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Optimal control for obstacle problems involving time-dependent variational inequalities with Liouville–Caputo fractional derivative

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Abstract

We consider an optimal control problem for a time-dependent obstacle variational inequality involving fractional Liouville–Caputo derivative. The obstacle is considered as the control, and the corresponding solution to the obstacle problem is regarded as the state. Our aim is to find the optimal control with the properties that the state is closed to a given target profile and the obstacle is not excessively large in terms of its norm. We prove existence results and establish necessary conditions of obstacle problems via the approximated time fractional-order partial differential equations and their adjoint problems. The result in this paper is a generalization of the obstacle problem for a parabolic variational inequalities as the Liouville–Caputo fractional derivatives were used instead of the classical derivatives.

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1 Introduction

Variational inequalities are extensively used in many applications in mathematical economics, finance, optimal control, and optimization. The main problem is minimizing some function that occurs in mathematical models. To analyze those mathematical models, a qualitative study of solutions of variational inequalities becomes important for describing the behavior of the models. One of important classes of variational inequalities is the obstacle problem motivated by the physical problem of finding the stable shape of an elastic membrane that is pressed by an obstacle on one side. Many processes in engineering science can be explained by solutions of obstacle problems. For example, the filtration to porous medium or dam problem is an obstacle problem of studying fluid flow through porous medium, which can be derived from Darcy's law. Other examples include a cavitation problem in hydrodynamic lubrication, which is the study of behavior of a lubricant contained inside the narrow hydraulic clearance between two eccentric cylindrical bodies in relative motion. In addition, the time-dependent (parabolic) obstacle problems are applicable to financial problems, for example, in the optimal stopping problem involving American option pricing with expiration at time T.

There are extensive results on the existence, uniqueness, and regularity of solutions of obstacle problems. In many applications, it is crucial to find an obstacle so that its corresponding solution is closed to a given target profile. This motivates the study of optimal control for obstacle problems. Mathematically, the problem is finding an optimal obstacle that minimizes a certain objective functional involving solutions to obstacle problems and target functions. For example, Adams et al. [1] treated the elliptic case with $H_0^1(\Omega)$ obstacle without source term. They found that the optimal obstacle is identical to the corresponding state. This result does not generally hold for the elliptic case where there is a source term [2]. We refer to [3–10] for the results on other types of optimal control problem for variational inequalities. Moreover, Adams and Lenhart [11] investigated the characterization of necessary conditions for the optimal control of parabolic variational inequalities using classical derivatives.

Over the past decades, there has been an extensive development in fractional calculus. Various definitions of fractional integrals and fractional derivatives have been proposed, including Riemann–Liouville, Liouville–Caputo, Hilfer, Riesz, Erdelyi–Kober, Hadamard, and so on. We refer the reader to a recent survey-cum-expository review article by Srivastava [12] for the theory of fractional integral and fractional derivative operators with their applications. Several research areas of fractional calculus extended the classical theory, especially, calculus of variations and optimal control [13–20]. There is also a development of numerical algorithms for fractional-order differential equations arising in physical problems such as astrophysics, vibration, and nuclear magnetic resonance [21–26]. This generalization involves the differential equations with fractional-order time derivatives instead of integer-order derivatives. The fractional optimal control problem for differential equations and variational inequalities can be investigated by fractional variation principle, the method of Lagrange multipliers, or the Euler–Lagrange first-order optimality condition based on the adjoint problem. Recent development includes the study of optimal control problem of time-fractional diffusion equation under various assumptions such as the linear problem, nonlocal and nonsingular kernels, and time delay problem [27–30]. However, there seems to be less results for optimal control problem of fractional variational inequalities.

Motivated by [11], it is interesting to investigate the optimal control for the obstacle problem involving time-dependent variational inequalities under fractional calculus framework since fractional derivatives can take into account the past memory and nonlocal properties of the system. In this work, we consider the optimal control for the obstacle problem involving time fractional-order derivative in a domain $Q = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded set with C^1 boundary $\partial\Omega$. Let $z \in L^2(Q)$ be a given target profile, and let $u_0 \in H_0^1(\Omega)$ with $u_0(x) \geq 0$ a.e. in Ω. The control set is

$$U = \left\{\psi \in L^2\big(0, T; H^2(\Omega) \cap H_0^1(\Omega)\big) \mid {}_0^{\mathrm{LC}}D_t^\alpha \psi \in L^2(Q), \psi(x, 0) = 0\right\},$$

where ${}_0^{\mathrm{LC}}D_t^\alpha$ is the Liouville–Caputo fractional derivative of order $0 < \alpha < 1$. For any $\psi \in U$, we define the fractional admissible set for solutions as

$$\mathcal{K}_\alpha(\psi) = \left\{v \in L^2\big(0, T; H_0^1(\Omega)\big) \mid {}_0^{\mathrm{LC}}D_t^\alpha v \in L^2(Q),\right.$$

$$v \geq \psi \text{ a.e. on } Q, \text{ and } v(x,0) = u_0(x)\big\}$$

for $0 < \alpha < 1$.

Let an obstacle $\psi \in U$, and let $f \in L^2(Q)$. We denote the state $u = \mathcal{T}(\psi)$ for the corresponding solution of the time-dependent fractional-order variational inequality

$$\begin{aligned} &u \in \mathcal{K}_\alpha(\psi), \\ &\int_Q \big[{}^{\mathrm{LC}}_0 D_t^\alpha u(v-u) + \nabla u \cdot \nabla(v-u)\,dx\,dt\big] \geq \int_Q f(v-u)\,dx\,dt \end{aligned} \tag{1}$$

for all $v \in \mathcal{K}_\alpha(\psi)$.

We want to find an obstacle ψ^* in U such that the state $u^* = \mathcal{T}(\psi^*)$, the solution of (1), is closed to a given target profile z and the norm of ψ^* is not excessively big. More precisely, we consider the objective functional

$$J(\psi) = \int_Q \big[\big(\mathcal{T}(\psi) - z\big)^2 + |\Delta\psi|^2 + \big|{}^{\mathrm{LC}}_0 D_t^\alpha \psi\big|^2\big]\,dx\,dt. \tag{2}$$

Therefore the optimal control problem can be regarded as finding a minimizer of the functional J, that is, $\psi^* \in U$ such that

$$J\big(\psi^*\big) = \inf_{\psi \in U} J(\psi).$$

For a control ψ^* and the corresponding state $u^* = \mathcal{T}(\psi^*)$, we call the pair $(\psi^*, \mathcal{T}(\psi^*))$ an optimal pair.

The main contribution of this paper is that we generalize parabolic variational inequality in [11] to consider time fractional-order derivatives. Our results contain both existence and necessary conditions of the optimal pair for the obstacle problem. Indeed, the optimal pair can be constructed from the approximated parabolic equations and their adjoint problems. The results also provide an extension of optimal control for fractional diffusion equations to the case of time fractional-order variational inequalities, which is new in the literature. This paper is structured as follows. In Sect. 2, we recall basic preliminaries and some known results about fractional calculus. Next, we prove the existence results for the state variational inequality by considering the approximate time fractional-order semilinear differential equations and establish the existence of an optimal control in Sect. 3. Finally, in Sect. 4, we give the characterization of necessary conditions for the optimal control and discuss the results in Sect. 5.

2 Preliminaries

In this section, we begin by presenting some definitions and properties of the fractional operators including fractional derivatives and integrals. We refer to the books [31–33] for further background.

Definition 2.1 (Left and right Riemann–Liouville fractional integrals) Let u be defined and integrable on an interval $[a,b]$, and let $\mathrm{Re}(\alpha) > 0$. Then

(i) The left Riemann–Liouville integral of order α is defined by

$$\left({}_aI^{\alpha}u\right)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}u(s)\,ds.$$

(ii) The right Riemann–Liouville integral of order α is defined by

$$\left(I_b^{\alpha}u\right)(t)=\frac{1}{\Gamma(\alpha)}\int_t^b(s-t)^{\alpha-1}u(s)\,ds.$$

Definition 2.2 (Left and right Riemann–Liouville fractional derivatives) Let u be defined on an interval $[a,b]$, and let $0<\alpha\leq 1$.

