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Existence results for a general class of sequential hybrid fractional differential equations

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Abstract

In this paper, we study a class of nonlinear boundary value problems (BVPs) consisting of a more general class of sequential hybrid fractional differential equations (SHFDEs) together with a class of nonlinear boundary conditions at both end points of the domain. The nonlinear functions involved depend explicitly on the fractional derivatives. We study the necessary conditions required for the unique solution to the suggested BVP under the Caratheodory conditions using the technique of measure of noncompactness and degree theory. We also develop conditions for uniqueness results and also on stability analysis.

MSC: 34A08; 35R11

Keywords: Boundary value problems; Sequential hybrid fractional differential equations; Three point boundary conditions; Existence and uniqueness results; Stability analysis

1 Introduction

The existence theory for solutions of BVPs of hybrid fractional differential equations and SHFDEs has attracted the attention of many researchers, we refer to [1–11] and the references therein for the recent development in this particular area of interest. In most of these studies, BVPs with lower order fractional derivatives together with either constant or linear boundary conditions are considered. However, in many situations, there are possibilities to have nonlinear conditions at the boundary, and the differential equations may be of higher order involving functions that depend explicitly on the fractional order derivatives. For example, in case of head flow problems, there are possibilities to have some source or sink on both sides of the boundary (at $x = 0$ and $x = 1$) which may be nonlinear functions and a controller at $x = \zeta_0$ ($0 < \zeta_0 < 1$). Such situation may have importance in application point of view and also in theoretical development. The purpose of this paper is to investigate existence results for BVPs involving nonlinear boundary conditions at both end points, that is, we study the following class of three point

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BVPs:

$$\begin{aligned}
 {}^c\mathcal{D}^\vartheta \left[\frac{{}^c\mathcal{D}^\omega u(t) - \sum_1^m I^{\beta_i} h_i(t, u(t), \mathcal{D}^{\omega-1} u(t))}{f(t, u(t), \mathcal{D}^{\omega-1} u(t))} \right] &= g(t, u(t), I^\gamma u(t)), \quad t \in I = [0, 1] \\
 {}^c\mathcal{D}^\omega u(0) = 0, \quad u(0) = \psi_1(u(\zeta_0)), \quad u(1) = \psi_2(u(\zeta_0)), &
 \end{aligned}
 \tag{1}$$

where the parameters are such that $0 < \vartheta \leq 1$, $1 < \omega \leq 2$, $0 < \zeta_0 < 1$, the functions $f : I \times \mathcal{R}_e \times \mathcal{R}_e \rightarrow \mathcal{R}_e - \{0\}$, $h_i : I \times \mathcal{R}_e \times \mathcal{R}_e \rightarrow \mathcal{R}_e$ ($i = 1, 2, \dots, m$), and $g : I \times \mathcal{R}_e \times \mathcal{R}_e \rightarrow \mathcal{R}_e$ satisfy the Caratheodory conditions, the boundary functions $\psi_1, \psi_2 : \mathcal{R}_e \rightarrow \mathcal{R}_e$ are non-linear, and \mathcal{R}_e represents the set of real numbers. To the best of our knowledge, existence, uniqueness, and stability results have never been studied for BVP (1) previously.

Choose Ω_0 a bounded subset of a Banach space \mathbb{E} , where $\mathbb{E} = \{u \in C(I) : \mathcal{D}^{\omega-1} u \in C(I)\}$, endowed with the norm $\|u\|_1 = \max_{0 \leq t \leq 1} |u(t)| + \max_{0 \leq t \leq 1} |\mathcal{D}^{\omega-1} u|$. We recall the following definition [12].

2 Preliminaries

Definition 2.1 The Kuratowski measure of noncompactness $\varrho^{**} : \mathbb{S} \rightarrow [0, \infty)$ of a set $\mathbb{S} \subseteq \mathbb{E}$ is defined as

$$\varrho^{**}(\mathbb{S}) = \inf\{d > 0 : \mathbb{S} \in \mathbb{B} \text{ admits a finite cover by sets of diameter } \leq d\},$$

where \mathbb{B} denotes the family of all bounded subsets of \mathbb{E} .

Recall the following definitions and propositions from [13].

Proposition 2.1 The Kuratowski measure ϱ^{**} satisfies the following properties:

- (i) $\varrho^{**}(\mathbb{S}) = 0$ iff \mathbb{S} is relatively compact;
- (ii) ϱ^{**} is a seminorm, i.e., $\varrho^{**}(\lambda^{**}\mathbb{S}) = |\lambda^{**}| \varrho^{**}(\mathbb{S})$, $\lambda^{**} \in \mathbb{R}$, and $\varrho^{**}(\mathbb{S}_1 + \mathbb{S}_2) \leq \varrho^{**}(\mathbb{S}_1) + \varrho^{**}(\mathbb{S}_2)$;
- (iii) $\mathbb{S}_1 \subset \mathbb{S}_2$ implies $\varrho^{**}(\mathbb{S}_1) \leq \varrho^{**}(\mathbb{S}_2)$; $\varrho^{**}(\mathbb{S}_1 \cup \mathbb{S}_2) = \max\{\varrho^{**}(\mathbb{S}_1), \varrho^{**}(\mathbb{S}_2)\}$;
- (iv) $\varrho^{**}(\overline{\mathbb{S}}) = \varrho^{**}(\text{conv } \mathbb{S}) = \varrho^{**}(\mathbb{S})$.

Definition 2.2 Let the function $T : \mathbb{S} \rightarrow \mathbb{E}$ be continuous and bounded, where $\mathbb{S} \subset \mathbb{E}$. Then T is ϱ^{**} -Lipschitz (k -set contraction) if there exists $k \geq 0$ such that

$$\varrho^{**}(T(\Omega_0)) \leq k \varrho^{**}(\Omega_0), \quad \forall \text{ bounded } \Omega_0 \subset \mathbb{S}.$$

Furthermore, if $k < 1$, then T will be a strict ϱ^{**} -contraction.

Definition 2.3 T is said to be ϱ^{**} -condensing if

$$\varrho^{**}(T(\Omega_0)) < \varrho^{**}(\Omega_0), \quad \forall \text{ bounded } \Omega_0 \subset \mathbb{S} \text{ with } \varrho^{**}(\Omega_0) > 0.$$

In other words, $\varrho^{**}(T(\Omega_0)) \geq \varrho^{**}(\Omega_0)$ implies $\varrho^{**}(\Omega_0) = 0$.

