirić–Reich–Rus type contractions in the sense of Wardowski [8]. Altun and Tasdemir [9] presented the study of best proximity points using interpolative proximal contraction inequalities. Along with the aforementioned studies, many other interesting studies on fixed point theory are available in [10–23]; they help readers to verify the existence of fixed points for self-mappings and best proximity points for nonself mappings.

Jleli *et al.* [23] introduced the concept of E-fixed point (also called ϕ-fixed point), which states that, for maps $V : K \to K$ and $E : K \to [0,\infty)$, a point $k \in K$ is called E-fixed point of $V : K \to K$ if $V(k) = k$ and $E(k) = 0$, and proved the existence of such points by using a single inequality involving both maps V and E. It is important to note that Jleli *et al.* [23] used the lower semicontinuity of E. This use of the lower semicontinuity of E by Jleli *et al.* [23] arises the question whether the condition of lower semicontinuity of E can be left and some other technique be adopted. In this research article, we try to investigate the results by applying the following conditions to the aforementioned study by Jleli *et al.* [23].

(I) If the condition of lower semicontinuity is not applied to E.

(II) And if two different inequalities are used for V and E instead of a single inequality as used by Jleli *et al.* [23].

As an application and consequence of the main results, we get some other interesting results like a common fixed point result, an E-fixed point result on a metric space equipped with graph, and an existence theorem for a solution of integral equations.

In the next section, we use the family of functions given by Wardowski [8]. Ω is a set of functions $F : (0,\infty) \to \mathbb{R}$ with the following properties:

(F1) $k_1 < k_2 \Leftrightarrow F(k_1) < F(k_2)$ $\forall k_1, k_2 \in (0,\infty)$;

(F2) for each $\{k_n : k_n > 0\}$, we get $\lim_{n\to\infty} k_n = 0$ if and only if $\lim_{n\to\infty} F(k_n) = -\infty$;

(F3) there exists $t \in (0,1)$ with $\lim_{k\to 0^+} (k)^t F(k) = 0$.

2 Main results

In this section we assume that (K, d_K) is a metric space equipped with a binary relation R_B and $V : K \to K, E : K \to [0,\infty)$ are two maps. The set of all natural numbers is represented by $\mathbb{N}$, and the set of all whole numbers is represented by $\mathbb{W}$.

Definition 2.1 A map V is called interpolative F-contraction with E shrink if the following inequalities exist:

$$\Lambda + F\big(d_K(Vk, Vz)\big) \le \tau_1 F\big(d_K(k,z)\big) + \tau_2 F\big(d_K(k, Vk)\big) + \tau_3 F\big(d_K(z, Vz)\big) \tag{2.1}$$

for each $k, z \in K$ with kR_Bz and

$$\min\big\{d_K(Vk, Vz), d_K(k, Vk), d_K(z, Vz)\big\} > 0,$$

where $F \in \Omega$, $\Lambda > 0$ and $\tau_1, \tau_2, \tau_3 \in (0,1)$ with $\tau_1 + \tau_2 + \tau_3 = 1$;

for each $k \in K$, we have

$$E(Vk) \le \lambda E(k), \tag{2.2}$$

where $\lambda \in [0,1)$.

Note that (K, d_K) equipped with a binary relation R_B is called complete in the sense of R_B if, for each sequence $\{k_n\}$ in K with $\lim_{n,m\to\infty} d_K(k_n, k_m) = 0$ and $k_n R_B k_{n+1}$ $\forall n \in \mathbb{N}$, we have $k_b \in K$ with $\lim_{n\to\infty} d_K(k_n, k_b) = 0$ and $k_n R_B k_b$ $\forall n \geq n_0$ (for some $n_0 \in \mathbb{N}$).

Now we are in a position to state and prove the first result.

Theorem 2.2 *Consider V to be an interpolative F-contraction with E shrink on a metric space (K, d_K) equipped with a binary relation R_B. Also, consider that*

(a) *there exist $k_0 \in K$ and $k_1 = Vk_0$ such that $k_0 R_B k_1$;*
(b) *if $k, l \in K$ with $kR_B l$, then $VkR_B Vl$, that is, V is R_B-preserving;*
(c) *(K, d_K) is complete in the sense of R_B;*
(d) *either V is continuous or F is continuous.*

Then there exists a point $b \in K$ with $Vb = b$ and $Eb = 0$.

