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On solutions of nonlinear BVPs with general boundary conditions by using a generalized Riesz–Caputo operator

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Abstract

In this work, we study the existence, uniqueness, and continuous dependence of solutions for a class of fractional differential equations by using a generalized Riesz fractional operator. One can view the results of this work as a refinement for the existence theory of fractional differential equations with Riemann–Liouville, Caputo, and classical Riesz derivative. Some special cases can be derived to obtain corresponding existence results for fractional differential equations. We provide an illustrated example for the unique solution of our main result.

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1 Introduction

Fractional differential equations are considered as prolongation of the concept of derivative operator from integer order to any real or complex order. Fractional differential equations usually describe the nonlocal effects. Over the last two decades, there has been a blistering growth in the field of fractional calculus. Owing to the vast amount of applications, many mathematicians focused their engrossment on fractional calculus.

There exist several definitions for fractional derivatives and fractional integrals in the literature like Riemann–Liouville, Caputo, Hadamard, Riesz, Grunwald–Letnikov, Marchaud, Erdelyi–Kober, etc. The process of developing these operators began with a series of stages ranging from exponential functions to different classes of functions. Having lately come into holocene Udita N. Katugampola [1] generalized the above mentioned integral and differential operators. Meanwhile the well-developed theory and many more applications of the said operators are still a spotlight area of research in applied sciences.

As we know, the existence theory is meat-and-potatoes in every field of science, as it is very applicative to comprehend whether there is a solution to a given differential equation beforehand; otherwise, all the attempts to find a numerical or analytic solution will become valueless. The analysis of fractional differential equations has been carried out by various authors (see, for example, [2–18]).

As most fractional derivatives are computed using the corresponding integrals, researchers describe the nonlocal effects in terms of left and the right derivative. Thus, many mathematicians are in a hunt to generalize the notions further. In this context, Riesz [19] demonstrated the two-sided fractional operators using both left and right Riemann–Liouville's fractional differential and integral operators.

Due to the two-sided nature of Riesz's differential operator, the interesting differential is specifically used for fractional modeling on a finite domain. Some optimality conditions are discussed by Almeida for fractional variational problems with Riesz–Caputo derivative [20]. Frederico et al. derived Noether's theorem for variational problems having Riesz–Caputo derivatives. In [21], Mandelbrot demonstrated that there is a close connection between Brownian motion and fractional calculus.

In [22], the authors solved the fractional Poisson equation having Riesz derivative using Fourier transform. Due to the validity of Riesz derivative operator on the whole domain, it appears in the fractional turbulent diffusion model. In [23], the authors numerically solved the advection-diffusion equation having Riesz derivative. For further applications of Riesz derivative on the anomalous diffusion, see [24–29].

In this work, we define the generalized Riesz–Caputo type derivative operator by using the generalized operators. We present basic perspectives on the existence and uniqueness of solutions of fractional differential equations. Motivated by [30, 31], we provide the analysis on existence of solutions for the following nonlinear fractional differential equation involving generalized Riesz–Caputo type derivative operator with general boundary conditions:

$$\begin{cases} {}_0^{RC}D_T^{\alpha,\rho}\phi(\mu) = g(\mu,\phi(\mu),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\mu)), \quad \mu \in [0,T], \\ \phi(0) = \phi_0, \qquad \phi(T) = \phi_T, \end{cases} \tag{1}$$

where ϕ_0 and ϕ_T are constants, while $g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous with $1 < \alpha \leq 2$, $0 < \alpha^* \leq 1$, and $1 < \rho < \infty$.

The rest of the paper is organized as follows: Sect. 2 presents some basic definitions and lemmas from literature. In Sect. 3 we introduce the generalized Riesz–Caputo's fractional operators and derived some useful results, while in Sect. 4 we establish some equivalence results for boundary value problem (1) and establish the results for the existence and uniqueness of solutions for BVP (1). The last section of this paper presents the stability of solutions for BVP (1) by means of continuous dependence on parameters.

2 Preliminaries

In this section we demonstrate some useful results including definitions and lemmas related to Riesz–Caputo derivatives and integrals that will help us in our later discussions. Following the same traditional definitions of Riesz–Caputo derivative and integral [19, 30, 32], we can generalize these definitions using a generalized Caputo type derivative operator. Some preliminary structural properties, which we will frequently use in our later discussion, are also introduced in this section. In 2010, Om Prakash Agrawal defined the generalized fractional in the following way.

Definition 2.1 ([33]) Let $\alpha > 0$. Then the generalized fractional integral operator $A^{\alpha}_{(a,T;r,s)}$ is defined as

$$A^{\alpha}_{(a,T;r,s)}\phi(\mu) := r\int_a^{\mu} K_\alpha(\mu,\eta)\phi(\eta)\,d\eta + s\int_{\mu}^{T} K_\alpha(\mu,\eta)\phi(\eta)\,d\eta,$$

where the kernel function $K_\alpha(\mu,\eta)$ may depend on α and $a < \mu < T$ and $r,s \in \mathbb{R}$.

This is the generalized fractional integral operator which, by using the specific kernel function, leads to the specific operator. For example, if $K_\alpha(\mu,\eta) = \frac{(\mu-\eta)^{\alpha-1}}{\Gamma(\alpha)}$ and by taking $T = 0$ leads to the left sided R-L integral operator and by taking $K_\alpha(\mu,\eta) = \frac{\eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}$ with $T = 0$ gives the left generalized integral defined below. Furthermore, the limits of integration a and T can be extended to $-\infty$ and ∞ respectively.

Definition 2.2 ([34]) Let $\alpha \in \mathbb{R}_+, c \in \mathbb{R}$, and $g \in X^p_c(a,b)$, where $X^p_c(a,b)$ is the space of Lebesgue measurable functions. Then corresponding generalized left- and right-sided fractional integrals $({}^\rho I^\alpha_{a+}g)(\mu)$ and $({}^\rho I^\alpha_{b-}g)(\mu)$ of order $\alpha \in \mathbb{C}(\mathrm{Re}(\alpha)) > 0$ are defined by

$$\left({}^\rho I^\alpha_{a+}g\right)(\mu) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\mu}\frac{\eta^{\rho-1}g(\eta)}{(\mu^\rho-\eta^\rho)^{1-\alpha}}\,d\eta, \quad \mu > a, \rho > 0,$$

$$\left({}^\rho I^\alpha_{b-}g\right)(\mu) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu}^{b}\frac{\eta^{\rho-1}g(\eta)}{(\eta^\rho-\mu^\rho)^{1-\alpha}}\,d\eta, \quad \mu < b, \rho > 0,$$

respectively, where $\Gamma(\cdot)$ is Euler's gamma function.

Theorem 2.3 ([35]) *Let $\alpha, \rho \in \mathbb{R}$ and $\rho, a > 0$. Then, for $\phi \in X^p_c(a,b)$, the following relation holds*:

$$\left({}^\rho_* D^\alpha_{a^+}\,{}^\rho I^\alpha_{a^+}\phi\right)(\mu) = \phi(\mu).$$

Similarly, the inverse property holds for a right-hand-sided integral and a derivative operator as well.

Lemma 2.4 ([35]) *Let $0 < \alpha < \beta < 1$ and $\rho, a > 0$. Then, for $\phi \in X^p_c(a,b)$, the following relation holds*:

$${}^\rho_* D^\alpha_{a^+}\,{}^\rho I^\beta_{a^+}\phi(\mu) = {}^\rho I^{\beta-\alpha}_{a^+}\phi(\mu).$$

Lemma 2.5 ([35]) *Let $\alpha, \rho \in \mathbb{R}_+$ and $g \in AC^n_\delta[0,T]$: the space of complex-valued functions g which have continuous derivatives up to order $(n-1)$ on $[a,b]$ such that $\delta^{(n-1)}_\rho g(\mu) \in AC[0,T]$ is absolutely continuous on $[0,T]$, where $\delta_\rho(g(\mu)) = \mu^{1-\rho}\frac{d}{\mu}(g(\mu))$. Then, for $0 \le \mu \le T$, the following relations hold*:

(i) $({}^\rho_0 I^\alpha_{\mu *} D^\alpha_{0,\mu} g)(\mu) = g(\mu) - \sum_{j=0}^{n}\frac{\delta^j_\rho g(0)}{j!}\left(\frac{\mu^\rho}{\rho}\right)^j$,

(ii) $({}^\rho_\mu I^\alpha_{T*}\,{}^\rho D^\alpha_{\mu,T} g)(T) = \{g(\mu) - \sum_{j=0}^{n}\frac{(-1)^j\delta^j_\rho g(T)}{j!}\left(\frac{T^\rho-\mu^\rho}{\rho}\right)^j\}$,

where $n = \lceil\alpha\rceil$ and $\delta^j_\rho = (\eta^{1-\rho}\frac{d}{d\eta})^j$.

3 Generalized Riesz–Caputo fractional operators

In this section we introduce the generalized Riesz–Caputo fractional integrals and derivative operators.

Definition 3.1 ([19]) For $g(\mu) \in C(0,T)$, the classical Riesz–Caputo derivative is defined by

$$\begin{aligned} {}^{R}_{*}D^{\alpha}_{0,T}g(\mu) &= \frac{1}{\Gamma(n-\alpha)}\int_0^T |\mu-\eta|^{n-\alpha-1}g^{(n)}(\eta)\,d\eta \\ &= \frac{1}{2}\big({}_{*}D^{\alpha}_{0,\mu} + (-1)^n {}_{*}D^{\alpha}_{\mu,T}\big)g(\mu), \end{aligned}$$

where ${}_{*}D^{\alpha}_{0,\mu}$ and ${}_{*}D^{\alpha}_{\mu,T}$ are left and right Caputo derivatives [36], respectively.

Following the same mechanism, we generalize the Riesz fractional integral by means of Definition 2.2 as follows.

Definition 3.2 Let $g(\mu) \in X^{p}_{c}(a,b)$ and $\alpha, \rho > 0$. Then, for $0 \le \mu \le T$, the generalized Riesz type integral is defined as

$$\begin{aligned} \big({}^{\rho}_{0}I^{\alpha}_{T}g\big)(\mu) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^T \eta^{\rho-1}\big|\big(\eta^{\rho}-\mu^{\rho}\big)\big|^{\alpha-1}g(\eta)\,d\eta \\ &= {}^{\rho}_{0}I^{\alpha}_{\mu}g(\mu) + {}^{\rho}_{\mu}I^{\alpha}_{T}g(\mu). \end{aligned}$$

Accordingly, the Riesz–Caputo derivative [19] can be generalized by means of generalized Caputo type derivative operators [1] as follows.

