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A numerical and analytical study of SE(Is)(Ih)AR epidemic fractional order COVID-19 model

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Abstract

This article describes the corona virus spread in a population under certain assumptions with the help of a fractional order mathematical model. The fractional order derivative is the well-known fractal fractional operator. We have given the existence results and numerical simulations with the help of the given data in the literature. Our results show similar behavior as the classical order ones. This characteristic shows the applicability and usefulness of the derivative and our numerical scheme.

Keywords: Fractal fractional derivatives; Existence and uniqueness of the solutions; Hyers–Ulam stability; Numerical scheme

1 Introduction

The end of year 2019 was shocking for the world, especially for Chinese people in Wuhan city where a novel corona virus (COVID-19) was identified with rapid transmission rate. Later this virus spread in almost all parts of the globe at pandemic level and caused 111 million infections with 2.6 million deaths. According to Johns Hopkins University, the biggest amount of cases were reported in the United States of America with a tally of 28.1 M infections and 497 K deaths. Initially, it was considered that this virus came from the local fish market in Wuhan city; however, the transmission was identified from people to people with a huge ratio. This transmission happened through water, food, air droplets, and through physical contact with an infected person. The symptoms of COVID-19 infection last for 14 days, and to overcome or to resist the spread of this infection, 20 seconds of hands wash, avoidance of social gathering, and wearing face masks was suggested by the World Health Organization (WHO). Many countries banned traveling of people from one place to another to minimize the spreading ratio and also defined policies which can uplift the balance between country economy and health sector [1]. The scientists analyzed and made different experiments to find the cure or any medicinal treatment of the COVID-19 infection. Different countries have endorsed various mitigation strategies; however, the world still awaits the arrival of vaccine which is the only tool to fight against this infection. Approximately, 100 vaccines are under development, and some of these are in Phase

3 stage of clinical trials [2]. The number of vaccines has been identified in this regard with different recovery percentage. Currently, three vaccines are authorized and recommended to prevent COVID-19 i.e. Pfizer-BioNTech, Moderna's, and China's Sinopharm COVID-19 vaccine. As of December 28, 2020, large-scale (Phase 3) clinical trials have been in progress or being planned for three COVID-19 vaccines in the United States, namely AstraZeneca's COVID-19 vaccine, Janssen's COVID-19 vaccine, and Novavax's COVID-19 vaccine [3]. Recently, scientists from the field of medical engineering acknowledged the importance of mathematical modeling of any pandemic disease. Many of such mathematical modeling examples have already contributed to the control of infections [4–7]. These models also can be used for the prediction of expected patients in the future and can define well the control strategies. The mathematical models are usually developed in ordinary (ODEs) or in partial differential equations (PDEs) having equations of integration of natural order (IDEs). Such types of equations are well utilized in various fields of science i.e. medicine, economic, business, engineering, and analysis of different infections [8–16]. Recently, the implementation and application of fractional calculus for different models got attention from researchers [17–20]. Fractional calculus is defined as various kinds of possibilities of defining real or complex number powers [21–23]. Fractional calculus of any disease model plays a vital role in making decisions and helping to control the spread of infections. The fractional calculus was first communicated between Leibnitz and L'Hospital for the nth derivative of y. Fractional derivative was first introduced by Lacroix [24]. Afterward, many of the researchers introduced fractional derivatives in different forms, among which the most valuable are Caputo fractional derivative [25], Riemann–Liouville fractional derivative [26], and Atangana–Baleanu derivative [27]. Recently, various models have been solved by using fractional differential equations in many fields such as dynamics, control theory, and biology. The existence, uniqueness, and stability of models have been studied deeply [28–33]. In recent research a new idea of differentiation i.e. that the operator has fractional order as well as fractal dimension if the operator is of order two was proposed [34]. Usually, nonlinear models need specific parameters which are not available from experiments. The possible solution of these problems has been addressed by using fractal fractional derivatives. The fractal fractional derivatives models have advantage over the standard integer order derivatives [35, 36].

We use fractal fractional deactivate for the following formulation of SE(Is)(Ih)AR epidemic model with the help of [37]:

$$\begin{cases} {}^{FF}{}_0D_\tau^{u_1,u_2}S(\tau) = b_1 - [b_2 + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A) + K_v]S + \eta R, \\ {}^{FF}{}_0D_\tau^{u_1,u_2}E(\tau) = -(b_2 + \gamma)E + \beta(I_s + \beta_{hr}I_h + \beta_(ar)A)S, \\ {}^{FF}{}_0D_\tau^{u_1,u_2}I_s(\tau) = -(b_2 + \tau_0)I_s + \gamma P_s E, \\ {}^{FF}{}_0D_\tau^{u_1,u_2}I_h(\tau) = -(b_2 + \alpha + \tau_0 + K_T)I_h + \gamma P_h E, \\ {}^{FF}{}_0D_\tau^{u_1,u_2}A(\tau) = -(b_2 + \tau_0)A + \gamma(1 - P_s - P_h)E, \\ {}^{FF}{}_0D_\tau^{u_1,u_2}R(\tau) = -(b_2 + \eta)R + \tau_0(I_s + I_h + A) + K_T I_h + K_v S, \end{cases} \tag{1}$$

where $t > 0$ with the initial conditions $S(0) = S_0$, $E(0) = E_0$, $I_s(0) = I_s(0)$, $I_h(0) = I_h(0)$, $A(0) = A_0$,and $R(0) = R_0$ subject to $\min(S_0, E_0, I_{s_0}, I_{h_0}, A_0, R_0) \geq 0$. It is clear that the dimensions of both sides of the model are equivalent. In this model, b_1 is the recruitment rate, b_2 is the natural average death rate, $\beta(t)$, $\beta_{hr}\beta(t)$, $\beta_{ar}\beta(t)$ are the rates of transmission to the susceptible, $\frac{1}{\eta}$ is the average time of transition from the recovered to the susceptibles, γ is the rate of transition from the exposed class to the infectious group, α is the average mortality of the symptomatic infectious population, τ_0 is the natural immune response rate for the infected people, $p_s, p_h, p_a = 1 - p_s - p_h$ are the fractions of the exposed that become slightly symptomatic, seriously symptomatic, and asymptomatic infected people, respectively.

We highlight some more related articles used for the definitions and applications of the following notions [38–51].

Definition 1.1 Suppose that $\psi\tau$ is a continuous function and fractal differentiable in the interval (a, b) of order u_2, then the fractal fractional derivative of $\psi\tau$ of order $u_1 \in (0,1)$ in the Caputo sense is given by

$$ {}^{FF}{}_0D_\tau^{u_1,u_2}\psi(\tau) = \frac{AB(u_1)}{1-u_1}\int_0^\tau \frac{d}{dt^{u_2}} E_{u_1}\left(-\frac{u_1}{1-u_1}(\tau-s)^{u_1}\right)\psi(s)\,ds, \tag{2} $$

where $AB(u_1) = 1 - u_1 + \frac{u_1}{\Gamma u_1}$.

Definition 1.2 Suppose that $\psi(\tau)$ is a continuous function in the interval (a, b), then the fractal fractional integral of $\psi(\tau)$ of order u_1 having a Mittag-Leffler type kernel is given by

$$ {}^{FF}{}_0I_\tau^{u_1,u_2}\psi(\tau) = \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^\tau s^{u_2-1}\psi(s)(\tau-s)^{u_1-1}\,ds + \frac{u_2(1-u_1)\tau^{u_2-1}}{AB(u_1)}\psi(\tau). \tag{3} $$

2 Existence criteria

With the help of fixed point procedure we check the existence of fractal fractional to SE(Is)(Ih)AR epidemic model (1). We have

$$ \begin{cases} S(t) - S(0) \\ \quad = \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}(b_1 - [b_2 + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A) + K_v]S + \eta R)\,ds \\ \qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}(b_1 - [b_2 + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A) + K_v]S + \eta R), \\ E(t) - E(0) \\ \quad = \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}(-(b_2+\gamma)E + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A)S)\,ds \\ \qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}(-(b_2+\gamma)E + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A)S), \\ I_s(t) - I_s(0) = \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}(-(b_2+\tau_0)I_s + \gamma P_sE)\,ds \\ \qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}(-(b_2+\tau_0)I_s + \gamma P_sE), \\ I_h(t) - I_h(0) = \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}(-(b_2+\alpha+\tau_0+K_T)I_h + \gamma P_hE)\,ds \\ \qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}(-(b_2+\alpha+\tau_0+K_T)I_h + \gamma P_hE), \\ A(t) - A(0) = \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}(-(b_2+\tau_0)A + \gamma(1-P_s-P_h)E)\,ds \\ \qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}(-(b_2+\tau_0)A + \gamma(1-P_s-P_h)E), \\ R(t) - R(0) \\ \quad = \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}(-(b_2+\eta)R + \tau_0(I_s+I_h+A) + K_TI_h + K_vS)\,ds \\ \qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}(-(b_2+\eta)R + \tau_0(I_s+I_h+A) + K_TI_h + K_vS). \end{cases} \tag{4} $$

Now, we define some functions Q_i and some constants η_i, $i \epsilon N_1^6$ as follows:

$$\begin{cases} Q_1(t,S) = b_1 - [b_2 + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A) + K_v]S + \eta R, \\ Q_2(t,E) = -(b_2 + \gamma)E + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A)S, \\ Q_3(t,I_s) = -(b_2 + \tau_0)I_s + \gamma P_s E, \\ Q_4(t,I_h) == -(b_2 + \alpha + \tau_0 + K_T)I_h + \gamma P_h E, \\ Q_5(t,A) = -(b_2 + \tau_0)A + \gamma(1 - P_s - P_h)E, \\ Q_6(t,R) = -(b_2 + \eta)R + \tau_0(I_s + I_h + A) + K_T I_h + K_v S. \end{cases} \tag{5}$$

(G^*): For proving our results, we assume the following assumptions: The continuous functions $S(t)$, $E(t)$, $I_s(t)$, $I_h(t)$, $A(t)$, $R(t)$ and $S^*(t)$, $E^*(t)$, $I_s^*(t)$, $I_h^*(t)$, $A^*(t)$, $R^*(t)$ all belong to $L[0,1]$ such that $\|I_s\| \leq \psi_1$, $\|I_h\| \leq \psi_2$, $\|A\| \leq \psi_3$ for $\psi_1, \phi_2, \psi_3 > 0$ and constants.