Proof The existence of axiom (a) implies $k_0 \in K$ and $k_1 = Vk_0$ such that $k_0 R_B k_1$. Then, by repeated application of axiom (b), we say that $V^n k_0 R_B V^{n+1} k_0$ $\forall n \in \mathbb{N}$. Consider $k_n = V^n k_0$ $\forall n \in \mathbb{W}$, then $k_n R_B k_{n+1}$ $\forall n \in \mathbb{W}$. Suppose that

$$\min\bigl\{d_K(Vk_{n_0}, Vk_{n_0+1}), d_K(k_{n_0}, Vk_{n_0}), d_K(k_{n_0+1}, Vk_{n_0+1})\bigr\} = 0 \quad \text{for some } n_0 \in \mathbb{W},$$

then either k_{n_0} or k_{n_0+1} is a fixed point of V. Say, it is k_{n_0}; by (2.2), we obtain $E(k_{n_0}) = E(Vk_{n_0}) \leq \lambda E(k_{n_0})$, that is, $E(k_{n_0}) = 0$. Hence, the conclusion is proved. Now assume that

$$\min\bigl\{d_K(Vk_n, Vk_{n+1}), d_K(k_n, Vk_n), d_K(k_{n+1}, Vk_{n+1})\bigr\} > 0 \quad \forall n \in \mathbb{W},$$

then (2.1) implies

$$\begin{aligned} \Lambda + F\bigl(d_K(Vk_n, Vk_{n+1})\bigr) &\leq \tau_1 F\bigl(d_K(k_n, k_{n+1})\bigr) + \tau_2 F\bigl(d_K(k_n, Vk_n)\bigr) \\ &\quad + \tau_3 F\bigl(d_K(k_{n+1}, Vk_{n+1})\bigr) \quad \forall n \in \mathbb{W}. \end{aligned} \tag{2.3}$$

That is,

$$\begin{aligned} \Lambda + F\bigl(d_K(k_{n+1}, k_{n+2})\bigr) &\leq \tau_1 F\bigl(d_K(k_n, k_{n+1})\bigr) + \tau_2 F\bigl(d_K(k_n, k_{n+1})\bigr) \\ &\quad + \tau_3 F\bigl(d_K(k_{n+1}, k_{n+2})\bigr) \quad \forall n \in \mathbb{W}. \end{aligned} \tag{2.4}$$

Here, we claim $d_K(k_{n+1}, k_{n+2}) < d_K(k_n, k_{n+1})$ for all $n \in \mathbb{W}$. Assume in contrast that there is $n_* \in W$ with $d_K(k_{n_*+1}, k_{n_*+2}) \geq d_K(k_{n_*}, k_{n_*+1})$. By keeping this fact and increasing natural of F, (2.4) becomes

$$\begin{aligned} \Lambda + F\bigl(d_K(k_{n_*+1}, k_{n_*+2})\bigr) &\leq \tau_1 F\bigl(d_K(k_{n_*}, k_{n_*+1})\bigr) + \tau_2 F\bigl(d_K(k_{n_*}, k_{n_*+1})\bigr) \\ &\quad + \tau_3 F\bigl(d_K(k_{n_*+1}, k_{n_*+2})\bigr) \\ &\leq \tau_1 F\bigl(d_K(k_{n_*+1}, k_{n_*+2})\bigr) + \tau_2 F\bigl(d_K(k_{n_*+1}, k_{n_*+2})\bigr) \\ &\quad + \tau_3 F\bigl(d_K(k_{n_*+1}, k_{n_*+2})\bigr) \\ &= F\bigl(d_K(k_{n_*+1}, k_{n_*+2})\bigr), \end{aligned} \tag{2.5}$$

which is not possible. Hence, the claim is true, that is, $d_K(k_{n+1},k_{n+2}) < d_K(k_n,k_{n+1})$ for all $n \in \mathbb{W}$. Hence, by (2.4), we obtain

$$\begin{aligned}\Lambda + F\bigl(d_K(k_{n+1},k_{n+2})\bigr) &\le \tau_1 F\bigl(d_K(k_n,k_{n+1})\bigr) + \tau_2 F\bigl(d_K(k_n,k_{n+1})\bigr)\\ &\quad + \tau_3 F\bigl(d_K(k_{n+1},k_{n+2})\bigr)\\ &\le F\bigl(d_K(k_n,k_{n+1})\bigr) \quad \forall n \in \mathbb{W}.\end{aligned} \tag{2.6}$$

This gives

$$F\bigl(d_K(k_{n+1},k_{n+2})\bigr) \le F\bigl(d_K(k_0,k_1)\bigr) - (n+1)\Lambda \quad \forall n \in \mathbb{W}. \tag{2.7}$$

From (2.7), we get $\lim_{n\to\infty} F(d_K(k_{n+1},k_{n+2})) = -\infty$. This implies $\lim_{n\to\infty} d_K(k_{n+1},k_{n+2}) = 0$ due to (F2). Setting $d_K(k_{n+1},k_{n+2}) = \xi_{n+1}$, thus, $\lim_{n\to\infty} \xi_{n+1} = 0$. Condition (F3) ensures that there is $t \in (0,1)$ such that $\lim_{n\to\infty} \xi_{n+1}^t F(\xi_{n+1}) = 0$. From (2.7), we obtain

$$\xi_{n+1}^t F(\xi_{n+1}) - \xi_{n+1}^t F(\xi_0) \le -\xi_{n+1}^t (n+1)\Lambda < 0 \quad \forall n \in \mathbb{W}. \tag{2.8}$$

Then $\lim_{n\to\infty} \xi_{n+1}^t (n+1) = 0$. Thus, there exists $N_* \in \mathbb{N}$ with $\xi_{n+1} \le \frac{1}{(n+1)^{\frac{1}{t}}}$ $\forall n \ge N_*$. For every $m,n \in \mathbb{N}$ with $m > n$, we obtain

$$\begin{aligned}d_K(k_n,k_m) &\le d_K(k_n,k_{n+1}) + d_K(k_{n+1},k_{n+2}) + \cdots + d_K(k_{m-1},k_m)\\ &\le \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{t}}}.\end{aligned}$$