Proposition 2.2 For ϱ^{**} -Lipschitz maps $\mathbb{A}, \mathbb{B} : \Omega_0 \rightarrow \mathbb{E}$ with constants k and k' , respectively, $\mathbb{A} + \mathbb{B} : \Omega_0 \rightarrow \mathbb{E}$ is ϱ^{**} -Lipschitz with constant $k + k'$.

Proposition 2.3 *Let $\mathbb{T} : \Omega_0 \rightarrow \mathbb{E}$ be compact, then \mathbb{T} is ϱ^{**} -Lipschitz with constant $K = 0$.*

Proposition 2.4 *If $\mathbb{T} : \Omega_0 \rightarrow \mathbb{E}$ satisfies Lipschitz with constant k , then \mathbb{T} is ϱ^{**} -Lipschitz with the same constant k .*

The following theorem [14] will be used in the sequel.

Theorem 2.4 *Let $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{E}$ be a ϱ^{**} condensing map and*

$$\Theta = \{u \in \mathbb{E} : \exists \lambda^{**} \in [0, 1] \text{ s.t } u = \lambda^{**}\mathbb{T}u\}.$$

If Θ is a bounded set in \mathbb{E} , that is, there exists $r > 0$ such that $\Theta \subset B_r(0)$, then the topological degree

$$\mathbb{D}(I - \lambda^{**}\mathbb{T}, B_r(0), 0) = 1, \quad \forall \lambda^{**} \in [0, 1],$$

that is, \mathbb{T} has a fixed point in $B_r(0)$.

Organization of the paper. This article consists of five sections. The first section explains the importance of the article and the related literature. In the second section, we study sufficient conditions for the existence and uniqueness of solutions to the hybrid fractional differential equations (1). Third section is reserved for the Hyers–Ulam stability of problem (1). Section 4 explains the application of the results, and finally the conclusion of the article is given in Sect. 5.

3 Existence criteria

This section of the article is reserved for the existence and uniqueness of solution of hybrid problem (1) with the help of the fixed point approach. For these, we first transform the suggested problem into an integral form of the problem.

Lemma 3.1 *For integrable functions f, g , and h_i on I , problem (1) has an integral representation given by*

$$u(t) = \int_0^1 \left(\sum_1^m K_{\beta_i}(s, t) h_i(s, u(s), \mathcal{D}^{\omega-1}u(s)) + K_0(s, t) \Psi(s, u(s), \mathcal{D}^{\omega-1}u(s)) \right) ds + t\psi_2(u(\zeta_0)) + (1-t)\psi_1(u(\zeta_0)), \tag{2}$$

where $\Psi(t, u(t), \mathcal{D}^{\omega-1}u(t)) = f(t, u(t), \mathcal{D}^{\omega-1}u(t))I^\vartheta g(t, u(t), I^\gamma u(t))$, and

$$K_{\beta_i}(s, t) = \frac{-1}{\Gamma(\omega + \beta_i)} \begin{cases} t(1-s)^{\omega-1+\beta_i}; & t \leq s, \\ t(1-s)^{\omega-1+\beta_i} - (t-s)^{\omega-1+\beta_i}; & s \leq t, \end{cases} \tag{3}$$

$$K_0(s, t) = \frac{-1}{\Gamma(\omega)} \begin{cases} t(1-s)^{\omega-1}; & t \leq s, \\ t(1-s)^{\omega-1} - (t-s)^{\omega-1}; & s \leq t. \end{cases} \tag{4}$$

Proof Applying the ϑ th integral (I^ϑ) to both sides of (1), we obtain

$${}^c\mathcal{D}^\omega u(t) - \sum_1^m I^{\beta_i} h_i(t, u(t), \mathcal{D}^{\omega-1} u(t)) = f(t, u(t), \mathcal{D}^{\omega-1} u(t)) I^\vartheta g(t, u(t), I^\gamma u(t)) + C_1.$$

The initial condition ${}^c\mathcal{D}^\omega u(0) = 0$ results in $C_1 = 0$, and hence we obtain

$$\begin{aligned} {}^c\mathcal{D}^\omega u(t) &= \sum_1^m I^{\beta_i} h_i(t, u(t), \mathcal{D}^{\omega-1} u(t)) + f(t, u(t), \mathcal{D}^{\omega-1} u(t)) I^\vartheta g(t, u(t), I^\gamma u(t)) \\ &= \sum_1^m I^{\beta_i} h_i(t, u(t), \mathcal{D}^{\omega-1} u(t)) + \Psi(t, u(t), \mathcal{D}^{\omega-1} u(t)), \end{aligned} \tag{5}$$

where $\Psi(t, u(t), \mathcal{D}^{\omega-1} u(t)) = f(t, u(t), \mathcal{D}^{\omega-1} u(t)) I^\vartheta g(t, u(t), I^\gamma u(t))$. Applying the ω th integral (I^ω) on (5) and using the semigroup property of the integrals, we obtain

$$u(t) = \sum_1^m I^{\omega+\beta_i} h_i(t, u(t), \mathcal{D}^{\omega-1} u(t)) + I^\omega \Psi(t, u(t), \mathcal{D}^{\omega-1} u(t)) + D_1 + D_2 t. \tag{6}$$

The boundary conditions $u(0) = \psi_1(u(\zeta_0))$, $u(1) = \psi_2(u(\zeta_0))$ respectively give $D_1 = \psi_1(u(\zeta_0))$ and

$$D_2 = \psi_2(u(\zeta_0)) - \psi_1(u(\zeta_0)) - I^\omega \Psi(1, u(1), \mathcal{D}^{\omega-1} u(1)) - \sum_1^m I^{\omega+\beta_i} h_i(1, u(1), \mathcal{D}^{\omega-1} u(1)),$$

where $I^\omega \Psi(1, u(1), \mathcal{D}^{\omega-1} u(1))$ denotes the value of the integral $I^\omega \Psi(t, u(t), \mathcal{D}^{\omega-1} u(t))$ at $t = 1$ and $I^{\omega+\beta_i} h_i(1, u(1), \mathcal{D}^{\omega-1} u(1))$ denotes the value of the integral $I^{\omega+\beta_i} h_i(t, u(t), \mathcal{D}^{\omega-1} u(t))$ at $t = 1$ for $i = 1, 2, 3, \dots, m$. Hence, it follows that

