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Existence of solutions for a coupled system of fractional differential equations by means of topological degree theory

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Abstract

This paper investigates the existence of solutions for a coupled system of fractional differential equations. The existence is proved by using the topological degree theory, and an example is given to show the applicability of our main result.

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Keywords: Fractional differential equations; Boundary value problems; Topological degree theory; Coupled systems

1 Introduction

In this manuscript, the following coupled system of fractional differential equations is discussed:

$$\begin{cases} D^{\theta}[x(t) - f(t, x(t))] = h(t, y(t), I^{\alpha}y(t)), & t \in [0,1], \\ D^{\theta}[y(t) - f(t, y(t))] = h(t, x(t), I^{\alpha}x(t)), & t \in [0,1], \\ x'(0) = y'(0) = 0, \\ a_1x(0) - b_1x(\eta) - c_1x(1) = \frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}\phi(s, x(s))\,ds, \\ a_2y(0) - b_2y(\xi) - c_2y(1) = \frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}\psi(s, y(s))\,ds, \end{cases} \tag{1.1}$$

where $1 < \theta \le 2$, $\alpha > 0$, $\eta, \xi \in (0,1]$, a_j, b_j, c_j $(j = 1,2)$ are real numbers with $a_j \ne b_j + c_j$ $(j = 1,2)$. Further $f, \phi, \psi : [0,1] \times \mathbb{R} \to [0,1]$, and $h : [0,1] \times \mathbb{R} \times \mathbb{R} \to [0,1]$ are continuous functions, $f(0, x(0)) = 0$, $\frac{\partial^i f(t,x(t))}{\partial t^i}|_{t=0} = 0$ for $i = 1, 2, \dots, n-1$. D^{θ} represents the Caputo fractional derivative of order θ.

Fractional differential equations are widely used in many fields such as chemistry, physics, biology, and optimization theory [1–4]. In addition, coupled systems of fractional differential equations have attracted particular concern from scholars considering their appearance in the mathematical modeling of physical phenomena like chaos synchronization [5], anomalous diffusion [6], disease models [7], and so on. The existence theory to fractional differential equations with integral boundary conditions has widespread applications in optimization theory, many researchers have studied [8–13], and the existence of solutions is the basis of studying the stability and numerical solutions of differential equations [14]. For the existence of solutions of fractional differential equations, the authors use diverse methods, such as fixed point theory [15–19], upper and lower solutions method [20], monotone iterative technique and Mawhin's continuation theorem [21], and topological degree theory [22]. When studying the existing literature, we find that fractional differential equations with integral boundary conditions are not properly tested via topological degree theory. Thus we investigate the existence result to a coupled system of fractional differential equations(1.1) through applying topological degree theory.

Bashiri et al. [23]investigated the existence of solutions for fractional differential equations by means of the coupled fixed point theorem of Krasnoselskii type

$$\begin{cases} D^{\theta}[x(t)-f(t,x(t))] = h(t,y(t),I^{\alpha}y(t)), \\ D^{\theta}[y(t)-f(t,y(t))] = h(t,x(t),I^{\alpha}x(t)), \\ x(0)=y(0)=0, \end{cases}$$

where D^{θ} denotes the Riemann–Liouville fractional derivative, $\theta \in (0,1)$, $\alpha > 0$.

Ahmad et al. [24]established existence results as well as studied qualitative aspects of the proposed coupled system of fractional hybrid delay differential equations

$$\begin{cases} {}^{C}D^{\kappa}_{+0}(r(t)-P_1(t,r(t),h(t))) = Q_1(t,r(\nu t),h(\nu t)), \quad t \in \mathcal{A}, \\ {}^{C}D^{\sigma}_{+0}(h(t)-P_2(t,r(t),h(t))) = Q_2(t,r(\nu t),h(\nu t)), \quad t \in \mathcal{A}, \\ r(t)|_{t=0} = r_0, \qquad h(t)|_{t=0} = h_0, \end{cases}$$

where $\mathcal{A} = [0,\tau]$, ${}^{C}D_{+0}, \tau > 0$ is Caputo's derivative, and r_0, h_0 are real numbers, while the delay parameter is denoted by $\nu \in (0,1)$.

Muthaiah et al. [25]considered the existence and Hyers–Ulam type stability results for the nonlinear coupled system of Caputo–Hadamard type fractional differential equations

$$\begin{cases} {}^{C}D^{\varrho}y(\tau) = f(\tau,y(\tau),z(\tau)), \quad \tau \in [1,T] := \mathcal{K}, \\ {}^{C}D^{\varsigma}z(\tau) = g(\tau,y(\tau),z(\tau)), \quad \tau \in [1,T] := \mathcal{K}, \\ y(1)=0, y'(1)=0, y(T) = \alpha_1 \sum_{j=1}^{k-2} \xi_j z(\zeta_j) + \beta_1 {}^{H}I^{\varsigma_1} z(\vartheta), \\ z(1)=0, z'(1)=0, z(T) = \alpha_2 \sum_{j=1}^{k-2} \upsilon_j z(\omega_j) + \beta_2 {}^{H}I^{\varrho_1} y(\varphi), \\ 1 < \vartheta < \varphi < \xi_1 < \omega_1 < \xi_2 < \omega_2 < \cdots < \xi_{k-2} < \omega_{k-2} < T, \end{cases}$$

where ${}^{C}D^{(\cdot)}$ denotes the Caputo–Hadamard fractional derivative, ${}^{H}I^{(\cdot)}$ denotes the Hadamard fractional integrals, $2 < \varrho, \varsigma \leq 3$, $0 < \varrho_1, \varsigma_1 < 1$, α_1, α_2, β_1, β_2 are real constants and ζ_j, υ_j, $j = 1,2,\ldots,k-2$, are positive real constants. The consequence of existence is obtained by employing the alternative of Leray–Schauder and Krasnoselskii's, whereas the uniqueness result is based on the principle of Banach contraction mapping.