Thus, $\lim_{n,m\to\infty} d_K(k_n,k_m) = 0$, since $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{t}}}$ is convergent. Hence, we get a Cauchy $\{k_n\}$ with $k_n R_B k_{n+1} \forall n \in \mathbb{W}$. As we have already assumed in the statement of the theorem that (K,d_K) is complete in the sense of R_B, thus, there is $k_* \in K$ with $k_n \to k_*$ and $k_n R_B k_*$ for each $n > m_0$ (for some m_0).

Suppose that V is continuous with respect to d_K, and we know $\lim_{n\to\infty} d_K(k_n,k_*) = 0$, thus, $0 = \lim_{n\to\infty} d_K(Vk_n, Vk_*) = \lim_{n\to\infty} d_K(k_{n+1}, Vk_*) = d_K(k_*, Vk_*)$. Hence, we conclude that $Vk_* = k_*$.

Suppose that F is continuous. Here, we claim $Vk_* = k_*$. Suppose, in contrast, that it is wrong. Then we conclude that

$$\min\bigl\{d_K(Vk_n, Vk_*), d_K(k_n, Vk_n), d_K(k_*, Vk_*)\bigr\} > 0 \quad \forall n \ge m_a \in \mathbb{N}.$$

As $k_n R_B k_*$ for each $n > m_0$, by (2.1), we get

$$\begin{aligned}\Lambda + F\bigl(d_K(Vk_n, Vk_*)\bigr) &\le \tau_1 F\bigl(d_K(k_n,k_*)\bigr) + \tau_2 F\bigl(d_K(k_n, Vk_n)\bigr)\\ &\quad + \tau_3 F\bigl(d_K(k_*, Vk_*)\bigr) \quad \forall n > m_b = \max\{m_a, m_0\}.\end{aligned} \tag{2.9}$$

As $\lim_{n\to\infty} d_K(k_n,k_*) = 0$, $\lim_{n\to\infty} d_K(k_n,k_{n+1}) = 0$, and $d_K(k_*, Vk_*) \neq 0$, thus, there is $N_0 \in \mathbb{N}$ with

$$\max\bigl\{d_K(k_n,k_*), d_K(k_n,k_{n+1}), d_K(k_*, Vk_*)\bigr\} = d_K(k_*, Vk_*) \quad \forall n \ge N_0.$$

Using it in (2.9), we obtain

$$\Lambda + F\big(d_K(k_{n+1}, Vk_*)\big) \leq (\tau_1 + \tau_2 + \tau_3) F\big(d_K(k_*, Vk_*)\big) \quad \forall n > \max\{m_b, N_0\}. \tag{2.10}$$

Assuming that $n \to \infty$ and using the continuity of F in (2.10), we obtain

$$\Lambda + F\big(d_K(k_*, Vk_*)\big) \leq (\tau_1 + \tau_2 + \tau_3) F\big(d_K(k_*, Vk_*)\big),$$

which is not possible for $d_K(k_*, Vk_*) \neq 0$, that is, a contradiction to our assumption. Hence, the claim is true, that is, $k_* = Vk_*$. Consider $k = k_*$ in (2.2), we get $E(Vk_*) \leq \lambda E(k_*)$, that is, $(1 - \lambda)E(k_*) \leq 0$. Hence, $E(k_*) = 0$. □

Now we present the notion of F-contraction with shrink map in the following way.

Definition 2.3 A map V is called F-contraction with E shrink if the following inequalities exist:

$$\Lambda + F\big(d_K(Vk, Vz)\big) \leq F\big(\tau_1 d_K(k, z) + \tau_2 d_K(k, Vk) + \tau_3 d_K(z, Vz)\big) \tag{2.11}$$

for each $k, z \in K$ with $kR_B z$ and $Vk \neq Vz$, where $F \in \Omega$, $\Lambda > 0$ and $\tau_1, \tau_2, \tau_3 \in [0, 1]$ with $\tau_1 + \tau_2 + \tau_3 = 1$;

for each $k \in K$, we have

$$E(Vk) \leq \lambda E(k), \tag{2.12}$$

where $\lambda \in [0, 1)$.

Now we shall discuss the second result of this section.

Theorem 2.4 *Consider V to be an F-contraction with E shrink on a metric space (K, d_K) with a binary relation R_B. Also, consider that*

(a) *there exist $k_0 \in K$ and $k_1 = Vk_0$ such that $k_0 R_B k_1$;*
(b) *if $k, l \in K$ with $kR_B l$, then $VkR_B Vl$, that is, V is R_B-preserving;*
(c) *(K, d_K) is complete in the sense of R_B;*
(d) *either V is continuous or F is continuous.*

Then there exists a point $b \in K$ with $Vb = b$ and $Eb = 0$.