$$\begin{aligned} u(t) &= \sum_1^m I^{\omega+\beta_i} h_i(t, u(t), \mathcal{D}^{\omega-1} u(t)) + I^\omega \Psi(t, u(t), \mathcal{D}^{\omega-1} u(t)) + \psi_1(u(\zeta_0)) + t(\psi_2(u(\zeta_0)) \\ &\quad - \psi_1(u(\zeta_0)) - I^\omega \Psi(1, u(1), \mathcal{D}^{\omega-1} u(1)) - \sum_1^m I^{\omega+\beta_i} h_i(1, u(1), \mathcal{D}^{\omega-1} u(1))) \\ &= \sum_1^m (I^{\omega+\beta_i} h_i(t, u(t), \mathcal{D}^{\omega-1} u(t)) - t I^{\omega+\beta_i} h_i(1, u(1), \mathcal{D}^{\omega-1} u(1))) \\ &\quad + I^\omega \Psi(t, u(t), \mathcal{D}^{\omega-1} u(t)) - t I^\omega \Psi(1, u(1), \mathcal{D}^{\omega-1} u(1)) + t \psi_2(u(\zeta_0)) \\ &\quad + (1-t) \psi_1(u(\zeta_0)), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} u(t) &= \int_0^1 \left[\sum_1^m K_{\beta_i}(s, t) h_i(s, u(s), \mathcal{D}^{\omega-1} u(s)) + K_0(s, t) \Psi(s, u(s), \mathcal{D}^{\omega-1} u(s)) \right] ds \\ &\quad + (1-t) \psi_1(u(\zeta_0)) + t \psi_2(u(\zeta_0)). \end{aligned} \tag{7}$$

□

From (7), it follows that

$$\begin{aligned} \mathcal{D}^{\omega-1}u(t) = & \int_0^1 \left[\sum_1^m G_{\beta_i}(s,t)h_i(s,u(s),\mathcal{D}^{\omega-1}u(s)) + G_0(s,t)\Psi(s,u(s),\mathcal{D}^{\omega-1}u(s)) \right] ds \\ & + \frac{t^{2-\omega}}{\Gamma(3-\omega)}(\psi_2(u(\zeta_0)) - \psi_1(u(\zeta_0))), \end{aligned} \tag{8}$$

where

$$\begin{aligned} G_{\beta_i}(s,t) = & \mathcal{D}^{\omega-1}K_{\beta_i}(s,t) \\ = & \frac{-1}{\Gamma(3-\omega)\Gamma(\omega+\beta_i)} \begin{cases} t^{2-\omega}(1-s)^{\omega-1+\beta_i}; & t \leq s, \\ t^{2-\omega}(1-s)^{\omega-1+\beta_i} - (t-s)^{\beta_i}; & s \leq t, \end{cases} \end{aligned} \tag{9}$$

$$G_0(s,t) = \mathcal{D}^{\omega-1}K_0(s,t) = \frac{-1}{\Gamma(3-\omega)\Gamma(\omega)} \begin{cases} t^{2-\omega}(1-s)^{\omega-1}; & t \leq s, \\ t^{2-\omega}(1-s)^{\omega-1} - \Gamma\omega; & s \leq t. \end{cases} \tag{10}$$

From (3), (4), (9), and (10), it follows that

$$\begin{aligned} \max_{t \in [0,1]} |K_{\beta_i}(s,t)| &= \frac{s(1-s)^{\omega-1+\beta_i}}{\Gamma(\omega+\beta_i)} \leq \frac{1}{\Gamma(\omega+\beta_i)}, \\ \max_{t \in [0,1]} |K_0(s,t)| &= \frac{s(1-s)^{\omega-1}}{\Gamma(\omega)} \leq \frac{1}{\Gamma(\omega)}, \\ \max_{t \in [0,1]} |G_{\beta_i}(s,t)| &\leq \frac{1}{\Gamma(3-\omega)\Gamma(\omega+\beta_i)}, \\ \max_{t \in [0,1]} |G_0(s,t)| &\leq \frac{1}{\Gamma(3-\omega)\Gamma(\omega)}. \end{aligned} \tag{11}$$

Define operators $\mathbb{A}, \mathbb{B} : E = C(I, \mathcal{R}_e) \rightarrow E$ by

$$\mathbb{A}(u) = \int_0^1 \left(\sum_1^m K_{\beta_i}(s,t)h_i(s,u(s),\mathcal{D}^{\omega-1}u(s)) + K_0(s,t)\Psi(s,u(s),\mathcal{D}^{\omega-1}u(s)) \right) ds, \tag{12}$$

$$\mathbb{B}(u) = (1-t)\psi_1(u(\zeta_0)) + t\psi_2(u(\zeta_0)),$$

then (7) takes the form of the operator equation

$$u(t) = \mathbb{A}u(t) + \mathbb{B}u(t) = \mathbb{T}u(t), \quad t \in I, \tag{13}$$

and fixed points of operator equation (13) are solutions of BVP (1). Now, we list the following hypotheses.

- (H₁) $f : I \times \mathcal{R}_e \times \mathcal{R}_e \rightarrow \mathcal{R}_e - \{0\}$, $h_i : I \times \mathcal{R}_e \times \mathcal{R}_e \rightarrow \mathcal{R}_e$ ($i = 1, 2, \dots, m$), and $g : I \times \mathcal{R}_e \times \mathcal{R}_e \rightarrow \mathcal{R}_e$ satisfy the Caratheodory conditions.
- (H₂) There exist positive constants $k_1, k_2 \in [0, 1]$, $q \in (0, 1)$, and d_1, d_2, e_1, e_2 such that, for $u, u_1, u_2 \in E$, we have

$$\begin{aligned} |\psi_1(u_2) - \psi_1(u_1)| &\leq k_1|u_2 - u_1|, & |\psi_1(u)| &\leq e_1|u|^q + e_2, \\ |\psi_2(u_2) - \psi_2(u_1)| &\leq k_2|u_2 - u_1|, & |\psi_2(u)| &\leq d_1|u|^q + d_2. \end{aligned}$$

(H₃) There exist positive continuous functions $\varrho^{**}, \rho : I \rightarrow \mathcal{R}_e$, parameters $0 < q, \delta < 1$, and positive constants θ_i, ν, ξ such that, for $u \in E$,

$$\begin{aligned} |h_i(t, u(t), \mathcal{D}^{\omega-1}u(t))| &\leq \theta_i, \\ |f(t, u(t), \mathcal{D}^{\omega-1}u(t))| &\leq \varrho^{**}(t)(|u(t)| + |\mathcal{D}^{\omega-1}u(t)|)^\delta + \xi, \\ |g(t, u(t), I^\nu u(t))| &\leq |\rho(t)| + \nu(|u|^q + |I^\nu u|^q). \end{aligned}$$