(i) The left Riemann–Liouville fractional derivative of order α is defined by

$$\left({}_aD_t^{\alpha}u\right)(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t(t-s)^{-\alpha}u(s)\,ds.$$

(ii) The right Riemann–Liouville fractional derivative of order α is defined by

$$\left({}_tD_b^{\alpha}u\right)(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b(s-t)^{-\alpha}u(s)\,ds.$$

Let $AC([a,b];\mathbb{R})$ be the set of absolutely continuous functions.

Definition 2.3 (Left and right Liouville–Caputo fractional derivatives) Let u be defined and absolutely continuous on an interval $[a,b]$, that is, $u\in AC([a,b];\mathbb{R})$, and let $0<\alpha\leq 1$.

(i) The left Liouville–Caputo fractional derivative of order α is defined by

$$\left({}_a^{\mathrm{LC}}D_t^{\alpha}u\right)(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t(t-s)^{-\alpha}u'(s)\,ds.$$

(ii) The right Liouville–Caputo fractional derivative of order α is defined by

$$\left({}_t^{\mathrm{LC}}D_b^{\alpha}u\right)(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^b(s-t)^{-\alpha}u'(s)\,ds.$$

Proposition 2.4 *Let $0<\alpha<1$ and $u\in AC([a,b];\mathbb{R})$. Then the Riemann–Liouville and Liouville–Caputo fractional derivatives satisfy*

$$\left({}_a^{\mathrm{LC}}D_t^{\alpha}u\right)(t)=\left({}_aD_t^{\alpha}u\right)(t)-\frac{u(a)}{(t-a)^{\alpha}\Gamma(1-\alpha)},$$
$$\left({}_t^{\mathrm{LC}}D_b^{\alpha}u\right)(t)=\left({}_tD_b^{\alpha}u\right)(t)-\frac{u(b)}{(b-t)^{\alpha}\Gamma(1-\alpha)}.$$

Lemma 2.5 ([34] Integration by parts for Liouville–Caputo fractional derivatives) *Let $0<\alpha<1$, $u\in L^p(a,b)$, and $v\in AC([a,b];\mathbb{R})$. Then*

$$\int_a^b u(t)\left({}_a^{\mathrm{LC}}D_t^{\alpha}v\right)(t)\,dt=\int_a^b v(t)\left({}_tD_b^{\alpha}u\right)(t)\,dt+\left(I_b^{1-\alpha}u\right)(t)v(t)\big|_a^b,$$
$$\int_a^b u(t)\left({}_t^{\mathrm{LC}}D_b^{\alpha}v\right)(t)\,dt=\int_a^b v(t)\left({}_aD_t^{\alpha}u\right)(t)\,dt-\left({}_aI^{1-\alpha}u\right)(t)v(t)\big|_a^b.$$

Lemma 2.6 ([34] Integration by parts for Riemann–Liouville fractional derivatives) *Let* $0 < \alpha \leq 1$, $u \in L^p(a,b)$, *and* $v \in AC([a,b];\mathbb{R})$. *Then*

$$\int_a^b u(t)\big({}_aD_t^\alpha v\big)(t)\,dt = \int_a^b v(t)\big({}_tD_b^\alpha u\big)(t)\,dt + v(b)\big(I_b^{1-\alpha}u\big)\big(b^-\big),$$

$$\int_a^b u(t)\big({}_tD_b^\alpha v\big)(t)\,dt = \int_a^b v(t)\big({}_aD_t^\alpha u\big)(t)\,dt + v(a)\big({}_aI^{1-\alpha}u\big)\big(a^+\big).$$

The following product rule can be obtained from the corresponding Riemann–Liouville version in [35, 36].

Lemma 2.7 *For the Liouville–Caputo derivative, we have*

$$\begin{aligned} {}_0^{\mathrm{LC}}D_t^\alpha(uv)(t) = {}& u(t)\big({}_0^{\mathrm{LC}}D_t^\alpha v\big)(t) + v(t)\big({}_0^{\mathrm{LC}}D_t^\alpha u\big)(t) \\ & - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{(u(s)-u(t))(v(s)-v(t))}{(t-s)^{\alpha+1}}\,ds \\ & + \frac{1}{t^\alpha\Gamma(1-\alpha)}\big(u(t)v(0)+v(t)u(0)-u(0)v(0)-u(t)v(t)\big). \end{aligned} \tag{3}$$

Moreover, if $u = v$, *then we have*

$$\begin{aligned} 2u(t)\big({}_0^{\mathrm{LC}}D_t^\alpha u\big)(t) = {}& {}_0^{\mathrm{LC}}D_t^\alpha u^2(t) + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{(u(t)-u(s))^2}{(t-s)^{\alpha+1}}\,ds \\ & + \frac{1}{t^\alpha\Gamma(1-\alpha)}\big(u(t)-u(0)\big)^2. \end{aligned} \tag{4}$$

Lemma 2.8 ([37]) *Let* $0 < \alpha < 1$, *and let u be absolutely continuous on* $[0,T]$. *Then*

$$2u(t){}_0^{\mathrm{LC}}D_t^\alpha u(t) \geq {}_0^{\mathrm{LC}}D_t^\alpha u^2(t).$$

Lemma 2.9 *Let* $u = u(x,t)$ *be a solution of the following problem subject to the Dirichlet boundary condition on a bounded open set* $\Omega \subset \mathbb{R}^n$ *for* $u : \Omega \times [0,T] \to \mathbb{R}$ *such that* $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ *with* ${}_0^{\mathrm{LC}}D_t^\alpha u \in L^2(Q)$ *and*

$$\begin{aligned} & {}_0^{\mathrm{LC}}D_t^\alpha u - \Delta u = f(x,t) \quad \textit{for } x \in \Omega \textit{ and } 0 < t \leq T, \\ & u(x,t) = 0 \quad \textit{for } x \in \partial\Omega \textit{ and } 0 < t \leq T, \\ & u(x,0) = u_0(x) \quad \textit{for } x \in \Omega, \end{aligned}$$

where $f : \Omega \times (0,T) \to \mathbb{R}$ *in* $L^2(Q)$ *and* u_0 *in* $H_0^1(\Omega)$ *are a given nonhomogeneous term and initial condition. Then there is a constant* $C > 0$ *such that*

$$\big\|{}_0^{\mathrm{LC}}D_t^\alpha u\big\|_{L^2(Q)} + \sup_{0\leq t\leq T}\|\nabla u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(Q)} \leq C\big(\|f\|_{L^2(Q)} + \|\nabla u_0\|_{L^2(\Omega)}\big).$$

Proof For $0 \leq t \leq T$, we use Lemma 2.8 to obtain

$$\int_Q f^2\,dx\,dt = \int_Q \big({}_0^{\mathrm{LC}}D_t^\alpha u\big)^2 - 2\Delta u\big({}_0^{\mathrm{LC}}D_t^\alpha u\big) + (\Delta u)^2\,dx\,dt$$

$$= \int_Q \left({}_0^{\mathrm{LC}}D_t^\alpha u\right)^2 + 2\nabla u \cdot \nabla\left({}_0^{\mathrm{LC}}D_t^\alpha u\right) + (\Delta u)^2\, dx\, dt$$
$$\geq \int_Q \left({}_0^{\mathrm{LC}}D_t^\alpha u\right)^2 + {}_0^{\mathrm{LC}}D_t^\alpha |\nabla u|^2 + (\Delta u)^2\, dx\, dt$$
$$= \left\|{}_0^{\mathrm{LC}}D_t^\alpha u\right\|_{L^2(Q)}^2 + \int_0^T {}_0^{\mathrm{LC}}D_t^\alpha \|\nabla u\|_{L^2(\Omega)}^2\, dt + \|\Delta u\|_{L^2(Q)}^2.$$

By Proposition 2.4 we have

$$\int_0^T {}_0^{\mathrm{LC}}D_t^\alpha \|\nabla u\|_{L^2(\Omega)}^2\, dt = \int_0^T \left[{}_0D_t^\alpha \|\nabla u\|_{L^2(\Omega)}^2 - \frac{t^{-\alpha}\|\nabla u_0\|_{L^2(\Omega)}^2}{\Gamma(1-\alpha)}\right] dt$$
$$= {}_0I_T^{1-\alpha}\|\nabla u\|_{L^2(\Omega)}^2 - \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\|\nabla u_0\|_{L^2(\Omega)}^2.$$