Proof Axiom (a) says that $k_0 \in K$ and $k_1 = Vk_0$ satisfy $k_0 R_B k_1$. From axiom (b), we conclude that $V^n k_0 R_B V^{n+1} k_0$ $\forall n \in \mathbb{N}$. Letting $k_n = V^n k_0$ $\forall n \in \mathbb{W}$, we get $k_n R_B k_{n+1}$ $\forall n \in \mathbb{W}$. Suppose that there is some $n_0 \in \mathbb{W}$ with $k_{n_0} = k_{n_0+1} = Vk_{n_0}$, then k_{n_0} is a fixed point of V. From (2.12), we obtain $E(k_{n_0}) = E(Vk_{n_0}) \leq \lambda E(k_{n_0})$, that is, $E(k_{n_0}) = 0$. Hence, the conclusion is proved. Now assume that $d_K(Vk_n, Vk_{n+1}) > 0$ $\forall n \in \mathbb{W}$, then (2.11) implies

$$\begin{aligned} \Lambda + F\big(d_K(Vk_n, Vk_{n+1})\big) \leq{}& F\big(\tau_1 d_K(k_n, k_{n+1}) + \tau_2 d_K(k_n, Vk_n) \\ & + \tau_3 d_K(k_{n+1}, Vk_{n+1})\big) \quad \forall n \in \mathbb{W}. \end{aligned} \tag{2.13}$$

That is,

$$\Lambda + F\big(d_K(k_{n+1},k_{n+2})\big) \le F\big(\tau_1 d_K(k_n,k_{n+1}) + \tau_2 d_K(k_n,k_{n+1}) + \tau_3 d_K(k_{n+1},k_{n+2})\big) \quad \forall n \in \mathbb{W}. \tag{2.14}$$

Here, we claim $d_K(k_{n+1},k_{n+2}) < d_K(k_n,k_{n+1})$ $\forall n \in \mathbb{W}$. Suppose, in contrast, that there is $n_* \in W$ with $d_K(k_{n_*+1},k_{n_*+2}) \ge d_K(k_{n_*},k_{n_*+1})$. By this fact and increasing natural of F, (2.14) becomes

$$\begin{aligned}\Lambda + F\big(d_K(k_{n_*+1},k_{n_*+2})\big) &\le F\big(\tau_1 d_K(k_{n_*},k_{n_*+1}) + \tau_2 d_K(k_{n_*},k_{n_*+1}) \\ &\quad + \tau_3 d_K(k_{n_*+1},k_{n_*+2})\big) \\ &\le F\big(\tau_1 d_K(k_{n_*+1},k_{n_*+2}) + \tau_2 d_K(k_{n_*+1},k_{n_*+2}) \\ &\quad + \tau_3 d_K(k_{n_*+1},k_{n_*+2})\big) \\ &= F\big(d_K(k_{n_*+1},k_{n_*+2})\big),\end{aligned} \tag{2.15}$$

which is not possible. Hence, the claim is valid, that is, $d_K(k_{n+1},k_{n+2}) < d_K(k_n,k_{n+1})$ $\forall n \in \mathbb{W}$. Hence, by (2.14), we obtain

$$\begin{aligned}\Lambda + F\big(d_K(k_{n+1},k_{n+2})\big) &\le F\big(\tau_1 d_K(k_n,k_{n+1}) + \tau_2 d_K(k_n,k_{n+1}) \\ &\quad + \tau_3 d_K(k_{n+1},k_{n+2})\big) \\ &\le F\big(d_K(k_n,k_{n+1})\big) \quad \forall n \in \mathbb{W}.\end{aligned} \tag{2.16}$$

This implies

$$F\big(d_K(k_{n+1},k_{n+2})\big) \le F\big(d_K(k_0,k_1)\big) - (n+1)\Lambda \quad \forall n \in \mathbb{W}. \tag{2.17}$$

By (2.17), we get $\lim_{n\to\infty} F(d_K(k_{n+1},k_{n+2})) = -\infty$. This implies $\lim_{n\to\infty} d_K(k_{n+1},k_{n+2}) = 0$ from (F2). Let $d_K(k_{n+1},k_{n+2}) = \xi_{n+1}$, thus, $\lim_{n\to\infty} \xi_{n+1} = 0$. (F3) ensures that there is $t \in (0,1)$ with $\lim_{n\to\infty} \xi_{n+1}^t F(\xi_{n+1}) = 0$. By (2.17), we get

$$\xi_{n+1}^t F(\xi_{n+1}) - \xi_{n+1}^t F(\xi_0) \le -\xi_{n+1}^t (n+1)\Lambda < 0 \quad \forall n \in \mathbb{W}. \tag{2.18}$$

Thus, $\lim_{n\to\infty} \xi_{n+1}^t (n+1) = 0$. Then there exists $N_* \in \mathbb{N}$ with $\xi_{n+1} \le \frac{1}{(n+1)^{\frac{1}{t}}}$ $\forall n \ge N_*$. By the triangle inequality, for each $m,n \in \mathbb{N}$ with $m > n$, we get

$$\begin{aligned}d_K(k_n,k_m) &\le d_K(k_n,k_{n+1}) + d_K(k_{n+1},k_{n+2}) + \cdots + d_K(k_{m-1},k_m) \\ &\le \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{t}}}.\end{aligned}$$

Thus, $\lim_{n,m\to\infty} d_K(k_n,k_m) = 0$, since $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{t}}}$ is convergent. Hence, we obtain a Cauchy sequence $\{k_n\}$ with $k_n R_B k_{n+1}$ $\forall n \in \mathbb{W}$. As (K,d_K) is complete in the sense of R_B, there is $k_* \in K$ with $k_n \to k_*$ and $k_n R_B k_*$ for each $n > m_0$ (for some m_0).