Motivated especially by the aforementioned work, we consider the existence of solutions to a coupled system of fractional differential equations (1.1). According to our literature review, no scholars have studied equation (1.1), the results are entirely new. The remainder of this paper is as follows. In the second part, we display some definitions, facts, and results. We confirm the existence of solutions for system (1.1) in the third part. Finally, we provide an example to prove our results.

2 Preliminaries

In this part, we recollect a number of facts, definitions, and conclusions. Let $C([0,1]\times\mathbb{R},[0,1])$ represent the space of all continuous functions $f,\phi,\psi:[0,1]\times\mathbb{R}\to[0,1]$, and let $C([0,1]\times\mathbb{R}\times\mathbb{R},[0,1])$ express the class of functions $h:[0,1]\times\mathbb{R}\times\mathbb{R}\to[0,1]$ such that

(1) the map $t\to h(t,x,y)$ is measurable for each $x,y\in\mathbb{R}$,

(2) the map $x\to h(t,x,y)$ is continuous for each $x\in\mathbb{R}]$,

(3) the map $y\to h(t,x,y)$ is continuous for each $y\in\mathbb{R}$.

Let X be a Banach space and $\mathbb{B}\subset P(X)$, where $P(X)$ stands for the family of all bounded subsets of X. Next, we introduce some concepts.

Definition 2.1 ([26]) The Kuratowski measure of noncompactness $\alpha:\mathbb{B}\to\mathbb{R}_+$ is defined as

$$\alpha(B)=\inf\{d>0,\text{where } B\in\mathbb{B} \text{ admits a finite cover by set of diameter } \leq d\}.$$

Definition 2.2 ([26]) Let $\mathcal{F}:\Omega\to X$ be a continuous bounded map, where $\Omega\subseteq X$. Then $\mathcal{F}$ is

(1) α-Lipschitz if there exists $k\geq 0$, therefore $\alpha(\mathcal{F}(S))\leq k\alpha(S)$ for all bounded subsets $S\subseteq\Omega$;

(2) strict α-contraction if there exists $0\leq k<1$ such that $\alpha(\mathcal{F}(S))\leq k\alpha(S)$ for all bounded subsets $S\subseteq\Omega$;

(3) α-condensing if $\alpha(\mathcal{F}(S))<\alpha(S)$ for all bounded subsets $S\subseteq\Omega$ with $\alpha(S)>0$. In other words, $\alpha(\mathcal{F}(S))\geq\alpha(S)$ implies $\alpha(S)=0$.

All classes of strict α-contraction $\mathcal{F}:\Omega\to X$ and all classes of α-condensing maps $\mathcal{F}:\Omega\to X$ are represented by $\Lambda C_\alpha(\Omega)$ and $C_\alpha(\Omega)$, respectively. Then $\Lambda C_\alpha(\Omega)\subset C_\alpha(\Omega)$ and each $\mathcal{F}\in C_\alpha(\Omega)$ is α-Lipschitz with constant $k=1$. Moreover, $\mathcal{F}:\Omega\to X$ is Lipschitz whenever there is $k>0$, therefore

$$\|\mathcal{F}(x)-\mathcal{F}(y)\|\leq k\|x-y\|\quad \text{for all } x,y\in\Omega.$$

Further, $\mathcal{F}$ will be a strict contraction if $k<1$.

Proposition 2.3 ([27]) *If $\mathcal{F},\mathcal{G}:\Omega\to X$ are α-Lipschitz with respective constants k_1 and k_2, then $\mathcal{F}+\mathcal{G}$ is α-Lipschitz with constant k_1+k_2.*

Proposition 2.4 ([27]) *If $\mathcal{F}:\Omega\to X$ is Lipschitz with constant k, then $\mathcal{F}$ is α-Lipschitz with the equal constant k.*

Proposition 2.5 ([27]) *If $\mathcal{F}:\Omega\to X$ is compact, then $\mathcal{F}$ is α-Lipschitz with constant $k=0$.*

Theorem 2.6 ([27]) *If $\mathcal{F}:X\to X$ is α-condensing and*

$$\Lambda=\{x\in X:\textit{there exists } 0\leq\nu\leq 1 \textit{ such that } x=\nu\mathcal{F}x\}.$$

If Λ is a bounded set in X, so we have $r>0$ such that $\Lambda\subset B_r(0)$, then

$$D\big(I-\nu\mathcal{F},B_r(0),0\big)=1\quad \textit{for all } \nu\in[0,1].$$

Consequently, $\mathcal{F}$ has at least one fixed point, and the set of the fixed points of $\mathcal{F}$ lies in $B_r(0)$.

Definition 2.7 ([28]) The fractional integral of order $\theta(\theta > 0)$ of function $f : [0,\infty) \to R$ is defined as

$$I^{\theta}f(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}f(s)\,ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.8 ([28]) The Caputo fractional derivative of order $\theta(\theta > 0)$ of the function $f : [0,\infty) \to R$ is defined by

$$D^{\theta}f(t) = \frac{1}{\Gamma(n-\theta)}\int_0^t (t-s)^{n-\theta-1}f^{(n)}(s)\,ds,$$

where $t > 0$, $n = [\theta] + 1$.

Lemma 2.9 ([28]) *Let $\theta > 0$, then the following result holds for fractional differential equations*:

$$I^{\theta}\bigl[D^{\theta}f(t)\bigr] = f(t) + C_0 + C_1t + C_2t^2 + \cdots + C_{n-1}t^{n-1}$$

for arbitrary $n = [\theta] + 1$, $[\theta]$ indicates the integer part of the real number $\theta > 0$, $C_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$. D^{θ} is a Caputo fractional derivative.

3 Main results

In this part, we discuss the existence result for (1.1).

The space $X = C([0,1],\mathbb{R})$ of all continuous functions is a Banach space under the topological norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$ and the product space $X \times X$ is a Banach space under the norm $\|(x,y)\| = \|x\| + \|y\|$ or $\|(x,y)\| = \max\{\|x\|,\|y\|\}$.

In order to get the result of our result, we need the following hypotheses.