Then we obtain

$$\left\|{}_0^{\mathrm{LC}}D_t^\alpha u\right\|_{L^2(Q)}^2 + {}_0I_T^{1-\alpha}\|\nabla u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(Q)}^2$$
$$\leq \int_Q f^2\, dx\, dt + \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\|\nabla u_0\|_{L^2(\Omega)}^2,$$

and hence

$$\left\|{}_0^{\mathrm{LC}}D_t^\alpha u\right\|_{L^2(Q)}^2 + \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \sup_{0\leq t\leq T}\|\nabla u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(Q)}^2$$
$$\leq \|f\|_{L^2(Q)}^2 + \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\|\nabla u_0\|_{L^2(\Omega)}^2.$$

It follows that

$$\left\|{}_0^{\mathrm{LC}}D_t^\alpha u\right\|_{L^2(Q)} + \sup_{0\leq t\leq T}\|\nabla u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(Q)} \leq C\left(\|f\|_{L^2(Q)} + \|\nabla u_0\|_{L^2(\Omega)}\right). \qquad \square$$

3 Existence of an optimal control

To establish the existence of an optimal control, we approximate the time fractional-order variational inequality (1) by a time fractional-order semilinear partial differential equation. The approximate equation will provide a priori estimates for solutions of the original variational inequality (1).

Define

$$V = \left\{u \in L^2\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right) \mid {}_0^{\mathrm{LC}}D_t^\alpha u \in L^2(Q) \text{ and } u(x,0) = u_0(x)\right\}.$$

Assume that the function f belongs to $L^2(Q)$. We first observe that for $u \in \mathcal{K}_\alpha(\psi)$ or $u \in V$, we have $u \in C([0,T];L^2(\Omega))$. Hence the initial condition can be considered as a function in V.

For $\delta > 0$, we consider the time fractional-order semilinear parabolic approximation problem:

$$\text{find } u^\delta \in V \text{ such that } {}_0^{\mathrm{LC}}D_t^\alpha u^\delta - \Delta u^\delta + \beta_\delta\left(u^\delta - \psi\right) = f \text{ in } Q, \tag{5}$$

where

$$\beta_\delta(s) = \frac{1}{\delta}\beta(s), \qquad \beta(s) = 0 \quad \text{for all } s \geq 0, \qquad 0 \leq \beta'(s) \leq 1 \quad \text{for all } s,$$

and $\beta \in C^1(\mathbb{R})$. We denote the solution of (5) by $u^\delta = \mathcal{T}^\delta(\psi)$. We next prove an estimate and convergence property of u^δ.

Proposition 3.1 *For $\psi \in U$, the solution $u^\delta = \mathcal{T}^\delta(\psi)$ of (5) satisfies*

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left\|\nabla u^\delta\right\|_{L^2(\Omega)} + \left\|{}_0^{\mathrm{LC}}D_t^\alpha u^\delta\right\|_{L^2(Q)} + \left\|\Delta u^\delta\right\|_{L^2(Q)} \\
&\quad\leq C\left[1 + \|\nabla u_0\|_{L^2(\Omega)} + \left\|\Delta\psi(t)\right\|_{L^2(Q)} + \left\|{}_0^{\mathrm{LC}}D_t^\alpha \psi(t)\right\|_{L^2(Q)} + \|f\|_{L^2(Q)}\right].
\end{aligned} \tag{6}$$

Proof We start by giving an estimation of β_δ term in the L^2-norm:

$$\begin{aligned}
&\int_\Omega \left[\beta_\delta\left(u^\delta - \psi\right)\right]^2(x,t)\,dx \\
&\quad= \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\left[f - {}_0^{\mathrm{LC}}D_t^\alpha u^\delta + \Delta u^\delta\right](x,t)\,dx \\
&\quad= \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\Delta\psi(x,t)\,dx + \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\left[f - {}_0^{\mathrm{LC}}D_t^\alpha \psi\right](x,t)\,dx \\
&\qquad+ \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\left[\Delta\left(u^\delta - \psi\right) - {}_0^{\mathrm{LC}}D_t^\alpha\left(u^\delta - \psi\right)\right](x,t)\,dx \\
&\quad\leq \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\Delta\psi(x,t)\,dx + \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\left[f - {}_0^{\mathrm{LC}}D_t^\alpha \psi\right](x,t)\,dx \\
&\qquad- \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\left[{}_0^{\mathrm{LC}}D_t^\alpha\left(u^\delta - \psi\right)\right]dx \\
&\quad\leq \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\Delta\psi(x,t)\,dx + \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\left[f - {}_0^{\mathrm{LC}}D_t^\alpha \psi\right](x,t)\,dx \\
&\qquad- \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\left[\left(u^\delta - \psi\right)(x,t) - \left(u^\delta - \psi\right)(x,0)\right]dx.
\end{aligned} \tag{7}$$

This implies

$$\begin{aligned}
\int_\Omega \left[\beta_\delta\left(u^\delta - \psi\right)\right]^2(x,t)\,dx \leq{}& \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\Delta\psi(x,t)\,dx \\
&+ \int_\Omega \beta_\delta\left(u^\delta - \psi\right)\left[f - {}_0^{\mathrm{LC}}D_t^\alpha \psi\right](x,t)\,dx \\
&+ \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\int_\Omega \beta_\delta\left(u^\delta - \psi\right)u^\delta(x,0)\,dx.
\end{aligned}$$

Thus we obtain

$$\left\|\beta_\delta\left(u^\delta - \psi\right)\right\|_{L^2(Q)} \leq C\left(1 + \|\Delta\psi\|_{L^2(Q)} + \left\|{}_0^{\mathrm{LC}}D_t^\alpha \psi\right\|_{L^2(Q)} + \|f\|_{L^2(Q)}\right). \tag{8}$$

By Lemma 2.9 it follows that

$$\sup_{0\leq t\leq T}\left\|\nabla u^\delta\right\|_{L^2(\Omega)} + \left\|{}_0^{\mathrm{LC}}D_t^\alpha u^\delta\right\|_{L^2(Q)} + \left\|\Delta u^\delta\right\|_{L^2(Q)}$$

$$\le C\left(\left\|\beta_\delta\left(u^\delta-\psi\right)\right\|_{L^2(Q)}+\|f\|_{L^2(Q)}+\|\nabla u_0\|_{L^2(\Omega)}\right), \tag{9}$$

which, together with (8), gives estimate (6). □

The main result for the existence and uniqueness of solution of the obstacle problem via the approximated problem can be proved by letting $\delta \to 0$ as shown in the theorem below.

Proposition 3.2 *For $\psi \in U$, there exists $u \in \mathcal{K}_\alpha$ such that $u = \mathcal{T}(\psi)$, and as $\delta \to 0$, the solutions $u^\delta = T_\delta(\psi)$ of (6) satisfy*

$$\begin{aligned}
&u^\delta \to u \quad \text{strongly in } L^2(Q),\\
&\nabla u^\delta \to \nabla u \quad \text{strongly in } L^2(Q),\\
&{}_0^{\mathrm{LC}}D_t^\alpha u^\delta \rightharpoonup {}_0^{\mathrm{LC}}D_t^\alpha u \quad \text{weakly in } L^2(Q),\\
&\Delta u^\delta \rightharpoonup \Delta u \quad \text{weakly in } L^2(Q).
\end{aligned}$$

Moreover, we obtain the estimate

$$\begin{aligned}
&\sup_{0\le t\le T}\left\|\nabla u^\delta(t)\right\|_{L^2(\Omega)}+\left\|{}_0^{\mathrm{LC}}D_t^\alpha u^\delta(t)\right\|_{L^2(Q)}+\left\|\Delta u^\delta(t)\right\|_{L^2(Q)}\\
&\quad\le C\left[1+\|\nabla u_0\|_{L^2(\Omega)}+\left\|\Delta\psi(t)\right\|_{L^2(Q)}+\left\|{}_0^{\mathrm{LC}}D_t^\alpha\psi(t)\right\|_{L^2(Q)}+\|f\|_{L^2(Q)}\right].
\end{aligned} \tag{10}$$