Definition 3.3 Let $\alpha, \rho \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\rho) > 0$ and $g(\mu) \in X^{\rho}_{c}(a,b)$ for $0 \le \mu \le T$. Then the generalized Riesz–Caputo type derivative operator is defined as

$$\begin{aligned} {}^{RC}_{0}D^{\alpha,\rho}_{T}g(\mu) &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_0^T \frac{\eta^{\rho-1}}{|(\mu^{\rho}-\eta^{\rho})|^{\alpha-n+1}}\left(\eta^{1-\rho}\frac{d}{d\eta}\right)^{n} g(\eta)\,d\eta \\ &= \frac{1}{2}\big({}^{\rho}_{*}D^{\alpha}_{0,\mu} + (-1)^{n\rho}{}^{\rho}_{*}D^{\alpha}_{\mu,T}\big)g(\mu), \end{aligned}$$

where ${}^{\rho}_{*}D^{\alpha}_{0,\mu}$ and ${}^{\rho}_{*}D^{\alpha}_{\mu,T}$ are left and right generalized Caputo type derivatives [37] as follows:

$${}^{\rho}_{*}D^{\alpha}_{0,\mu} = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_0^{\mu} \frac{\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{\alpha-n+1}}\left(\eta^{1-\rho}\frac{d}{d\eta}\right)^{n} g(\eta)\,d\eta$$

and

$${}^{\rho}_{*}D^{\alpha}_{\mu,T} = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_{\mu}^{T} \frac{\eta^{\rho-1}}{(\eta^{\rho}-\mu^{\rho})^{\alpha-n+1}}\left(-\eta^{1-\rho}\frac{d}{d\eta}\right)^{n} g(\eta)\,d\eta,$$

where $n = \lceil \alpha \rceil$.

Since for $\alpha = 1$ the right generalized derivative is the negative of the left generalized derivative, so for integer values of α, the generalized Riesz–Caputo type derivative defined above comes to term with the conventional definitions of derivative.

Lemma 3.4 *Let $g \in AC_{\delta}^{n}[0,T]$ with $0 \le \mu \le T$. Then the following relation is true*:

$$ {}_0^{\rho}I_{T0}^{\alpha}{}^{RC}D_T^{\alpha,\rho}g(\mu) = \frac{1}{2}\big({}_0^{\rho}I_{\mu *}^{\alpha}{}^{\rho}D_{0,\mu}^{\alpha} + (-1)^n {}_{\mu}^{\rho}I_{T*}^{\alpha}{}^{\rho}D_{\mu,T}^{\alpha}\big)g(\mu). \tag{$*$} $$

Proof Using the above definitions, we can write

$$ \begin{aligned} {}_0^{\rho}I_{T0}^{\alpha}{}^{RC}D_T^{\alpha,\rho}g(\mu) &= \frac{1}{2}{}_0^{\rho}I_T^{\alpha}\big({}_*^{\rho}D_{0,\mu}^{\alpha} + (-1)^n {}_*^{\rho}D_{\mu,T}^{\alpha}\big)g(\mu) \\ &= \frac{1}{2}{}_0^{\rho}I_{T*}^{\alpha}{}^{\rho}D_{0,\mu}^{\alpha}g(\mu) + \frac{(-1)^n}{2}{}_0^{\rho}I_{T*}^{\alpha}{}^{\rho}D_{\mu,T}^{\alpha}g(\mu) \\ &= \frac{1}{2}\big({}_0^{\rho}I_{\mu *}^{\alpha}{}^{\rho}D_{0,\mu}^{\alpha} + {}_{\mu}^{\rho}I_{T*}^{\alpha}{}^{\rho}D_{0,\mu}^{\alpha}\big)g(\mu) + \frac{(-1)^n}{2}\big({}_0^{\rho}I_{\mu *}^{\alpha}{}^{\rho}D_{\mu,T}^{\alpha} + {}_{\mu}^{\rho}I_{T*}^{\alpha}{}^{\rho}D_{\mu,T}^{\alpha}\big)g(\mu) \\ &= \frac{1}{2}\big({}_0^{\rho}I_{\mu *}^{\alpha}{}^{\rho}D_{0,\mu}^{\alpha} + (-1)^n {}_{\mu}^{\rho}I_{T*}^{\alpha}{}^{\rho}D_{\mu,T}^{\alpha}\big)g(\mu), \end{aligned} $$

and the proof is finished. □

Remark 3.5 If $0 < \alpha \le 1$, then for $g(\mu) \in C[0,T]$ the relation illustrated in $(*)$ becomes

$$ {}_0^{\rho}I_{T0}^{\alpha}{}^{RC}D_T^{\alpha,\rho}g(\mu) = g(0) - \frac{1}{2}\big(g(0) + g(T)\big). $$

Proof The proof simply follows by using $n = 1$ in Lemma 3.4 and Lemma 2.5, which yields the required result. □

Theorem 3.6 *Let $\alpha > 0$ and $\{\phi_j\}_{j=1}^{\infty}$ be a uniformly convergent sequence of continuous functions on $[a,b]$. Then we can interchange the generalized fractional integral operator and the limit, i.e.,*

$$ \big({}^{\rho}I_{a+}^{\alpha}\lim_{j\to 0}\phi_j\big)(\mu) = \big(\lim_{j\to\infty}{}^{\rho}I_{a+}^{\alpha}\phi_j\big)(\mu). $$

Proof Let ϕ be the limit of the sequence $\{\phi_j\}$. Since $\{\phi_j\}$ is the convergent sequence of continuous functions, so ϕ is also continuous. To prove that under the given conditions we can interchange fractional integral and limit, it is enough to show that the sequence $\{{}^{\rho}I_{a+}^{\alpha}\phi_j\}_{j=1}^{\infty}$ is also uniformly convergent. That is, $|{}^{\rho}I_{a+}^{\alpha}\phi_j(\mu) - {}^{\rho}I_{a+}^{\alpha}\phi(\mu)| \to 0$ as $j \to \infty$. For this, consider

$$ \begin{aligned} &\big|{}^{\rho}I_{a+}^{\alpha}\phi_j(\mu) - {}^{\rho}I_{a+}^{\alpha}\phi(\mu)\big| \\ &\quad= \left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\mu}\frac{\phi_j(\eta)\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha}}\,d\eta - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\mu}\frac{\phi_j(\eta)\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha}}\,d\eta\right| \\ &\quad\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\mu}\left|\frac{(\phi_j(\eta)-\phi(\eta))\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha}}\right|d\eta \\ &\quad\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\|\phi_j-\phi\|_{\infty}\int_a^{\eta}\eta^{\rho-1}\big(\mu^{\rho}-\eta^{\rho}\big)^{\alpha-1}\,d\eta. \end{aligned} \tag{2} $$

Now, we first shall evaluate the integral

$$\int_a^\eta \eta^{\rho-1}\left(\mu^\rho-\eta^\rho\right)^{\alpha-1}d\eta=\mu^{\alpha\rho-\rho}\int_a^\eta \eta^{\rho-1}\left(1-\frac{\eta^\rho}{\mu^\rho}\right)^{\alpha-1}d\eta.$$

Substituting $\frac{\eta^\rho}{\mu^\rho}=u$, we have

$$\begin{aligned}&\int_a^\eta \eta^{\rho-1}\left(\mu^\rho-\eta^\rho\right)^{\alpha-1}d\eta\\&\quad=\frac{\mu^{\alpha\rho}}{\rho}\int_{\frac{\eta^\rho}{\mu^\rho}}^1 u^0(1-u)^{\alpha-1}\,du=\frac{\mu^{\alpha\rho}}{\rho}\int_{\frac{\eta^\rho}{\mu^\rho}}^1\left(u-\frac{\eta^\rho}{\mu^\rho}\right)^{1-1}(1-u)^{\alpha-1}\,du.\end{aligned}$$

Now, using the result $\int_{\xi_1}^{\xi_2}(u-\xi_1)^{\alpha-1}(\xi_2-u)^{\beta-1}\,du=(\xi_2-\xi_1)^{\alpha+\beta-1}B(\alpha,\beta)$, the above equation leads to

$$\begin{aligned}&\int_a^\eta \eta^{\rho-1}\left(\mu^\rho-\eta^\rho\right)^{\alpha-1}d\eta\\&\quad=\frac{\mu^{\alpha\rho}}{\rho}\left\{\frac{(\mu^\rho-a^\rho)^\alpha B(1,\alpha)}{\mu^{\alpha\rho}}\right\}=\frac{(\mu^\rho-a^\rho)^\alpha B(1,\alpha)}{\rho}=\frac{(\mu^\rho-a^\rho)^\alpha}{\alpha\rho}.\end{aligned}$$

Consequently, from equation (2), we arrive at

$$\left|{}^\rho I_{a+}^\alpha\phi_j(\mu)-{}^\rho I_{a+}^\alpha\phi(\mu)\right|\le\frac{(\mu^\rho-a^\rho)^\alpha}{\rho\Gamma(\alpha+1)}\|\phi_j-\phi\|_\infty.$$

Since (ϕ_j) is a uniformly convergent sequence, thus

$$\left|{}^\rho I_{a+}^\alpha\phi_j(\mu)-{}^\rho I_{a+}^\alpha\phi(\mu)\right|\to 0\quad\text{as } j\to\infty.$$

Therefore, the sequence $\{{}^\rho I_{a+}^\alpha\phi_j\}_{j=1}^\infty$ is also uniformly convergent, and hence the result follows. □

The similar result holds true for the right-sided generalized fractional integral as well.

Lemma 3.7 *If $\phi(\mu)$ is an analytic function in $(a_0-\xi,a_0+\xi)$, where $t>0$ and $\alpha,a_0>0$, then*

$$\left({}^\rho I_{a_0}^\alpha\phi\right)(\xi)=\sum_{j=0}^\infty\frac{\Gamma(\frac{j}{\rho}+1)\xi^{\alpha(\rho-1)}(\xi-a_0)^{j+\alpha}}{j!\Gamma(\frac{j}{\rho}+\alpha+1)\rho^\alpha}\phi^j(a_0).$$

In particular, ${}^\rho I_{a_0}^\alpha\phi$ is also analytic.