(H_1) For each $(t,x),(t,\bar{x}),(t,y),(t,\bar{y}) \in [0,1] \times \mathbb{R}$, there exist constants $\lambda_1,\lambda_2 \in [0,1)$ such that

$$\bigl|f(t,x) - f(t,\bar{x})\bigr| \le \lambda_1\|x - \bar{x}\|,$$
$$\bigl|f(t,y) - f(t,\bar{y})\bigr| \le \lambda_2\|y - \bar{y}\|.$$

(H_2) For each $(t,x,y) \in \mathbb{R}$, there exist positive constants l_h^1, l_h^2, M_h and $q_1 \in [0,1)$ such that

$$\bigl|h(t,x,y)\bigr| \le l_h^1\|x\|^{q_1} + l_h^2\|y\|^{q_1} + M_h.$$

(H_3) For each $(t,x) \in [0,1] \times \mathbb{R}$, there exist positive constants l_f, M_f and $q_2 \in [0,1)$ such that

$$\bigl|f\bigl(t,x(t)\bigr)\bigr| \le l_f\|x\|^{q_2} + M_f.$$

(H_4) For each $(t,x),(t,y) \in [0,1] \times \mathbb{R}$, there exist positive constants c_ϕ, c_ψ, M_ϕ, M_ψ, and $q_2 \in [0,1)$ such that

$$\begin{aligned}&\left|\phi(t,x)\right| \le c_\phi \|x\|^{q_2} + M_\phi,\\&\left|\psi(t,y)\right| \le c_\psi \|y\|^{q_2} + M_\psi.\end{aligned}$$

(H_5) For each $(t,x),(t,\bar{x}),(t,y),(t,\bar{y}) \in [0,1] \times \mathbb{R}$, we have positive constants $b_\phi, b_\psi \in [0,1)$ such that

$$\begin{aligned}&\left|\phi(t,x) - \phi(t,\bar{x})\right| \le b_\phi \|x - \bar{x}\|,\\&\left|\psi(t,y) - \psi(t,\bar{y})\right| \le b_\psi \|y - \bar{y}\|.\end{aligned}$$

Lemma 3.1 *If* $f(0,x(0)) = 0$, $\frac{\partial^i f(t,x(t))}{\partial t^i}|_{t=0} = 0$ *for* $i = 1,2,\dots,n-1$, *then the consequence of fractional differential equations* (1.1) *is a conclusion of the following system of integral equations*:

$$\begin{cases} x(t) = f(t,x(t)) + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} h(s,y(s),I^\alpha y(s))\,ds \\ \qquad + \frac{1}{a_1-(b_1+c_1)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}\phi(s,x(s))\,ds \\ \qquad + \frac{b_1}{a_1-(b_1+c_1)} f(\eta,x(\eta)) + \frac{c_1}{a_1-(b_1+c_1)} f(1,x(1)) \\ \qquad + \frac{b_1}{a_1-(b_1+c_1)}\frac{1}{\Gamma(\theta)}\int_0^\eta (\eta-s)^{\theta-1} h(s,y(s),I^\alpha y(s))\,ds \\ \qquad + \frac{c_1}{a_1-(b_1+c_1)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1} h(s,y(s),I^\alpha y(s))\,ds, \\ y(t) = f(t,y(t)) + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} h(s,x(s),I^\alpha x(s))\,ds \\ \qquad + \frac{1}{a_2-(b_2+c_2)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}\phi(s,y(s))\,ds \\ \qquad + \frac{b_2}{a_2-(b_2+c_2)} f(\xi,y(\xi)) + \frac{c_2}{a_2-(b_2+c_2)} f(1,y(1)) \\ \qquad + \frac{b_2}{a_2-(b_2+c_2)}\frac{1}{\Gamma(\theta)}\int_0^\xi (\xi-s)^{\theta-1} h(s,x(s),I^\alpha x(s))\,ds \\ \qquad + \frac{c_2}{a_2-(b_2+c_2)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1} h(s,x(s),I^\alpha x(s))\,ds. \end{cases} \tag{3.1}$$

Proof Applying the fractional integrable operator I^θ on the equation of system (1.1) and through applying Lemma 2.9, we get

$$x(t) = f\bigl(t,x(t)\bigr) + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} h\bigl(s,y(s),I^\alpha y(s)\bigr)\,ds + C_0 + C_1 t.$$

By applying the initial conditions $x'(0) = 0$ and $\frac{\partial^i f(t,x(t))}{\partial t^i}|_{t=0} = 0$, we obtain $C_1 = 0$ and

$$x(t) = f\bigl(t,x(t)\bigr) + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} h\bigl(s,y(s),I^\alpha y(s)\bigr)\,ds + C_0. \tag{3.2}$$

Now, applying the boundary conditions $a_1 x(0) - b_1 x(\eta) - c_1 x(1) = \frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}\phi(s, x(s))\,ds$ to (3.2), we have

$$\begin{aligned}(a_1 - b_1 - c_1)C_0 &= b_1 f\bigl(\eta,x(\eta)\bigr) + c_1 f\bigl(1,x(1)\bigr) + \frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}\phi\bigl(s,x(s)\bigr)\\&\quad + \frac{b_1}{\Gamma(\theta)}\int_0^\eta (\eta-s)^{\theta-1} h\bigl(s,y(s),I^\alpha y(s)\bigr)\,ds\end{aligned}$$

$$+\frac{c_1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}h\big(s,y(s),I^{\alpha}y(s)\big)\,ds.$$

By rearranging, we obtain

$$\begin{aligned}
C_0 &= \frac{b_1}{a_1-(b_1+c_1)}f\big(\eta,x(\eta)\big)+\frac{c_1}{a_1-(b_1+c_1)}f\big(1,x(1)\big)\\
&\quad+\frac{1}{a_1-(b_1+c_1)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}\phi\big(s,x(s)\big)\\
&\quad+\frac{b_1}{a_1-(b_1+c_1)}\frac{1}{\Gamma(\theta)}\int_0^{\eta} (\eta-s)^{\theta-1}h\big(s,y(s),I^{\alpha}y(s)\big)\,ds\\
&\quad+\frac{c_1}{a_1-(b_1+c_1)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}h\big(s,y(s),I^{\alpha}y(s)\big)\,ds.
\end{aligned}$$