Theorem 2.1 *The kernels Q_i for $i = 1,2,3,\ldots,6$ satisfy Lipschitz conditions if the assumption (G^*) holds and satisfies $\phi_i < 1$ for $i \in N_1^6$.*

Proof First, we prove that $Q_1(t,S)$ satisfies the Lipschitz condition. Using $S(t)$, $S^*(t)$, we have

$$\begin{aligned} \left\|Q_1(t,S) - Q_1(t,S^*)\right\| &= \left\|\left(b_1 - \left[b_2 + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A) + K_v\right]S + \eta R\right)\right. \\ &\quad \left. - \left(b_1 - \left[b_2 + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A) + K_v\right]S + \eta R\right)\right\| \\ &= \left\|(b_2 + \beta I_s + \beta_{hr}I_h + \beta_{ar}A + K_v)\left(S^* - S\right)\right\| \\ &\leq \left(b_2 + \beta\|I_s\| + \beta_{hr}\|I_h\| + \beta_{ar}\|A\| + K_v\right)\left\|\left(S^* - S\right)\right\| \\ &\leq (b_2 + \beta\psi_1 + \beta_{hr}\psi_2) + \beta_{ar}\psi_3 + K_v\left\|(S - S*)\right\| \\ &\leq \phi_1\left\|\left(S - S^*\right)\right\|. \end{aligned}$$

Hence Q_1 satisfies the Lipschitz condition and $\phi_1 < 1$. Next we prove that $Q_2(t,E)$ satisfies the Lipschitz condition. Now, using $E(t)$, $E^*(t)$, we have

$$\begin{aligned} \left\|Q_2(t,E) - Q_2(t,E^*)\right\| &= \left\|\left(-(b_2 + \gamma)E + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A)S\right)\right. \\ &\quad \left. - \left(-(b_2 + \gamma)E^* + \beta(I_s + \beta_{hr}I_h + \beta_{ar}A)S\right)\right\| \\ &= \left\|(b_2 + \lambda)\left(E^* - E\right)\right\| \\ &\leq (b_2 + \lambda)\left\|\left(E^* - E\right)\right\| \\ &\leq \phi_2\left\|E - E^*\right\|. \end{aligned}$$

Hence Q_2 satisfies the Lipschitz condition and $\phi_2 < 1$. Next we prove that $Q_3(t,I_s)$ satisfies the Lipschitz condition. Using $I_s(t)$, $I_s^*(t)$, we have

$$\begin{aligned} \left\|Q_3(t,I_s) - Q_3(t,I_s^*)\right\| &= \left\|\left(-(b_2 + \tau_0)I_s + \gamma P_s E\right) - \left(-(b_2 + \tau_0)I_s + \gamma P_s E\right)\right\| \\ &= \left\|(b_2 + \tau_0)\left(I_s^* - I_s\right)\right\| \end{aligned}$$

$$
\begin{aligned}
&\leq (b_2 + \tau_0)\left\| \left(I_s^* - I_s\right) \right\| \\
&\leq \phi_3 \left\| I_s - I_s^* \right\|.
\end{aligned}
$$

Hence Q_3 satisfies the Lipschitz condition and $\phi_3 < 1$. Next we prove that $Q_4(t, I_h)$ satisfies the Lipschitz condition. Using $I_h(t)$, $I_h^*(t)$, we have

$$
\begin{aligned}
&\left\| Q_4(t, I_h) - Q_4\left(t, I_h^*\right) \right\| \\
&\quad = \left\| \left(-(b_2 + \alpha + \tau_0 + K_T)I_h + \gamma P_h E\right) \right. \\
&\qquad \left. - \left(-(b_2 + \alpha + \tau_0 + K_T)I_h^* + \gamma P_h E\right) \right\| \\
&\quad = \|((b_2 + \alpha + \tau_0 + K_T)\left(I_h^* - I_h\right)\| \\
&\quad \leq ((b_2 + \alpha + \tau_0 + K_T)\left\| \left(I_h^* - I_h\right) \right\| \\
&\quad \leq \phi_4 \left\| I_h - I_h^* \right\|.
\end{aligned}
$$

Hence Q_4 satisfies the Lipschitz condition and $\phi_4 < 1$. Next we prove that $Q_5(t, A)$ satisfies the Lipschitz condition. Using $A(t)$, $A^*(t)$, we have

$$
\begin{aligned}
&\left\| Q_5(t, A) - Q_5\left(t, A^*\right) \right\| \\
&\quad = \left\| \left(-(b_2 + \tau_0)A + \gamma(1 - P_s - P_h)E\right) \right. \\
&\qquad \left. - \left(-(b_2 + \tau_0)A^* + \gamma(1 - P_s - P_h)E\right) \right\| \\
&\quad = \left\| (b_2 + \tau_0)\left(A^* - A\right) \right\| \\
&\quad \leq \|(b_2 + \tau_0 \left\| \left(A^* - A\right) \right\| \\
&\quad \leq \phi_5 \left\| A - A^* \right\|.
\end{aligned}
$$

Hence Q_5 satisfies the Lipschitz condition and $\phi_5 < 1$. Next we prove that $Q_6(t, R)$ satisfies the Lipschitz condition. Using $R(t)$, $R^*(t)$, we have

$$
\begin{aligned}
&\left\| Q_6(t, E_M) - Q_6\left(t, R^*\right) \right\| \\
&\quad = \left\| (\beta S_M I - \nu E_M - \mu_M E_M) - \left(\beta S_M I - \nu E_M^* - \mu_M E_M^*\right) \right\| \\
&\quad = \left\| (b_2 + \eta)\left(R^* - R\right) \right\| \\
&\quad \leq (b_2 + \eta)\left\| \left(R^* - R\right) \right\| \\
&\quad \leq \phi_6 \left\| R - R^* \right\|.
\end{aligned}
$$

Hence Q_6 satisfies the Lipschitz condition and $\phi_6 < 1$. Ultimately all the functions satisfy Lipschitz conditions and are contractions with $\phi_i < 1$ for $i \in N_1^6$. Hence this completes the proof. □

We rewrite the system of equations (4) in the following form by using the kernels Q_i, $i \in N_1^6$ and the initial conditions $S(0) = E(0) = I_s(0) = I_h(0) = A(0) = R(0) = 0$, we have

$$\begin{cases} S(t) = \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1(s, S(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1(t, S(t)), \\ E(t) = \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2(s, E(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2(t, E(t)), \\ I_s(t) = \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3(s, I_s(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3(t, I_s(t)), \\ I_h(t) = \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_4(s, I_h(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_4(t, I_h(t)), \\ A(t) = \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_5(s, A(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_5(t, A(t)), \\ R(t) = \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6(s, R(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6(t, R(t)). \end{cases} \tag{6}$$

Now we define the following recursive formulas:

$$\begin{aligned} S_n(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma(u_1)} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1\bigl(s, S_{n-1}(s)\bigr)\, ds \\ &\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1\bigl(t, S_{n-1}(t)\bigr), \\ E_n(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2\bigl(s, E_{n-1}(s)\bigr)\, ds \\ &\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\bigl(t, E_{n-1}(t)\bigr), \\ I_{s_n}(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\bigl(s, I_{s_{n-1}}(s)\bigr)\, ds \\ &\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\bigl(t, I_{s_{n-1}}(t)\bigr), \\ I_{h_n}(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_4\bigl(s, I_{h_{n-1}}(s)\bigr)\, ds \\ &\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_4\bigl(t, I_{h_{n-1}}(t)\bigr), \\ A_n(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_5\bigl(s, A_{n-1}(s)\bigr)\, ds \\ &\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_5\bigl(t, A_{n-1}(t)\bigr), \\ R_n(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6\bigl(s, R_{n-1}(s)\bigr)\, ds \\ &\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6\bigl(t, R_{n-1}(t)\bigr). \end{aligned}$$

Now we consider the following differences:

$$\begin{aligned} DS_{n+1}(t) &= S_{n+1} - S_n \\ &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1\bigl(s, S_n(s)\bigr)\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1\bigl(t, S_n(t)\bigr) \\ &\quad - \biggl(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1\bigl(s, S_{n-1}(s)\bigr)\, ds \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1\big(t, S_{n-1}(t)\big)\Bigg) \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1}\big(Q_1\big(s, S_n(s)\big) - Q_1\big(s, S_{n-1}(s)\big)\big)\, ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big(Q_1\big(t, S_n(t)\big) - Q_1\big(t, S_{n-1}(t)\big)\big), \\
DE_{n+1}(t) &= E_{n+1} - E_n \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2\big(s, E_n(s)\big)\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\big(t, E_n(t)\big) \\
&\quad - \Bigg(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2\big(s, E_{n-1}(s)\big)\, ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\big(t, E_{n-1}(t)\big)\Bigg) \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1}\big(Q_2\big(s, E_n(s)\big) - Q_1\big(s, E_{n-1}(s)\big)\big)\, ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big(Q_2\big(t, E_n(t)\big) - Q_2\big(t, E_{n-1}(t)\big)\big), \\
DI_{s_{n+1}}(t) &= I_{s_{n+1}} - I_{H_n} \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\big(s, I_{s_n}(s)\big)\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\big(t, I_{s_n}(t)\big) \\
&\quad - \Bigg(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\big(s, I_{s_{n-1}}(s)\big)\, ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\big(t, I_{s_{n-1}}(t)\big)\Bigg) \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1}\big(Q_3\big(s, I_{s_n}(s)\big) - Q_3\big(s, I_{s_{n-1}}(s)\big)\big)\, ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big(Q_3\big(t, I_{s_n}(t)\big) - Q_3\big(t, I_{s_{n-1}}(t)\big)\big), \\
DI_{h_{n+1}}(t) &= I_{h_{n+1}} - I_{h_n} \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_4\big(s, I_{h_n}(s)\big)\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_4\big(t, I_{h_n}(t)\big) \\
&\quad - \Bigg(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_4\big(s, I_{h_{n-1}}(s)\big)\, ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_4\big(t, I_{h_{n-1}}(t)\big)\Bigg) \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1}\big(Q_4\big(s, I_{h_n}(s)\big) - Q_4\big(s, I_{h_{n-1}}(s)\big)\big)\, ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big(Q_4\big(t, I_{h_n}(t)\big) - Q_4\big(t, I_{h_{n-1}}(t)\big)\big), \\
DA_{n+1}(t) &= A_{n+1} - A_n
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_5\bigl(s, A_n(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_5\bigl(t, A_n(t)\bigr) \\
&\quad - \biggl(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_5\bigl(s, A_{n-1}(s)\bigr)\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_5\bigl(t, A_{n-1}(t)\bigr)\biggr) \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \bigl(Q_5\bigl(s, A_n(s)\bigr) - Q_5\bigl(s, A_{n-1}(s)\bigr)\bigr)\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \bigl(Q_5\bigl(t, A_n(t)\bigr) - Q_5\bigl(t, A_{n-1}(t)\bigr)\bigr),
\end{aligned}
$$

$$
\begin{aligned}
DR_{n+1}(t) &= R_{n+1} - R_n \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6\bigl(s, R_n(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6\bigl(t, R_n(t)\bigr) \\
&\quad - \biggl(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6\bigl(s, R_{n-1}(s)\bigr)\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6\bigl(t, R_{n-1}(t)\bigr)\biggr) \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \bigl(Q_6\bigl(s, R_n(s)\bigr) - Q_6\bigl(s, R_{n-1}(s)\bigr)\bigr)\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \bigl(Q_6\bigl(t, R_n(t)\bigr) - Q_6\bigl(t, R_{n-1}(t)\bigr)\bigr).
\end{aligned}
$$