Proof Since ϕ is an analytic function, thus it can be written in the form of convergent power series, i.e.,

$$\phi(\xi)=\sum_{j=0}^\infty\frac{(\xi-a_0)^j}{j!}\phi^j(a_0).$$

Using Definition 2.2, we get

$$\left({}^{\rho}I_{a_0}^{\alpha}\phi\right)(\xi) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{a_0}^{\xi}\eta^{\rho-1}\left(\xi^{\rho}-\eta^{\rho}\right)^{\alpha-1}\sum_{j=0}^{\infty}\frac{(\eta-a_0)^j}{j!}\phi^j(a_0)\,d\eta.$$

Using Theorem 3.6, the summation and integral sign are interchanged as follows:

$$\begin{aligned}\left({}^{\rho}I_{a_0}^{\alpha}\phi\right)(\xi) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{\phi^j(a_0)}{j!}\int_{a_0}^{\xi}\eta^{\rho-1}(\eta-a_0)^j\left(\xi^{\rho}-\eta^{\rho}\right)^{\alpha-1}d\eta, \left({}^{\rho}I_{a_0}^{\alpha}\phi\right)(\xi)\\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{\phi^j(a_0)\xi^{\alpha(\rho-1)}(\xi-a_0)^{j+\alpha}}{\rho j!}B\left(\frac{j}{\rho}+1,\alpha\right)\\ &= \sum_{j=0}^{\infty}\frac{\Gamma(\frac{j}{\rho}+1)\xi^{\alpha(\rho-1)}(\xi-a_0)^{j+\alpha}}{j!\Gamma(\frac{j}{\rho}+\alpha+1)\rho^{\alpha}}\phi^j(a_0).\end{aligned}$$

□

Theorem 3.8 *Let $\phi \in X_c^p(a,b)$ and $\{\lambda_j\}_{j=1}^{\infty}$ be a convergent sequence of nonnegative real numbers with limit λ. Then*

$$\lim_{j\to 0}\left({}^{\rho}I_{a+}^{\lambda_j}\phi\right)(\mu) = \left({}^{\rho}I_{a+}^{\lambda}\phi\right)(\mu),$$

where convergence of the sequence $\{{}^{\rho}I_{a+}^{\lambda_j}\phi\}_{j=1}^{\infty}$ is signified in terms of $X_c^p(a,b)$ norm with $1 \le p \le \infty$, $p,c \in \mathbb{R}$, $\rho > 0$, and $c \le \rho + 1$.

Proof Let the sequence $\{\lambda_j\}_{j=1}^{\infty}$ converge to the limit λ. Then, by definition,

$$\left({}^{\rho}I_{a+}^{\lambda_j}\phi\right)(\mu) = \frac{\rho^{1-\lambda_j}}{\Gamma(\lambda_j)}\int_a^{\mu}\frac{\eta^{\rho-1}\phi(\eta)}{(\mu^{\rho}-\eta^{\rho})^{1-\lambda_j}}\,d\eta, \quad \mu > a, \rho > 0,$$

and by taking limit on both sides and by using Theorem 3.6, we have

$$\begin{aligned}\lim_{j\to 0}\left({}^{\rho}I_{a+}^{\lambda_j}\phi\right)(\mu) &= \frac{\rho^{\lim_{j\to 0}(1-\lambda_j)}}{\lim_{j\to 0}\{(\lambda_j-1)!\}}\int_a^{\mu}\eta^{\rho-1}\phi(\eta)\left\{\lim_{j\to 0}\left(\mu^{\rho}-\eta^{\rho}\right)^{\lambda_j-1}\right\}d\eta\\ &= \frac{\rho^{(1-\lambda)}}{(\lambda-1)!}\int_a^{\mu}\eta^{\rho-1}\phi(\eta)\left(\mu^{\rho}-\eta^{\rho}\right)^{\lambda-1}d\eta\\ &= \left({}^{\rho}I_{a+}^{\lambda}\phi\right)(\mu),\end{aligned}$$

and this ends the proof. □

Theorem 3.9 *Let $\alpha > 0$ and $\{\phi_j\}_{j=1}^{\infty}$ be a uniformly convergent sequence of continuous functions on $[a,b]$. Then we can interchange the generalized Riesz fractional integral operator and the limit, i.e.,*

$$\left({}_0^{\rho}I_T^{\alpha}\lim_{j\to 0}\phi_j\right)(\mu) = \left(\lim_{j\to\infty}{}_0^{\rho}I_T^{\alpha}\phi_j\right)(\mu).$$

Proof The result follows taking into account Definition 3.2, Theorem 3.6, and the fact that sum of two convergent sequence is convergent. □

Lemma 3.10 ([38]) *Let $\alpha > 0$, $g(\mu)$ and $u_1(\mu)$ be locally integrable, nonnegative, and nondecreasing functions with $\mu \in [0, T]$. Also, assume that $v_1(\mu)$ is a nondecreasing continuous function such that $0 \le v_1(\mu) < L$, where L is a constant. Furthermore, if*

$$g(\mu) \le u_1(\mu) + \rho^{1-\alpha} v_1(\mu) \int_0^{\mu} \eta^{\rho-1} \left(\mu^{\rho} - \eta^{\rho}\right)^{\alpha-1} g(\eta)\, d\eta \quad (0 \le \mu \le T),$$

then the following inequality is true:

$$g(\mu) \le u_1(\mu) + \int_0^{\mu} \left[\sum_{j=1}^{\infty} \frac{\rho^{1-n\alpha} (v_1(T)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \eta^{\rho-1} u_1(\eta) \left(\mu^{\rho} - \eta^{\rho}\right)^{n\alpha-1} \right] d\eta.$$

Corollary 3.11 ([38]) *Let $\alpha > 0$ and assume that $g(\mu)$, $u_1(\mu)$, and $v_1(\mu)$ are defined in the same way as in Lemma 3.10. Furthermore, if g satisfies*

$$g(\mu) \le u_1(\mu) + \rho^{1-\alpha} v_1(\mu) \int_0^{\mu} \eta^{\rho-1} \left(\mu^{\rho} - \eta^{\rho}\right)^{\alpha-1} g(\eta)\, d\eta, \quad 0 \le \eta \le \mu,$$

on $\mu \in [0, T]$, then

$$g(\mu) \le u_1(\mu) E_{\alpha,1}\left(\rho^{-\alpha} v_1(\mu) \Gamma(\alpha) \mu^{\alpha\rho}\right),$$

where $E_{\alpha,1}(\cdot)$ is a Mittag-Leffler function [12].

Likewise, the Gronwall inequality for generalized right-sided generalized fractional operator is expressed as follows.

Lemma 3.12 ([38]) *Let $\alpha > 0$, $\mu \in [0, T]$ and assume that $g(\mu)$, $u_2(\mu)$, and $v_2(\mu)$ are defined in the same way as in Lemma 3.10. Furthermore, if*

$$g(\mu) \le u_2(\mu) + \rho^{1-\alpha} v_2(\mu) \int_{\mu}^{T} \eta^{\rho-1} \left(\eta^{\rho} - \mu^{\rho}\right)^{\alpha-1} g(\eta)\, d\eta, \quad 0 \le \mu \le \eta,$$

then the following inequality holds true:

$$g(\mu) \le u_2(\mu) + \int_{\mu}^{T} \left[\sum_{j=1}^{\infty} \frac{\rho^{1-n\alpha} (v_2(T)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \eta^{\rho-1} u_2(\eta) \left(\eta^{\rho} - \mu^{\rho}\right)^{n\alpha-1} \right] d\eta.$$

Lemma 3.13 *Let $\alpha > 0$ and assume that $g(\mu)$, $u_1(\mu)$, and $v_1(\mu)$ are defined in the same way as in Lemma 3.10. Furthermore, if*

$$g(\mu) \le u_2(\mu) + \rho^{1-\alpha} v_2(\mu) \int_{\mu}^{T} \eta^{\rho-1} \left(\eta^{\rho} - \mu^{\rho}\right)^{\alpha-1} g(\eta)\, d\eta, \quad 0 \le \mu \le \eta,$$

on $\mu \in [0, T]$, then

$$g(\mu) \le u_2(\mu) E_{\alpha,1}\left(\rho^{-\alpha} v_2(\mu) \Gamma(\alpha) \left(T^{\rho} - \mu^{\rho}\right)^{\alpha}\right).$$

Proof From Lemma 3.12,

$$g(\mu) \leq u_2(\mu) + \int_{\mu}^{T} \left[\sum_{j=1}^{\infty} \frac{\rho^{1-n\alpha}(\nu_2(T)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \eta^{\rho-1} u_2(\eta)\left(\eta^{\rho} - \mu^{\rho}\right)^{n\alpha-1} \right] d\eta.$$

Since u_2 is a nondecreasing function, therefore $u_2(\mu) \leq u_2(\eta)$ for all $\eta \in [0, T]$, and hence

$$\begin{aligned} g(\mu) &\leq u_2(\mu)\left\{ 1 + \int_{\mu}^{T} \sum_{j=1}^{\infty} \frac{\rho^{1-n\alpha}(\nu_2(T)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \eta^{\rho-1} u_2(\eta)\left(\eta^{\rho} - \mu^{\rho}\right)^{n\alpha-1} d\eta \right\} \\ &= u_2(\mu)\left\{ 1 + \sum_{j=1}^{\infty} \frac{\rho^{-n\alpha}(\nu_2(T)\Gamma(\alpha))^n}{\Gamma(n\alpha+1)} \left(T^{\rho} - \mu^{\rho}\right)^{n\alpha} \right\} \\ &= u_2(\mu)\left\{ \sum_{j=0}^{\infty} \frac{(\rho^{-\alpha}\nu_2(T)\Gamma(\alpha)(T^{\rho} - \mu^{\rho})^{\alpha})^n}{\Gamma(n\alpha+1)} \right\} \\ &= u_2(\mu) E_{\alpha,1}\left(\rho^{-\alpha}\nu_2(\mu)\Gamma(\alpha)\left(T^{\rho} - \mu^{\rho}\right)^{\alpha}\right), \end{aligned}$$

and the proof is ended. □

Lemma 3.14 *Let $\alpha > 0, 0 < \mu < T$, and assume that $g(\mu), u_1(\mu), u_2(\mu), \nu_1(\mu)$, and $\nu_2(\mu)$ are defined in the same way as in Lemma 3.10 and Lemma 3.12. Furthermore, if $g(\mu)$ satisfies the inequality*

$$\begin{aligned} g(\mu) \leq u_1(\mu) + \rho^{1-\alpha}\nu_1(\mu) \int_0^{\mu} \eta^{\rho-1}\left(\mu^{\rho} - \eta^{\rho}\right)^{\alpha-1} g(\eta)\, d\eta + u_2(\mu) + \rho^{1-\alpha}\nu_2(\mu) \\ \times \int_{\mu}^{T} \eta^{\rho-1}\left(\eta^{\rho} - \mu^{\rho}\right)^{\alpha-1} g(\eta)\, d\eta, \end{aligned}$$

then the following inequality holds true:

$$g(\mu) \leq \left(u_1(\mu) + u_2(\mu)\right) E_{\alpha,1}\left(\rho^{-\alpha}\nu_2(\mu)\Gamma(\alpha)\left(T^{\rho} - \mu^{\rho}\right)^{\alpha}\right) E_{\alpha,1}\left(\rho^{-\alpha}\nu_1(\mu)\Gamma(\alpha)\mu^{\alpha\rho}\right),$$

where $E_{\alpha,1}(\cdot)$ is a Mittag-Leffler function.