Proof Using estimate (6), we have the weak convergences above. However, for each t, we have

$$\left\|\nabla\left(u^\delta-u\right)\right\|_{L^2(\Omega)}^2 \le \left\|\Delta\left(u^\delta-u\right)\right\|_{L^2(\Omega)}\left\|u^\delta-u\right\|_{L^2(\Omega)}. \tag{11}$$

Then the $H^1(Q)$ estimate on the $\{u^\delta\}$ approximations implies the strong convergence $u^\delta \to u$ in $L^2(Q)$, and inequality (11) implies the strong convergence $\nabla u^\delta \to \nabla u$ in $L^2(Q)$. Next, we can see that the approximation u^δ satisfies

$$\int_Q\left[{}_0^{\mathrm{LC}}D_t^\alpha u\left(v-u^\delta\right)+\nabla u\cdot\nabla\left(v-u^\delta\right)dx\,dt\right]\ge\int_Q f\left(v-u^\delta\right)dx\,dt$$

for all $v \in \mathcal{K}_\alpha(\psi)$. It follows from the strong convergence of $\{u^\delta\}$ and the weak convergences that

$$\int_Q\left[{}_0^{\mathrm{LC}}D_t^\alpha u(v-u)+\nabla u\cdot\nabla(v-u)\,dx\,dt\right]\ge\int_Q f(v-u)\,dx\,dt$$

for all $v \in \mathcal{K}_\alpha(\psi)$. Using

$$\frac{1}{\delta}\left\|\beta\left(u^\delta-\psi\right)\right\|_{L^2(Q)}\le C(\alpha,\psi,f),$$

we obtain $\|\beta(u^\delta-\psi)\|_{L^2(Q)} = 0$ and $u \ge \psi$ a.e. on Q. Hence we conclude that $u = \mathcal{T}(\psi)$ and u satisfies (10). □

We now prove that there exists an optimal control that minimizes the objective functional (2).

Theorem 3.3 *There exists an optimal control $\psi^* \in U$ such that $J(\psi^*)$ is the minimal value of the objective functional* (2).

Proof Let $\{\psi_k\}_{k=1}^{\infty}$ be a minimizing sequence of obstacles in U, that is,

$$\inf_{\psi\in U} J(\psi) = \lim_{k\to\infty} J(\psi_k)$$

For the functional (2), bounds on $\{J(\psi_k)\}_{k=1}^{\infty}$ imply that there is an obstacle $\psi^* \in U$ such that for a subsequence denoted again by $\{\psi_k\}$),

$$\begin{aligned}
&\psi_k \to \psi^* \quad \text{strongly in } L^2(Q),\\
&\psi_k \rightharpoonup \psi^* \quad \text{weakly in } L^2\bigl(0,T;H_0^1(\Omega)\bigr),\\
&{}_0^{\mathrm{LC}}D_t^{\alpha}\psi_k \rightharpoonup {}_0^{\mathrm{LC}}D_t^{\alpha}\psi^* \quad \text{weakly in } L^2(Q).
\end{aligned}$$

Using estimate (10) for ψ_k and $u_k = \mathcal{T}(\psi_k)$, there is $u^* \in V$ such that

$$\begin{aligned}
&u_k \to u^* \quad \text{strongly in } L^2(Q),\\
&{}_0^{\mathrm{LC}}D_t^{\alpha}u_k \rightharpoonup {}_0^{\mathrm{LC}}D_t^{\alpha}u^* \quad \text{weakly in } L^2(Q),\\
&\Delta u_k \rightharpoonup \Delta u^* \quad \text{weakly in } L^2(Q).
\end{aligned}$$

By a similar argument as in Proposition 3.1 we see that $\nabla u_k \to \nabla u^*$ strongly in $L^2(Q)$. Let $v \in \mathcal{K}_\alpha(\psi^*)$ and set $v_k = \max(v,\psi_k)$. We observe that $v_k \in \mathcal{K}(\psi_k)$ are such that $v_k \to v$ weakly in $L^2(0,T;H_0^1(\Omega))$ and

$$\int_Q \bigl[{}_0^{\mathrm{LC}}D_t^{\alpha}u(v_k-u_k) + \nabla u_k\cdot\nabla(v_k-u_k)\,dx\,dt\bigr] \geq \int_Q f(v_k-u_k)\,dx\,dt.$$

Letting $k\to\infty$ in this inequality, we have

$$\int_Q \bigl[{}_0^{\mathrm{LC}}D_t^{\alpha}u(v-u) + \nabla u\cdot\nabla(v-u)\,dx\,dt\bigr] \geq \int_Q f(v-u)\,dx\,dt$$

for all $v \in \mathcal{K}_\alpha(\psi^*)$.

Since $\psi_k \leq u_k$ a.e. on Ω, the strong convergence with respect to L^2 yields $\psi^* \leq u^*$. Hence $u^* \in \mathcal{K}_\alpha(\psi^*)$ and $u^* = \mathcal{T}(\psi^*)$.

By the lower semicontinuity of the functional J with respect to weak L^2 convergence and $\lim_{k\to\infty}\mathcal{T}(\psi_k) = \mathcal{T}(\psi^*)$, we have $J(\psi^*) \leq \lim_{k\to\infty} J(\psi_k)$. Thus ψ^* is an optimal control that minimizes the functional (2). □

4 Necessary conditions

In this section, we identify necessary conditions of the optimal pair (ψ^*,u^*) where $u^* = \mathcal{T}(\psi^*)$. For this, we establish conditions on the approximations $u^\delta = \mathcal{T}_\delta(\psi^\delta)$ and then derive conditions for (ψ^*,u^*) by passing to the limit as $\delta\to 0$.

Theorem 4.1 *For $\delta > 0$, the solution map $\psi \to u^\delta = \mathcal{T}_\delta(\psi)$ of* (5) *in V is differentiable in the following sense*:

Given $\psi \in U$ and $l \in L^2(Q)$ with $\psi + \varepsilon l \in U$, there is ξ^δ in $L^2(0,T;H_0^1(\Omega))$ such that

$$\frac{u^\delta(\psi + \varepsilon l) - u^\delta(\psi)}{\varepsilon} \rightharpoonup \xi^\delta \quad \text{weakly in } L^2\big(0,T;H_0^1(\Omega)\big)$$

as $\varepsilon \to 0$. In addition, we have ${}_0^{\mathrm{LC}}D_t^\alpha \xi^\delta \in L^2(0,T;H^{-1}(\Omega))$, and ξ^δ satisfies

$$\begin{aligned} &{}_0^{\mathrm{LC}}D_t^\alpha \xi^\delta - \Delta\xi^\delta + \beta_\delta'\big(u^\delta - \psi\big)\big(\xi^\delta - l\big) = 0 \quad \text{in } Q, \\ &\xi^\delta(x,0) = 0 \quad \text{in } \Omega. \end{aligned} \tag{12}$$

Proof Denote $u^{\delta,\varepsilon} = \mathcal{T}_\delta(\psi + \varepsilon l)$ and $u^\delta = \mathcal{T}_\delta(\psi)$. Consider the fractional PDE on $Q_t = \Omega \times (0,t)$,

$$ {}_0^{\mathrm{LC}}D_t^\alpha u^{\delta,\varepsilon} - {}_0^{\mathrm{LC}}D_t^\alpha u^\delta - \Delta\big(u^{\delta,\varepsilon} - u^\delta\big) + \beta_\delta\big(u^{\delta,\varepsilon} - (\psi + \varepsilon l)\big) - \beta\big(u^\delta - \psi\big) = 0. \tag{13}$$