Thus equation (3.2) becomes

$$\begin{aligned}
x(t) &= f\big(t,x(t)\big)+\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}h\big(s,y(s),I^{\alpha}y(s)\big)\,ds\\
&\quad+\frac{1}{a_1-(b_1+c_1)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}\phi\big(s,x(s)\big)\,ds\\
&\quad+\frac{b_1}{a_1-(b_1+c_1)}f\big(\eta,x(\eta)\big)+\frac{c_1}{a_1-(b_1+c_1)}f\big(1,x(1)\big)\\
&\quad+\frac{b_1}{a_1-(b_1+c_1)}\frac{1}{\Gamma(\theta)}\int_0^{\eta} (\eta-s)^{\theta-1}h\big(s,y(s),I^{\alpha}y(s)\big)\,ds\\
&\quad+\frac{c_1}{a_1-(b_1+c_1)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}h\big(s,y(s),I^{\alpha}y(s)\big)\,ds.
\end{aligned}$$

Analogously, following the same steps in the process for the second equation of system (1.1), we get

$$\begin{aligned}
y(t) &= f\big(t,y(t)\big)+\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}h\big(s,x(s),I^{\alpha}x(s)\big)\,ds\\
&\quad+\frac{1}{a_2-(b_2+c_2)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}\psi\big(s,y(s)\big)\,ds\\
&\quad+\frac{b_2}{a_2-(b_2+c_2)}f\big(\xi,y(\xi)\big)+\frac{c_2}{a_2-(b_2+c_2)}f\big(1,y(1)\big)\\
&\quad+\frac{b_2}{a_2-(b_2+c_2)}\frac{1}{\Gamma(\theta)}\int_0^{\xi} (\xi-s)^{\theta-1}h\big(s,x(s),I^{\alpha}x(s)\big)\,ds\\
&\quad+\frac{c_2}{a_2-(b_2+c_2)}\frac{1}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}h\big(s,x(s),I^{\alpha}x(s)\big)\,ds.
\end{aligned}$$

□

Define the operator $F,H,T:X\times X\to X\times X$ by

$$\begin{aligned}
&F(x,y)(t)=\big(F_1x(t),F_2y(t)\big),\\
&H(x,y)(t)=\big(H_2y(t),H_1x(t)\big),\\
&T(x,y)(t)=F(x,y)(t)+H(x,y)(t),
\end{aligned}$$

here $F_1, F_2, H_1, H_2 : X \to X$ are

$$\begin{aligned} F_1x(t) &= f\bigl(t,x(t)\bigr) + \frac{b_1}{a_1-(b_1+c_1)} f\bigl(\eta,x(\eta)\bigr) + \frac{c_1}{a_1-(b_1+c_1)} f\bigl(1,x(1)\bigr) \\ &\quad + \frac{1}{a_1-(b_1+c_1)} \frac{1}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} \phi\bigl(s,x(s)\bigr)\,ds, \\ F_2y(t) &= f\bigl(t,y(t)\bigr) + \frac{b_2}{a_2-(b_2+c_2)} f\bigl(\xi,y(\xi)\bigr) + \frac{c_2}{a_2-(b_2+c_2)} f\bigl(1,y(1)\bigr) \\ &\quad + \frac{1}{a_2-(b_2+c_2)} \frac{1}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} \psi\bigl(s,y(s)\bigr)\,ds, \\ H_1x(t) &= \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} h\bigl(s,x(s),I^\alpha x(s)\bigr)\,ds \\ &\quad + \frac{b_2}{a_2-(b_2+c_2)} \frac{1}{\Gamma(\theta)} \int_0^\xi (\xi-s)^{\theta-1} h\bigl(s,x(s),I^\alpha x(s)\bigr)\,ds \\ &\quad + \frac{c_2}{a_2-(b_2+c_2)} \frac{1}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} h\bigl(s,x(s),I^\alpha x(s)\bigr)\,ds, \\ H_2y(t) &= \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} h\bigl(s,y(s),I^\alpha y(s)\bigr)\,ds \\ &\quad + \frac{b_1}{a_1-(b_1+c_1)} \frac{1}{\Gamma(\theta)} \int_0^\eta (\eta-s)^{\theta-1} h\bigl(s,y(s),I^\alpha y(s)\bigr)\,ds \\ &\quad + \frac{c_1}{a_1-(b_1+c_1)} \frac{1}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} h\bigl(s,y(s),I^\alpha y(s)\bigr)\,ds. \end{aligned}$$

Then the system of integral equations (3.1) can be written as an operator equation

$$(x,y) = T(x,y) = F(x,y) + H(x,y),$$

and fixed points of the operator equation are results of system (1.1).

Theorem 3.2 *The operator F is Lipschitz with constant k. Therefore F is α-Lipschitz with the equal constant k and meets the following growth condition*:

$$\bigl\|F\bigl(x(t),y(t)\bigr)\bigr\| \le L_F \bigl\|(x,y)\bigr\|^{q_2} + M_F.$$