Suppose that V is continuous with respect to d_K, then it is trivial that $Vk_* = k_*$.

Suppose that F is continuous. Here, we claim $Vk_* = k_*$. Assume in contrast that it is wrong. Then we have $d_K(Vk_n, Vk_*) > 0\ \forall n \geq m_a$. As $k_n R_B k_*$ for each $n > m_b = \max\{m_a, m_0\}$, by (2.11), we obtain

$$\Lambda + F\bigl(d_K(Vk_n, Vk_*)\bigr) \leq F\bigl(\tau_1 d_K(k_n, k_*) + \tau_2 d_K(k_n, Vk_n) + \tau_3 d_K(k_*, Vk_*)\bigr) \quad \forall n > m_b. \tag{2.19}$$

As $\lim_{n\to\infty} d_K(k_n, k_*) = 0$, $\lim_{n\to\infty} d_K(k_n, k_{n+1}) = 0$, and $d_K(k_*, Vk_*) \neq 0$, thus, there is $N_0 \in \mathbb{N}$ with

$$\max\bigl\{d_K(k_n, k_*), d_K(k_n, k_{n+1}), d_K(k_*, Vk_*)\bigr\} = d_K(k_*, Vk_*) \quad \forall n \geq N_0.$$

By this fact and (2.19), we get

$$\Lambda + F\bigl(d_K(k_{n+1}, Vk_*)\bigr) \leq F\bigl((\tau_1 + \tau_2 + \tau_3) d_K(k_*, Vk_*)\bigr) \quad \forall n > \max\{m_b, N_0\}. \tag{2.20}$$

Let $n \to \infty$ in (2.20), by the continuity of F, we get

$$\Lambda + F\bigl(d_K(k_*, Vk_*)\bigr) \leq F\bigl((\tau_1 + \tau_2 + \tau_3) d_K(k_*, Vk_*)\bigr),$$

which is not possible if $d_K(k_*, Vk_*) \neq 0$, that is, a contradiction to our assumption. Hence, the claim is true, that is, $k_* = Vk_*$. Consider $k = k_*$ in (2.12), we get $E(Vk_*) \leq \lambda E(k_*)$, that is, $(1 - \lambda)E(k_*) \leq 0$. Hence, $E(k_*) = 0$. □

3 Consequences

In this section, we list some results that are the consequences of the main results.

3.1 Common fixed point result

As a consequence of the above results, we get the following common fixed point result by defining $E(k) = d_K(k, Sk)\ \forall k \in K$, where $S : K \to K$ is any map.

Corollary 3.1 *Consider a metric space (K, d_K) equipped with a binary relation R_B, and consider two maps $V, S : K \to K$ that satisfy the following two inequalities:*

(1) *either*

$$\Lambda + F\bigl(d_K(Vk, Vz)\bigr) \leq \tau_1 F\bigl(d_K(k, z)\bigr) + \tau_2 F\bigl(d_K(k, Vk)\bigr) + \tau_3 F\bigl(d_K(z, Vz)\bigr) \tag{3.1}$$

for each $k, z \in K$ with $kR_B z$ and

$$\min\bigl\{d_K(Vk, Vz), d_K(k, Vk), d_K(z, Vz)\bigr\} > 0,$$

where $F \in \Omega$, $\Lambda > 0$ and $\tau_1, \tau_2, \tau_3 \in (0, 1)$ with $\tau_1 + \tau_2 + \tau_3 = 1$,

or

$$\Lambda + F\bigl(d_K(Vk, Vz)\bigr) \leq F\bigl(\tau_1 d_K(k, z) + \tau_2 d_K(k, Vk) + \tau_3 d_K(z, Vz)\bigr) \tag{3.2}$$

for each $k, z \in K$ with $kR_B z$ and $Vk \neq Vz$, where $F \in \Omega$, $\Lambda > 0$ and $\tau_1, \tau_2, \tau_3 \in [0, 1]$ with $\tau_1 + \tau_2 + \tau_3 = 1$.

(2)

$$d_K(Vk, SVk) \leq \lambda d_K(k, Sk) \quad \forall k \in K, \tag{3.3}$$

where $\lambda \in [0,1)$. *Also, consider that*

(a) *there exist* $k_0 \in K$ *and* $k_1 = Vk_0$ *such that* $k_0 R_B k_1$;

(b) *if* $k, l \in K$ *with* $kR_B l$, *then* $VkR_B Vl$, *that is,* V *is* R_B*-preserving*;

(c) (K, d_K) *is complete in the sense of* R_B;

(d) *either* V *is continuous or* F *is continuous.*

Then there exists a point $b \in K$ *with* $Vb = b$ *and* $Sb = b$.