(H₄) There exist positive constants λ^{**}_i for $i = 1, 2, \dots, m$ such that, for $u, \bar{u} \in E$ and $\varrho^{**}_0 = \max_{t \in I} \varrho^{**}(t), \rho_0 = \max_{t \in I} \rho(t)$,

$$\begin{aligned} &|h_i(t, u(t), \mathcal{D}^{\omega-1}u(t)) - h_i(t, \bar{u}(t), \mathcal{D}^{\omega-1}\bar{u}(t))| \\ &\leq \lambda^{**}_i(|u - \bar{u}| + |\mathcal{D}^{\omega-1}u - \mathcal{D}^{\omega-1}\bar{u}|), \\ &|f(t, u(t), \mathcal{D}^{\omega-1}u(t)) - f(t, \bar{u}(t), \mathcal{D}^{\omega-1}\bar{u}(t))| \\ &\leq \varrho^{**}_0(|u - \bar{u}| + |\mathcal{D}^{\omega-1}u - \mathcal{D}^{\omega-1}\bar{u}|), \\ &|g(t, u(t), I^\nu u(t)) - g(t, \bar{u}(t), I^\nu \bar{u}(t))| \leq \rho_0|u - \bar{u}|. \end{aligned}$$

Lemma 3.2 *Under condition (H₂), the operator \mathbb{B} is ϱ^{**} -Lipschitz with constant $k = \max\{(1 - t + \frac{t^{2-\omega}}{\Gamma(3-\omega)})k_1 + (t + \frac{t^{2-\omega}}{\Gamma(3-\omega)})k_2, t \in I\}$. Further, \mathbb{B} satisfies the following growth condition:*

$$\|\mathbb{B}u(t)\|_1 \leq d\|u\|_1^q + e, \tag{14}$$

where $d = \max\{(t + \frac{t^{2-\omega}}{\Gamma(3-\omega)})d_1 + (1 - t + \frac{t^{2-\omega}}{\Gamma(3-\omega)})e_1, t \in I\}$, $e = \max\{(t + \frac{t^{2-\omega}}{\Gamma(3-\omega)})d_2 + (1 - t + \frac{t^{2-\omega}}{\Gamma(3-\omega)})e_2, t \in I\}$.

Proof For $u_1, u_2 \in E$ such that $u_1 < u_2$, using (H₂), we obtain

$$\begin{aligned} |\mathbb{B}(u_1) - \mathbb{B}(u_2)| &\leq (1 - t)|\psi_1(u_1(\zeta_0)) - \psi_1(u_2(\zeta_0))| + t|\psi_2(u_1(\zeta_0)) - \psi_2(u_2(\zeta_0))| \\ &\leq (1 - t)k_1|u_1(\zeta_0) - u_2(\zeta_0)| + tk_2|u_1(\zeta_0) - u_2(\zeta_0)| \\ &= ((1 - t)k_1 + tk_2)|u_1(\zeta_0) - u_2(\zeta_0)|, \end{aligned} \tag{15}$$

$$\begin{aligned} &|\mathcal{D}^{\omega-1}\mathbb{B}(u_1) - \mathcal{D}^{\omega-1}\mathbb{B}(u_2)| \\ &\leq \frac{t^{2-\omega}}{\Gamma(3-\omega)}|\psi_1(u_1(\zeta_0)) - \psi_1(u_2(\zeta_0))| + |\psi_2(u_1(\zeta_0)) - \psi_2(u_2(\zeta_0))| \\ &\leq t^{2-\omega} \frac{k_1 + k_2}{\Gamma(3-\omega)}|u_1(\zeta_0) - u_2(\zeta_0)|. \end{aligned} \tag{16}$$

Hence, from (15) and (16), it follows that

$$\|\mathbb{B}(u_1) - \mathbb{B}(u_2)\|_1 \leq k\|u_1 - u_2\| \leq k\|u_1 - u_2\|_1. \tag{17}$$

By Proposition 2.4, \mathbb{B} is ϱ^{**} -Lipschitz with constant k . Further,

$$\begin{aligned}
 |\mathbb{B}(u)| &\leq (1-t)|\psi_1(u(\zeta_0))| + t|\psi_2(u(\zeta_0))| \\
 &\leq (1-t)(d_1|u|^q + d_2) + t(e_1|u|^q + e_2) \\
 &= (td_1 + (1-t)e_1)|u|^q + (td_2 + (1-t)e_2),
 \end{aligned}
 \tag{18}$$

$$\begin{aligned}
 |\mathcal{D}^{\omega-1}\mathbb{B}(u)| &\leq \frac{t^{2-\omega}}{\Gamma(3-\omega)}(|\psi_1(u(\zeta_0))| + |\psi_2(u(\zeta_0))|) \\
 &\leq \frac{t^{2-\omega}}{\Gamma(3-\omega)}((d_1 + e_1)|u|^q + (d_2 + e_2)).
 \end{aligned}
 \tag{19}$$

From (18) and (19), it follows that

$$\|\mathbb{B}(u)\|_1 \leq d\|u\|^q + e \leq d\|u\|_1^q + e. \quad \square$$

Lemma 3.3 *Under conditions (H_1) and (H_3) , the operator \mathbb{A} is ϱ^{**} -Lipschitz with zero constant. Further \mathbb{A} satisfies the following growth condition:*

$$\|\mathbb{A}u(t)\| \leq \mathcal{E}_0 + \mathcal{E}_1\|u\|_1^\delta + \mathcal{E}_2\|u\|_1^q + \mathcal{E}_3\|u\|_1^{q+\delta}, \quad u \in \mathbb{E}. \tag{20}$$

Proof By (H_1) , the continuity of h_i, Ψ with respect to u for each fixed $t \in I$ implies the continuity of the operator \mathbb{A} for each fixed $t \in I$. Moreover, for each $u \in E$, using (H_3) , we obtain

$$\begin{aligned}
 |f(t, u(t), \mathcal{D}^{\omega-1}u(t))| &\leq \varrho^{**}(t)\|u\|^\delta + \xi, \\
 |g(t, u, I^\gamma u)| &\leq |\rho(t)| + \nu(\|u\|^q + \|I^\gamma u\|^q).
 \end{aligned}$$