Taking norm of the above differences, we have

$$
\begin{aligned}
&\bigl\|DS_{n+1}(t)\bigr\| \\
&\quad = \|S_{n+1} - S_n\| \\
&\quad = \biggl\|\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1\bigl(s, S_n(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1\bigl(t, S_n(t)\bigr) \\
&\qquad - \biggl(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1\bigl(s, S_{n-1}(s)\bigr)\,ds \\
&\qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1\bigl(t, S_{n-1}(t)\bigr)\biggr)\biggr\| \\
&\quad = \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \bigl\|Q_1\bigl(s, S_n(s)\bigr) - Q_1\bigl(s, S_{n-1}(s)\bigr)\bigr\|\,ds \\
&\qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \bigl\|Q_1\bigl(t, S_n(t)\bigr) - Q_1\bigl(t, S_{n-1}(t)\bigr)\bigr\|,
\end{aligned}
$$

$$
\begin{aligned}
&\bigl\|DE_{n+1}(t)\bigr\| \\
&\quad = \|E_{n+1} - E_n\| \\
&\quad = \biggl\|\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2\bigl(s, E_n(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\bigl(t, E_n(t)\bigr) \\
&\qquad - \biggl(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2\bigl(s, E_{n-1}(s)\bigr)\,ds
\end{aligned}
$$

$$+ \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\big(t, E_{n-1}(t)\big)\bigg)\bigg\|$$

$$= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \big\| \big(Q_2\big(s, E_n(s)\big) - Q_1\big(s, E_{n-1}(s)\big)\big) \big\| \, ds$$

$$+ \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \|(Q_2\big(t, E_n(t)\big) - Q_2\big(t, E_{n-1}(t)\big)\|,$$

$$\big\| DI_{s_{n+1}}(t) \big\|$$

$$= \|I_{s_{n+1}} - I_{s_n}\|$$

$$= \bigg\| \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\big(s, I_{s_n}(s)\big) \, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\big(t, I_{s_n}(t)\big)$$

$$- \bigg(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\big(s, I_{s_{n-1}}(s)\big) \, ds$$

$$+ \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\big(t, I_{s_{n-1}}(t)\big)\bigg)\bigg\|$$

$$= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \| \big(Q_3\big(s, I_{s_n}(s)\big) - Q_3\big(s, I_{s_{n-1}}(s)\big) \|\big) \, ds$$

$$+ \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \big\| Q_3\big(t, I_{s_n}(t)\big) - Q_3\big(t, I_{s_{n-1}}(t)\big) \big\|,$$

$$\big\| DI_{h_{n+1}}(t) \big\|$$

$$= \|I_{h_{n+1}} - I_{h_n}\|$$

$$= \bigg\| \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_4\big(s, I_{h_n}(s)\big) \, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_4\big(t, I_{h_n}(t)\big)$$

$$- \bigg(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_4\big(s, I_{h_{n-1}}(s)\big) \, ds$$

$$+ \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_4\big(t, I_{h_{n-1}}(t)\big)\bigg)\bigg\|$$

$$= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \big\| Q_4\big(s, I_{h_n}(s)\big) - Q_4\big(s, I_{h_{n-1}}(s)\big) \big\| \, ds$$

$$+ \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \big\| Q_4\big(t, I_{h_n}(t)\big) - Q_4\big(t, I_{h_{n-1}}(t)\big) \big\|,$$

$$\big\| DA_{n+1}(t) \big\|$$

$$= \|A_{n+1} - A_n\|$$

$$= \bigg\| \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_5\big(s, A_n(s)\big) \, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_5\big(t, A_n(t)\big)$$

$$- \bigg(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_5\big(s, A_{n-1}(s)\big) \, ds$$

$$+ \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_5\big(t, A_{n-1}(t)\big)\bigg)\bigg\|$$

$$= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \big\| Q_5\big(s, A_n(s)\big) - Q_5\big(s, A_{n-1}(s)\big) \big\| \, ds$$

$$+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left\|Q_5\big(t,A_n(t)\big)-Q_5\big(t,A_{n-1}(t)\big)\right\|,$$

$$\begin{aligned}
&\left\|DR_{n+1}(t)\right\|\\
&=\|R_{n+1}-R_n\|\\
&=\left\|\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_6\big(s,R_n(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_6\big(t,R_n(t)\big)\right.\\
&\quad-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_6\big(s,R_{n-1}(s)\big)\,ds\right.\\
&\quad\left.\left.+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_6\big(t,R_{n-1}(t)\big)\right)\right\|\\
&=\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left\|Q_6\big(s,R_n(s)\big)-Q_6\big(s,R_{n-1}(s)\big)\right\|ds\\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left\|Q_6\big(t,R_n(t)\big)-Q_6\big(t,R_{n-1}(t)\big)\right\|.
\end{aligned}$$

Theorem 2.2 *The fractal fractional of diffusion model SE(Is)(Ih)AR epidemic has a solution if the following holds true*:

$$\sigma=\max\{\phi_1,\phi_2,\dots,\phi_6\}<1.$$

Proof Let us define the following functions:

$$\begin{cases}
G_1n(t)=S_{n+1}(t)-S(t),\\
G_2n(t)=E_{n+1}(t)-E(t),\\
G_3n(t)=I_{s_{n+1}}(t)-I_s(t),\\
G_4n(t)=I_{h_{n+1}}(t)-I_h(t),\\
G_5n(t)=A_{n+1}(t)-A(t),\\
G_6n(t)=R_{n+1}(t)-R(t).
\end{cases}\tag{7}$$

Taking norm of the above system, we have

$$\begin{aligned}
&\left\|G_1n(t)\right\|\\
&=\left\|S_{n+1}(t)-S(t)\right\|\\
&=\left\|\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_1\big(s,S_n(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_1\big(t,S_n(t)\big)\right.\\
&\quad\left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_1(s,S)(s)\right)ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_1(t,S)(t))\right\|\\
&=\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left\|Q_1\big(s,S_n(s)\big)-Q_1(s,S)(s)\right\|ds\\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left\|Q_1\big(t,S_n(t)\big)-Q_1(t,S)(t)\right\|\\
&\le\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_1\|S_n-S\|
\end{aligned}$$

$$\leq \left(\frac{u_1 u_2 \Gamma u_2}{AB(u_1 \Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_1 \|S_n - S\|$$
$$\leq \left(\frac{u_1 u_2 \Gamma u_2}{AB(u_1 \Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\right)^n \sigma^n \|S_1 - S\|,$$

where $\sigma < 1$ and as $n \to \infty$ so $S_n \to S$, and using the formula $B(u,v) = (b-a)^{-u+v+1} \int_a^b (s-a)^{u-1}(b-s)^{v-1}\,ds$ and as $t \in [0,1]$ so $t^{-1-u_1+u_2} \leq 1$ and $t^{u_2} \leq 1$,

$$\begin{aligned}
\|G_2 n(t)\| &= \|E_{n+1}(t) - E(t)\| \\
&= \bigg\| \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2\big(s, E_n(s)\big)\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\big(t, E_n(t)\big) \\
&\quad - \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2(s,E)(s)\right) ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\big(t, E(t)\big)\bigg\| \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \big\| Q_2\big(s, E_n(s)\big) - Q_2(s,E)(s)\big\|\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \big\| Q_2\big(t, E_n(t)\big) - Q_2(t,E)(t)\big\| \\
&\leq \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_2 \|E_n - E\| \\
&\leq \left(\frac{u_1 u_2 \Gamma u_2}{AB(u_1 \Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_2 \|E_n - E\| \\
&\leq \left(\frac{u_1 u_2 \Gamma u_2}{AB(u_1 \Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\right)^n \sigma^n \|E_1 - E\|,
\end{aligned}$$

where $\sigma < 1$ and as $n \to \infty$ so $E_{H_n} \to E$.

$$\begin{aligned}
&\|G_3 n(t)\| \\
&\quad = \|I_{s_{n+1}}(t) - I_s(t)\| \\
&\quad = \bigg\| \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\big(s, I_{s_n}(s)\big)\,ds \\
&\qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\big(t, I_{s_n}(t)\big) \\
&\qquad - \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\big(s, I_s(s)\big)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\big(t, I_s(t)\big)\right)\bigg\| \\
&\quad = \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \big\| Q_3\big(s, I_{s_n}(s)\big) - Q_3\big(s, I_s(s)\big)\big\|\,ds \\
&\qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \big\| Q_3\big(t, I_{s_n}(t)\big) - Q_3\big(t, I_s(t)\big)\big\| \\
&\quad \leq \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_3 \|I_{s_n} - I_s\|
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_3\|I_{s_n}-I_s\| \\
&\leq \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)^n\sigma^n\|I_{s_1}-I_s\|,
\end{aligned}$$

where $\sigma < 1$ and as $n\to\infty$ so $I_{s_n}\to I_s$.

$$\begin{aligned}
&\|G_4n(t)\| \\
&\quad = \|I_{h_{n+1}}(t)-I_h(t)\| \\
&\quad = \left\|\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_4\big(s,I_{h_n}(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_4\big(t,I_{h_n}(t)\big)\right. \\
&\qquad \left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_4\big(s,I_h(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_4\big(t,I_h(t)\big)\right)\right\| \\
&\quad = \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\big\|Q_4\big(s,I_{h_n}(s)\big)-Q_4\big(s,I_h(s)\big)\big\|\,ds \\
&\qquad +\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big\|Q_4\big(t,I_{h_n}(t)\big)-Q_4\big(t,I_h(t)\big)\big\| \\
&\quad \leq \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_4\|I_{h_n}-I_h\| \\
&\quad \leq \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_4\|I_{h_n}-I_h\| \\
&\quad \leq \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)^n\sigma^n\|I_{h_1}-I_h\|,
\end{aligned}$$

where $\sigma < 1$ and as $n\to\infty$ so $I_{h_n}\to I_h$.

$$\begin{aligned}
&\|G_5n(t)\| \\
&\quad = \|A_{n+1}(t)-A(t)\| \\
&\quad = \left\|\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_5\big(s,A_n(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_5\big(t,A_n(t)\big)\right. \\
&\qquad \left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_5(s,A)(s)\right)ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_5(t,A)(t)))\right\| \\
&\quad = \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\|Q_5\big(s,A_n(s)\big)-Q_5(s,A)(s))\Big\|\,ds \\
&\qquad +\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\Big\|Q_5\big(t,A_n(t)\big)-Q_5(t,A)(t))\| \\
&\quad \leq \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_5\|A_n-A\| \\
&\quad \leq \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_5\|A_n-A\| \\
&\quad \leq \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)^n\sigma^n\|A_1-A\|,
\end{aligned}$$

where $\sigma < 1$ and as $n \to \infty$ so $A_n \to A$.