Proof Now, we shall display that the operator F is Lipschitz with constant k. Let $x_1, x_2 \in X$, then we get

$$\begin{aligned} &\bigl|F_1x_1(t) - F_1x_2(t)\bigr| \\ &\quad = \biggl|\bigl(f\bigl(t,x_1(t)\bigr) - f\bigl(t,x_2(t)\bigr)\bigr) + \frac{b_1}{a_1-(b_1+c_1)}\bigl(f\bigl(\eta,x_1(\eta)\bigr) - f\bigl(\eta,x_2(\eta)\bigr)\bigr) \\ &\qquad + \frac{c_1}{a_1-(b_1+c_1)}\bigl(f\bigl(1,x_1(1)\bigr) - f\bigl(1,x_2(1)\bigr)\bigr) \\ &\qquad + \frac{1}{a_1-(b_1+c_1)} \frac{1}{\Gamma(\theta)} \biggl(\int_0^1 (1-s)^{\theta-1} \phi\bigl(s,x_1(s)\bigr)\,ds - \int_0^1 (1-s)^{\theta-1} \phi\bigl(s,x_2(s)\bigr)\,ds\biggr)\biggr| \\ &\quad \le \bigl|f\bigl(t,x_1(t)\bigr) - f\bigl(t,x_2(t)\bigr)\bigr| \end{aligned}$$

$$+ \frac{|b_1|}{|a_1-(b_1+c_1)|}\bigl|f\bigl(\eta,x_1(\eta)\bigr)-f\bigl(\eta,x_2(\eta)\bigr)\bigr|$$
$$+ \frac{|c_1|}{|a_1-(b_1+c_1)|}\bigl|f\bigl(1,x_1(1)\bigr)-f\bigl(1,x_2(1)\bigr)\bigr|$$
$$+ \frac{1}{|a_1-(b_1+c_1)|\Gamma(\theta)}\int_0^1(1-s)^{\theta-1}\bigl|\phi\bigl(s,x_1(s)\bigr)-\phi\bigl(s,x_2(s)\bigr)\bigr|\,ds.$$

By using conditions (H_1) and (H_5), we can write

$$\begin{aligned}\bigl\|F_1x_1(t)-F_1x_2(t)\bigr\| &\le \lambda_1\|x_1-x_2\| + \frac{|b_1|\lambda_1}{|a_1-(b_1+c_1)|}\|x_1-x_2\| \\ &\quad + \frac{|c_1|\lambda_1}{|a_1-(b_1+c_1)|}\|x_1-x_2\| + \frac{b_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}\|x_1-x_2\| \\ &= \biggl[\lambda_1 + \frac{(|b_1|+|c_1|)\lambda_1}{|a_1-(b_1+c_1)|} + \frac{b_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}\biggr]\|x_1-x_2\| \\ &= k_1\|x_1-x_2\|,\end{aligned}$$

where $k_1 = \lambda_1 + \frac{(|b_1|+|c_1|)\lambda_1}{|a_1-(b_1+c_1)|} + \frac{b_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}$.

Similarly,

$$\begin{aligned}\bigl\|F_2y_1(t)-F_2y_2(t)\bigr\| &\le \biggl[\lambda_2 + \frac{(|b_2|+|c_2|)\lambda_2}{|a_2-(b_2+c_2)|} + \frac{b_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\biggr]\|y_1-y_2\| \\ &= k_2\|y_1-y_2\|,\end{aligned}$$

where $k_2 = \lambda_2 + \frac{(|b_2|+|c_2|)\lambda_2}{|a_2-(b_2+c_2)|} + \frac{b_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}$. Thus

$$\begin{aligned}\bigl\|F(x_1,y_1)-F(x_2,y_2)\bigr\| &= \bigl\|F_1x_1(t)-F_1x_2(t)\bigr\| + \bigl\|F_2y_1(t)-F_2y_2(t)\bigr\| \\ &\le k_1\|x_1-x_2\| + k_2\|y_1-y_2\| \\ &\le k\bigl\|(x_1,y_1)-(x_2,y_2)\bigr\|,\end{aligned}$$

where $k = \max(\lambda_1 + \frac{(|b_1|+|c_1|)\lambda_1}{|a_1-(b_1+c_1)|} + \frac{b_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}, \lambda_2 + \frac{(|b_2|+|c_2|)\lambda_2}{|a_2-(b_2+c_2)|} + \frac{b_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)})$. Then F satisfies the Lipschitz condition, thus F is Lipschitz with constant k. According to Proposition 2.4, F is α-Lipschitz with constant k.

Moreover, we get

$$\begin{aligned}\bigl|F_1x(t)\bigr| &\le \bigl|f\bigl(t,x(t)\bigr)\bigr| + \frac{|b_1|}{|a_1-(b_1+c_1)|}\bigl|f\bigl(\eta,x(\eta)\bigr)\bigr| + \frac{|c_1|}{|a_1-(b_1+c_1)|}\bigl|f\bigl(1,x(1)\bigr)\bigr| \\ &\quad + \frac{1}{|a_1-(b_1+c_1)|}\frac{1}{\Gamma(\theta)}\int_0^1(1-s)^{\theta-1}\bigl|\phi\bigl(s,x(s)\bigr)\bigr|\,ds.\end{aligned}$$

By (H_3) and (H_4), we have

$$\begin{aligned}\bigl|F_1x(t)\bigr| &\le l_f\|x\|^{q_2} + M_f + \frac{|b_1|}{|a_1-(b_1+c_1)|}\bigl(l_f\|x\|^{q_2} + M_f\bigr) \\ &\quad + \frac{|c_1|}{|a_1-(b_1+c_1)|}\bigl(l_f\|x\|^{q_2} + M_f\bigr)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{|a_1-(b_1+c_1)|}\frac{1}{\Gamma(\theta+1)}\bigl(c_\phi\|x\|^{q_2}+M_\phi\bigr)\\
&= \biggl[l_f + \frac{(|b_1|+|c_1|)l_f}{|a_1-(b_1+c_1)|} + \frac{c_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}\biggr]\|x\|^{q_2}\\
&\quad + \biggl[M_f + \frac{(|b_1|+|c_1|)M_f}{|a_1-(b_1+c_1)|} + \frac{M_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}\biggr].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\bigl|F_2y(t)\bigr| \le{}& \biggl[l_f + \frac{(|b_2|+|c_2|)l_f}{|a_2-(b_2+c_2)|} + \frac{c_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\biggr]\|y\|^{q_2}\\
&+ \biggl[M_f + \frac{(|b_2|+|c_2|)M_f}{|a_2-(b_2+c_2)|} + \frac{M_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\biggr].
\end{aligned}
$$