Multiplying this equation by $u^{\delta,\varepsilon} - u^\delta$ and integrating both sides with respect to x, we get

$$\begin{aligned} &\int_\Omega \big[{}_0^{\mathrm{LC}}D_t^\alpha\big(u^{\delta,\varepsilon} - u^\delta\big)\big(u^{\delta,\varepsilon} - u^\delta\big) + \big|\nabla\big(u^{\delta,\varepsilon} - u^\delta\big)\big|^2\big]\,dx \\ &\quad = -\frac{1}{\delta}\int_\Omega \big[\beta\big(u^{\delta,\varepsilon} - (\psi + \varepsilon l)\big) - \beta\big(u^\delta - \psi\big)\big]\big(u^{\delta,\varepsilon} - u^\delta\big)\,dx \\ &\quad = -\frac{1}{\delta}\int_\Omega \int_0^1 \beta'(\tilde\theta)\big(u^{\delta,\varepsilon} - \psi - \varepsilon l\big)\big(u^{\delta,\varepsilon} - u^\delta\big)\,d\theta\,dx, \end{aligned}$$

where $\tilde\theta := \theta(u^{\delta,\varepsilon} - \psi - \varepsilon l) + (1-\theta)(u^\delta - \varepsilon l)$. Applying equality (4), we have

$$\begin{aligned} &\int_\Omega {}_0^{\mathrm{LC}}D_t^\alpha\big(u^{\delta,\varepsilon} - u^\delta\big)\big(u^{\delta,\varepsilon} - u^\delta\big)\,dx \\ &\quad = \frac{1}{2}{}_0^{\mathrm{LC}}D_t^\alpha \big\|u^{\delta,\varepsilon}(\cdot,t) - u^\delta(\cdot,t)\big\|_{L^2(\Omega)}^2 \\ &\qquad + \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha-1}\int_\Omega \big|\big(u^{\delta,\varepsilon}(x,t) - u^\delta(x,t)\big) - \big(u^{\delta,\varepsilon}(x,s) - u^\delta(x,s)\big)\big|^2\,dx\,ds \\ &\qquad + \frac{t^{-\alpha}}{2\Gamma(1-\alpha)}\int_\Omega \big|\big(u^{\delta,\varepsilon}(x,t) - u^\delta(x,t)\big)\big|^2\,dx. \end{aligned}$$

It follows that

$$\begin{aligned} &\int_\Omega \big|u^{\delta,\varepsilon}(x,t) - u^\delta(x,t)\big|^2\,ds + 2t^\alpha\Gamma(1-\alpha)\int_\Omega \big|\nabla\big(u^{\delta,\varepsilon} - u^\delta\big)\big|^2\,dx \\ &\quad = -\frac{2t^\alpha\Gamma(1-\alpha)}{\delta}\int_\Omega\int_0^1 \beta'(\tilde\theta)\big(u^{\delta,\varepsilon} - u^\delta - \varepsilon l\big)\big(u^{\delta,\varepsilon} - u^\delta\big)\,d\theta\,dx \\ &\qquad - t^\alpha\Gamma(1-\alpha){}_0^{\mathrm{LC}}D_t^\alpha\big\|u^{\delta,\varepsilon}(\cdot,t) - u^\delta(\cdot,t)\big\|_{L^2(\Omega)}^2 \\ &\qquad - \alpha t^\alpha\int_0^t (t-s)^{-\alpha-1}\int_\Omega \big|\big(u^{\delta,\varepsilon}(x,t) - u^\delta(x,t)\big) - \big(u^{\delta,\varepsilon}(x,s) - u^\delta(x,s)\big)\big|^2\,dx\,ds \\ &\quad \le \frac{2\varepsilon t^\alpha\Gamma(1-\alpha)}{\delta}\int_\Omega\int_0^1 \beta'(\tilde\theta)l(x,t)\big(u^{\delta,\varepsilon} - u^\delta\big)\,d\theta\,dx \end{aligned}$$

$$
\begin{aligned}
&- t^{\alpha}\Gamma(1-\alpha)\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{d}{ds}\left\|u^{\delta,\varepsilon}(\cdot,s)-u^{\delta}(\cdot,s)\right\|^2_{L^2(\Omega)}\,ds \\
&\leq \frac{2\varepsilon t^{\alpha}\Gamma(1-\alpha)}{\delta}\int_{\Omega} l(x,t)\big(u^{\delta,\varepsilon}-u^{\delta}\big)\,dx - \left\|u^{\delta,\varepsilon}(\cdot,t)-u^{\delta}(\cdot,t)\right\|^2_{L^2(\Omega)} \\
&\leq \frac{2\varepsilon t^{\alpha}\Gamma(1-\alpha)}{\delta}\left\|u^{\delta,\varepsilon}(\cdot,t)-u^{\delta}(\cdot,t)\right\|_{L^2(\Omega)}\left\|l(\cdot,t)\right\|_{L^2(\Omega)} - \left\|u^{\delta,\varepsilon}(\cdot,t)-u^{\delta}(\cdot,t)\right\|^2_{L^2(\Omega)} \\
&\leq \frac{\varepsilon^2 t^{2\alpha}(\Gamma(1-\alpha))^2}{\delta^2}\left\|l(\cdot,t)\right\|^2_{L^2(\Omega)}.
\end{aligned}
$$

This implies the following two inequalities:

$$
\begin{aligned}
&\int_0^t\int_{\Omega}\left|u^{\delta,\varepsilon}(x,t)-u^{\delta}(x,t)\right|^2\,ds\,dt, \leq \frac{\varepsilon^2 T^2(\Gamma(1-\alpha))^2}{\delta^2}\|l\|^2_{L^2(Q_t)}, \\
&\int_0^t\int_{\Omega}\left|\nabla\big(u^{\delta,\varepsilon}-u^{\delta}\big)\right|^2\,dx\,ds \leq \frac{\varepsilon^2 T\Gamma(1-\alpha)}{2\delta^2}\|l\|^2_{L^2(Q_t)}.
\end{aligned}
$$

Thus we obtain

$$
\left\|\frac{u^{\delta,\varepsilon}-u^{\delta}}{\varepsilon}\right\|_{L^2(0,T;H_0^1(\Omega))} \leq \frac{C}{\delta}\|l\|_{L^2(Q)} \tag{14}
$$

for all $\varepsilon > 0$, which implies that there is ξ^{δ} in $L^2(0,T;H_0^1(\Omega))$ such that

$$
\frac{u^{\delta,\varepsilon}-u^{\delta}}{\varepsilon} \rightharpoonup \xi^{\delta} \quad \text{weakly in } L^2\big(0,T;H_0^1(\Omega)\big)
$$

and

$$
\left\|\nabla\xi^{\delta}\right\|_{L^2(Q)} \leq \frac{C}{\delta}\|l\|_{L^2(Q_t)}. \tag{15}
$$

To estimate ${}_0^{\mathrm{LC}}D_t^{\alpha}\xi^{\delta}$, consider

$$
\begin{aligned}
&\int_Q\left[\beta_\delta\big(u^{\delta,\varepsilon}-\psi-\varepsilon l\big)-\beta_\delta\big(u^{\delta}-\psi\big)\right]^2\frac{1}{\varepsilon^2}\,dx\,dt \\
&= \frac{1}{\delta^2\varepsilon^2}\int_Q\left[\int_0^1\beta'(\tilde{\theta})\,d\theta\big(u^{\delta,\varepsilon}-u^{\delta}-\varepsilon l\big)\right]^2\,dx\,dt \\
&\leq \frac{1}{\delta^2}\int_Q\left(\frac{u^{\delta,\varepsilon}-u^{\delta}-\varepsilon l}{\varepsilon}\right)^2\,dx\,dt \\
&\leq \frac{2}{\delta^2}\int_Q\left[\left(\frac{u^{\delta,\varepsilon}-u^{\delta}}{\varepsilon}\right)^2+l^2\right]\,dx\,dt \leq \frac{C}{\delta^2}\|l\|^2_{L^2(Q)}.
\end{aligned} \tag{16}
$$

From (13) we obtain

$$
\begin{aligned}
&\left\|{}_0^{\mathrm{LC}}D_t^{\alpha}\left(\frac{u^{\delta,\varepsilon}-u^{\delta}}{\varepsilon}\right)\right\|_{L^2(0,T;H^{-1}(\Omega))} \\
&\quad\leq \left\|\Delta\left(\frac{u^{\delta,\varepsilon}-u^{\delta}}{\varepsilon}\right)\right\|_{L^2(0,T;H^{-1}(\Omega))} + \left\|\frac{\beta_\delta(u^{\delta,\varepsilon}-\psi-\varepsilon l)-\beta_\delta(u^{\delta}-\psi)}{\varepsilon}\right\|_{L^2(Q)}.
\end{aligned}
$$

By estimates (14)–(16) we conclude

$$\left\|{}_0^{\mathrm{LC}}D_t^{\alpha}\xi^{\delta}\right\|_{L^2(0,T;H^{-1}(\Omega))} \leq \left\|\nabla\xi^{\delta}\right\|_{L^2(0,T;H^{-1}(\Omega))} + \frac{C}{\delta}\|l\|_{L^2(Q)}^2 \leq \frac{C_1}{\delta}\|l\|_{L^2(Q)}^2.$$

Therefore the function ξ^{δ} satisfies (12). □

Let $W = \{v \in H^1(Q) | v = 0 \text{ on } (\partial\Omega \times (0,T)) \cup (\Omega \times \{0\})\}$, and let W' be the dual space of W.