$$\begin{aligned}
&\|G_6 n(t)\| \\
&= \|R_{n+1}(t) - R(t)\| \\
&= \Bigg\| \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6(s, R_n(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6(t, R_n(t)) \\
&\quad - \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6(s, R)(s) \right) ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6(t, R)(t))) \Bigg\| \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \| Q_6(s, R_n(s)) - Q_6(s, R)(s)) \Big\| \, ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \Big\| Q_6(t, R_n(t)) - Q_6(t, R)(t)) \| \\
&\leq \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \right) \phi_6 \|R_n - R\| \\
&\leq \left(\frac{u_1 u_2 \Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)} \right) \phi_6 \|R_n - R\| \\
&\leq \left(\frac{u_1 u_2 \Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)} \right)^n \sigma^n \|R_1 - R\|,
\end{aligned}$$

where $\sigma < 1$ and as $n \to \infty$ so $R_n \to R$. Thus we find that $G_i n(t) \to 0$ as $n \to \infty$ for $i \in N_1^6$ AND $\sigma < 1$. Hence this completes the proof. □

2.1 Uniqueness of the solution

Theorem 2.3 *The fractal fractional model* (1) *has a unique solution if the following inequalities hold true*:

$$\left(\frac{u_1 u_2 \Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)} \right) \phi_i \leq 1, \quad i \in N_1^6. \tag{8}$$

Proof Let us consider the contradiction that there exists another solution of fractal fractional model (1) such that S^*, E^*, I_s^*, I_h^*, A^*, R^* satisfying the given model. We have

$$\begin{aligned}
S^*(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1(s, S^*(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1(t, S^*(t)), \\
E^*(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2(s, E^*(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2(t, E^*(t)), \\
I_s^*(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3(s, I_s^*(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3(t, I_s^*(t)), \\
I_h^*(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_4(s, I_h^*(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_4(t, I_h^*(t)), \\
A^*(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_5(s, A^*(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_5(t, A^*(t)), \\
R^*(t) &= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6(s, R^*(s))\, ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6(t, R^*(t)).
\end{aligned}$$

Now, taking norm of the difference of $S(t)$, $S^*(t)$, we have

$$\begin{aligned}
&\left\|S(t)-S^*(t)\right\| \\
&= \left\|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_1\bigl(s,S(s)\bigr)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_1\bigl(t,S(t)\bigr)\right)\right.\\
&\quad\left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_1\bigl(s,S^*(s)\bigr)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_1\bigl(t,S^*(t)\bigr)\right)\right\|\\
&= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left\|Q_1(s,S(s)-Q_1\bigl(s,S^*(s)\bigr)\right\|\,ds\\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left\|Q_1(t,S(t)-Q_1\bigl(t,S^*(t)\bigr)\right\|\,ds\\
&\le \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_1\left\|S-S^*\right\|\\
&\quad\times\left[1-\left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_1\right]\left\|S-S^*\right\|\le 0.
\end{aligned}$$

The above inequality is true if $\|S-S^*\|=0$, which implies $S=S^*$. Similarly, taking norm of the difference of $E(t)$, $E^*(t)$, we have

$$\begin{aligned}
&\left\|E(t)-E^*(t)\right\| \\
&= \left\|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_2\bigl(s,E(s)\bigr)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_2\bigl(t,E(t)\bigr)\right)\right.\\
&\quad\left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_2\bigl(s,E^*(s)\bigr)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_2\bigl(t,E^*(t)\bigr)\right)\right\|\\
&= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left\|Q_2(s,E(s)-Q_2\bigl(s,E^*(s)\bigr)\right\|\,ds\\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left\|Q_2(t,E(t)-Q_2\bigl(t,E^*(t)\bigr)\right\|\,ds\\
&\le \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_2\left\|E-E^*\right\|\\
&\quad\times\left[1-\left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_2\right]\left\|E-E^*\right\|\le 0.
\end{aligned}$$

The above inequality is true if $\|E-E^*\|=0$, which implies $E=E^*$. Similarly, taking norm of the difference of $I_s(t)$, $I_s^*(t)$, we have

$$\begin{aligned}
&\left\|I_s(t)-I_s^*(t)\right\| \\
&= \left\|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_3\bigl(s,I_s(s)\bigr)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_3\bigl(t,I_s(t)\bigr)\right)\right.\\
&\quad\left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_3\bigl(s,I_s^*(s)\bigr)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_3\bigl(t,I_s^*(t)\bigr)\right)\right\|\\
&= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left\|Q_3(s,I_s(s)-Q_3\bigl(s,I_s^*(s)\bigr)\right\|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left\|Q_3(t,I_s(t)-Q_3\big(t,I_s^*(t)\big)\right\|ds\\
&\le\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_3\left\|I_s-I_s^*\right\|\\
&\times\left[1-\left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_3\right]\left\|I_s-I_s^*\right\|\le 0.
\end{aligned}
$$

The above inequality is true if $\|I_s-I_s^*\|=0$, which implies $I_s=I_s^*$. Similarly, taking norm of the difference of $I_h(t)$, $I_h^*(t)$, we have

$$
\begin{aligned}
&\left\|I_h(t)-I_h^*(t)\right\|\\
&=\left\|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_4\big(s,I_h(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_4\big(t,I_h(t)\big)\right)\right.\\
&\quad\left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_4\big(s,I_h^*(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_4\big(t,I_h^*(t)\big)\right)\right\|\\
&=\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left\|Q_4(s,I_h(s)-Q_4\big(s,I_h^*(s)\big)\right\|ds\\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left\|Q_4(t,I_h(t)-Q_4\big(t,I_h^*(t)\big)\right\|ds\\
&\le\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_4\left\|I_h-I_h^*\right\|\\
&\quad\times\left[1-\left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_4\right]\left\|I_h-I_h^*\right\|\le 0.
\end{aligned}
$$

The above inequality is true if $\|I_h-I_h^*\|=0$, which implies $I_h=I_h^*$. Similarly, taking norm of the difference of $A(t)$, $A^*(t)$, we have

$$
\begin{aligned}
&\left\|A(t)-A^*(t)\right\|\\
&=\left\|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_5\big(s,A(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_5\big(t,A(t)\big)\right)\right.\\
&\quad-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_5\big(s,A^*(s)\big)\,ds\right.\\
&\quad\left.\left.+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_5\big(t,A^*(t)\big)\right)\right\|\\
&=\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left\|Q_5(s,A(s)-Q_5\big(s,A^*(s)\big)\right\|ds\\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left\|Q_5(t,A(t)-Q_5\big(t,A^*(t)\big)\right\|ds\\
&\le\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_5\left\|A-A^*\right\|\\
&\quad\times\left[1-\left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_5\right]\left\|A-A^*\right\|\le 0.
\end{aligned}
$$

The above inequality is true if $\|A - A^*\| = 0$, which implies $A = A^*$. Similarly, taking norm of the difference of $R(t)$, $R^*(t)$, we have

$$\begin{aligned}
&\bigl\|R(t) - R^*(t)\bigr\| \\
&= \biggl\| \biggl(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6\bigl(s, R(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6\bigl(t, R(t)\bigr) \biggr) \\
&\quad - \biggl(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6\bigl(s, R^*(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6\bigl(t, R^*(t)\bigr) \biggr) \biggr\| \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} \| Q_6(s, R(s) - Q_2\bigl(s, R^*(s)\bigr) \biggr\|\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \biggl\| Q_6(t, R(t) - Q_6\bigl(t, R^*(t)\bigr) \|\,ds \\
&\leq \biggl(\frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} \biggr) \phi_6 \bigl\|R - R^*\bigr\| \\
&\quad \times \biggl[1 - \biggl(\frac{u_1 u_2 \Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)} \biggr) \phi_6 \biggr] \bigl\|R - R^*\bigr\| \leq 0.
\end{aligned}$$

The above inequality is true if $\|R - R^*\| = 0$, which implies $R(t)$, $R^*(t)$. Hence we see that $S = S^*$, $E = E^*$, $I_s = I_s^*$, $rI_h = I_h^*$, $A = A^*$, $R = R^*$, so our supposition is wrong and the theorem has a unique solution. □

Hyers–Ulam stability

Definition 2.4 The fractal fractional integrals (6) are said to be Hyers–Ulam stable if there exist constants $\alpha_i > 0$, $i \in N_1^6$ satisfying, for every $\beta_i > 0$, $i \in N_1^6$, the following:

$$\begin{aligned}
&\biggl| S(t) - \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1\bigl(s, S(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1\bigl(t, S(t)\bigr) \biggr| \leq \beta_1, \\
&\biggl| E(t) - \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2\bigl(s, E(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\bigl(t, E(t)\bigr) \biggr| \leq \beta_2, \\
&\biggl| I_s(t) - \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\bigl(s, I_s(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\bigl(t, I_s(t)\bigr) \biggr| \\
&\quad \leq \beta_3, \\
&\biggl| I_h(t) - \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_4\bigl(s, I_h(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_4\bigl(t, I_h(t)\bigr) \biggr| \\
&\quad \leq \beta_4, \\
&\biggl| A(t) - \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_5\bigl(s, A(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_5\bigl(t, A(t)\bigr) \biggr| \\
&\quad \leq \beta_5, \\
&\biggl| R(t) - \frac{u_1 u_2}{AB(u_1)\Gamma u_1} \int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_6\bigl(s, R(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_6\bigl(t, R(t)\bigr) \biggr| \leq \beta_6.
\end{aligned}$$

There exists an approximate solution of model (1) $S^*(t)$, $E^*(t)$, $I_s^*(t)$, $I_h^*(t)$, $A^*(t)$, $R^*(t)$ that satisfies the given model, such that

$$\begin{aligned}
&\left|S(t) - S^*(t)\right| \\
&= \left|\left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1\bigl(s, S(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1\bigl(t, S(t)\bigr)\right)\right. \\
&\quad \left. - \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_1\bigl(s, S^*(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_1\bigl(t, S^*(t)\bigr)\right)\right| \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left|Q_1(s, S(s) - Q_1\bigl(s, S^*(s)\bigr)\right|\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left|Q_1(t, S(t) - Q_1\bigl(t, S^*(t)\bigr)\right|\,ds \\
&\leq \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_1\left\|S - S^*\right\| \\
&\leq \left(\frac{u_1 u_2 \Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_1\left\|S - S^*\right\|.
\end{aligned}$$

Let $\zeta_1 = (\frac{u_1 u_2 \Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)})\|S - S^*\|$, $\eta_1 = \phi_1$,so the above inequality becomes $|S - S^*| \leq \zeta_1\eta_1$.