Hence it follows that

$$
\begin{aligned}
\bigl\|F\bigl(x(t),y(t)\bigr)\bigr\| &= \bigl\|\bigl(F_1(x),F_2(y)\bigr)\bigr\|\\
&= \bigl\|F_1(x)\bigr\| + \bigl\|F_2(y)\bigr\|\\
&\le \biggl[l_f + \frac{(|b_1|+|c_1|)l_f}{|a_1-(b_1+c_1)|} + \frac{c_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}\biggr]\|x\|^{q_2}\\
&\quad + \biggl[l_f + \frac{(|b_2|+|c_2|)l_f}{|a_2-(b_2+c_2)|} + \frac{c_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\biggr]\|y\|^{q_2}\\
&\quad + \biggl[M_f + \frac{(|b_1|+|c_1|)M_f}{|a_1-(b_1+c_1)|} + \frac{M_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}\biggr]\\
&\quad + \biggl[M_f + \frac{(|b_2|+|c_2|)M_f}{|a_2-(b_2+c_2)|} + \frac{M_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\biggr]\\
&\le L_F\bigl\|(x,y)\bigr\|^{q_2} + M_F,
\end{aligned}
$$

where

$$
\begin{aligned}
L_F ={}& \max\biggl(l_f + \frac{(|b_1|+|c_1|)l_f}{|a_1-(b_1+c_1)|} + \frac{c_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)},\\
&l_f + \frac{(|b_2|+|c_2|)l_f}{|a_2-(b_2+c_2)|} + \frac{c_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\biggr),\\
M_F ={}& 2\max\biggl(M_f + \frac{(|b_1|+|c_1|)M_f}{|a_1-(b_1+c_1)|} + \frac{M_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)},\\
&M_f + \frac{(|b_2|+|c_2|)M_f}{|a_2-(b_2+c_2)|} + \frac{M_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\biggr).
\end{aligned}
$$

□

Theorem 3.3 *The operator $H : X \times X \to X \times X$ is continuous and meets the following growth condition*:

$$
\bigl\|H\bigl(x(t),y(t)\bigr)\bigr\| \le L_H\bigl\|(x,y)\bigr\|^{q_1} + M_H.
$$

Proof Consider a bounded subset of $X \times X$ as

$$B_r = \{ \| (x,y) \| \leq r : (x,y) \in X \times X \} \subseteq X \times X.$$

Let $\{(x_n, y_n)\}$ be a sequence in B_r such that $(x_n, y_n) \to (x,y)$ as $n \to \infty$. To show that H is continuous, we consider

$$\begin{aligned}
&\left| H_1(x_n) - H_1(x) \right| \\
&\quad \leq \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} \left| h(s, x_n(s), I^\alpha x_n(s)) - h(s, x(s), I^\alpha x(s)) \right| ds \\
&\qquad + \frac{|b_2|}{|a_2 - (b_2 + c_2)|} \frac{1}{\Gamma(\theta)} \int_0^\xi (\xi - s)^{\theta-1} \left| h(s, x_n(s), I^\alpha x_n(s)) - h(s, x(s), I^\alpha x(s)) \right| ds \\
&\qquad + \frac{|c_2|}{|a_2 - (b_2 + c_2)|} \frac{1}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} \left| h(s, x_n(s), I^\alpha x_n(s)) - h(s, x(s), I^\alpha x(s)) \right| ds.
\end{aligned}$$

From the continuity of h, it follows that

$$h(s, x_n(s), I^\alpha x_n(s)) \to h(s, x(s), I^\alpha x(s)) \quad \text{as } n \to \infty.$$

For every $t \in [0,1]$, by applying (H_2), and the Lebesgue dominated convergent theorem, we can get

$$\int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} \left| h(s, x_n(s), I^\alpha x_n(s)) - h(s, x(s), I^\alpha x(s)) \right| ds \to 0 \quad \text{as } n \to \infty.$$

The same as the other terms approach 0 as $n \to \infty$, thus

$$\left\| H_1(x_n) - H_1(x) \right\| \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Then H_1 is continuous. By the same steps as above, one lightly gets that

$$\left\| H_2(y_n) - H_2(y) \right\| \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

That is, H_2 is continuous. From (3.3) and (3.4), we have

$$\left\| H(x_n, y_n) - H(x,y) \right\| \to 0 \quad \text{as } n \to \infty,$$

which means that H is continuous.

Moreover, by (H_2), we have

$$\begin{aligned}
\left| H_1 x(t) \right| = \bigg| &\frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} h(s, x(s), I^\alpha x(s)) \, ds \\
&+ \frac{b_2}{a_2 - (b_2 + c_2)} \frac{1}{\Gamma(\theta)} \int_0^\xi (\xi - s)^{\theta-1} h(s, x(s), I^\alpha x(s)) \, ds \\
&+ \frac{c_2}{a_2 - (b_2 + c_2)} \frac{1}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} h(s, x(s), I^\alpha x(s)) \, ds \bigg|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{t^\theta}{\Gamma(\theta+1)}\bigl(l_h^1\|x\|^{q_1} + l_h^2\bigl\|I^\alpha x\bigr\|^{q_1} + M_h\bigr) \\
&\quad + \frac{|b_2|\xi^\theta}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\bigl(l_h^1\|x\|^{q_1} + l_h^2\bigl\|I^\alpha x\bigr\|^{q_1} + M_h\bigr) \\
&\quad + \frac{|c_2|}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\bigl(l_h^1\|x\|^{q_1} + l_h^2\bigl\|I^\alpha x\bigr\|^{q_1} + M_h\bigr) \\
&\le \left[\frac{1}{\Gamma(\theta+1)} + \frac{|b_2|+|c_2|}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\right]\bigl(l_h^1\|x\|^{q_1} + l_h^2\bigl\|I^\alpha x\bigr\|^{q_1} + M_h\bigr).
\end{aligned}
$$

Similarly,

$$
\bigl|H_2y(t)\bigr| \le \left[\frac{1}{\Gamma(\theta+1)} + \frac{|b_1|+|c_1|}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}\right]\bigl(l_h^1\|y\|^{q_1} + l_h^2\bigl\|I^\alpha y\bigr\|^{q_1} + M_h\bigr). \tag{3.5}
$$