Proof Conflating Lemma 3.10 and Lemma 3.13 gives

$$\begin{aligned} g(\mu) &\leq \left(u_1(\mu) + u_2(\mu) + \rho^{1-\alpha}\nu_2(\mu) \int_{\mu}^{T} \eta^{\rho-1}\left(\eta^{\rho} - \mu^{\rho}\right)^{\alpha-1} d\eta \right) \\ &\quad \times E_{\alpha,1}\left(\rho^{-\alpha}\nu_1(\mu)\Gamma(\alpha)\mu^{\alpha\rho}\right) \\ &\leq \left(u_1(\mu) + u_2(\mu)\right) E_{\alpha,1}\left(\rho^{-\alpha}\nu_2(\mu)\Gamma(\alpha)\left(T^{\rho} - \mu^{\rho}\right)^{\alpha}\right) \\ &\quad \times E_{\alpha,1}\left(\rho^{-\alpha}\nu_1(\mu)\Gamma(\alpha)\mu^{\alpha\rho}\right). \end{aligned}$$

□

4 Existence and stability

For the upcoming existence results and discussion for boundary value (1), we use the following conditions. Let $J = [0, T]$ and $C(J)$ be the space of all continuous functions defined

on J. We define the space

$$X = \left\{\phi(\mu)|\phi(\mu) \in C(J) \text{ and } {}^{\rho}_{*}D^{\alpha^*}\phi(\mu) \in C(J)\right\}$$

characterized by the norm $\|\phi(\mu)\|_X = \max_{\mu\in J}|\phi(\mu)| + \max_{\mu\in J}|{}^{\rho}_{*}D^{\alpha^*}\phi(\mu)|$.

Lemma 4.1 *$(X, \|\cdot\|_X)$ is a Banach space.*

Proof Let $\{\phi_j\}_{j=0}^{\infty}$ be a Cauchy sequence in $(X, \|\cdot\|_X)$. Then clearly $\{{}^{\rho}_{*}D^{\alpha^*}\phi_j\}_{j=0}^{\infty}$ is also a Cauchy sequence in the space $C(J)$. Therefore both $\{\phi_j(\mu)\}_{j=0}^{\infty}$ and $\{{}^{\rho}_{*}D^{\alpha^*}\phi_j(\mu)\}_{j=0}^{\infty}$ converge uniformly, say $u(\mu)$ and $v(\mu)$, respectively, in the space $C(J)$. We just have to show that $v = {}^{\rho}_{*}D^{\alpha^*}u$. For this, consider

$$\begin{aligned}
&\left|{}^{\rho}I^{\beta^*}_{0^+}\,{}^{\rho}_{*}D^{\alpha^*}_{0^+}\phi_j(\mu) - {}^{\rho}I^{\alpha^*}_{0^+}v(\mu)\right| \\
&\quad= \left|\frac{\rho^{1-\alpha^*}}{\Gamma(\alpha^*)}\int_0^{\mu}\frac{{}^{\rho}_{*}D^{\alpha^*}_{0^+}\phi_j(\eta)\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha^*}}\,d\eta - \frac{\rho^{1-\alpha^*}}{\Gamma(\alpha^*)}\int_0^{\mu}\frac{v(\eta)\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha^*}}\,d\eta\right| \\
&\quad\le \frac{\rho^{1-\alpha^*}}{\Gamma(\alpha^*)}\int_0^{\mu}\left|\frac{({}^{\rho}_{*}D^{\alpha^*}_{0^+}\phi_j(\eta) - v(\eta))\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha^*}}\right|d\eta \\
&\quad\le \frac{\mu^{\rho\alpha^*}}{\rho\Gamma(\alpha^*+1)}\max_{\mu\in J}\left|{}^{\rho}_{*}D^{\alpha^*}_{0^+}\phi_j(\mu) - v(\mu)\right|.
\end{aligned}$$

Since $\{{}^{\rho}_{*}D^{\alpha^*}\phi_j(\mu)\}_{j=0}^{\infty}$ converges uniformly to $v(\mu)$ for $\mu \in J$, hence

$$\left|{}^{\rho}I^{\alpha^*}_{0^+}\,{}^{\rho}_{*}D^{\alpha^*}_{0^+}\phi_j(\mu) - {}^{\rho}I^{\beta^*}_{0^+}v(\mu)\right| \to 0$$

as $j \to \infty$, i.e., $\lim_{j\to\infty} {}^{\rho}I^{\alpha^*}_{0^+}\,{}^{\rho}_{*}D^{\alpha^*}_{0^+}\phi_j(\mu) \cong {}^{\rho}I^{\alpha^*}_{0^+}v(\mu)$. Now considering

$${}^{\rho}_{*}D^{\alpha^*}_{0^+}\Big(\lim_{j\to\infty}{}^{\rho}I^{\alpha^*}_{0^+}\,{}^{\rho}_{*}D^{\beta^*}_{0^+}\phi_j(\mu)\Big) = {}^{\rho}_{*}D^{\alpha^*}_{0^+}\,{}^{\rho}I^{\alpha^*}_{0^+}v(\mu)$$

and taking into account Theorem 3.6 and Theorem 2.3, we get $v(\mu) = {}^{\rho}_{*}D^{\alpha^*}u(\mu)$. This completes the proof. □

Lemma 4.2 *Let $\alpha \in (1,2)$, $\alpha^* \in (0,1)$, and $g \in C(J)$. Then problem (1) is equivalent to the following integral equation*:

$$\begin{aligned}
\phi(\mu) &= \frac{1}{2}(\phi_0 + \phi_T) + \left(\frac{\phi_T - \phi_0}{2T^{\rho}}\right)\left(2\mu^{\rho} - T^{\rho}\right) \\
&\quad + \frac{\rho^{1-\alpha}(\mu^{\rho} - T^{\rho})}{T^{\rho}\Gamma(\alpha)}\int_0^{T}\eta^{\alpha\rho-1}g\left(\eta,\phi(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\eta)\right)d\eta \\
&\quad - \frac{\mu^{\rho}\rho^{1-\alpha}}{T^{\rho}\Gamma(\alpha)}\int_0^{T}\eta^{\rho-1}\left(T^{\rho} - \eta^{\rho}\right)^{\alpha-1}g\left(\eta,\phi(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\eta)\right)d\eta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{T}\eta^{\rho-1}\left|\eta^{\rho} - \mu^{\rho}\right|^{\alpha-1}g\left(\eta,\phi(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\eta)\right)d\eta \\
&= \frac{1}{2}(\phi_0 + \phi_T) + \psi(\mu), \qquad (3)
\end{aligned}$$

where, for $\mu > \eta$,

$$\psi(\mu) = \frac{(\phi_T - \phi_0)\mu^\rho}{2T^\rho} - \frac{\mu^\rho \rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^T \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(T^\rho - \eta^\rho)^{1-\alpha}}\,d\eta + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\mu^\rho - \eta^\rho)^{1-\alpha}}\,d\eta,$$

and for $\eta > \mu$,

$$\psi(\mu) = \frac{(\phi_T - \phi_0)(\mu^\rho - T^\rho)}{2T^\rho} + \frac{(\mu^\rho - T^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^T \frac{g(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}\,d\eta + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^T \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho - \mu^\rho)^{1-\alpha}}\,d\eta.$$

Proof Let $\phi(\mu) \in X$ be a solution of boundary value problem (1). Then, by applying the generalized Riesz-type integral operator on both sides of equation (1) and using Definition 3.2, Lemma 2.5, and Lemma 3.4, we obtain

$$\frac{1}{2}\phi(\mu) - \frac{1}{2}\phi(0) - c_0\frac{\mu^\rho}{2\rho} + \frac{1}{2}\phi(\mu) - \frac{1}{2}\phi(T) - \frac{1}{2}c_1\left(\frac{\mu^\rho - T^\rho}{\rho}\right) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^T \eta^{\rho-1}\left|\left(\eta^\rho - \mu^\rho\right)\right|^{\alpha-1} g\left(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*}\phi(\eta)\right)d\eta$$

or

$$\phi(\mu) = \frac{1}{2}(\phi_0 + \phi_T) + \frac{c_0\mu^\rho}{2\rho} + \frac{1}{2}c_1\left(\frac{\mu^\rho - T^\rho}{\rho}\right) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu \eta^{\rho-1}\left(\mu^\rho - \eta^\rho\right)^{\alpha-1} g\left(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*}\phi(\eta)\right)d\eta + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^T \eta^{\rho-1}\left(\eta^\rho - \mu^\rho\right)^{\alpha-1} g\left(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*}\phi(\eta)\right)d\eta.$$

Using the boundary conditions $\phi(0) = \phi_0$ and $\phi(T) = \phi_T$ into the above equation, we get

$$c_0 = \frac{\rho(\phi_T - \phi_0)}{T^\rho} - \frac{2\rho^{2-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^T \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(T^\rho - \eta^\rho)^{1-\alpha}}\,d\eta$$

and

$$c_1 = \frac{\rho(\phi_T - \phi_0)}{T^\rho} + \frac{2\rho^{2-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^T \frac{g(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}\,d\eta.$$

Now again substituting these values of constants into the above equation, we get

$$\phi(\mu) = \frac{1}{2}(\phi_0 + \phi_T)\frac{(\phi_T - \phi_0)\mu^\rho}{2T^\rho} - \frac{\mu^\rho \rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^T \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(T^\rho - \eta^\rho)^{1-\alpha}}\,d\eta$$

$$
\begin{aligned}
&+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\mu^\rho-\eta^\rho)^{1-\alpha}}\,d\eta+\frac{(\phi_T-\phi_0)(\mu^\rho-T^\rho)}{2T^\rho}\\
&+\frac{(\mu^\rho-T^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^{T}\frac{g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}\,d\eta\\
&+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu}^{T}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu^\rho)^{1-\alpha}}\,d\eta\\
=\;&\frac{1}{2}(\phi_0+\phi_T)+\left(\frac{\phi_T-\phi_0}{2T^\rho}\right)\left(2\mu^\rho-T^\rho\right)\\
&+\frac{\rho^{1-\alpha}(\mu^\rho-T^\rho)}{T^\rho\Gamma(\alpha)}\int_0^{T}\eta^{\alpha\rho-1}g\left(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta)\right)d\eta\\
&-\frac{\mu^\rho\rho^{1-\alpha}}{T^\rho\Gamma(\alpha)}\int_0^{T}\eta^{\rho-1}\left(T^\rho-\eta^\rho\right)^{\alpha-1}g\left(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta)\right)d\eta\\
&+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{T}\eta^{\rho-1}\left|\eta^\rho-\mu^\rho\right|^{\alpha-1}g\left(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta)\right)d\eta\\
=\;&\frac{1}{2}(\phi_0+\phi_T)+\psi(\mu),
\end{aligned}
$$

where, for $\mu>\eta$,

$$
\begin{aligned}
\psi(\mu)=\;&\frac{(\phi_T-\phi_0)\mu^\rho}{2T^\rho}-\frac{\mu^\rho\rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^{T}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(T^\rho-\eta^\rho)^{1-\alpha}}\,d\eta\\
&+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\mu^\rho-\eta^\rho)^{1-\alpha}}\,d\eta,
\end{aligned}
$$

and for $\eta>\mu$,

$$
\begin{aligned}
\psi(\mu)=\;&\frac{(\phi_T-\phi_0)(\mu^\rho-T^\rho)}{2T^\rho}+\frac{(\mu^\rho-T^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^{T}\frac{g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}\,d\eta\\
&+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu}^{T}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu^\rho)^{1-\alpha}}\,d\eta.
\end{aligned}
$$