$$\begin{aligned}
&\left|E(t) - E^*(t)\right| \\
&= \left|\left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2\bigl(s, E(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\bigl(t, E(t)\bigr)\right)\right. \\
&\quad \left. - \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_2\bigl(s, E^*(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_2\bigl(t, E^*(t)\bigr)\right)\right| \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left|Q_2(s, E(s) - Q_2\bigl(s, E^*(s)\bigr)\right|\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left|Q_2(t, E(t) - Q_2\bigl(t, E^*(t)\bigr)\right|\,ds \\
&\leq \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_2\left\|E - E^*\right\| \\
&\leq \left(\frac{u_1 u_2 \Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_2\left\|E - E^*\right\|.
\end{aligned}$$

Let $\zeta_2 = (\frac{u_1 u_2 \Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)})\|E - E^*\|$, $\eta_2 = \phi_2$,so the above inequality becomes $|E - E^*| \leq \zeta_2\eta_2$.

$$\begin{aligned}
&\left|I_s(t) - I_s^*(t)\right| \\
&= \left|\left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\bigl(s, I_s(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\bigl(t, I_s(t)\bigr)\right)\right. \\
&\quad \left. - \left(\frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} Q_3\bigl(s, I_s^*(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)} Q_3\bigl(t, I_s^*(t)\bigr)\right)\right| \\
&= \frac{u_1 u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left|Q_3(s, I_s(s) - Q_3\bigl(s, I_s^*(s)\bigr)\right|\,ds
\end{aligned}$$

$$\begin{aligned}
&+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big|Q_3(t,I_s(t)-Q_3\big(t,I_s^*(t)\big)\big|\,ds\\
&\le\left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_3\big\|I_s-I_s^*\big\|.
\end{aligned}$$

Let $\zeta_3=(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)})\|I_s-I_s^*\|$, $\eta_3=\phi_3$, so the above inequality becomes $|I_s-I_s^*|\le\zeta_3\eta_3$.

$$\begin{aligned}
&\big|I_h(t)-I_h^*(t)\big|\\
&=\left|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_4\big(s,I_h(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_4\big(t,I_h(t)\big)\right)\right.\\
&\quad\left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_4\big(s,I_h^*(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_4\big(t,I_h^*(t)\big)\right)\right|\\
&=\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\big|Q_4(s,I_h(s)-Q_4\big(s,I_h^*(s)\big)\big|\,ds\\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big|Q_4(t,I_h(t)-Q_4\big(t,I_h^*(t)\big)\big|\,ds\\
&\le\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_4\big\|I_h-I_h^*\big\|\\
&\le\left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_4\big\|I_h-I_h^*\big\|.
\end{aligned}$$

Let $\zeta_4=(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)})\|I_h-I_h^*\|$, $\eta_4=\phi_4$, so the above inequality becomes $|I_h-I_h^*|\le\zeta_4\eta_4$.

$$\begin{aligned}
&\big|A(t)-A^*(t)\big|\\
&=\left|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_5\big(s,A(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_5\big(t,A(t)\big)\right)\right.\\
&\quad\left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_5\big(s,A^*(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_5\big(t,A^*(t)\big)\right)\right|\\
&=\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\big|Q_5(s,A(s)-Q_5\big(s,A^*(s)\big)\big|\,ds\\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big|Q_5(t,A(t)-Q_5\big(t,A^*(t)\big)\big|\,ds\\
&\le\left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_5\big\|A-A^*\big\|.
\end{aligned}$$

Let $\zeta_5=(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)})\|A-A^*\|$, $\eta_5=\phi_5$, so the above inequality becomes $|A-A^*|\le\zeta_5\eta_5$.

$$\begin{aligned}
&\big|R(t)-R^*(t)\big|\\
&=\left|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_6\big(s,R(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_6\big(t,R(t)\big)\right)\right.\\
&\quad\left.-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_6\big(s,R^*(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_6\big(t,R^*(t)\big)\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}|Q_6(s,R(s)-Q_6\big(s,R^*(s)\big)\Big|\,ds \\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\Big|Q_6(t,R(t)-Q_6\big(t,R^*(t)\big)|\,ds \\
&\le \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_6\big\|R-R^*\big\|.
\end{aligned}
$$

Let $\zeta_6 = (\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)})\|R-R^*\|$, $\eta_6 = \phi_6$, so the above inequality becomes $|R - R^*| \le \zeta_6\eta_6$.

Theorem 2.5 *With assumption* (G^*), *the fractal fractional model* (1) *is Hyers–Ulam stable.*

Proof We know that the fractal fractional model (1) has a unique solution. Let there exist an approximate solution of model (1) $S^*(t)$, $E^*(t)$, $I_s^*(t)$, $I_h^*(t)$, $A^*(t)$, $R^*(t)$ that satisfies the given model, such that

$$
\begin{aligned}
&\big|S(t)-S^*(t)\big| \\
&= \Bigg|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_1\big(s,S(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_1\big(t,S(t)\big)\right) \\
&\quad-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_1\big(s,S^*(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_1\big(t,S^*(t)\big)\right)\Bigg| \\
&= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\big|Q_1(s,S(s)-Q_1\big(s,S^*(s)\big)\big|\,ds \\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big|Q_1(t,S(t)-Q_1\big(t,S^*(t)\big)\big|\,ds \\
&\le \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_1\big\|S-S^*\big\| \\
&\le \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))}+\frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_1\big\|S-S^*\big\|.
\end{aligned}
$$

Let $\alpha_1 = (\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)})\|S-S^*\|$, $\Delta_1 = \phi_1$, so the above inequality becomes $|S - S^*| \le \alpha_1\Delta_1$.

$$
\begin{aligned}
&\big|E(t)-E^*(t)\big| \\
&= \Bigg|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_2\big(s,E(s)\big)\,ds+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_2\big(t,E(t)\big)\right) \\
&\quad-\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_2\big(s,E^*(s)\big)\,ds \right.\\
&\quad\left.+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_2\big(t,E^*(t)\big)\right)\Bigg| \\
&= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\big|Q_2(s,E(s)-Q_2\big(s,E^*(s)\big)\big|\,ds \\
&\quad+\frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\big|Q_2(t,E(t)-Q_2\big(t,E^*(t)\big)\big|\,ds
\end{aligned}
$$

$$\leq \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_2\|E-E^*\|$$
$$\leq \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_2\|E-E^*\|.$$

Let $\alpha_2 = (\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)})\|E-E^*\|$, $\Delta_2 = \phi_2$, so the above inequality becomes $|E - E^*| \leq \alpha_2\Delta_2$.

$$\begin{aligned}
&\left|I_s(t) - I_s^*(t)\right| \\
&= \left|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_3\big(s,I_s(s)\big)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_3\big(t,I_s(t)\big)\right)\right. \\
&\quad - \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_3\big(s,I_s^*(s)\big)\,ds\right. \\
&\quad \left.\left. + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_3\big(t,I_s^*(t)\big)\right)\right| \\
&= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left|Q_3(s,I_s(s) - Q_3\big(s,I_s^*(s)\big)\right|\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left|Q_3(t,I_s(t) - Q_3\big(t,I_s^*(t)\big)\right|\,ds \\
&\leq \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_3\|I_s - I_s^*\| \\
&\leq \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_3\|I_s - I_s^*\|.
\end{aligned}$$

Let $\alpha_3 = (\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)})\|I_s - I_s^*\|$, $\Delta_3 = \phi_3$, so the above inequality becomes $|I_s - I_s^*| \leq \alpha_3\Delta_3$.

$$\begin{aligned}
&\left|I_h(t) - I_h^*(t)\right| \\
&= \left|\left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_4\big(s,I_h(s)\big)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_4\big(t,I_h(t)\big)\right)\right. \\
&\quad - \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_4\big(s,I_h^*(s)\big)\,ds\right. \\
&\quad \left.\left. + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_4\big(t,I_h^*(t)\big)\right)\right| \\
&= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\left|Q_4(s,I_h(s) - Q_4\big(s,I_h^*(s)\big)\right|\,ds \\
&\quad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\left|Q_4(t,I_h(t) - Q_4\big(t,I_h^*(t)\big)\right|\,ds \\
&\leq \left(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\right)\phi_4\|I_h - I_h^*\| \\
&\leq \left(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\right)\phi_4\|I_h - I_h^*\|.
\end{aligned}$$

Let $\alpha_4 = (\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)})\|I_h - I_h^*\|$, $\Delta_4 = \phi_4$, so the above inequality becomes $|I_h - I_h^*| \le \alpha_4\Delta_4$.

$$
\begin{aligned}
&\bigl|A(t) - A^*(t)\bigr| \\
&\quad= \biggl|\biggl(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_5\bigl(s,A(s)\bigr)\,ds + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_5\bigl(t,A(t)\bigr)\biggr) \\
&\qquad - \biggl(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_5\bigl(s,A^*(s)\bigr)\,ds \\
&\qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_5\bigl(t,A^*(t)\bigr)\biggr)\biggr| \\
&\quad= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\bigl|Q_5(s,A(s) - Q_5\bigl(s,A^*(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\bigl|Q_5(t,A(t) - Q_5\bigl(t,A^*(t)\bigr)\bigr|\,ds \\
&\quad\le \biggl(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\biggr)\phi_5\bigl\|A - A^*\bigr\| \\
&\quad\le \biggl(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\biggr)\phi_5\bigl\|A - A^*\bigr\|.
\end{aligned}
$$

Let $\alpha_5 = (\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)})\|A - A^*\|$, $\Delta_5 = \phi_5$, so the above inequality becomes $|A - A^*| \le \alpha_5\Delta_5$.

$$
\begin{aligned}
&\bigl|R(t) - R^*(t)\bigr| \\
&\quad= \biggl|\biggl(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_6\bigl(s,R(s)\bigr)\,ds \\
&\qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_6\bigl(t,R(t)\bigr)\biggr) \\
&\qquad - \biggl(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}Q_6\bigl(s,R^*(s)\bigr)\,ds \\
&\qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}Q_6\bigl(t,R^*(t)\bigr)\biggr)\biggr| \\
&\quad= \frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1}\bigl|Q_6(s,R(s) - Q_6\bigl(s,R^*(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\bigl|Q_6(t,R(t) - Q_6\bigl(t,R^*(t)\bigr)\bigr|\,ds \\
&\quad\le \biggl(\frac{u_1u_2}{AB(u_1)\Gamma u_1}\int_0^t s^{u_2-1}(t-s)^{u_1-1} + \frac{u_2(1-u_1)t^{u_2-1}}{AB(u_1)}\biggr)\phi_6\bigl\|R - R^*\bigr\| \\
&\quad\le \biggl(\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)}\biggr)\phi_6\bigl\|R - R^*\bigr\|.
\end{aligned}
$$

Let $\alpha_6 = (\frac{u_1u_2\Gamma u_2}{AB(u_1\Gamma(u_1+u_2))} + \frac{u_2(1-u_1)}{AB(u_1)})\|R - R^*\|$, $\Delta_6 = \phi_6$, so the above inequality becomes $|R - R^*| \le \alpha_6\Delta_6$. Consequently, by definition the fractal fractional model (1) is Hyers–Ulam stable. This completes the proof. □

3 Numerical scheme

Numerical scheme for the fractal fractional order SE(Is)(Ih)AR epidemic model.