Thus

$$
\begin{aligned}
\bigl\|H\bigl(x(t),y(t)\bigr)\bigr\| &= \bigl\|H_1x(t)\bigr\| + \bigl\|H_2y(t)\bigr\| \\
&\le \left[\frac{1}{\Gamma(\theta+1)} + \frac{|b_2|+|c_2|}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\right]\bigl(l_h^1\|x\|^{q_1} + l_h^2\bigl\|I^\alpha x\bigr\|^{q_1} + M_h\bigr) \\
&\quad + \left[\frac{1}{\Gamma(\theta+1)} + \frac{|b_1|+|c_1|}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}\right]\bigl(l_h^1\|y\|^{q_1} + l_h^2\bigl\|I^\alpha y\bigr\|^{q_1} + M_h\bigr) \\
&\le \left[\frac{1}{\Gamma(\theta+1)} + \frac{|b_2|+|c_2|}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\right]\bigl[l\bigl(\|x\|^{q_1} + \bigl\|I^\alpha x\bigr\|^{q_1}\bigr) + M_h\bigr] \\
&\quad + \left[\frac{1}{\Gamma(\theta+1)} + \frac{|b_1|+|c_1|}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}\right]\bigl[l\bigl(\|y\|^{q_1} + \bigl\|I^\alpha y\bigr\|^{q_1}\bigr) + M_h\bigr] \\
&\le L_h\bigl[\|x\|^{q_1} + \|y\|^{q_1} + \bigl\|I^\alpha x\bigr\|^{q_1} + \bigl\|I^\alpha y\bigr\|^{q_1} + 2M_h\bigr] \\
&\le L_H\bigl\|(x,y)\bigr\|^{q_1} + M_H,
\end{aligned}
$$

where $l = \max\{l_h^1, l_h^2\} \in [0,1)$, $L_h = \max(\frac{1}{\Gamma(\theta+1)} + \frac{|b_2|+|c_2|}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}, \frac{1}{\Gamma(\theta+1)} + \frac{|b_1|+|c_1|}{|a_1-(b_1+c_1)|\Gamma(\theta+1)})$, $L_H = (1 + (\frac{1}{\Gamma(\alpha+1)})^{q_1})L_h$, $M_H = 2L_hM_h$. Hence H satisfies the growth condition. □

Theorem 3.4 *The operator $H : X \times X \to X \times X$ is compact.*

Proof Let Ω be a bounded subset of $B_r \subseteq X \times X$ and $\{(x_n, y_n)\}$ be a sequence in Ω, through applying the growth condition of H, it is obvious that $H(\Omega)$ is uniformly bounded in $X \times X$. Now, we need to reveal that H is equicontinuous. Let $0 \le t \le \tau \le 1$, then we obtain

$$
\begin{aligned}
\bigl|H_1x_n(t) - H_1x_n(\tau)\bigr| &= \left|\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}h\bigl(s,x_n(s),I^\alpha x_n(s)\bigr)\,ds\right. \\
&\quad \left. - \frac{1}{\Gamma(\theta)}\int_0^\tau (\tau-s)^{\theta-1}h\bigl(s,x_n(s),I^\alpha x_n(s)\bigr)\,ds\right| \\
&= \left|\frac{1}{\Gamma(\theta)}\int_0^t \bigl[(t-s)^{\theta-1} - (\tau-s)^{\theta-1}\bigr]h\bigl(s,x_n(s),I^\alpha x_n(s)\bigr)\,ds\right. \\
&\quad \left. - \frac{1}{\Gamma(\theta)}\int_t^\tau (\tau-s)^{\theta-1}h\bigl(s,x_n(s),I^\alpha x_n(s)\bigr)\,ds\right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\theta)}\int_0^t \bigl[(t-s)^{\theta-1}-(\tau-s)^{\theta-1}\bigr]\bigl|h\bigl(s,x_n(s),I^\alpha x_n(s)\bigr)\bigr|\,ds \\
&\quad+\frac{1}{\Gamma(\theta)}\int_t^\tau (\tau-s)^{\theta-1}\bigl|h\bigl(s,x_n(s),I^\alpha x_n(s)\bigr)\bigr|\,ds \\
&\leq \frac{l_h^1\|x_n\|^{q_1}+l_h^2\|I^\alpha x_n\|^{q_1}+M_h}{\Gamma(\theta+1)}\bigl[t^\theta-\tau^\theta+2(\tau-t)^\theta\bigr].
\end{aligned}$$

Similarly,

$$\bigl|H_2y_n(t)-H_2y_n(\tau)\bigr|\leq \frac{l_h^1\|y_n\|^{q_1}+l_h^2\|I^\alpha y_n\|^{q_1}+M_h}{\Gamma(\theta+1)}\bigl[t^\theta-\tau^\theta+2(\tau-t)^\theta\bigr].$$

Taking limit as $t\to\tau$, we get

$$\bigl\|H_1x_n(t)-H_1x_n(\tau)\bigr\|\to 0$$

and

$$\bigl\|H_2y_n(t)-H_2y_n(\tau)\bigr\|\to 0,$$

which implies that

$$\bigl\|H(x_n,y_n)(t)-H(x_n,y_n)(\tau)\bigr\|\to 0.$$

This reveals that $H(x,y)$ is equicontinuous. $H(x,y)$ is compact by the Arzelá–Ascoli theorem. Hence, according to Proposition 2.5, H is α-Lipschitz with constant zero. □

Theorem 3.5 *If* (H_1)–(H_5) *hold and*

$$\begin{aligned}
k = \max\biggl(&\lambda_1+\frac{(|b_1|+|c_1|)\lambda_1}{|a_1-(b_1+c_1)|}+\frac{b_\phi}{|a_1-(b_1+c_1)|\Gamma(\theta+1)}, \\
&\lambda_2+\frac{(|b_2|+|c_2|)\lambda_2}{|a_2-(b_2+c_2)|}+\frac{b_\psi}{|a_2-(b_2+c_2)|\Gamma(\theta+1)}\biggr)\in[0,1),
\end{aligned}$$

then coupled system (1.1) *has at least one solution* $(x,y)\in X\times X$. *And the solution set of* (1.1) *is bounded in* $X\times X$.