Hence it follows that

$$\begin{aligned}
 &|\Psi(t, u(t), \mathcal{D}^{\omega-1}u(t))| \\
 &= |f(t, u(t), \mathcal{D}^{\omega-1}u(t))| |I^\vartheta g(t, u, I^\gamma u)| \\
 &\leq \frac{1}{\Gamma(\vartheta + 1)} \left[\varrho^{**}_0 \rho_0 \|u\|^\delta \right. \\
 &\quad \left. + \varrho^{**}_0 \nu \left(1 + \frac{1}{(\Gamma(\gamma + 1))^q} \right) \|u\|^{\delta+q} + \xi \nu \left(1 + \frac{1}{(\Gamma(\gamma + 1))^q} \right) \|u\|^q + \xi \rho_0 \right],
 \end{aligned}
 \tag{21}$$

where $\varrho^{**}_0 = \max_{t \in I} |\varrho^{**}(t)|$, $\rho_0 = \max_{t \in I} |\rho(t)|$. Thus \mathbb{A} satisfies the following growth condition:

$$\begin{aligned}
 |\mathbb{A}u(t)| + |\mathcal{D}^{\omega-1}\mathbb{A}u(t)| &\leq \int_0^1 \sum_1^m (|K_{\beta_i}(s, t)| + |G_{\beta_i}(s, t)| |h_i(s, u(s), \mathcal{D}^{\omega-1}u(s))| \\
 &\quad + (|K_0(s, t)| + |G_0(s, t)| |\Psi(s, u(s), \mathcal{D}^{\omega-1}u(s))|) ds,
 \end{aligned}
 \tag{22}$$

which in view of (11) and (21) implies that

$$\begin{aligned}
 & |\mathbb{A}u(t)| + |\mathcal{D}^{\omega-1}\mathbb{A}u(t)| \\
 & \leq \left(1 + \frac{1}{\Gamma(3-\omega)}\right) \left(\sum_1^m \frac{\|\theta_i\|}{\Gamma(\omega + \beta_i)} + \frac{1}{\Gamma(\omega)} \|\Psi(s, u(s), \mathcal{D}^{\omega-1}u(s))\|\right) \\
 & \leq \left(1 + \frac{1}{\Gamma(3-\omega)}\right) \left(\sum_1^m \frac{\|\theta_i\|}{\Gamma(\omega + \beta_i)} + \frac{1}{\Gamma(\vartheta + 1)\Gamma(\omega)} \left(\varrho^{**} \rho_0 \|u\|^\delta \right. \right. \\
 & \quad \left. \left. + \varrho^{**} \nu \left(1 + \frac{1}{(\Gamma(\gamma + 1))^q}\right) \|u\|^{\delta+q} + \xi \nu \left(1 + \frac{1}{(\Gamma(\gamma + 1))^q}\right) \|u\|^q + \xi \rho_0\right)\right).
 \end{aligned}$$

Hence, it follows that

$$\begin{aligned}
 \|\mathbb{A}u(t)\|_1 & \leq \mathcal{E}_0 + \mathcal{E}_1 \|u\|^\delta + \mathcal{E}_2 \|u\|^q + \mathcal{E}_3 \|u\|^{q+\delta} \\
 & \leq \mathcal{E}_0 + \mathcal{E}_1 \|u\|_1^\delta + \mathcal{E}_2 \|u\|_1^q + \mathcal{E}_3 \|u\|_1^{q+\delta}, \quad u \in \mathbb{E},
 \end{aligned} \tag{23}$$

where $\mathcal{E}_0 = (1 + \frac{1}{\Gamma(3-\omega)}) (\sum_1^m \frac{\|\theta_i\|}{\Gamma(\omega + \beta_i)} + \frac{\varrho_0 \xi}{\Gamma(\omega)\Gamma(\vartheta+1)})$, $\mathcal{E}_1 = (1 + \frac{1}{\Gamma(3-\omega)}) \frac{\varrho^{**} \rho_0}{\Gamma(\omega)\Gamma(\vartheta+1)}$, $\mathcal{E}_2 = \frac{\nu \xi}{\Gamma(\omega)\Gamma(\vartheta+1)} (1 + \frac{1}{\Gamma(3-\omega)}) (1 + \frac{1}{(\Gamma(\gamma+1))^q})$, and $\mathcal{E}_3 = \frac{\nu \varrho^{**} \rho_0}{\Gamma(\omega)\Gamma(\vartheta+1)} (1 + \frac{1}{\Gamma(3-\omega)}) (1 + \frac{1}{(\Gamma(\gamma+1))^q})$. From (23), it also follows that \mathbb{A} is uniformly bounded on any bounded subset Ω_0 of E . Now, for $u \in \Omega_0$ and $t_1, t_2 \in I$ such that $t_1 < t_2$, consider

$$\begin{aligned}
 & |\mathbb{A}u(t_2) - \mathbb{A}u(t_1)| \\
 & \leq \int_0^1 \left(\sum_1^m |K_{\beta_i}(s, t_2) - K_{\beta_i}(s, t_1)| |h_i(s, u, \mathcal{D}^{\omega-1}u(s))| \right. \\
 & \quad \left. + |K_0(s, t_2) - K_0(s, t_1)| |\Psi(s, u, \mathcal{D}^{\omega-1}u(s))| \right) ds,
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 & |\mathcal{D}^{\omega-1}\mathbb{A}u(t_2) - \mathcal{D}^{\omega-1}\mathbb{A}u(t_1)| \\
 & \leq \int_0^1 \left(\sum_1^m |G_{\beta_i}(s, t_2) - G_{\beta_i}(s, t_1)| |h_i(s, u, \mathcal{D}^{\omega-1}u(s))| \right. \\
 & \quad \left. + |G_0(s, t_2) - G_0(s, t_1)| |\Psi(s, u, \mathcal{D}^{\omega-1}u(s))| \right) ds.
 \end{aligned} \tag{25}$$