Definition 3.1 Suppose that $\psi(t)$ is continuous and fractal differentiable on the interval (u,v) with order Υ_2, then the fractal fractional derivative of $\psi(t)$ with order Υ_1 in the Riemann–Liouville sense having power law type kernel is given by

$$^{\mathrm{FFP}}{}_0D_t^{\Upsilon_1,\Upsilon_2}\psi(t) = \frac{1}{\Gamma(p-\Upsilon_1}\frac{d}{dt^{\Upsilon_2}}\int_0^t (t-s)^{p-\Upsilon_1-1}\psi(s)\,ds,$$

where $p-1<\Upsilon_1,\ \Upsilon_2 \le p \in \mathcal{N}$, and $\frac{d}{dt^{\Upsilon_2}} = \lim_{t\to s}\frac{\psi(t)-\psi(s)}{t^{\Upsilon_2}-s^{\Upsilon_2}}$.

Definition 3.2 Suppose that $\psi(t)$ is continuous on the interval (u,v), then the fractal fractional integral of $\psi(t)$ with order Υ_1 having Mittag-Leffler type kernel is given by

$$\begin{aligned}^{\mathrm{FFM}}{}_0I_t^{\Upsilon_1,\Upsilon_2}\psi(t) = {}&\frac{\Upsilon_1\Upsilon_2}{AB(\Upsilon_1)\Gamma\Upsilon_1}\int_0^t s^{\Upsilon_2-1}\psi(s)(t-s)^{\Upsilon_1-1}\,ds\\ &+\frac{\Upsilon_2(1-\Upsilon_1)t^{(}\Upsilon_2-1)}{AB(\Upsilon_1)}\psi(t).\end{aligned}$$

Let us consider $^{\mathrm{FFM}}{}_0D_t^{\Upsilon_1,\Upsilon_2}\eta(t) = \mathcal{H}(t,\eta(t))R$, where $\eta(0)=\eta_0$ The above equation can be written in fractal fractional derivative as follows:

$$^{\mathrm{FFR}}{}_0D_t^{\Upsilon_1}\eta(t) = \Upsilon_2 t^{\Upsilon_2-1}\mathcal{L}\bigl(t,\eta(t)\bigr) = \mathcal{H}\bigl(t,\eta(t)\bigr).$$

With the help of integral, we get

$$\begin{aligned}\eta(t) = {}&\eta(0) + \frac{1-\Upsilon_1}{AB(\Upsilon_1)}\mathcal{H}\bigl(t,\eta(t)\bigr)\\ &+\frac{\Upsilon_1}{AB(\Upsilon_1)\Gamma\Upsilon_1}\int_0^t \zeta^{\Upsilon_2-1}(t-\zeta)^{\Upsilon_1-1}\mathcal{H}\bigl(\zeta,\eta(\zeta)\bigr)\,d\zeta.\end{aligned}$$

Replacing (t) with t_{n+1}, we have

$$\begin{aligned}\eta^{n+1} = {}&\eta(0) + \frac{1-\Upsilon_1}{AB(\Upsilon_1)}\mathcal{H}\bigl(t_n,\eta(t_n)\bigr)\\ &+\frac{\Upsilon_1}{AB(\Upsilon_1)\Gamma\Upsilon_1}\int_0^{t^{n+1}} \zeta^{\Upsilon_2-1}(t_{n+1}-\zeta)^{\Upsilon_1-1}\mathcal{H}\bigl(\zeta,\eta(\zeta)\bigr)\,d\zeta.\end{aligned}\tag{9}$$

By applying two-step Lagrange polynomial, we obtain

$$\begin{aligned}\theta\bigl(y,\eta(y)\bigr) &= \frac{(y-t_{k-1})H(t_k,\eta(t_k))}{t_k-t_{k-1}} - \frac{(y-t_k)H(t_{k-1},\eta(t_{k-1}))}{t_k-t_{k-1}}\\ &= \frac{H(t_k,\eta(t_k)(y-t_{k-1})}{t_k-t_{k-1}} - \frac{H(t_{k-1},\eta(t_{k-1}))(y-t_k)}{t_k-t_{k-1}}\\ &= \frac{H(t_k,\eta_k)(y-t_{k-1})}{h} - \frac{H(t_{k-1},\eta_{k-1})(y-t_k)}{h}.\end{aligned}$$

Applying Lagrange polynomial to equation (9), we get

$$\begin{aligned}\eta^{n+1} &= \eta(0) + \frac{1-\Upsilon_1}{AB(\Upsilon_1)}\mathcal{H}\big(t_n,\eta(t_n)\big)\\&\quad + \frac{\Upsilon_1}{AB(\Upsilon_1)\Gamma\theta_1}\sum_{i=1}^{n}\biggl[\frac{\mathcal{H}(t_i,\eta(t_i))}{h}\int_{t_k}^{t^{k+1}}(\zeta-t_{i-1})(t_{n+1}-\zeta)^{\Upsilon_1-1}\,d\zeta\\&\quad - \frac{\mathcal{H}(t_{i-1},\eta(t_{i-1}))}{h}\int_{t_k}^{t^{n+1}}(\zeta-t_i)(t_{n+1}-\zeta)^{\Upsilon_1-1}\,d\zeta\biggr].\end{aligned}$$

Now, solving the integral, we get

$$\begin{aligned}\eta^{n+1} &= \eta(0) + \frac{1-\Upsilon_1}{AB(\Upsilon_1)}\mathcal{H}\big(t_n,\eta(t_n)\big)\\&\quad + \frac{\Upsilon_1 h^{\Upsilon_1}}{\Gamma(\Upsilon_1+2)}\sum_{i=1}^{n}\bigl[\mathcal{H}\big(t_i,\eta(t_i)\big)\big((n+1-i)_1^{\Upsilon}(n-i+2+\Upsilon_1)\\&\quad - (n-i)_1^{\Upsilon}(n-i+2+2\Upsilon_1)\big)\\&\quad - \mathcal{H}(t_{i-1},\eta_{i-1})\big((n+1-i)^{\Upsilon_1+1}-(n-i+1+\Upsilon_1)(n-i)^{\Upsilon_1}\big)\bigr].\end{aligned}$$

Replacing the value of $\mathcal{H}(t,\eta(t))$, we have

$$\begin{aligned}\eta^{n+1} &= \eta(0) + \Upsilon_2 t^{\Upsilon_2-1}\frac{1-\Upsilon_1}{AB(\Upsilon_1)}\mathcal{I}\big(t_n,\eta(t_n)\big)\\&\quad + \Upsilon_2 t^{\Upsilon_2-1}\frac{\Upsilon_2 h^{\Upsilon_1}}{\Gamma(\Upsilon_1+2)}\sum_{i=1}^{n}\bigl[\mathcal{I}\big(t_i,\eta(t_i)\big)\big((n+1-i)_1^{\Upsilon}(n-i+2+\Upsilon_1)\\&\quad - (n-i)_1^{\Upsilon}(n-i+2+2\Upsilon_1)\big)\\&\quad - \mathcal{I}(t_{i-1},\eta_{i-1})\big((n+1-i)^{\Upsilon_1+1}-(n-i+1+\Upsilon_1)(n-i)^{\Upsilon_1}\big)\bigr].\end{aligned}$$

Now the system of equations with kernels $\mathcal{H}_i$, $i\in\mathcal{N}_1^6$ with initial conditions $S(0)=E(0)=I_s(0)=I_h(0)=A(0)=R(0)=0$:

$$\begin{aligned}S(t) &= S(0) + \frac{\Upsilon_1\Upsilon_2}{AB(\Upsilon_1)\Gamma\Upsilon_1}\int_0^t s^{\Upsilon_2-1}(t-s)^{\Upsilon_1-1}H_1\big(s,S(s)\big)\,ds\\&\quad + \frac{\Upsilon_2(1-\Upsilon_1)t^{\Upsilon_2-1}}{AB(\Upsilon_1)}H_1\big(t,S(t)\big),\\E(t) &= E(0) + \frac{\Upsilon_1\Upsilon_2}{AB(\Upsilon_1)\Gamma\Upsilon_1}\int_0^t s^{\Upsilon_2-1}(t-s)^{\Upsilon_1-1}H_2\big(s,E(s)\big)\,ds\\&\quad + \frac{\Upsilon_2(1-\Upsilon_1)t^{\Upsilon_2-1}}{AB(\Upsilon_1)}H_2\big(t,E(t)\big),\\I_s(t) &= I_s(0) + \frac{\Upsilon_1\Upsilon_2}{AB(\Upsilon_1)\Gamma\Upsilon_1}\int_0^t s^{\Upsilon_2-1}(t-s)^{\Upsilon_1-1}H_3\big(s,I_s(s)\big)\,ds\\&\quad + \frac{\Upsilon_2(1-\Upsilon_1)t^{\Upsilon_2-1}}{AB(\Upsilon_1)}H_3\big(t,I_s(t)\big),\\I_h(t) &= I_h(0) + \frac{\Upsilon_1\Upsilon_2}{AB(\Upsilon_1)\Gamma\Upsilon_1}\int_0^t s^{\Upsilon_2-1}(t-s)^{\Upsilon_1-1}H_4\big(s,I_h(s)\big)\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\Upsilon_2(1-\Upsilon_1)t^{\Upsilon_2-1}}{AB(\Upsilon_1)} H_4\big(t, I_h(t)\big), \\
A(t) &= A(0) + \frac{\Upsilon_1\Upsilon_2}{AB(\Upsilon_1)\Gamma\Upsilon_1}\int_0^t s^{\Upsilon_2-1}(t-s)^{\Upsilon_1-1} H_5\big(s, A(s)\big)\,ds \\
&+ \frac{\Upsilon_2(1-\Upsilon_1)t^{\Upsilon_2-1}}{AB(\Upsilon_1)} H_5\big(t, A(t)\big), \\
R(t) &= R(0) + \frac{\Upsilon_1\Upsilon_2}{AB(\Upsilon_1)\Gamma\Upsilon_1}\int_0^t s^{\Upsilon_2-1}(t-s)^{\Upsilon_1-1} H_6\big(s, R(s)\big)\,ds \\
&+ \frac{\Upsilon_2(1-\Upsilon_1)t^{\Upsilon_2-1}}{AB(\Upsilon_1)} H_6\big(t, R(t)\big).
\end{aligned}
$$