3.2 Result on a metric space equipped with graph

Suppose that $G = (V_E, E_D)$ denotes a directed graph with a vertex set $V_E = K$ and an edge set $E_D \subset K \times K$ such that E_D does not contain parallel edges but contains each loop, that is, $(k,k) \in E_D$ $\forall k \in K$.

The following stated result is an explicit consequence of Theorem 2.2 and Theorem 2.4 by defining a binary relation R_B on K as $kR_B l$ if $(k,l) \in E_D$.

Corollary 3.2 *Consider a metric space* (K, d_K) *equipped with the graph* G. *Also, consider maps* $V : K \to K$ *and* $E : K \to [0,\infty)$ *that satisfy the following two inequalities*:

(1) *either*

$$\Lambda + F\big(d_K(Vk, Vz)\big) \leq \tau_1 F\big(d_K(k,z)\big) + \tau_2 F\big(d_K(k, Vk)\big) + \tau_3 F\big(d_K(z, Vz)\big) \tag{3.4}$$

for each $k, z \in K$ *with* $(k,z) \in E_D$ *and*

$$\min\big\{d_K(Vk, Vz), d_K(k, Vk), d_K(z, Vz)\big\} > 0,$$

where $F \in \Omega$, $\Lambda > 0$ *and* $\tau_1, \tau_2, \tau_3 \in (0,1)$ *with* $\tau_1 + \tau_2 + \tau_3 = 1$,

or

$$\Lambda + F\big(d_K(Vk, Vz)\big) \leq F\big(\tau_1 d_K(k,z) + \tau_2 d_K(k, Vk) + \tau_3 d_K(z, Vz)\big) \tag{3.5}$$

for each $k, z \in K$ *with* $(k,z) \in E_D$ *and* $Vk \neq Vz$, *where* $F \in \Omega$, $\Lambda > 0$ *and* $\tau_1, \tau_2, \tau_3 \in [0,1]$ *with* $\tau_1 + \tau_2 + \tau_3 = 1$.

(2)

$$E(Vk) \leq \lambda E(k) \quad \forall k \in K, \tag{3.6}$$

where $\lambda \in [0,1)$. *Also, consider that*

(a) *there exist* $k_0 \in K$ *and* $k_1 = Vk_0$ *such that* $(k_0, k_1) \in E_D$;

(b) *if* $k, l \in K$ *with* $(k,l) \in E_D$, *then* $(Vk, Vl) \in E_D$, *that is,* V *is* E_D*-preserving*;

(c) (K, d_K) *is complete in the sense of* G, *that is, for each sequence* $\{k_n\}$ *with* $\lim_{n,m\to\infty} d_K(k_n, k_m) = 0$ *and* $(k_n, k_{n+1}) \in E_D$ $\forall n \in \mathbb{N}$, *we have* $k_b \in K$ *with* $\lim_{n\to\infty} d_K(k_n, k_b) = 0$ *and* $(k_n, k_b) \in E_D$ $\forall n \geq n_0$ (*for some* $n_0 \in \mathbb{N}$);

(d) *either* V *is continuous or* F *is continuous.*

Then there exists a point $b \in K$ *with* $Vb = b$ *and* $Eb = 0$.

3.3 Result on a metric space with a function α

The result stated below has been obtained from Theorems 2.2 and 2.4 by taking a binary relation R_B on K as kR_Bl if $\alpha(k,l) \geq 1$, where $\alpha : K \times K \to [0,\infty)$ is a function.

Corollary 3.3 *Consider a metric space (K,d_K) and a function $\alpha : K \times K \to [0,\infty)$. Also, consider the maps $V : K \to K$ and $E : K \to [0,\infty)$ that satisfy the following two inequalities*:

(1) *either*

$$\Lambda + F\big(d_K(Vk,Vz)\big) \leq \tau_1 F\big(d_K(k,z)\big) + \tau_2 F\big(d_K(k,Vk)\big) + \tau_3 F\big(d_K(z,Vz)\big) \tag{3.7}$$

for each $k,z \in K$ with $\alpha(k,z) \geq 1$ and

$$\min\big\{d_K(Vk,Vz),d_K(k,Vk),d_K(z,Vz)\big\} > 0$$

where $F \in \Omega$, $\Lambda > 0$ and $\tau_1,\tau_2,\tau_3 \in (0,1)$ with $\tau_1 + \tau_2 + \tau_3 = 1$,
or

$$\Lambda + F\big(d_K(Vk,Vz)\big) \leq F\big(\tau_1 d_K(k,z) + \tau_2 d_K(k,Vk) + \tau_3 d_K(z,Vz)\big) \tag{3.8}$$

for each $k,z \in K$ with $\alpha(k,z) \geq 1$ and $Vk \neq Vz$, where $F \in \Omega$, $\Lambda > 0$ and $\tau_1,\tau_2,\tau_3 \in [0,1]$ with $\tau_1 + \tau_2 + \tau_3 = 1$.