But

$$\begin{aligned}
 & |K_{\beta_i}(s, t_2) - K_{\beta_i}(s, t_1)| \\
 & = \frac{1}{\Gamma(\omega + \beta_i)} \begin{cases} (1-s)^{\omega+\beta_i-1} (t_2 - t_1); & t \leq s, \\ (1-s)^{\omega+\beta_i-1} (t_2 - t_1) + (t_2 - s)^{\omega-1+\beta_i} - (t_1 - s)^{\omega-1+\beta_i}; & s \leq t, \end{cases} \\
 & |K_0(s, t_2) - K_0(s, t_1)| = \frac{1}{\Gamma(\omega)} \begin{cases} (1-s)^{\omega-1} (t_2 - t_1); & t \leq s, \\ (1-s)^{\omega-1} (t_2 - t_1) + (t_2 - s)^{\omega-1} - (t_1 - s)^{\omega-1}; & s \leq t, \end{cases} \\
 & |G_{\beta_i}(s, t_2) - G_{\beta_i}(s, t_1)|
 \end{aligned}$$

$$= \frac{1}{\Gamma(\omega)\Gamma(\omega + \beta_i)} \begin{cases} (1 - s)^{\omega + \beta_i - 1}(t_2^{2-\omega} - t_1^{2-\omega}); & t \leq s, \\ (1 - s)^{\omega + \beta_i - 1}(t_2^{2-\omega} - t_1^{2-\omega}) + (t_2 - s)^{\beta_i} - (t_1 - s)^{\beta_i}; & s \leq t, \end{cases}$$

$$|G_0(s, t_2) - G_0(s, t_1)| = \frac{1}{\Gamma(\omega)\Gamma(3 - \omega)} \begin{cases} (1 - s)^{\omega - 1}(t_2^{2-\omega} - t_1^{2-\omega}); & t \leq s, \\ (1 - s)^{\omega - 1}(t_2^{2-\omega} - t_1^{2-\omega}); & s \leq t. \end{cases}$$

Hence, using the relation $\int_0^1 a(t, s) ds = \int_0^t a(t, s) ds + \int_t^1 a(t, s) ds$ and the notation $\Delta F(s, t) = F(s, t_2) - F(s, t_1)$ for the difference, we obtain

$$\begin{aligned} \int_0^1 |\Delta K_{\beta_i}(s, t)| ds &= \frac{[(t_2 - t_1) + (t_2^{\omega + \beta_i} - t_1^{\omega + \beta_i}) - (t_2 - t)^{\omega + \beta_i} + (t_1 - t)^{\omega + \beta_i}]}{\Gamma(\omega + 1 + \beta_i)}, \\ \int_0^1 |\Delta K_0(s, t)| ds &= \frac{1}{\Gamma(\omega + 1)} [(t_2 - t_1) + (t_2^\omega - t_1^\omega) - (t_2 - t)^\omega + (t_1 - t)^\omega], \\ \int_0^1 |\Delta G_{\beta_i}(s, t)| ds &= \frac{[(t_2^{2-\omega} - t_1^{2-\omega}) + t_2^{\beta_i + 1} - t_1^{\beta_i + 1} - (t_2 - t)^{\beta_i + 1} + (t_1 - t)^{\beta_i + 1}]}{\Gamma(3 - \omega)\Gamma(\omega + 1 + \beta_i)}, \\ \int_0^1 |\Delta G_0(s, t)| ds &= \frac{1}{\Gamma(3 - \omega)\Gamma(\omega + 1)} [t_2^{2-\omega} - t_1^{2-\omega}]. \end{aligned} \tag{26}$$

Using (21), the assumption $|h_i(s, u, \mathcal{D}^{\omega-1}u(s))| \leq \|\theta_i\|$ on Ω_0 , and (26) in (24) and (25), it follows that

$$|\mathbb{A}u(t_2) - \mathbb{A}u(t_1)| \rightarrow 0, \quad |\mathcal{D}^{\omega-1}\mathbb{A}u(t_2) - \mathcal{D}^{\omega-1}\mathbb{A}u(t_1)| \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2. \tag{27}$$

Therefore \mathbb{A} is equicontinuous, and by Arzela–Ascoli theorem \mathbb{A} is compact. By Proposition 2.3, the operator \mathbb{A} is ϱ^{**} -Lipschitz with zero constant. \square

Theorem 3.1 *Under assumptions (H₁)–(H₃), system (13) has at least one solution $u \in E$ provided that $q \leq 1 - \delta$, $\mathcal{L}_3 < 1$. Also, the set of solutions of (13) is bounded in \mathbb{E} .*

Proof By Lemma 3.2, the operator \mathbb{B} is ϱ^{**} -Lipschitz for $k \in [0, 1)$, and by Lemma 3.3, the operator \mathbb{A} is ϱ^{**} -Lipschitz with zero constant. It follows by Proposition 2.2 that \mathbb{T} is ϱ^{**} -Lipschitz with constant $k \in [0, 1)$. Define

$$\mathbb{G} = \{u \in \mathbb{E} : \exists \hbar \in [0, 1] \text{ such that } u = \hbar \mathbb{T}u\}.$$

For $u \in \mathbb{G}$, using the growth conditions (20) and (23), we obtain

$$\begin{aligned} \|u\|_1 &\leq \hbar(\|\mathbb{A}u\|_1 + \|\mathbb{B}u\|_1) \\ &\leq \hbar(d\|u\|_1^{q_1} + e + \mathcal{L}_0 + \mathcal{L}_1\|u\|_1^\delta + \mathcal{L}_2\|u\|_1^q + \mathcal{L}_3\|u\|_1^{q+\delta}) \\ &= \hbar(d\|u\|_1^{q_1} + \mathcal{L}_1\|u\|_1^\delta + \mathcal{L}_2\|u\|_1^q + \mathcal{L}_3\|u\|_1^{q+\delta}) + \hbar(e + \mathcal{L}_0). \end{aligned} \tag{28}$$

Since $q \leq 1 - \delta$ and $\mathcal{L}_3 = \frac{\nu \varrho^{**}_0}{\Gamma(\omega)\Gamma(\vartheta+1)}(1 + \frac{1}{\Gamma(3-\omega)})(1 + \frac{1}{(\Gamma(\gamma+1))^q}) < 1$, it follows that the set \mathbb{G} is bounded. Hence, by Theorem (2.4), BVP (1) has at least one solution. \square

Choose $0 < R < 1$ and consider a closed bounded and convex subset $\bar{B} = \{z \in E : \|z\|_1 \leq R\} \subseteq \mathbb{E}$.