To establish necessary conditions for the optimal control and state for the functional (2), we study the approximation problem

$$\inf_{\psi\in U} J_{\delta}(\psi) \quad \text{with } J_{\delta}(\psi) = \int_Q \left[\left(\mathcal{T}_{\delta}(\psi) - z\right)^2 + |\Delta\psi|^2 + \left({}_0^{\mathrm{LC}}D_t^{\alpha}\psi\right)^2\right] dx\, dt,$$

where $\mathcal{T}_{\delta}(\psi)$ is the solution of the time fractional partial differential equation (5) for obstacle ψ. We obtain the existence of an obstacle ψ^{δ} and the corresponding state $u^{\delta} = \mathcal{T}_{\delta}(\psi^{\delta})$, which is the minimizer of $J_{\delta}(\psi)$ by using a similar argument as in the proof of Theorem 3.3.

To establish the PDE for which ψ^{δ} is a solution, we state the following:

Definition 4.2 The function ψ^{δ} in U is a weak solution of

$$\begin{aligned}
&{}_tD_{T0}^{\alpha}{}^{\mathrm{LC}}D_t^{\alpha}\psi^{\delta} + \Delta^2\psi^{\delta} + \beta_{\delta}'(u^{\delta} - \psi^{\delta})p^{\delta} = 0 \quad \text{in } Q,\\
&\psi^{\delta}(x,0) = 0 \quad \text{in } \Omega,\\
&{}_0^{\mathrm{LC}}D_t^{\alpha}\psi^{\delta}(x,T) = 0 \quad \text{in } \Omega,\\
&\Delta\psi^{\delta} = 0 \quad \text{on } \partial\Omega \times (0,T)
\end{aligned} \tag{17}$$

if $\psi^{\delta} \in L^2(0,T;H^3(\Omega) \cap H_0^1(\Omega))$, ${}_0^{\mathrm{LC}}D_t^{\alpha}\psi^{\delta} \in L^2(Q)$, ${}_tD_{T0}^{\alpha}{}^{\mathrm{LC}}D_t^{\alpha}\psi^{\delta} \in W'$, and for all $\phi \in W$,

$$\int_Q \left[\left({}_0^{\mathrm{LC}}D_t^{\alpha}\psi^{\delta}\right)\left({}_0^{\mathrm{LC}}D_t^{\alpha}\phi\right) + \nabla\Delta\psi^{\delta} \cdot \nabla\phi + \beta_{\delta}'(u^{\delta} - \psi^{\delta})p^{\delta}\phi\right] dx\, dt = 0.$$

Theorem 4.3 *Let $\psi^{\delta} \in U$ be an optimal control that minimizes $J_{\delta}(\psi)$. There exists p^{δ} in $L^2(0,T;H_0^1(\Omega))$ with ${}_t^{\mathrm{LC}}D_T^{\alpha}p^{\delta} \in L^2(0,T;H^{-1}(\Omega))$ such that*

$$\begin{aligned}
&{}_t^{\mathrm{LC}}D_T^{\alpha}p^{\delta} - \Delta p^{\delta} + \beta_{\delta}'(u^{\delta} - \psi^{\delta})p^{\delta} = u^{\delta} - z \quad \text{in } Q,\\
&p^{\delta}(x,T) = 0 \quad \text{in } \Omega.
\end{aligned} \tag{18}$$

Furthermore ψ^{δ} satisfies (17) in the sense defined above with $u^{\delta} = \mathcal{T}_{\delta}(\psi^{\delta})$, and

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left\|p^{\delta}(t)\right\|_{L^2(\Omega)} + \left\|\nabla p^{\delta}(t)\right\|_{L^2(\Omega)} + \left\|\beta_{\delta}'(u^{\delta} - \psi^{\delta})p^{\delta}\right\|_{W'} + \left\|{}_t^{\mathrm{LC}}D_T^{\alpha}p^{\delta}\right\|_{W'} \leq C_1,\\
&\left\|\nabla\left({}_0^{\mathrm{LC}}D_t^{\alpha}\psi^{\delta}\right)\right\|_{L^2(Q)} + \left\|\nabla\Delta\psi^{\delta}\right\|_{L^2(Q)} + \left\|{}_tD_{T0}^{\alpha}{}^{\mathrm{LC}}D_t^{\alpha}\psi^{\delta}\right\|_{W'} \leq C_2,
\end{aligned}$$

where C_1 and C_2 are independent of δ.

Proof Since the approximate adjoint time fractional partial differential equation (18) is linear in p^δ, there exists a solution such that $p^\delta \in L^2(0,T;H_0^1(\Omega))$, ${}_t^{\mathrm{LC}}D_T^\alpha p^\delta \in L^2(0,T;H^{-1}(\Omega))$, and $p^\delta(x,T)=0$ in Ω.

Consider the estimate of ∇p^δ on $Q_t = \Omega \times (t,T)$,

$$\int_{Q_t} \bigl[\bigl({}_t^{\mathrm{LC}}D_T^\alpha p^\delta\bigr)p^\delta + \bigl|\nabla p^\delta\bigr|^2 + \beta_\delta'\bigl(u^\delta - \psi\bigr)\bigl(p^\delta\bigr)^2\bigr]\,dx\,dt = \int_{Q_t} p^\delta\bigl(u^\delta - z\bigr)\,dx\,dt.$$

By Proposition 2.4 and $p^\delta(x,T)=0$ we get by integration by parts that

$$\begin{aligned}
&\int_t^T \bigl({}_t^{\mathrm{LC}}D_T^\alpha p^\delta\bigr)p^\delta\,d\tau \\
&\quad= \int_t^T \bigl({}_tD_T^\alpha p^\delta\bigr)p^\delta\,d\tau \\
&\quad= -\frac{1}{\Gamma(1-\alpha)}\int_t^T p^\delta(\tau)\biggl(\frac{d}{d\tau}\int_\tau^T (s-\tau)^{-\alpha}p^\delta(s)\,ds\biggr)\,d\tau \\
&\quad= \frac{1}{\Gamma(1-\alpha)}\biggl[p^\delta(t)\int_t^T (s-t)^{-\alpha}p^\delta(s)\,ds + \int_t^T p_\tau^\delta(\tau)\int_\tau^T (s-\tau)^{-\alpha}p^\delta(s)\,ds d\tau\biggr] \\
&\quad\ge \frac{1}{\Gamma(1-\alpha)}\biggl[p^\delta(t)(T-t)^{-\alpha}\int_t^T p^\delta(s)\,ds + \int_t^T (T-\tau)^{-\alpha}p_\tau^\delta(\tau)\int_\tau^T p^\delta(s)\,ds\,d\tau\biggr] \\
&\quad\ge \frac{1}{\Gamma(1-\alpha)}\biggl[p^\delta(t)(T-t)^{-\alpha}\int_t^T p^\delta(s)\,ds + (T-t)^{-\alpha}\int_t^T p_\tau^\delta(\tau)\int_\tau^T p^\delta(s)\,ds d\tau\biggr] \\
&\quad\ge \frac{(T-t)^{-\alpha}}{\Gamma(1-\alpha)}\int_t^T \bigl(p^\delta(\tau)\bigr)^2\,d\tau.
\end{aligned}$$