Now from the numerical scheme for fractal-fractional order model (1), we have

$$
\begin{aligned}
S_{n+1} &= \eta(0) + \Upsilon_2 t^{\Upsilon_2-1}\frac{1-\Upsilon_1}{AB(\Upsilon_1)} F_1\big(t_n, S(t_n)\big) + \Upsilon_2 t^{\Upsilon_2-1}\frac{\Upsilon_2 h^{\Upsilon_1}}{\Gamma(\Upsilon_1+2)} \\
&\times \sum_{i=1}^{n}\big[F_1\big(t_i, S(t_i)\big)\big((n+1-i)_1^{\Upsilon}(n-i+2+\Upsilon_1) \\
&- (n-i)_1^{\Upsilon}(n-i+2+2\Upsilon_1)\big) \\
&- F_1(t_{i-1}, S_{i-1})\big((n+1-i)^{\Upsilon_1+1} - (n-i+1+\Upsilon_1)(n-i)^{\Upsilon_1}\big)\big], \\
E_{n+1} &= \eta(0) + \Upsilon_2 t^{\Upsilon_2-1}\frac{1-\Upsilon_1}{AB(\Upsilon_1)} F_2\big(t_n, E(t_n)\big) + \Upsilon_2 t^{\Upsilon_2-1}\frac{\Upsilon_2 h^{\Upsilon_1}}{\Gamma(\Upsilon_1+2)} \\
&\times \sum_{i=1}^{n}\big[F_2\big(t_i, E(t_i)\big)\big((n+1-i)_1^{\Upsilon}(n-i+2+\Upsilon_1) \\
&- (n-i)_1^{\Upsilon}(n-i+2+2\Upsilon_1)\big) \\
&- F_2(t_{i-1}, E_{i-1})\big((n+1-i)^{\Upsilon_1+1} - (n-i+1+\Upsilon_1)(n-i)^{\Upsilon_1}\big)\big], \\
I_{s_{n+1}} &= \eta(0) + \Upsilon_2 t^{\Upsilon_2-1}\frac{1-\Upsilon_1}{AB(\Upsilon_1)} F_3\big(t_n, I_s(t_n)\big) + \Upsilon_2 t^{\Upsilon_2-1}\frac{\Upsilon_2 h^{\Upsilon_1}}{\Gamma(\Upsilon_1+2)} \\
&\times \sum_{i=1}^{n}\big[F_3\big(t_i, I_s(t_i)\big)\big((n+1-i)_1^{\Upsilon}(n-i+2+\Upsilon_1) - (n-i)_1^{\Upsilon}(n-i+2+2\Upsilon_1)\big) \\
&- F_3(t_{i-1}, I_{s_{i-1}})\big((n+1-i)^{\Upsilon_1+1} - (n-i+1+\Upsilon_1)(n-i)^{\Upsilon_1}\big)\big], \\
I_{h_{n+1}} &= \eta(0) + \Upsilon_2 t^{\Upsilon_2-1}\frac{1-\Upsilon_1}{AB(\Upsilon_1)} F_4\big(t_n, I_h(t_n)\big) + \Upsilon_2 t^{\Upsilon_2-1}\frac{\Upsilon_2 h^{\Upsilon_1}}{\Gamma(\Upsilon_1+2)} \\
&\times \sum_{i=1}^{n}\big[F_4\big(t_i, I_h(t_i)\big)\big((n+1-i)_1^{\Upsilon}(n-i+2+\Upsilon_1) - (n-i)_1^{\Upsilon}(n-i+2+2\Upsilon_1)\big) \\
&- F_4(t_{i-1}, I_h i-1)\big((n+1-i)^{\Upsilon_1+1} - (n-i+1+\Upsilon_1)(n-i)^{\Upsilon_1}\big)\big], \\
A_{n+1} &= \eta(0) + \Upsilon_2 t^{\Upsilon_2-1}\frac{1-\Upsilon_1}{AB(\Upsilon_1)} F_5\big(t_n, A(t_n)\big) + \Upsilon_2 t^{\Upsilon_2-1}\frac{\Upsilon_2 h^{\Upsilon_1}}{\Gamma(\Upsilon_1+2)} \\
&\times \sum_{i=1}^{n}\big[F_5\big(t_i, A(t_i)\big)\big((n+1-i)_1^{\Upsilon}(n-i+2+\Upsilon_1) - (n-i)_1^{\Upsilon}(n-i+2+2\Upsilon_1)\big) \\
&- F_5(t_{i-1}, A_{i-1})\big((n+1-i)^{\Upsilon_1+1} - (n-i+1+\Upsilon_1)(n-i)^{\Upsilon_1}\big)\big],
\end{aligned}
$$

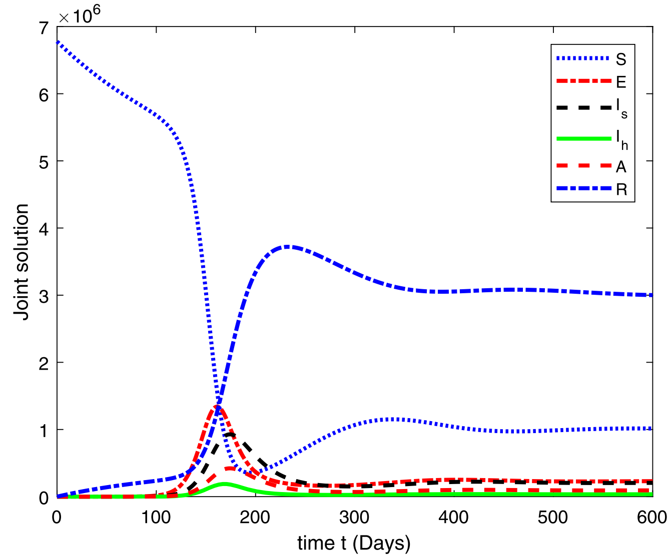

Figure 1 Joint solution of COVID 19 model (1) for the order 1.0

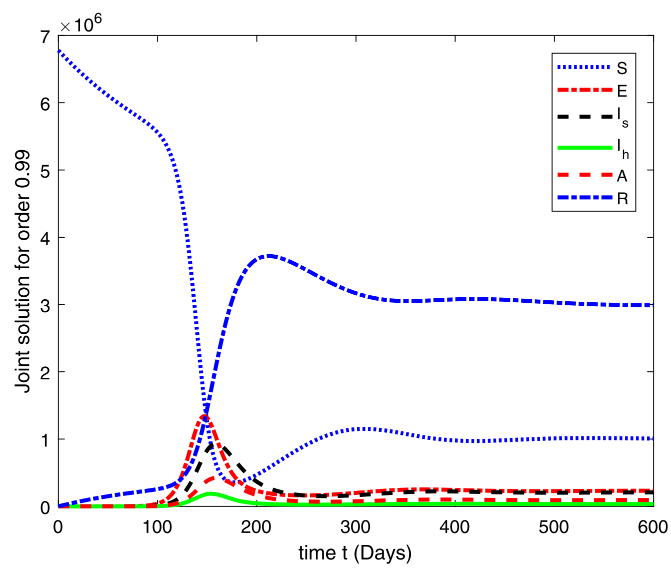

Figure 2 Joint solution of COVID 19 model (1) for the order 0.99

$$\begin{aligned} R_{n+1} = {} & \eta(0) + \Upsilon_2 t^{\Upsilon_2-1}\frac{1-\Upsilon_1}{AB(\Upsilon_1)}F_6\bigl(t_n, R(t_n)\bigr) + \Upsilon_2 t^{\Upsilon_2-1}\frac{\Upsilon_2 h^{\Upsilon_1}}{\Gamma(\Upsilon_1+2)} \\ & \times \sum_{i=1}^{n}\bigl[F_6\bigl(t_i, R(t_i)\bigr)\bigl((n+1-i)_1^{\Upsilon}(n-i+2+\Upsilon_1) \\ & - (n-i)_1^{\Upsilon}(n-i+2+2\Upsilon_1)\bigr) \\ & - F_6(t_{i-1}, R_{i-1})\bigl((n+1-i)^{\Upsilon_1+1} - (n-i+1+\Upsilon_1)(n-i)^{\Upsilon_1}\bigr)\bigr]. \end{aligned}$$

3.1 Computational results

Here, we present the computational results based on the literature. The initial values for the population are: $S(0) = 6{,}778{,}382$, $E(0) = 1$, $I_s(0) = 0$, $I_h(0) = 0$, $A(0) = 0$, $R(0) = 0$, and the parametric values are: $b_1 = 57{,}554$, $b_2 = 1/85$, $N = 6{,}778{,}383$, $\beta = 1/N$, $\beta_a r = 1$, $\beta_h r = 1/80$, $\gamma = 1/5.5$, $\eta = 0$, $\alpha = 0.12$, $\tau_0 = 1/10$, $p_s = 0.55$, $p_h = 0.20$, $k_v = 0.001$, $k_T = 0.004$ [37].

The computational results are given via ten graphs. In the Fig. 1, we have given the numerical solution of the suggested model for order 1 which is compared with the numerical solution of the model for orders 0.99, 0.98, 0.97 in Figs. 2, 3 and 4, respectively. Figure 5 is