(2)

$$E(Vk) \leq \lambda E(k) \quad \forall k \in K, \tag{3.9}$$

where $\lambda \in [0,1)$. Also, consider that

(a) *there exist $k_0 \in K$ and $k_1 = Vk_0$ such that $\alpha(k_0,k_1) \geq 1$;*
(b) *if $k,l \in K$ with $\alpha(k,l) \geq 1$, then $\alpha(Vk,Vl) \geq 1$, that is, V is α-preserving/admissible;*
(c) *(K,d_K) is complete in the sense of α, that is, for each sequence $\{k_n\}$ with $\lim_{n,m\to\infty} d_K(k_n,k_m) = 0$ and $\alpha(k_n,k_{n+1}) \geq 1$ $\forall n \in \mathbb{N}$, we have $k_b \in K$ with $\lim_{n\to\infty} d_K(k_n,k_b) = 0$ and $\alpha(k_n,k_b) \geq 1$ $\forall n \geq n_0$ (for some $n_0 \in \mathbb{N}$);*
(d) *either V is continuous, or F is continuous.*

Then there exists a point $b \in K$ with $Vb = b$ and $Eb = 0$.

4 Application and examples

Consider an integral equation

$$k(w) = h(w) + \mu \int_0^{f(w)} M\big(w,q,k(q)\big)\,dq, \quad w \in I_N = [0,\infty), \tag{4.1}$$

where μ is constant, $h : I_N \to \mathbb{R}$, $f : I_N \to \mathbb{R}^+ = [0,\infty)$, and $M : I_N \times I_N \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Also, consider $K = C(I_N,\mathbb{R})$ to be the set that contains all real-valued continuous functions with the domain set I_N and $d_K(k,l) = \max_{w\in I_N} |k(w) - l(w)| = \|k - l\|$.

Theorem 4.1 *Consider* $K = C(I_N,\mathbb{R})$ *and consider an operator* $V : K \to K$ *defined by*

$$Vk(w) = h(w) + \mu \int_0^{f(w)} M\bigl(w,q,k(q)\bigr)\,dq, \quad w \in I_N = [0,\infty), \tag{4.2}$$

where μ *is constant,* $h : I_N \to \mathbb{R}$, $f : I_N \to \mathbb{R}^+ = [0,\infty)$, *and* $M : I_N \times I_N \times \mathbb{R} \to \mathbb{R}$ *are continuous functions. Also, assume that there are* $\Lambda > 0$ *and* $\tau_1,\tau_2,\tau_3 \in (0,1)$ *with* $\tau_1 + \tau_2 + \tau_3 = 1$ *satisfying*

$$\bigl|M\bigl(w,q,k(q)\bigr) - M\bigl(w,q,l(q)\bigr)\bigr| \le \frac{|k(q)-l(q)|}{[\Lambda\sqrt{\|k-l\|} + \tau_1 + \tau_2\sqrt{\frac{\|k-l\|}{\|k-Vk\|}} + \tau_3\sqrt{\frac{\|k-l\|}{\|l-Vl\|}}]^2} \tag{4.3}$$

for all $w,q \in I_N$ *and for each* $k,l \in K$ *with* $\min\{\|Vk-Vl\|,\|k-Vk\|,\|l-Vl\|\} > 0$; *moreover,*

$$\sup_{w\in I_N} \int_0^{f(w)} dq \le \frac{1}{|\mu|}.$$

Then (4.1) *possesses a solution.*

Proof By (4.2) and (4.3), for each $k,l \in K$ with $\min\{\|Vk-Vl\|,\|k-Vk\|,\|l-Vl\|\} > 0$, we reach

$$\begin{aligned}
\bigl|Vk(w) - Vl(w)\bigr| &\le |\mu| \int_0^{f(w)} \bigl|M\bigl(w,q,k(q)\bigr) - M\bigl(w,q,l(q)\bigr)\bigr|\,dq \\
&\le |\mu| \int_0^{f(w)} \frac{|k(q)-l(q)|}{[\Lambda\sqrt{\|k-l\|} + \tau_1 + \tau_2\sqrt{\frac{\|k-l\|}{\|k-Vk\|}} + \tau_3\sqrt{\frac{\|k-l\|}{\|l-Vl\|}}]^2}\,dq \\
&\le \frac{\|k-l\|}{[\Lambda\sqrt{\|k-l\|} + \tau_1 + \tau_2\sqrt{\frac{\|k-l\|}{\|k-Vk\|}} + \tau_3\sqrt{\frac{\|k-l\|}{\|l-Vl\|}}]^2} |\mu| \int_0^{f(w)} dq \\
&\le \frac{\|k-l\|}{[\Lambda\sqrt{\|k-l\|} + \tau_1 + \tau_2\sqrt{\frac{\|k-l\|}{\|k-Vk\|}} + \tau_3\sqrt{\frac{\|k-l\|}{\|l-Vl\|}}]^2} \quad \forall w \in I_N.
\end{aligned}$$