Proof From Theorem 3.2 and $k\in[0,1)$, F is α-Lipschitz with constant $k\in[0,1)$, according to Theorem 3.4, H is α-Lipschitz with constant 0. By Proposition 2.3 and Definition 2.2, T is a strict α-contraction with constant k. Hence, T is α-condensing. Then, we think over the following set:

$$R=\bigl\{(x,y)\in X\times X:\text{there exist }\zeta\in[0,1],(x,y)=\zeta T(x,y)\bigr\}.$$

We have to reveal that R is bounded in $X\times X$. Let $(x,y)\in R$, then by applying the growth conditions of Theorem 3.2 and Theorem 3.3, we have

$$(x,y)=\zeta T(x,y)=\zeta\bigl(F(x,y)+H(x,y)\bigr),$$

thus

$$\begin{aligned}\|(x,y)\| &= \zeta \|T(x,y)\| \\ &\le \zeta \left(\|F(x,y)\| + \|H(x,y)\|\right) \\ &\le \zeta \left[L_F \|(x,y)\|^{q_2} + M_F + L_H \|(x,y)\|^{q_1} + M_H\right] \\ &= \zeta \left(L_F \|(x,y)\|^{q_2} + L_H \|(x,y)\|^{q_1}\right) + \zeta (M_F + M_H),\end{aligned}$$

where $q_1, q_2 \in [0,1)$. Thus R is bounded in $X \times X$. According to Theorem 2.6, there exists $r > 0$ such that $R \subset B_r(0)$, then

$$D\big(I - \zeta T, B_r(0), 0\big) = 1, \quad \text{for all } \zeta \in [0,1].$$

Therefore, T has at least one fixed point, then coupled system (1.1) has at least one solution. □

4 Examples

This part, we have the following example account for our main results.

Example 4.1 Give thought to the following equation:

$$\begin{cases} D^{\frac{3}{2}}[x(t) - \frac{\sin^2(t)|x(t)|}{5(2+|x(t)|)}] = \frac{e^{-\pi t}}{10} + \frac{\cos|y(t)|+\sin|y(t)|}{51+t^2}, & t \in [0,1], \\ D^{\frac{3}{2}}[y(t) - \frac{\sin^2(t)|y(t)|}{5(2+|y(t)|)}] = \frac{e^{-\pi t}}{10} + \frac{\cos|x(t)|+\sin|x(t)|}{51+t^2}, & t \in [0,1], \\ x'(0) = y'(0) = 0, \\ \frac{1}{4}x(0) - \frac{1}{2}x(\frac{1}{2}) - 6x(1) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 (1-s)^{\frac{1}{2}} \frac{\sin x(s)}{2}\, ds, \\ \frac{1}{5}y(0) - \frac{1}{7}y(\frac{1}{2}) - 8y(1) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 (1-s)^{\frac{1}{2}} \frac{\cos y(s)}{5}\, ds, \end{cases} \tag{4.1}$$

where $h = \frac{e^{-\pi t}}{10} + \frac{\cos|y(t)|+\sin|y(t)|}{51+t^2}$, $\theta = \frac{3}{2}$, $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{2}$, $c_1 = 6$, $a_2 = \frac{1}{5}$, $b_2 = \frac{1}{7}$, $c_2 = 8$, $\eta = \xi = \frac{1}{2}$. Let $\zeta = \frac{1}{5}$, then by routine calculation, we can have $c_\phi = b_\phi = \frac{1}{2}$, $c_\psi = b_\psi = \frac{1}{5}$, $M_\phi = M_\psi = 0$, $l_h^1 = l_h^2 = \frac{1}{51}$, $M_h = \frac{1}{10}$, $l_f = \frac{1}{10}$, $M_f = 0$, $\lambda_1 = \lambda_2 = \frac{1}{10}$. Thus assumptions $(H_1)-(H_5)$ hold, and

$$\begin{aligned}&\lambda_1 + \frac{(|b_1| + |c_1|)\lambda_1}{|a_1 - (b_1 + c_1)|} + \frac{b_\phi}{|a_1 - (b_1 + c_1)|\Gamma(\theta+1)} \approx 0.389, \\ &\lambda_2 + \frac{(|b_2| + |c_2|)\lambda_2}{|a_2 - (b_2 + c_2)|} + \frac{b_\psi}{|a_2 - (b_2 + c_2)|\Gamma(\theta+1)}) \approx 0.221.\end{aligned}$$

Next,

$$\begin{aligned}\left|F(x_1,y_1)(t) - F(x_2,y_2)(t)\right| &\le \frac{1}{10}\left|x_1(t) - x_2(t)\right| + \frac{1}{10}\left|y_1(t) - y_2(t)\right| \\ &\quad + \frac{1}{11.078}\int_0^1 (1-s)^{\frac{1}{2}} \left|\sin(x_1) - \sin(x_2)\right| ds \\ &\quad + \frac{1}{35.196}\int_0^1 (1-s)^{\frac{1}{2}} \left|\cos(y_1) - \cos(y_2)\right| ds \\ &\le 0.160\|x_1 - x_2\| + 0.119\|y_1 - y_2\|\end{aligned}$$

$$\leq 0.160 \| (x_1, y_1) - (x_2, y_2) \|,$$

thus F is α-Lipschitz with the constant 0.160, and thus H is α-Lipschitz with the constant zero, which means that T is α-Lipschitz with the constant 0.160. Since

$$R = \{(x, y) \in X \times X : \text{there exist } \zeta \in [0, 1], (x, y) = \zeta T(x, y)\},$$

then, by routine calculation, we obtain $L_F = 0.216$, $L_H = 2.380$, $M_F = 0$, $M_H = 0.238$.

Thus

$$\| (x, y) \| \cong 0.703 \leq 1,$$

then R is bounded, through Theorem 3.5, then problem (4.1) has at least one solution.

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