Theorem 3.2 *Under assumptions (H₁)–(H₄), system (13) has a unique solution in \bar{B} provided that*

$$k + \sum_1^m \frac{\lambda^{**}_i (1 + \frac{1}{\Gamma(3-\omega)})}{\Gamma(\omega + \beta_i)} + \frac{\varrho^{**}_0 (1 + \frac{1}{\Gamma(3-\omega)})}{\Gamma(\vartheta + 1)\Gamma\omega} \left(\rho_0 + \nu \left(1 + \frac{1}{(\Gamma(\gamma + 1))^q} \right) R^q + \rho_0 (\varrho^{**}_0 R^\delta + \xi) \right) < 1.$$

Proof For $u \in \bar{B}$, using H_3 , we obtain

$$\begin{aligned} |f(t, u(t), \mathcal{D}^{\omega-1}u(t))| &\leq \varrho^{**}_0 R^\delta + \xi, \\ |I^\vartheta g(t, u(t), I^\gamma u(t))| &\leq \frac{(\rho_0 + \nu(1 + \frac{1}{(\Gamma(\gamma+1))^q})R^q)}{\Gamma(\vartheta + 1)}. \end{aligned} \tag{29}$$

For $u_1, u_2 \in \bar{B}$, using H_4 , we obtain

$$\begin{aligned} |h_i(t, u_1(t), \mathcal{D}^{\omega-1}u_1(t)) - h_i(t, u_2(t), \mathcal{D}^{\omega-1}u_2(t))| &\leq \lambda^{**}_i \|u_1 - u_2\|_1, \\ |f(t, u_1(t), \mathcal{D}^{\omega-1}u_1(t)) - f(t, u_2(t), \mathcal{D}^{\omega-1}u_2(t))| &\leq \varrho^{**}_0 \|u_1 - u_2\|_1, \\ |I^\vartheta g(t, u_1(t), I^\gamma u_1(t)) - I^\vartheta g(t, u_2(t), I^\gamma u_2(t))| &\leq \frac{\rho_0 \|u_1 - u_2\|_1}{\Gamma(\vartheta + 1)} \leq \frac{\rho_0 \|u_1 - u_2\|_1}{\Gamma(\vartheta + 1)}. \end{aligned} \tag{30}$$

Further,

$$\begin{aligned} &|\Psi(t, u_1, \mathcal{D}^{\omega-1}u_1) - \Psi(t, u_2, \mathcal{D}^{\omega-1}u_2)| \\ &\leq |I^\vartheta g(t, u_1, I^\gamma u_1)| |f(t, u_1, \mathcal{D}^{\omega-1}u_1) - f(t, u_2, \mathcal{D}^{\omega-1}u_2)| \\ &\quad + |f(t, u_2, \mathcal{D}^{\omega-1}u_2)| |I^\vartheta g(t, u_1, I^\gamma u_1) - I^\vartheta g(t, u_2, I^\gamma u_2(t))|, \end{aligned}$$

which in view of (29) implies that

$$\begin{aligned} &|\Psi(t, u_1, \mathcal{D}^{\omega-1}u_1) - \Psi(t, u_2, \mathcal{D}^{\omega-1}u_2)| \\ &\leq \left(\frac{(\rho_0 + \nu(1 + \frac{1}{(\Gamma(\gamma+1))^q})R^q)\varrho^{**}_0}{\Gamma(\vartheta + 1)} + \frac{(\varrho^{**}_0 R^\delta + \xi)\rho_0}{\Gamma(\vartheta + 1)} \right) \|u_1 - u_2\|_1. \end{aligned} \tag{31}$$

Now, using definition (12), we obtain

$$\begin{aligned} &\|\mathbb{A}(u_1) - \mathbb{A}(u_2)\|_1 \\ &\leq \int_0^1 \left(\sum_1^m |K_{\beta_i}(s, t) + G_{\beta_i}(s, t)| |h_i(s, u_1, \mathcal{D}^{\omega-1}u_1) - h_i(s, u_2, \mathcal{D}^{\omega-1}u_2)| \right. \\ &\quad \left. + (|K_0(s, t) + G_0(s, t)|) |\Psi(s, u_1, \mathcal{D}^{\omega-1}u_1) - \Psi(s, u_2, \mathcal{D}^{\omega-1}u_2)| \right) ds, \end{aligned}$$

which in view of (30) and (31) implies that

$$\begin{aligned} \|\mathbb{A}(u_1) - \mathbb{A}(u_2)\|_1 &\leq \left[\sum_1^m \frac{\lambda^{**}_i(1 + \frac{1}{\Gamma(3-\omega)})}{\Gamma(\omega + \beta_i)} + \frac{(1 + \frac{1}{\Gamma(3-\omega)})}{\Gamma(\omega)\Gamma(\vartheta + 1)} \left(\varrho^{**}_0 \left(\rho_0 \right. \right. \right. \\ &\quad \left. \left. \left. + \nu \left(1 + \frac{1}{(\Gamma(\gamma + 1))^q} \right) R^q \right) + \rho_0 (\varrho^{**}_0 R^\delta + \xi) \right) \right] \|u_1 - u_2\|_1 \quad (32) \\ &= k_1 \|u_1 - u_2\|_1, \end{aligned}$$

where

$$\begin{aligned} k_1 &= \sum_1^m \frac{\lambda^{**}_i(1 + \frac{1}{\Gamma(3-\omega)})}{\Gamma(\omega + \beta_i)} \\ &\quad + \frac{(1 + \frac{1}{\Gamma(3-\omega)})}{\Gamma(\omega)\Gamma(\vartheta + 1)} \left(\varrho^{**}_0 \left(\rho_0 + \nu \left(1 + \frac{1}{(\Gamma(\gamma + 1))^q} \right) R^q \right) + \rho_0 (\varrho^{**}_0 R^\delta + \xi) \right). \end{aligned}$$

Hence, using (17) and (32), it follows that

$$\begin{aligned} \|\mathbb{T}(u_1) - \mathbb{T}(u_2)\|_1 &\leq \|\mathbb{A}(u_1) - \mathbb{A}(u_2)\|_1 + \|\mathbb{B}(u_1) - \mathbb{B}(u_2)\|_1 \\ &\leq (k + k_1) \|u_1 - u_2\|_1, \end{aligned} \quad (33)$$

and uniqueness follows by the Banach contraction principle. □

4 Hyers–Ulam stability

In this section, we present the Hyers–Ulam stability analysis for the hybrid fractional differential equation (1). For more related problems to the Hyers–Ulam stability, the readers may take help from the references in [15–20] and the literature.

Definition 4.1 The fractional integral system (13) is said to be Hyers–Ulam stable if there exists a constant $\zeta > 0$ such that, for given $\varphi > 0$ and for each solution u of the inequality

$$\|u(t) - (\mathbb{A} + \mathbb{B})u(t)\|_1 < \varphi, \quad (34)$$

there exists a solution $\bar{u}(t)$ of the integral system (13)

$$\bar{u}(t) = (\mathbb{A} + \mathbb{B})\bar{u}(t)$$

such that

$$\|u(t) - \bar{u}(t)\|_1 < \varphi \zeta.$$

Theorem 4.2 Under assumptions (H_2) and (H_4) , the fractional order hybrid differential equation (1) is Hyers–Ulam stable provided $k + k_1 < 1$.