Conversely, let $\phi(\mu)\in X$ be a solution of the fractional integral operator (3), and we denote the right-hand side of equation (3) by $\Phi(\mu)$, i.e.,

$$
\Phi(\mu)=\frac{1}{2}(\phi_0+\phi_T)+\psi(\mu).
$$

Now taking the left and the right generalized Caputo derivative on both sides of the above equation, we get

$$
\begin{aligned}
{}_*^{\rho}D_{0,\mu}^{\alpha}\Phi(\mu)=\;&{}_*^{\rho}D_{0,\mu}^{\alpha}\left(\frac{1}{2}(\phi_0+\phi_T)\right)+\frac{(\phi_T-\phi_0)}{2T^\rho}{}_*^{\rho}D_{0,\mu}^{\alpha}\left(\mu^\rho\right)\\
&-\frac{{}_0^{\rho}I_T^{\alpha}g(T,\phi(T),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(T))}{T^\rho}{}_*^{\rho}D_{0,\mu}^{\alpha}\left(\mu^\rho\right)\\
&+{}_*^{\rho}D_{0,\mu}^{\alpha}\,{}_0^{\rho}I_{\mu}^{\alpha}g\left(\mu,\phi(\mu),{}_0^{RC}D_T^{\alpha^*}\phi(\mu)\right)\\
=\;&g\left(\mu,\phi(\mu),{}_0^{RC}D_T^{\alpha^*}\phi(\mu)\right)
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
{}^{\rho}_{*}D^{\alpha}_{\mu,T}\Phi(\mu) &= {}^{\rho}_{*}D^{\alpha}_{\mu,T}\left(\frac{1}{2}(\phi_0+\phi_T)\right)+\frac{(\phi_T-\phi_0)}{2T^{\rho}}{}^{\rho}_{*}D^{\alpha}_{\mu,T}\left(\mu^{\rho}-T^{\rho}\right)\\
&\quad+\frac{{}^{\rho}_{0}I^{\alpha}_{T}g(T,\phi(T),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(T))}{T^{\rho}}{}^{\rho}_{*}D^{\alpha}_{\mu,T}\left(\mu^{\rho}-T^{\rho}\right)\\
&\quad+{}^{\rho}_{*}D^{\alpha}_{\mu,T}{}^{\rho}_{\mu}I^{\alpha}_{T}g\left(\mu,\phi(\mu),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\mu)\right)\\
&= g\left(\mu,\phi(\mu),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\mu)\right). \qquad (5)
\end{aligned}
$$

Here, we have used Theorem 2.3, and some simple calculation leads to the facts that ${}^{\rho}_{*}D^{\alpha}_{0,\mu}(\mu^{\rho})=0$ and ${}^{\rho}_{*}D^{\alpha}_{\mu,T}(\mu^{\rho}-T^{\rho})=0$. Consequently, from equations (4), (5) and Definition 3.3, the required result follows, i.e.,

$$
\frac{1}{2}\left({}^{\rho}_{*}D^{\alpha}_{0,\mu}\Phi(\mu)+{}^{\rho}_{*}D^{\alpha}_{\mu,T}\Phi(\mu)\right)={}^{RC}_{0}D^{\alpha,\rho}_{T}\Phi(\mu)=g\left(\mu,\phi(\mu),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\mu)\right),
$$

and the proof is completed. □

Now we present the existence and uniqueness results for the nonlinear boundary value problem (1). We define an operator $\tilde{T}:X\to X$ by

$$
\begin{aligned}
\tilde{T}\left(\phi(\mu)\right) &= \frac{1}{2}(\phi_0+\phi_T)+\left(\frac{\phi_T-\phi_0}{2T^{\rho}}\right)\left(2\mu^{\rho}-T^{\rho}\right)\\
&\quad+\frac{\rho^{1-\alpha}(\mu^{\rho}-T^{\rho})}{T^{\rho}\Gamma(\alpha)}\int_0^T\eta^{\alpha\rho-1}g\left(\eta,\phi(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\eta)\right)d\eta\\
&\quad-\frac{\mu^{\rho}\rho^{1-\alpha}}{T^{\rho}\Gamma(\alpha)}\int_0^T\eta^{\rho-1}\left(T^{\rho}-\eta^{\rho}\right)^{\alpha-1}g\left(\eta,\phi(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\eta)\right)d\eta\\
&\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^T\eta^{\rho-1}\left|\eta^{\rho}-\mu^{\rho}\right|^{\alpha-1}g\left(\eta,\phi(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\eta)\right)d\eta. \qquad (6)
\end{aligned}
$$

Lemma 4.2 signifies that solutions of problem (1) coincide with the fixed points of the operator $T(\phi(\mu))$. Ahead of the detailed existence results, let us have the following considerations first:

(H_1^*) Let $1<\alpha<2$, $0<\alpha^*<1$, and $g:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function and $U(\mu)\in L^1[J,\mathbb{R}_+]$ be a nonnegative function such that $U(\mu)\le\phi(\mu)$. Furthermore, g satisfies

$$
\left|g\left(\mu,\phi(\mu),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\mu)\right)\right|\le\rho^{\alpha}\left(a_1\left|\phi(\mu)\right|+a_2\left|{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi(\mu)\right|\right)+\frac{b\rho^{\alpha}}{T^{\rho}}U(\mu),
$$

where $a_1,a_2,b\in\mathbb{R}_+$.

(H_2^*) Let $1<\alpha<2$, $0<\alpha^*<1$, and $g:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function, and g satisfies the Lipschitz condition, i.e.,

$$
\begin{aligned}
&\left|g\left(\eta,\phi_1(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_1(\eta)\right)-g\left(\eta,\phi_2(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_2(\eta)\right)\right|\\
&\quad\le\lambda_1\left(\left|\phi_1(\mu)-\phi_2(\mu)\right|+\left|{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_1(\eta)\right)-{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_2(\eta))\right|),
\end{aligned}
$$

where $0<\lambda_1<\frac{1}{2}\max\{K_1,K_2\}$.

Let $M_1 = \max_{\mu\in J}\{h_1(\mu) : |h_1(\mu)| \le d_2\}$ and $M^* = \max_{\mu\in J}\{f(\mu) : |f(\mu)| \le d_1\}$, where

$$h_1(\mu) = \frac{\rho^{\alpha^*}|\phi_T - \phi_0|\mu^{\rho(1-\alpha^*)}}{2T^\rho\Gamma(2-\alpha^*)} + \frac{2a_3K\mu^{\rho(\alpha-\alpha^*-1)}}{T^\rho\rho^{1-\alpha^*}\Gamma(2-\alpha^*)} + \frac{2a_3\rho^{\alpha-1}K\mu^{\rho(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*+1)} + \frac{b\rho^{1+\alpha^*}K^*\mu^{\rho(1-\alpha^*)}}{\Gamma(2-\alpha^*)T^{2\rho}}$$

and

$$f(\mu) = T^{\rho(\alpha-1)} + \mu^{\alpha\rho} + \left(T^\rho - \mu^\rho\right)^\alpha.$$

Furthermore, let

$$K^* := \frac{\rho}{\Gamma(\alpha)}\max\left(\int_0^T \eta^{\rho-1}\left(T^\rho - \eta^\rho\right)^{\alpha-1}U(\eta)\,d\eta, \int_0^T \eta^{\alpha\rho-1}U(\eta)\,d\eta\right),$$

$$L_1 := \sup\left(\max_{\mu\in J}\left(T^{\alpha\rho} + \mu^{\alpha\rho} + \left(T^\rho - \mu^\rho\right)^\alpha\right)\right)$$

and

$$L_2 := \sup\left[\max_{\mu\in J}\left\{\frac{T^{\rho(\alpha-1)}\mu^{\rho(1-\alpha^*)}}{\Gamma(\alpha+1)\Gamma(2-\alpha^*)} + \frac{\mu^{(\alpha-\alpha^*)}}{\Gamma(2-\alpha^*+1)} + \frac{T^{\rho(\alpha-1)}(T^\rho-\mu^\rho)^{1-\alpha^*}}{\Gamma(\alpha+1)\Gamma(2-\alpha^*)} + \frac{(T^\rho-\mu^\rho)^{\alpha-\alpha^*}}{\Gamma(\alpha-\alpha^*+1)}\right\}\right].$$

By means of local integrability of $U(\mu)$, K^* exists certainly. Define a set

$$A_r = \left\{\phi \in C(J) : \|\phi\| < r\right\},$$

where $r = \{4\max(|\phi_T|, |\phi_0|, \frac{2bK^*}{T^\rho}, \frac{2Ka_3M^*}{\Gamma(\alpha+1)})\}E^2_{\alpha,1}(b)$. Then manifestly the set A_r is a closed, bounded, and convex subset of the above defined Banach space $(X, \|\cdot\|_X)$.

Theorem 4.3 *Assume that condition* (H_1^*) *holds. Then problem* (1) *has a solution in* A_r.

Proof We prove this result using the Schauder fixed point theorem. First we show that the operator $\tilde{T} : A_r \to A_r$ is a self-map. Suppose $\phi \in A_r$, and for $L \in (0,1)$, the operator (6) satisfies $\phi(\mu) = L\tilde{T}(\phi(\mu))$. Then from (6) and using condition (H_1^*), we have that

$$\begin{aligned}\left|\phi(\mu)\right| &\le \left|\tilde{T}\phi(\mu)\right| \\ &\le \frac{1}{2}\left|(\phi_0+\phi_T)\right| + \frac{|\phi_T-\phi_0|\mu^\rho}{2T^\rho} + \frac{|\phi_0-\phi_T|(T^\rho-\mu^\rho)}{2T^\rho} \\ &\quad + \frac{a_1\rho\mu^\rho}{T^\rho\Gamma(\alpha)}\int_0^T \eta^{\rho-1}\left(T^\rho-\eta^\rho\right)^{\alpha-1}\left|\phi(\eta)\right|d\eta \\ &\quad + \frac{a_2\rho\mu^\rho}{T^\rho\Gamma(\alpha)}\int_0^T \eta^{\rho-1}\left(T^\rho-\eta^\rho\right)^{\alpha-1}\left|{}^{RC}_0D_T^{\alpha^*,\rho}\phi(\eta)\right|d\eta \\ &\quad + \frac{b\rho\mu^\rho}{T^{2\rho}\Gamma(\alpha)}\int_0^T \eta^{\rho-1}\left(T^\rho-\eta^\rho\right)^{\alpha-1}U(\eta)\,d\eta\end{aligned}$$