This gives

$$\|Vk - Vl\| \le \frac{\|k-l\|}{[\Lambda\sqrt{\|k-l\|} + \tau_1 + \tau_2\sqrt{\frac{\|k-l\|}{\|k-Vk\|}} + \tau_3\sqrt{\frac{\|k-l\|}{\|l-Vl\|}}]^2}.$$

This implies

$$\begin{aligned}
\frac{1}{\sqrt{\|Vk-Vl\|}} &\ge \frac{[\Lambda\sqrt{\|k-l\|} + \tau_1 + \tau_2\sqrt{\frac{\|k-l\|}{\|k-Vk\|}} + \tau_3\sqrt{\frac{\|k-l\|}{\|l-Vl\|}}]}{\sqrt{\|k-l\|}} \\
&= \Lambda + \frac{\tau_1}{\sqrt{\|k-l\|}} + \frac{\tau_2}{\sqrt{\|k-Vk\|}} + \frac{\tau_3}{\sqrt{\|l-Vl\|}}.
\end{aligned}$$

So, we get

$$\Lambda - \frac{1}{\sqrt{\|Vk-Vl\|}} \le \frac{-\tau_1}{\sqrt{\|k-l\|}} + \frac{-\tau_2}{\sqrt{\|k-Vk\|}} + \frac{-\tau_3}{\sqrt{\|l-Vl\|}}.$$

Hence,

$$\Lambda + F\big(d_K(Vk,Vl)\big) \le \tau_1 F\big(d_K(k,l)\big) + \tau_2 F\big(d_K(k,Vk)\big) + \tau_3 F\big(d_K(l,Vl)\big) \tag{4.4}$$

for each $k,l \in K$ with $\min\{d_K(Vk,Vl), d_K(k,Vk), d_K(l,Vl)\} > 0$, where $F(w) = \frac{-1}{\sqrt{w}}$. By defining a binary relation R_B on K as kR_Bl if $(k,l) \in K \times K$ and $E(k) = 0$ for each $k \in K$, we see that the axioms of Theorem 2.2 become true. Hence, we say that V contains a fixed point in K, that is, (4.1) possesses a solution. □

Illustration by examples

Example 4.2 Consider $K = \mathbb{R}$ and define

$$d_K(k,l) = \begin{cases} 0, & k = l, \\ \max\{|k|,|l|\} + 1, & \text{otherwise}. \end{cases}$$

Define a binary relation R_B on K by kR_Bl if $k,l \in [0,1]$. Define $V : K \to K$ and $E : K \to [0,\infty)$ by

$$V(k) = \begin{cases} 2k, & K \setminus [0,1], \\ 0, & k \in [0,1/3], \\ \frac{1}{9}, & k \in (1/3,1] \end{cases}$$

and

$$E(k) = \begin{cases} k, & k \in [0,1], \\ \frac{1}{|k|}, & \text{otherwise}. \end{cases}$$

In this example, one can see that the axioms of Theorem 2.2 are valid by taking $F(k) = \ln(k)$ and $\Lambda = 0.001$. Thus, there exists a point $b \in K$ such that $Vb = b$ and $Eb = 0$.

Remark 4.3 ([7, Theorem 2]) is not applicable to the above defined d_K and V.

Example 4.4 Consider $K = [0,2]$ and define $d_K(k,l) = |k - l|\ \forall k,l \in K$. Define a binary relation R_B on K by kR_Bl if $k,l \in \{0,1/3,2/3,1,2\}$. Define $V : K \to K$ and $E : K \to [0,\infty)$ by

$$V(k) = \frac{\lceil k - 0.9 \rceil}{3} \quad \forall k \in K$$

and

$$E(k) = \lfloor k \rfloor \quad \forall k \in K.$$

One can verify that the axioms of Theorem 2.4 are valid by taking $F(k) = \ln(k)$ and $\Lambda = 0.0001$, thus there exists a point $b \in K$ such that $Vb = b$ and $Eb = 0$.

Example 4.5 Consider $K = [0,2]$ and define $d_K(k,l) = |k - l|\ \forall k,l \in K$. Define a binary relation R_B on K by kR_Bl if $k,l \in \{0, 1/3, 2/3, 1, 2\}$. Define $V : K \to K$ and $E : K \to [0,\infty)$ by

$$V(k) = \begin{cases} \frac{\lceil k-0.9\rceil}{3}, & k \in \{0, 1/3, 2/3, 1, 2\}, \\ \lceil k - 0.9\rceil, & \text{otherwise} \end{cases}$$

and

$$E(k) = 0 \quad \forall k \in K.$$

One can verify that the axioms of Theorem 2.4 are valid by taking $F(k) = \ln(k)$ and $\Lambda = 0.0001$, thus there exists a point $b \in K$ such that $Vb = b$ and $Eb = 0$.

Remark 4.6 Note that for the function V defined in the last example inequality (2.11) is valid only for $k,l \in K$ with kR_Bl.

5 Conclusion

The notions of interpolative F-contractions with shrink map and F-contractions with shrink map have been defined and used to study the existence of E-fixed points on a metric space endowed with a binary relation. As an application and consequence of the main results, we have obtained a common fixed point result, an E-fixed point result on a metric space with graph, and an existence theorem for a solution of integral equations.

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