This implies

$$\frac{(T-t)^{-\alpha}}{\Gamma(1-\alpha)}\int_{Q_t} \bigl(p^\delta(x,t)\bigr)^2\,dx\,dt + \int_{Q_t} \bigl|\nabla p^\delta\bigr|^2(x,t)\,dx\,dt \le \int_{Q_t} p^\delta\bigl(u^\delta - z\bigr)\,dx\,dt$$

since the term β' is nonnegative. It follows that

$$\int_Q \bigl(p^\delta(x,t)\bigr)^2\,dx\,dt + \int_Q \bigl|\nabla p^\delta\bigr|^2(x,t)\,dx\,dt \le C\bigl\|u^\delta - z\bigr\|_{L^2(Q)}^2. \tag{19}$$

Next, we consider the estimate of $\beta'(\cdot)p^\delta$ in W': for $\phi \in W$,

$$\begin{aligned}
\biggl|\int_Q \beta_\delta'\bigl(u^\delta - \psi\bigr)p^\delta\phi\,dx\,dt\biggr| &= \biggl|\int_Q \bigl[\bigl(u^\delta - z\bigr)\phi - \bigl({}_t^{\mathrm{LC}}D_T^\alpha p^\delta\bigr)\phi + \nabla p^\delta\cdot\nabla\phi\bigr]\,dx\,dt\biggr| \\
&= \biggl|\int_Q \bigl[\bigl(u^\delta - z\bigr)\phi + p^\delta\,{}_0^{\mathrm{LC}}D_t^\alpha\phi + \nabla p^\delta\cdot\nabla\phi\bigr]\,dx\,dt\biggr| \\
&\le C\bigl\|u^\delta - z\bigr\|_{L^2(Q)}\|\phi\|_W
\end{aligned}$$

using estimate (19). We conclude

$$\bigl\|\beta_\delta'\bigl(u^\delta - \phi^\delta\bigr)p^\delta\bigr\|_{W'} \le C\bigl\|u^\delta - z\bigr\|_{L^2(Q)}, \tag{20}$$

which gives an estimate $\|{}_t^{\mathrm{LC}}D_T^\alpha p^\delta\|_{W'}$.

Denote $u^\delta = \mathcal{T}_\delta(\psi^\delta)$ and $u^{\delta,\varepsilon} = \mathcal{T}_\delta(\psi^\delta + \varepsilon l)$ with $l \in W \cap U$. Since ψ^δ is a minimizer for $J_\delta(\psi)$,

$$\begin{aligned}
0 &\leq \lim_{\varepsilon\to 0^+} \frac{J_\delta(\psi^\delta+\varepsilon l)-J_\delta(\psi^\delta)}{\varepsilon} \\
&= 2\int_Q \bigl[\xi^\delta\bigl(u^\delta - z\bigr) + \Delta\psi^\delta\cdot\Delta l + \bigl({}_0^{\mathrm{LC}}D_t^\alpha\psi^\delta\bigr)\bigl({}_0^{\mathrm{LC}}D_t^\alpha l\bigr)\bigr]\,dx\,dt \\
&= 2\int_Q \bigl[\xi^\delta\bigl({}_t^{\mathrm{LC}}D_T^\alpha p^\delta\bigr) + \nabla\xi^\delta\cdot\nabla p^\delta + \beta_\delta'\bigl(u^\delta-\psi^\delta\bigr)p^\delta\xi^\delta \\
&\quad + \Delta\psi^\delta\Delta l + \bigl({}_0^{\mathrm{LC}}D_t^\alpha\psi^\delta\bigr)\bigl({}_0^{\mathrm{LC}}D_t^\alpha l\bigr)\bigr]\,dx\,dt \\
&= 2\int_Q \bigl[\bigl({}_0^{\mathrm{LC}}D_t^\alpha\xi^\delta\bigr)p^\delta + \nabla\xi^\delta\cdot\nabla p^\delta + \beta_\delta'\bigl(u^\delta-\psi^\delta\bigr)p^\delta\xi^\delta \\
&\quad + \Delta\psi^\delta\Delta l + \bigl({}_0^{\mathrm{LC}}D_t^\alpha\psi^\delta\bigr)\bigl({}_0^{\mathrm{LC}}D_t^\alpha l\bigr)\bigr]\,dx\,dt \\
&= 2\int_Q \bigl[\beta_\delta'\bigl(u^\delta-\psi^\delta\bigr)lp^\delta + \Delta\psi^\delta\Delta l + \bigl({}_0^{\mathrm{LC}}D_t^\alpha\psi^\delta\bigr)\bigl({}_0^{\mathrm{LC}}D_t^\alpha l\bigr)\bigr]\,dx\,dt
\end{aligned}$$

by using the p^δ and ξ^δ PDE, (17) and (18). We conclude that ψ^δ satisfies PDE (17) subject to the indicated boundary conditions. Indeed, the condition $l(x,0) = 0$ is used in the step of integration by parts on $\psi_t^\delta l_t$ term. Moreover, $\Delta\psi^\delta = 0$ on $\partial\Omega \times (0,T)$ arrives from the weak formulation of the solution ψ^δ for the time fractional PDE and integration by parts on $\Delta\psi^\delta\Delta l$.

For the estimate of $\nabla\Delta\psi^\delta$ and ${}_tD_{T0}^{\alpha}{}^{\mathrm{LC}}D_t^\alpha\psi^\delta$, we have

$$\begin{aligned}
&\int_Q \bigl[\bigl(-{}_tD_{T0}^{\alpha}{}^{\mathrm{LC}}D_t^\alpha\psi^\delta\bigr)\Delta\psi^\delta + \bigl|\nabla\Delta\psi^\delta\bigr|^2\bigr]\,dx\,dt = \int_Q \beta_\delta'\bigl(u^\delta-\psi^\delta\bigr)p^\delta\Delta\psi^\delta\,dx\,dt, \\
&\int_Q \bigl[\bigl|\nabla\bigl({}_0^{\mathrm{LC}}D_t^\alpha\psi^\delta\bigr)\bigr|^2 + \bigl|\nabla\Delta\psi^\delta\bigr|^2\bigr]\,dx\,dt = \int_Q \bigl(u^\delta - z - {}_t^{\mathrm{LC}}D_T^\alpha p^\delta + \Delta p^\delta\bigr)\Delta\psi^\delta\,dx\,dt.
\end{aligned}$$

Using the Poincaré inequality to estimate

$$\int_Q \bigl|\Delta\psi^\delta\bigr|^2\,dx\,dt \leq C_1\int_Q \bigl|\nabla\Delta\psi^\delta\bigr|^2\,dx\,dt,$$

we obtain

$$\begin{aligned}
&\int_Q \bigl[\bigl|\nabla\bigl({}_0^{\mathrm{LC}}D_t^\alpha\psi^\delta\bigr)\bigr|^2 + \bigl|\nabla\Delta\psi^\delta\bigr|^2\bigr]\,dx\,dt \\
&\quad \leq C\biggl(\int_Q \bigl(\bigl(u^\delta - z\bigr)^2 + \bigl|\nabla p^\delta\bigr|^2\bigr)\,dx\,dt + \bigl\|{}_t^{\mathrm{LC}}D_T^\alpha p^\delta\bigr\|_{W'}^2\biggr).
\end{aligned}$$

For the solution ψ^δ of the time fractional PDE (17), the estimate of $|\nabla\Delta\psi^\delta|$ in L^2 and the estimate of $\beta'(u^\delta-\psi^\delta)p^\delta$ under W' norm in (20) give the required estimate of ${}_tD_{T0}^{\alpha}{}^{\mathrm{LC}}D_t^\alpha\psi$ in W' norm. □

Letting $\delta \to 0$, we establish necessary conditions on ψ^*, an optimal control for $J(\psi)$. We state our definition of a solution for the equation for ψ^* and the limiting adjoint function p.