$$
\begin{aligned}
&+\frac{a_1\rho}{\Gamma(\alpha)}\int_0^{\mu}\eta^{\rho-1}\left(\mu^{\rho}-\eta^{\rho}\right)^{\alpha-1}\left|\phi(\eta)\right|d\eta\\
&+\frac{a_2\rho}{\Gamma(\alpha)}\int_0^{\mu}\eta^{\rho-1}\left(\mu^{\rho}-\eta^{\rho}\right)^{\alpha-1}\left|{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta)\right|d\eta\\
&+\frac{b\rho}{T^{\rho}\Gamma(\alpha)}\int_0^{\mu}\eta^{\rho-1}\left(\mu^{\rho}-\eta^{\rho}\right)^{\alpha-1}U(\eta)\,d\eta\\
&+\frac{a_1\rho(T^{\rho}-\mu^{\rho})}{\Gamma(\alpha)T^{\rho}}\int_0^{T}\eta^{\alpha\rho-1}\left|\phi(\eta)\right|d\eta\\
&+\frac{a_2\rho(T^{\rho}-\mu^{\rho})}{\Gamma(\alpha)T^{\rho}}\int_0^{T}\eta^{\alpha\rho-1}\left|{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta)\right|d\eta\\
&+\frac{b\rho(T^{\rho}-\mu^{\rho})}{\Gamma(\alpha)T^{2\rho}}\int_0^{T}\eta^{\alpha\rho-1}U(\eta)\,d\eta+\frac{a_1\rho}{\Gamma(\alpha)}\int_{\mu}^{T}\eta^{\rho-1}\left(\eta^{\rho}-\mu^{\rho}\right)^{\alpha-1}\left|\phi(\eta)\right|d\eta\\
&+\frac{a_2\rho}{\Gamma(\alpha)}\int_{\mu}^{T}\eta^{\rho-1}\left(\eta^{\rho}-\mu^{\rho}\right)^{\alpha-1}\left|{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta)\right|d\eta\\
&+\frac{b\rho}{T^{\rho}\Gamma(\alpha)}\int_{\mu}^{T}\eta^{\rho-1}\left(\eta^{\rho}-\mu^{\rho}\right)^{\alpha-1}U(\eta)\,d\eta\\
\le{}&|\phi_T|+|\phi_0|+\frac{2a_3K\rho\mu^{\rho}}{T^{\rho}\Gamma(\alpha)}\int_0^{T}\eta^{\rho-1}\left(T^{\rho}-\eta^{\rho}\right)^{\alpha-1}d\eta\\
&+\frac{b\rho\mu^{\rho}}{T^{2\rho}\Gamma(\alpha)}\int_0^{T}\eta^{\rho-1}\left(T^{\rho}-\eta^{\rho}\right)^{\alpha-1}U(\eta)\,d\eta\\
&+\frac{2a_3K\rho}{\Gamma(\alpha)}\int_0^{\mu}\eta^{\rho-1}\left(\mu^{\rho}-\eta^{\rho}\right)^{\alpha-1}d\eta+\frac{b\rho}{T^{\rho}\Gamma(\alpha)}\int_0^{\mu}\eta^{\rho-1}\left(\mu^{\rho}-\eta^{\rho}\right)^{\alpha-1}U(\eta)\,d\eta\\
&+\frac{2a_3K\rho(T^{\rho}-\mu^{\rho})}{\Gamma(\alpha)T^{\rho}}\int_0^{T}\eta^{\alpha\rho-1}d\eta+\frac{b\rho(T^{\rho}-\mu^{\rho})}{\Gamma(\alpha)T^{2\rho}}\int_0^{T}\eta^{\alpha\rho-1}U(\eta)\,d\eta\\
&+\frac{2a_3K\rho}{\Gamma(\alpha)}\int_{\mu}^{T}\eta^{\rho-1}\left(\eta^{\rho}-\mu^{\rho}\right)^{\alpha-1}d\eta+\frac{b\rho}{T^{\rho}\Gamma(\alpha)}\int_{\mu}^{T}\eta^{\rho-1}\left(\eta^{\rho}-\mu^{\rho}\right)^{\alpha-1}U(\eta)\,d\eta,
\end{aligned}
$$

where $K=\max_{\mu\in[J}(|\phi(\mu)|,|{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\mu)|)$, and so

$$
\begin{aligned}
\left|\tilde{T}\phi(\mu)\right|\le{}&|\phi_T|+|\phi_0|+\frac{2a_3K\mu^{\rho}}{T^{\rho(2-\alpha)}\Gamma(\alpha+1)}+\frac{bK^*}{T^{\rho}}\\
&+\frac{2a_3K\mu^{\alpha\rho}}{\Gamma(\alpha+1)}+\frac{2a_3K(T^{\rho}-\mu^{\rho})}{T^{\rho(2-\alpha)}\Gamma(\alpha+1)}+\frac{2a_3K(T^{\rho}-\mu^{\rho})^{\alpha}}{\Gamma(\alpha+1)}\\
&+\frac{b\rho}{T^{\rho}\Gamma(\alpha)}\int_0^{\mu}\eta^{\rho-1}\left(\mu^{\rho}-\eta^{\rho}\right)^{\alpha-1}U(\eta)\,d\eta\\
&+\frac{b\rho}{T^{\rho}\Gamma(\alpha)}\int_{\mu}^{T}\eta^{\rho-1}\left(\eta^{\rho}-\mu^{\rho}\right)^{\alpha-1}U(\eta)\,d\eta\\
={}&|\phi_T|+|\phi_0|+\frac{bK^*}{T^{\rho}}+\frac{2Ka_3}{\Gamma(\alpha+1)}\left\{T^{\rho(\alpha-1)}+\mu^{\alpha\rho}+\left(T^{\rho}-\mu^{\rho}\right)^{\alpha}\right\}\\
&+\frac{b\rho}{T^{\rho}\Gamma(\alpha)}\int_0^{\mu}\eta^{\rho-1}\left(\mu^{\rho}-\eta^{\rho}\right)^{\alpha-1}U(\eta)\,d\eta\\
&+\frac{b\rho}{T^{\rho}\Gamma(\alpha)}\int_{\mu}^{T}\eta^{\rho-1}\left(\eta^{\rho}-\mu^{\rho}\right)^{\alpha-1}U(\eta)\,d\eta.
\end{aligned}
$$

Since the functions $\mu^{\alpha\rho}$ and $(T^\rho - \mu^\rho)^\alpha$ are integrable, uniformly continuous, and non-negative for $\mu \in [0, T]$ and also $U(\mu) \le \phi(\mu)$, hence applying Lemma 3.14 gives

$$\begin{aligned}\left|\tilde{T}\phi(\mu)\right| &\le \left\{|\phi_T| + |\phi_0| + \frac{2bK^*}{T^\rho} + \frac{2Ka_3}{\Gamma(\alpha+1)}\left\{T^{\rho(\alpha-1)} + \mu^{\alpha\rho} + \left(T^\rho - \mu^\rho\right)^\alpha\right\}\right\} \\ &\quad \times E_{\alpha,1}\left(\frac{b(T^\rho - \mu^\rho)^\alpha}{T^\rho}\right)E_{\alpha,1}\left(\frac{b\mu^{\alpha\rho}}{T^\rho}\right) \\ &\le \left\{|\phi_T| + |\phi_0| + \frac{bK^*}{T^\rho} + \frac{2Ka_3M^*}{\Gamma(\alpha+1)}\right\}E_{\alpha,1}\left(\frac{b(T^\rho - \mu^\rho)^\alpha}{T^\rho}\right)E_{\alpha,1}\left(\frac{b\mu^{\alpha\rho}}{T^\rho}\right).\end{aligned}$$

Thus

$$\left|\tilde{T}\phi(\mu)\right| \le \left\{|\phi_T| + |\phi_0| + \frac{2bK^*}{T^\rho} + \frac{2Ka_3M^*}{\Gamma(\alpha+1)}\right\}E^2_{\alpha,1}(b) < r.$$

Also,

$$\begin{aligned}&\left|{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}\left(\tilde{T}\phi(\mu)\right)\right| \\ &\quad \le {}^{\rho}_{*}D^{\alpha^*}_{0,\mu}\left(\frac{1}{2}(\phi_0 + \phi_T)\right) \\ &\qquad + \frac{(\phi_T - \phi_0)}{2T^\rho}\left({}^{\rho}_{*}D^{\alpha^*}_{0,\mu}\left(\mu^\rho\right)\right) + \left(\frac{2a_3\rho^{\alpha\rho}\,{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}(\mu^\rho)}{T^\rho}\right){}^{\rho}_{0}I^{\alpha}_{T}K \\ &\qquad + \left(\frac{b\rho^\alpha}{T^{2\rho}}\,{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}\left(\mu^\rho\right)\right){}^{\rho}_{0}I^{\alpha}_{T}U(\mu) + 2a_3\rho^{\alpha\rho}\,{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}\,{}^{\rho}_{0}I^{\alpha}_{\mu}K + \frac{b\rho^\alpha}{T^\rho}\,{}^{\rho}_{*}D^{\alpha}_{0,\mu}\,{}^{\rho}_{0}I^{\alpha}_{\mu}U(\eta).\end{aligned}$$

Using Theorem 2.3 and Lemma 2.5, we get

$$\begin{aligned}&\left|{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}\left(\tilde{T}\phi(\mu)\right)\right| \\ &\quad \le \frac{\rho^{\alpha^*}|\phi_T - \phi_0|\mu^{\rho(1-\alpha^*)}}{2T^\rho\Gamma(2-\alpha^*)} + \frac{2a_3K\mu^{\rho(\alpha-\alpha^*-1)}}{T^\rho\rho^{1-\alpha^*}\Gamma(2-\alpha^*)} \\ &\qquad + \frac{b\rho^{\alpha+\alpha^*}\mu^{\rho(1-\alpha^*)}\,{}^{\rho}_{0}I^{\alpha}_{T}U(\eta)}{\Gamma(2-\alpha^*)T^{2\rho}} + 2a_3\rho^{\alpha\rho}\,{}^{\rho}_{0}I^{\alpha-\alpha^*}_{\mu}K + \frac{b\rho^\alpha}{T^\rho}\,{}^{\rho}_{0}I^{\alpha-\alpha^*}_{\mu}U(\eta) \\ &\quad = \frac{\rho^{\alpha^*}|\phi_T - \phi_0|\mu^{\rho(1-\alpha^*)}}{2T^\rho\Gamma(2-\alpha^*)} + \frac{2a_3K\mu^{\rho(\alpha-\alpha^*-1)}}{T^\rho\rho^{1-\alpha^*}\Gamma(2-\alpha^*)} + \frac{2a_3\rho^{\alpha-1}K\mu^{\rho(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*+1)} \\ &\qquad + \frac{b\rho^{1+\alpha^*}K^*\mu^{\rho(1-\alpha^*)}}{\Gamma(2-\alpha^*)T^{2\rho}} + \frac{b\rho^{1+\alpha^*}}{T^\rho\Gamma(\alpha-\alpha^*)}\int_0^\mu \eta^{\rho-1}\left(\mu^\rho - \eta^\rho\right)^{\alpha-\alpha^*-1}U(\eta)\,d\eta.\end{aligned}$$