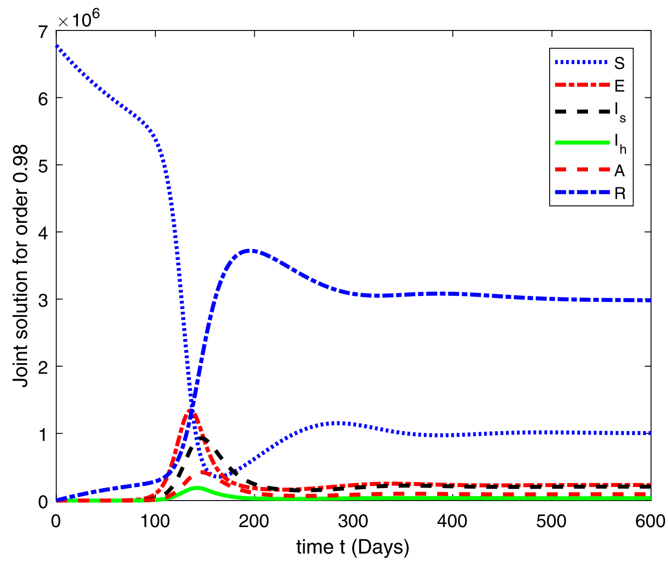

Figure 3 Joint solution of COVID 19 model (1) for the order 0.98

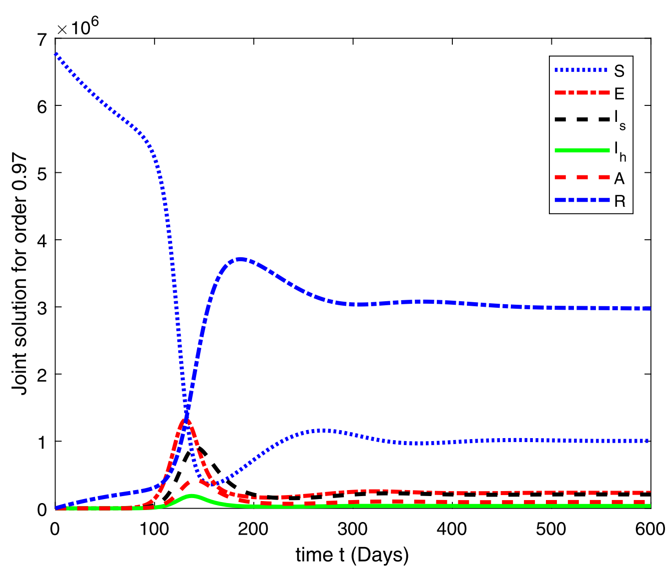

Figure 4 Joint solution of COVID 19 model (1) for the order 0.97

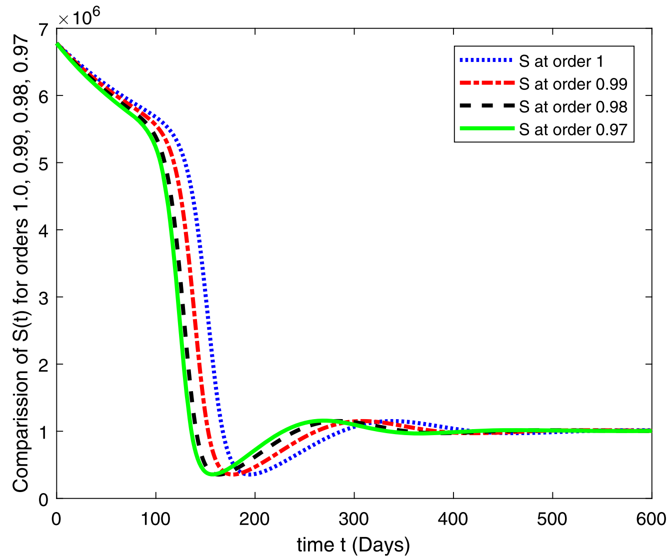

Figure 5 Computational results for *S*(*t*) while the fractional orders are 1.0, 0.99, 0.98, 0.97

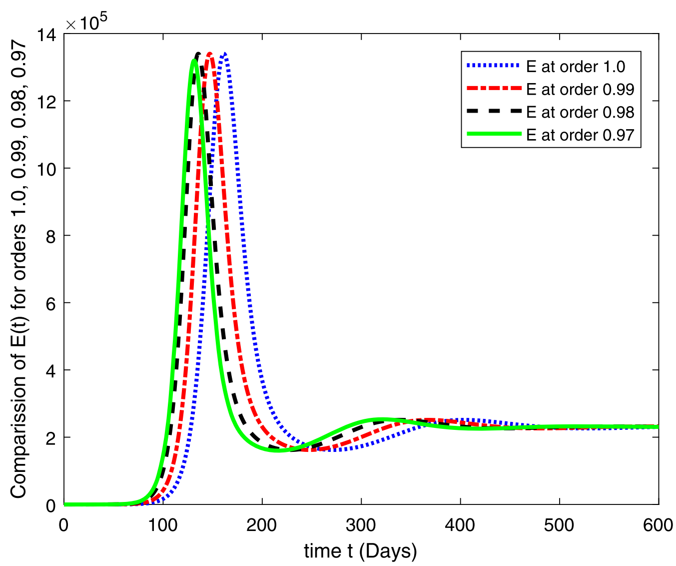

Figure 6 A comparative analysis of *E*(*t*) for the fractional orders 1.0, 0.99, 0.98, 0.97

for the comparative study of $S(t)$ for the different orders. Similarly, $E(t)$, $I_s(t)$, $I_h(t)$, $A(t)$ and $R(t)$ are analysed for different fractional orders in the Figs. 6, 7, 8, 9, and 10 respectively.

4 Conclusion

In current manuscript, we have established a detailed analysis related to the results about existence and uniqueness results with Ulam stability. The subjective problem is nonlocal multipoint BVPs involving delay term of FDEs. The respective analysis has been established via using classical fixed point theory and some results of nonlinear functional analysis. The whole analysis has been demonstrated via computational results based

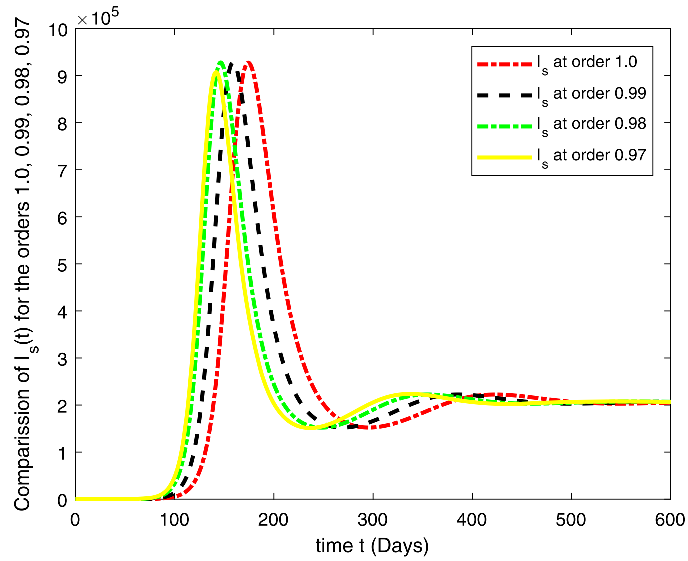

Figure 7 Computational results for $I_s(t)$ for the fractional orders 1.0, 0.99, 0.98, 0.97

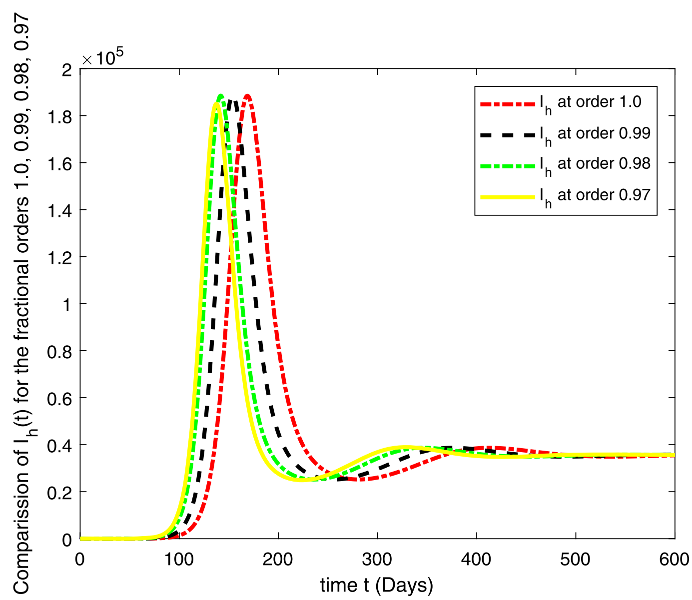

Figure 8 A comparative analysis of $I_h(t)$ for the fractional orders 1.0, 0.99, 0.98, 0.97

on the literature. The initial values for the population are: $S(0) = 6{,}778{,}382$, $E(0) = 1$, $I_s(0) = 0$, $I_h(0) = 0$, $A(0) = 0$, $R(0) = 0$, and the parametric values are: $b_1 = 57{,}554$, $b_2 = 1/85$, $N = 6{,}778{,}383$, $\beta = 1/N$, $\beta_a r = 1$, $\beta_h r = 1/80$, $\gamma = 1/5.5$, $\eta = 0$, $\alpha = 0.12$, $\tau_0 = 1/10$, $p_s = 0.55$, $p_h = 0.20$, $k_\nu = 0.001$, $k_T = 0.004$ [37]. The results are more realistic and of the same behavior as the classical ones. Our results are getting more similar to the integer order ones for the orders closer to 1.0.

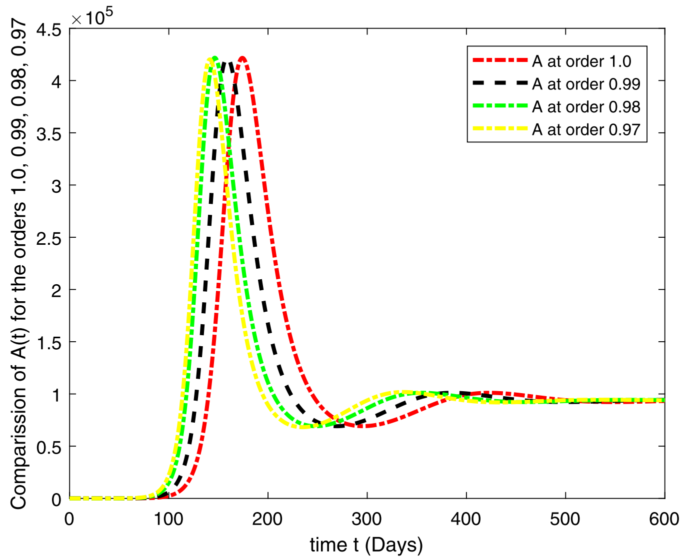

Figure 9 A comparative analysis of $A(t)$ for the fractional orders 1.0, 0.99, 0.98, 0.97

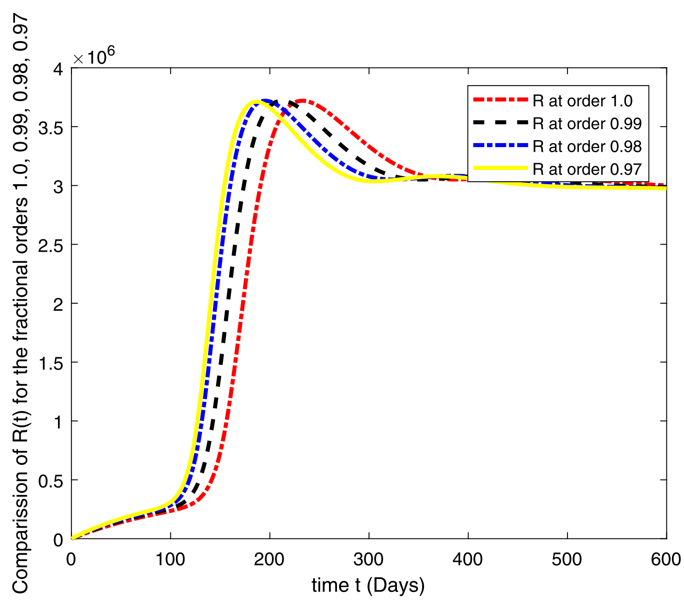

Figure 10 A comparative analysis of $R(t)$ for the fractional orders 1.0, 0.99, 0.98, 0.97

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Authors' contributions
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