Definition 4.4 Functions ψ^* and p satisfy

$$\begin{aligned} &{}^{\mathrm{LC}}_{t}D^{\alpha}_{T}p - \Delta p - {}_{t}D^{\alpha}_{T0}{}^{\mathrm{LC}}D^{\alpha}_{t}\psi^* - \Delta^2\psi^* = u^* - z \quad \text{in } Q, \\ &\psi^*(x,0) = {}^{\mathrm{LC}}_{0}D^{\alpha}_{t}\psi^*(x,T) = 0 \quad \text{in } \Omega, \\ &p(x,T) = 0 \quad \text{in } \Omega, \\ &\Delta\psi^* = 0 \quad \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{21}$$

where $u^* = T(\psi^*)$, if $\psi^* \in U, \psi^* \in L^2(0,T;H^3(\Omega)), {}^{\mathrm{LC}}_{0}D^{\alpha}_{t}\psi^* \in L^2(0,T;H^1_0(\Omega))$, ${}_{t}D^{\alpha}_{T0}{}^{\mathrm{LC}}D^{\alpha}_{t}\psi^* \in W', \Delta^2\psi^* \in L^2(0,T;H^{-1}(\Omega)), p \in L^2(0,T;H^1_0(\Omega)), {}^{\mathrm{LC}}_{t}D^{\alpha}_{T}p \in W'$, and for all $\phi \in W$,

$$\begin{aligned} &\int_Q \bigl[p\bigl({}^{\mathrm{LC}}_{t}D^{\alpha}_{T}\phi\bigr) + \nabla\phi\cdot\nabla p + \nabla\Delta\psi^*\cdot\nabla\phi - \bigl({}^{\mathrm{LC}}_{0}D^{\alpha}_{t}\psi^*\bigr)\bigl({}^{\mathrm{LC}}_{0}D^{\alpha}_{t}\phi\bigr)\bigr]\,dx\,dt \\ &\quad = \int_Q \bigl(u^* - z\bigr)\phi\,dx\,dt. \end{aligned}$$

Theorem 4.5 *There exist a sequence of minimizers $\{\psi^{\delta_n}\}$ in U for the functionals $J_{\delta_n}(\psi)$, the corresponding adjoint $\{p^{\delta_n}\}, \psi^* \in U$ with the corresponding state $u^* = \mathcal{T}(\psi^*)$, and the adjoint p in $L^2(0,T;H^1_0(\Omega))$ such that, as $\delta_n \to 0$,*

$$\begin{aligned} &\psi^{\delta_n} \to \psi^* \quad \textit{weakly in } L^2\bigl(0,T;H^3(\Omega)\bigr), \\ &{}_{t}D^{\alpha}_{T0}{}^{\mathrm{LC}}D^{\alpha}_{t}\psi^{\delta_n} \to {}_{t}D^{\alpha}_{T0}{}^{\mathrm{LC}}D^{\alpha}_{t}\psi^* \quad \textit{weak* in } W', \\ &p^{\delta_n} \to p \quad \textit{weakly in } L^2\bigl(0,T;H^1_0(\Omega)\bigr), \\ &{}^{\mathrm{LC}}_{0}D^{\alpha}_{t}p^{\delta_n} \to {}^{\mathrm{LC}}_{0}D^{\alpha}_{t}p \quad \textit{weak* in } W'. \end{aligned}$$

Moreover, ψ^* is an optimal control for $J(\psi)$, ψ^* and p satisfy (21) in the sense defined above, and the state $u^* = \mathcal{T}(\psi^*)$ satisfies the time fractional variational inequality (1).

Proof The a priori estimates obtained in Theorem 4.3 imply the convergences and the existence of ψ^*, u^*, and p. It remains to verify that $u^* = \mathcal{T}(\psi^*)$. For $v \in \mathcal{K}_\alpha(\psi^*)$, we see that

$$\int_Q \bigl[{}^{\mathrm{LC}}_{0}D^{\alpha}_{t}u^\delta\bigl(v^\delta - u^\delta\bigr) + \nabla u^\delta\cdot\nabla\bigl(v^\delta - u^\delta\bigr)\bigr]\,dx\,dt \ge \int_Q f\bigl(v^\delta - u^\delta\bigr)\,dx\,dt$$

for $v^\delta = \max(v, \psi^\delta)$. Letting $\delta_n \to 0$, we have $v^{\delta_n} \to v$ weakly in $L^2(0,T;H^1_0(\Omega))$ and

$$\int_Q \bigl[{}^{\mathrm{LC}}_{0}D^{\alpha}_{t}u^*\bigl(v - u^*\bigr) + \nabla u^*\cdot\nabla\bigl(v - u^*\bigr)\bigr]\,dx\,dt \ge \int_Q f\bigl(v - u^*\bigr)\,dx\,dt.$$

By estimate (8) we get

$$\bigl\|\beta_\delta\bigl(u^{\delta_n} - \psi^{\delta_n}\bigr)\bigr\|_{L^2(Q)} \le C \ne C(\delta),$$

which gives $u^* \in \mathcal{K}_\alpha(\psi^*)$. We conclude $u^* = \mathcal{T}(\psi^*)$. The convergences of ψ^{δ_n} and p^{δ_n} from Theorem 4.3 justify that ψ^* and p satisfy (21) in the defined weak sense. Finally, we show

ψ^* is an optimal control for $J(\psi)$. As ψ^{δ_n} is a minimizer for $J_\delta(\psi)$,

$$J_\delta(\psi^*) \geq J_\delta(\psi^{\delta_n}) \quad \text{for all } \delta > 0.$$

By the strong convergence $\mathcal{T}_{\delta_n}(\psi^*) \to u^* = \mathcal{T}(\psi^*)$ in $L^2(Q)$, as $\delta_n \to 0$, we get

$$J(\psi^*) = \overline{\lim_{\delta_n \to 0}} J_\delta(\psi^*) \geq \overline{\lim_{\delta_n \to 0}} J_\delta(\psi^\delta).$$

Since the functional is lower semicontinuous with respect to the weak L^2 convergence and $u^* = \mathcal{T}(\psi^*)$, it follows that

$$\underline{\lim_{\delta_n \to 0}} J_{\delta_n}(\psi^{\delta_n}) \geq J(\psi^*).$$

Therefore the functional $J(\psi)$ attains its minimum value over $\psi \in U$ at $J(\psi^*)$, and hence ψ^* is an optimal control. □

5 Discussion

To illustrate our main results, we consider the following example. Let $Q = \Omega \times (0, T)$, where Ω is an open bounded set in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$. Consider the fractional diffusion inequality with Liouville–Caputo fractional derivatives (similar to fractional diffusion equation given in Example 2 in [15])

$$\begin{aligned} &{}_0^{\mathrm{LC}}D_t^\alpha u(x,t) + \Delta u(x,t) \geq f(x,t), \quad (x,t) \in Q, \\ &u(x,t) = 0, \quad x \in \partial\Omega, t \in (0,T), \\ &u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{22}$$

where $u_0 \in H_0^1(\Omega)$ and $f \in L^2(Q)$. We seek for the solution u of (22) that lies above an obstacle $\psi \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ with ${}_0^{\mathrm{LC}}D_t^\alpha(\psi) \in L^2(Q)$ and $\psi(x,0) = 0$, that is,

$$u(x,t) \geq \psi(x,t) \quad \text{for a.e. } (x,t) \in Q. \tag{23}$$

The solution u satisfying (22) and (23) can be formulated as a solution of variational inequality (1), denoted by $u = \mathcal{T}(\psi)$.

Hence, given a target profile $z \in L^2(Q)$, we get from Theorem 3.3 that the optimal control problem for the obstacle equation with objective functional (2) has a solution $(\psi^*, u^* = \mathcal{T}(\psi^*))$. Moreover, by Theorem 4.5 the optimal solution ψ^* satisfies the adjoint problem (21).

Analogous results could be obtained in the context of Riemann–Liouville fractional derivative by using the relation between the Riemann–Liouville and Liouville–Caputo derivatives. In particular, the a priori estimate for a solution of the time fractional-order diffusion equation in Lemma 2.9 needs to be adjusted. Moreover, the initial condition should be replaced by some nonlocal conditions.

6 Conclusion

In this work, we establish the existence of the optimal control for obstacle variational inequalities involving Liouville–Caputo fractional derivatives. Necessary conditions of the optimal solutions were obtained through the adjoint problem in weak formulation. Future research in optimal control of obstacle problem for time fractional-order variational inequalities can be extended under various assumptions such as the time delay problem, nonlocal and nonsingular kernels, or under different boundary conditions, which would extend the results for fractional diffusion equations in the literature. In addition, numerical algorithms should be investigated to visualize the optimal solution.

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Authors' contributions
The main idea and background of this paper was proposed and mainly proved by PSN, whereas AS and PK performed some proofs and edited the manuscript. All authors read and approved the final manuscript.

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