Since $U(\mu) \le \phi(\mu)$ and $\mu^{\rho(1-\alpha^*)}$, $\mu^{\rho(\alpha-\alpha^*-1)}$ are measurable and continuous functions for $\mu \in [0, T]$, therefore by using the assumption $\phi(\mu) = L\tilde{T}(\phi(\mu))$ and Corollary 3.11, we get

$$\left|{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}\left(\tilde{T}\phi(\mu)\right)\right| \le M_1E_{(\alpha-\alpha^*),1}\left(\frac{\rho^{\alpha^*}b}{T^\rho}\mu^{(\alpha-\alpha^*)\rho}\right) < r_1. \tag{7}$$

Moreover,

$$\left|{}^{\rho}_{*}D^{\alpha^*}_{\mu,T}\left(\tilde{T}\phi(\mu)\right)\right|$$

$$
\begin{aligned}
&\leq {}^{\rho}_{*}D^{\alpha^*}_{\mu,T}\left(\frac{1}{2}(\phi_0+\phi_T)\right)\\
&\quad+\frac{|\phi_T-\phi_0|}{2T^{\rho}}{}^{\rho}_{*}D^{\alpha^*}_{\mu,T}\left(T^{\rho}-\mu^{\rho}\right)+\frac{2a_3K}{\Gamma(\alpha)T^{\rho(1-\alpha)}}{}^{\rho}_{*}D^{\alpha^*}_{\mu,T}\left(T^{\rho}-\mu^{\rho}\right)\\
&\quad+\frac{bK^*}{T^{2\rho}}{}^{\rho}_{*}D^{\alpha^*}_{\mu,T}\left(T^{\rho}-\mu^{\rho}\right)+2a_3\rho^{\alpha}{}^{\rho}_{*}D^{\alpha^*}_{\mu,T}{}^{\rho}_{\mu}I^{\alpha}_{T}K+\frac{b\rho^{\alpha}}{T^{\rho}}{}^{\rho}_{*}D^{\alpha^*}_{\mu,T}I^{\alpha}_{T}U(\eta).
\end{aligned}
$$

Using Theorem 2.3 and Lemma 2.5, we get

$$
\begin{aligned}
&\left|{}^{\rho}_{*}D^{\alpha^*}_{\mu,T}\left(\tilde{T}\phi(\mu)\right)\right|\\
&\leq\frac{\rho^{\alpha^*}|\phi_T-\phi_0|(T^{\rho}-\mu^{\rho})^{1-\alpha^*}}{2T^{\rho}\Gamma(2-\alpha^*)}+\frac{2a_3K\rho^{\alpha^*}(T^{\rho}-\mu^{\rho})^{1-\alpha^*}}{\Gamma(\alpha)\Gamma(2-\alpha^*)}\\
&\quad+\frac{bK^*\rho^{\alpha^*}(T^{\rho}-\mu^{\rho})^{1-\alpha^*}}{\Gamma(2-\alpha^*)T^{2\rho}}+2a_3\rho^{\alpha}{}^{\rho}_{\mu}I^{\alpha-\alpha^*}_{T}K+\frac{b\rho^{\alpha}}{T^{\rho}}{}^{\rho}_{\mu}I^{\alpha-\alpha^*}_{T}U(\eta)\\
&=\frac{\rho^{\alpha^*}|\phi_T-\phi_0|(T^{\rho}-\mu^{\rho})^{1-\alpha^*}}{2T^{\rho}\Gamma(2-\alpha^*)}+\frac{2a_3K\rho^{\alpha^*}(T^{\rho}-\mu^{\rho})^{1-\alpha^*}}{\Gamma(\alpha)\Gamma(2-\alpha^*)}\\
&\quad+\frac{bK^*\rho^{\alpha^*}(T^{\rho}-\mu^{\rho})^{1-\alpha^*}}{\Gamma(2-\alpha^*)T^{2\rho}}+\frac{2a_3K\rho^{\alpha^*}(T^{\rho}-\mu^{\rho})^{\alpha-\alpha^*}}{\Gamma(\alpha-\alpha^*+1)}\\
&\quad+\frac{b\rho^{1+\alpha^*}}{T^{\rho}\Gamma(\alpha-\alpha^*)}\int_{\mu}^{T}\eta^{\rho-1}\left(\eta^{\rho}-\mu^{\rho}\right)^{\alpha-\alpha^*-1}U(\eta)\,d\eta.
\end{aligned}
$$

Since $U(\mu)\leq\phi(\mu)$ and $(T^{\rho}-\mu^{\rho})^{\alpha-\alpha^*}\in L^1[J,\mathbb{R}_+]$ for $\mu\in[0,T]$, therefore taking into account the assumption that $\phi(\mu)=L\tilde{T}(\phi(\mu))$ with $L\in(0,1)$ and Corollary 3.11 yields

$$
\left|{}^{\rho}_{*}D^{\alpha^*}_{\mu,T}\left(\tilde{T}\phi(\mu)\right)\right|\leq M_1E_{(\alpha-\alpha^*),1}\left(\frac{\rho^{\alpha^*}b}{T^{\rho}}\left(T^{\rho}-\mu^{\rho}\right)^{\alpha-\alpha^*}\right)<r_2. \tag{8}
$$

From Definition 3.3, inequalities (7) and (8) yield

$$
\left|{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\left(\tilde{T}\phi(\mu)\right)\right|=\frac{1}{2}\left|{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}\left(\tilde{T}\phi(\mu)\right)-{}^{\rho}_{*}D^{\alpha^*}_{\mu,T}\left(\tilde{T}\phi(\mu)\right)\right|<r,
$$

which implies $\tilde{T}\phi\in A_r$; that is, the operator $\tilde{T}:A_r\to A_r$ is a self-map. Next we show that operator (6) is continuous. For this, let $\phi_1(\mu),\phi_2(\mu)\in A_r$. Then we have

$$
\begin{aligned}
&\left|\tilde{T}\phi_1(\mu)-\tilde{T}\phi_2(\mu)\right|\\
&\leq\frac{\mu^{\rho}\rho^{1-\alpha}}{\Gamma(\alpha)T^{\rho}}\int_0^T\frac{\eta^{\rho-1}|g(\eta,\phi_1(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_2(\eta))|}{(T^{\rho}-\eta^{\rho})^{1-\alpha}}\,d\eta\\
&\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu}\frac{\eta^{\rho-1}|g(\eta,\phi_1(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_2(\eta))|}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha}}\,d\eta\\
&\quad+\frac{(T^{\rho}-\mu^{\rho})\rho^{1-\alpha}}{\Gamma(\alpha)T^{\rho}}\int_0^T\frac{|g(\eta,\phi_1(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_2(\eta))|}{\eta^{1-\alpha\rho}}\,d\eta\\
&\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu}^T\frac{\eta^{\rho-1}|g(\eta,\phi_1(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_2(\eta))|}{(\eta^{\rho}-\mu^{\rho})^{1-\alpha}}\,d\eta\\
&=\frac{\{T^{\alpha\rho}+(T^{\rho}-\mu^{\rho})^{\alpha}\}}{\rho^{\alpha}\Gamma(\alpha+1)}\left|g\left(\eta,\phi_1(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_1(\eta)\right)-g\left(\eta,\phi_2(\eta),{}^{RC}_{0}D^{\alpha^*,\rho}_{T}\phi_2(\eta)\right)\right|.
\end{aligned}
$$

Since g is continuous on A_r, hence for all $\mu \in [0,T]$ there exists $\delta > 0$ such that $\|\phi_1(\eta) - \phi_2(\eta)\| < \delta$, and for any $\epsilon > 0$,

$$\left|g\left(\eta,\phi_1(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta)\right) - g\left(\eta,\phi_2(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta)\right)\right| < \frac{\rho^\alpha \Gamma(\alpha+1)}{T^{\alpha\rho}}\varepsilon.$$

Therefore,

$$\begin{aligned}
&\left|\tilde{T}\phi_1(\mu) - \tilde{T}\phi_2(\mu)\right| \\
&\quad\le \frac{\{T^{\alpha\rho} + (T^\rho - \mu^\rho)^\alpha\}}{\rho^\alpha \Gamma(\alpha+1)} \\
&\qquad\times \left|g\left(\eta,\phi_1(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta)\right) - g\left(\eta,\phi_2(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta)\right)\right| \\
&\quad\le \frac{\varepsilon}{2} + \frac{(T^\rho - \mu^\rho)^\alpha}{2T^{\alpha\rho}}\varepsilon < \varepsilon.
\end{aligned}$$

Likewise one can prove ${}_0^{RC}D_T^{\alpha^*,\rho}(\tilde{T}\phi(\mu))$ is continuous on A_r. Moreover, we show that operator (6) is completely continuous. For this, let $\eta_1, \eta_2 \in J$ with $\eta_1 < \eta_2$ and $\phi \in A_r$. Then we have

$$\begin{aligned}
&\left|\tilde{T}\phi(\mu_1) - \tilde{T}\phi(\mu_2)\right| \\
&\quad\le \frac{|\phi_T - \phi_0|(\mu_1^\rho - \mu_2^\rho)}{T^\rho} \\
&\qquad+ \left|\frac{(\mu_1^\rho - \mu_2^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^T \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(T^\rho - \eta^\rho)^{1-\alpha}}\,d\eta\right| \\
&\qquad+ \left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu_1} \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\mu_1^\rho - \eta^\rho)^{1-\alpha}}\,d\eta\right. \\
&\qquad\left.- \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu_2} \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\mu_2^\rho - \eta^\rho)^{1-\alpha}}\,d\eta\right| \\
&\qquad+ \left|\frac{(\mu_1^\rho - \mu_2^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^T \frac{g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}\,d\eta\right| \\
&\qquad+ \left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu_1}^T \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho - \mu_1^\rho)^{1-\alpha}}\,d\eta\right. \\
&\qquad\left.- \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu_2}^T \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho - \mu_2^\rho)^{1-\alpha}}\,d\eta\right| \\
&\quad\le \frac{|\phi_T - \phi_0|(\mu_1^\rho - \mu_2^\rho)}{T^\rho} + \frac{2a_3KT^{\alpha\rho}(\mu_1^\rho - \mu_2^\rho)}{\Gamma(\alpha+1)} \\
&\qquad+ \frac{b\rho K^*(\mu_1^\rho - \mu_2^\rho)}{T^{2\rho}\Gamma(\alpha)} + \frac{2Ka_3\rho}{\Gamma(\alpha)}\int_0^{\mu_1}\left\{\eta^{\rho-1}\left(\mu_1^\rho - \eta^\rho\right)^{\alpha-1} - \eta^{\rho-1}\left(\mu_2^\rho - \eta^\rho\right)^{\alpha-1}\right\}d\eta \\
&\qquad+ \frac{b\rho}{T^\rho\Gamma(\alpha)}\int_0^{\mu_1}\left\{\eta^{\rho-1}\left(\mu_1^\rho - \eta^\rho\right)^{\alpha-1} - \eta^{\rho-1}\left(\mu_2^\rho - \eta^\rho\right)^{\alpha-1}\right\}U(\eta)\,d\eta \\
&\qquad+ \frac{2Ka_3\rho}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\eta^{\rho-1}\left(\mu_2^\rho - \eta^\rho\right)^{\alpha-1}d\eta + \frac{b\rho}{T^\rho\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\eta^{\rho-1}\left(\mu_2^\rho - \eta^\rho\right)^{\alpha-1}U(\eta)\,d\eta
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2Ka_3(\mu_1^\rho-\mu_2^\rho)T^{\rho(\alpha-1)}}{\Gamma(\alpha+1)}+\frac{(\mu_1^\rho-\mu_2^\rho)K^*}{T^\rho}\\
&+\frac{2Ka_3\rho}{\Gamma(\alpha)}\int_{\mu_2}^{T}\left\{\eta^{\rho-1}\left(\eta^\rho-\mu_1^\rho\right)^{1-\alpha}-\eta^{\rho-1}\left(\eta^\rho-\mu_2^\rho\right)^{1-\alpha}\right\}d\eta\\
&+\frac{b\rho}{T^\rho\Gamma(\alpha)}\int_{\mu_2}^{T}\left\{\eta^{\rho-1}\left(\eta^\rho-\mu_1^\rho\right)^{1-\alpha}-\eta^{\rho-1}\left(\eta^\rho-\mu_2^\rho\right)^{1-\alpha}\right\}U(\eta)\,d\eta\\
&+\frac{2Ka_3\rho}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\frac{\eta^{\rho-1}}{\left(\eta^\rho-\mu_1^\rho\right)^{1-\alpha}}\,d\eta+\frac{b\rho}{T^\rho\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\frac{\eta^{\rho-1}U(\eta)}{\left(\eta^\rho-\mu_1^\rho\right)^{1-\alpha}}\,d\eta.
\end{aligned}
$$