Proof Let $u \in E$ satisfy inequality (34) and $\bar{u} \in E$ be a solution of BVP (1) satisfying the integral system (13). Then consider

$$\begin{aligned} \|u(t) - \bar{u}(t)\|_1 &= \|u(t) - (\mathbb{A} + \mathbb{B})\bar{u}(t)\|_1 \\ &\leq \|u(t) - (\mathbb{A} + \mathbb{B})u(t)\|_1 \\ &\quad + \|(\mathbb{A} + \mathbb{B})u(t) - (\mathbb{A} + \mathbb{B})\bar{u}(t)\|_1 \\ &< \varphi + \|(\mathbb{A} + \mathbb{B})u(t) - (\mathbb{A} + \mathbb{B})\bar{u}(t)\|_1. \end{aligned} \tag{35}$$

Now

$$\|(\mathbb{A} + \mathbb{B})u(t) - (\mathbb{A} + \mathbb{B})\bar{u}(t)\|_1 \leq \|\mathbb{A}u(t) - \mathbb{A}\bar{u}(t)\|_1 + \|\mathbb{B}u(t) - \mathbb{B}\bar{u}(t)\|_1,$$

which in view of H_2 and H_4 (that is, (17) and (32)) implies that

$$\|(\mathbb{A} + \mathbb{B})u(t) - (\mathbb{A} + \mathbb{B})\bar{u}(t)\|_1 \leq k_1 \|u(t) - \bar{u}(t)\|_1 + k \|u(t) - \bar{u}(t)\|_1. \tag{36}$$

Hence, from (35), it follows that

$$\|u(t) - \bar{u}(t)\|_1 < \varphi + (k_1 + k) \|u(t) - \bar{u}(t)\|_1,$$

which implies that

$$\|u(t) - \bar{u}(t)\|_1 < \varphi \zeta, \quad \text{where } \zeta = \frac{1}{1 - (k_1 + k)}. \quad \square$$

5 Application

In this section, we present an example in the application of the results we studied in the previous sections.

Example 1 We consider

$$\begin{aligned} {}^c\mathcal{D}^\vartheta \left[\frac{{}^c\mathcal{D}^\omega u(t) - \sum_1^m I^{\beta_i} h_i(t, u(t), \mathcal{D}^{\omega-1} u(t))}{f(t, u(t), \mathcal{D}^{\omega-1} u(t))} \right] &= g(t, u(t), I^\nu u(t)), \quad t \in I = [0, 1] \\ {}^c\mathcal{D}^\omega u(0) = 0, \quad u(0) &= \psi_1(u(\zeta_0)), \quad u(1) = \psi_2(u(\zeta_0)), \end{aligned} \tag{37}$$

where the parameters are such that $0 < \vartheta \leq 1, 1 < \omega \leq 2, 0 < \zeta_0 < 1$, the functions $f : I \times \mathcal{R}_e \times \mathcal{R}_e \rightarrow \mathcal{R}_e - \{0\}, h_i : I \times \mathcal{R}_e \times \mathcal{R}_e \rightarrow \mathcal{R}_e (i = 1, 2, \dots, m)$, and $g : I \times \mathcal{R}_e \times \mathcal{R}_e \rightarrow \mathcal{R}_e$ such that $\vartheta = 0.5, \omega = 1.5, \zeta_0 = 0.5, \beta_i = 0.5$ for $i = 1, 2, \dots, m. q = 0.2, \delta = 0.7$ and $\psi_1(u(t)) = \psi_2(u(t)) = \frac{2 + \sin(u(t))}{20}, k_1 = k_2 = \frac{1}{10}, h_i(t, u(t), \mathcal{D}^{\omega-1} u(t)) = \frac{1 + u(t) + \mathcal{D}^{\omega-1} u(t)}{50} = f_i(t, u(t), \mathcal{D}^{\omega-1} u(t)), g(t, u(t), I^\nu u(t)) = \frac{1 + t + u(t) I^\nu u(t)}{20}$. It is easy to see that $\lambda^{**}_i = \frac{1}{50}$ for $i = 1, 2, \dots, m, \rho = 1 + t, \nu = \frac{1}{20}$. And

$$K_{\beta_i}(s, t) = \frac{-1}{\Gamma(\omega + \beta_i)} \begin{cases} t(1-s)^{\omega-1+\beta_i}; & t \leq s, \\ t(1-s)^{\omega-1+\beta_i} - (t-s)^{\omega-1+\beta_i}; & s \leq t, \end{cases} \tag{38}$$

$$K_0(s, t) = \frac{-1}{\Gamma(\omega)} \begin{cases} t(1-s)^{\omega-1}; & t \leq s, \\ t(1-s)^{\omega-1} - (t-s)^{\omega-1}; & s \leq t. \end{cases} \tag{39}$$

It is easy to see that (H_1) – (H_4) are satisfied, also the inequality

$$k + \sum_1^m \frac{\lambda^{**}_i \left(1 + \frac{1}{\Gamma(3-\omega)}\right)}{\Gamma(\omega + \beta_i)} + \frac{\varrho^{**}_0 \left(1 + \frac{1}{\Gamma(3-\omega)}\right)}{\Gamma(\vartheta + 1)\Gamma\omega} \left(\rho_0 + \nu \left(1 + \frac{1}{(\Gamma(\gamma + 1))^q}\right) R^q + \rho_0 (\varrho^{**}_0 R^\delta + \xi) \right) < 1,$$

holds true. Thus, problem (37) has a unique solution. For more applications of the results, we refer the readers to the work in [21–29].

6 Conclusion

In this article, we have studied a general class of hybrid fractional differentials for the existence, uniqueness, and Hyers–Ulam stability. We have seen that under certain assumptions of (H_1) – (H_4) , the FDEs of the kind (1) have unique solutions and they are Hyers–Ulam stable too, subject to the inequalities given in the statements. At the end, we also presented an example as an application of the work. We suggest the readers for re-consideration of the suggested problem for the ABC-fractional order derivative and others too.

Acknowledgements

All the authors are thankful to the reviewers and the editor for their comments which have improved the quality of the article.

Funding

There is no source of funding this article.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors have equal contributions in this article. All authors read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 March 2021 Accepted: 31 May 2021 Published online: 11 June 2021

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