Since $U(\mu)\in L^1[J,\mathbb{R}_+]$, therefore the functions

$$
\left(\eta^{\rho-1}\left(\mu_1^\rho-\eta^\rho\right)^{\alpha-1}-\eta^{\rho-1}\left(\mu_2^\rho-\eta^\rho\right)^{\alpha-1}\right)U(\eta),
$$

$(\eta^{\rho-1}(\mu_2^\rho-\eta^\rho)^{\alpha-1}U(\eta))$, and $(\eta^{\rho-1}(\eta^\rho-\mu_1^\rho)^{1-\alpha}-\eta^{\rho-1}(\eta^\rho-\mu_2^\rho)^{1-\alpha})U(\eta)$ are Lebesgue integrable in η. Also $(\mu_1^\rho-\mu_2^\rho)T^{\rho(\alpha-1)}$ and $T^{\alpha\rho}(\mu_1^\rho-\mu_2^\rho)$ are uniformly continuous for $\mu_1,\mu_2\in J$. So we see through that the right-hand side of the above inequality tends to zero as $\mu_1\to\mu_2$. Furthermore, we prove that $|{}_0^{RC}D_T^{\alpha^*,\rho}(\tilde{T}\phi(\mu_1))-{}_0^{RC}D_T^{\alpha^*,\rho}(\tilde{T}\phi(\mu_2))|\to 0$ as $\mu_1\to\mu_2$ for all $\mu_1,\mu_2\in[0,T]$ with $\mu_1<\mu_2$. For this, let us compute first the left and the right generalized Caputo derivatives of operator (6).

$$
\begin{aligned}
{}_*^\rho D_{0,\mu}^{\alpha^*}\left(\tilde{T}\phi(\mu)\right)&=\frac{\rho^{\alpha^*}(\phi_T-\phi_0)\mu_1^{\rho(1-\alpha^*)}}{2T^\rho\Gamma(2-\alpha^*)}\\
&\quad-\frac{\rho^{1-(\alpha-\alpha^*)}\mu_1^{\rho(1-\alpha^*)}}{T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^T\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(T^\rho-\eta^\rho)^{1-\alpha}}\,d\eta\\
&\quad+\frac{\rho^{1-(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*)}\int_0^{\mu_1}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\mu_1^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}\,d\eta
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
{}_*^\rho D_{\mu,T}^{\alpha^*}\left(\tilde{T}\phi(\mu)\right)&=\frac{\rho^{\alpha^*}(\phi_0-\phi_T)(T^\rho-\mu_1^\rho)^{1-\alpha^*}}{2T^\rho\Gamma(2-\alpha^*)}\\
&\quad-\frac{\rho^{1-(\alpha-\alpha^*)}(T^\rho-\mu_1^\rho)^{1-\alpha^*}}{T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^T\frac{g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}\,d\eta\\
&\quad+\frac{\rho^{1-(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*)}\int_{\mu_1}^T\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu_1^\rho)^{1-(\alpha-\alpha^*)}}\,d\eta.
\end{aligned}
\tag{10}
$$

From Definition 3.3 and equations (9) and (10), we have

$$
\begin{aligned}
&{}_0^{RC}D_T^{\alpha^*,\rho}\left(\tilde{T}\phi(\mu_1)\right)\\
&=\frac{1}{2}\left({}_*^\rho D_{0,\mu_1}^{\alpha^*}\left(\tilde{T}\phi(\mu_1)\right)-{}_*^\rho D_{\mu_1,T}^{\alpha^*}\left(\tilde{T}\phi(\mu_1)\right)\right)\\
&=\frac{1}{2}\left[\frac{\rho^{\alpha^*}(\phi_T-\phi_0)\mu_1^{\rho(1-\alpha^*)}}{2T^\rho\Gamma(2-\alpha^*)}-\frac{\rho^{1-(\alpha-\alpha^*)}\mu_1^{\rho(1-\alpha^*)}}{T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^T\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(T^\rho-\eta^\rho)^{1-\alpha}}\,d\eta\right.\\
&\quad\left.+\frac{\rho^{1-(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*)}\int_0^{\mu_1}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\mu_1^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}\,d\eta+\frac{\rho^{\alpha^*}(\phi_T-\phi_0)(T^\rho-\mu_1^\rho)^{1-\alpha^*}}{2T^\rho\Gamma(2-\alpha^*)}\right.
\end{aligned}
$$

$$+\frac{\rho^{1-(\alpha-\alpha^*)}(T^\rho-\mu_1^\rho)^{1-\alpha^*}}{T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^T\frac{g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}\,d\eta$$
$$-\frac{\rho^{1-(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*)}\int_{\mu_1}^T\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu_1^\rho)^{1-(\alpha-\alpha^*)}}\,d\eta\Bigg].$$

Therefore, by using the above equation, we establish that

$$\begin{aligned}
&\left|{}_0^{RC}D_T^{\alpha^*,\rho}\big(\tilde{T}\phi(\mu_1)\big)-{}_0^{RC}D_T^{\alpha^*,\rho}\big(\tilde{T}\phi(\mu_2)\big)\right|\\
&\quad\le\left|\frac{\rho^{\alpha^*}(\phi_T-\phi_0)(\mu_1^{\rho(1-\alpha^*)}-\mu_1^{\rho(1-\alpha^*)})}{4T^\rho\Gamma(2-\alpha^*)}\right|\\
&\qquad+\left|\frac{\rho^{1-(\alpha-\alpha^*)}(\mu_2^{\rho(1-\alpha^*)}-\mu_1^{\rho(1-\alpha^*)})}{2T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^T\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(T^\rho-\eta^\rho)^{1-\alpha}}\,d\eta\right|\\
&\qquad+\left|\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)}\int_0^{\mu_1}\left\{\frac{\eta^{\rho-1}}{(\mu_1^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}-\frac{\eta^{\rho-1}}{(\mu_2^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}\right\}\right.\\
&\qquad\left.\times g\big(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta)\big)\,d\eta\right|\\
&\qquad+\left|\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)}\int_{\mu_1}^{\mu_2}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\mu_2^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}\,d\eta\right|\\
&\qquad+\left|\frac{\rho^{\alpha^*}(\phi_T-\phi_0)\{(T^\rho-\mu_1^\rho)^{1-\alpha^*}-(T^\rho-\mu_2^\rho)^{1-\alpha^*}\}}{4T^\rho\Gamma(2-\alpha^*)}\right|\\
&\qquad+\left|\frac{\rho^{1-(\alpha-\alpha^*)}\{(T^\rho-\mu_1^\rho)^{1-\alpha^*}-(T^\rho-\mu_2^\rho)^{1-\alpha^*}\}}{2T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^T\frac{g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}\,d\eta\right|\\
&\qquad+\left|\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)}\int_{\mu_1}^{\mu_2}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu_1^\rho)^{1-(\alpha-\alpha^*)}}\,d\eta\right|\\
&\qquad+\left|\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)}\left\{\int_{\mu_2}^T\frac{\eta^{\rho-1}}{(\eta^\rho-\mu_2^\rho)^{1-(\alpha-\alpha^*)}}-\int_{\mu_2}^T\frac{\eta^{\rho-1}}{(\eta^\rho-\mu_1^\rho)^{1-(\alpha-\alpha^*)}}\right\}\right.\\
&\qquad\left.\times g\big(\eta,\phi(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi(\eta)\big)\,d\eta\right|.
\end{aligned}$$

Using condition (H_1^*),

$$\begin{aligned}
&\left|{}_0^{RC}D_T^{\alpha^*,\rho}\big(\tilde{T}\phi(\mu_1)\big)-{}_0^{RC}D_T^{\alpha^*,\rho}\big(\tilde{T}\phi(\mu_2)\big)\right|\\
&\quad\le\frac{\rho^{\alpha^*}|\phi_T-\phi_0|(\mu_1^{\rho(1-\alpha^*)}-\mu_2^{\rho(1-\alpha^*)})}{4T^\rho\Gamma(2-\alpha^*)}\\
&\qquad+\frac{\rho^{1+\alpha^*}a_3K(\mu_1^{\rho(1-\alpha^*)}-\mu_2^{\rho(1-\alpha^*)})}{T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^T\eta^{\rho-1}\big(T^\rho-\eta^\rho\big)^{\alpha-1}\,d\eta\\
&\qquad+\frac{b\rho^{1+\alpha^*}(\mu_1^{\rho(1-\alpha^*)}-\mu_2^{\rho(1-\alpha^*)}}{2T^{2\rho}\Gamma(\alpha)}\int_0^T\eta^{\rho-1}\big(T^\rho-\eta^\rho\big)^{\alpha-1}U(\eta)\,d\eta\\
&\qquad+\frac{\rho^{1+\alpha^*}a_3K}{\Gamma(\alpha-\alpha^*)}\int_0^{\mu_1}\big\{\big(\mu_1^\rho-\eta^\rho\big)^{(\alpha-\alpha^*)-1}\eta^{\rho-1}-\eta^{\rho-1}\big(\mu_2^\rho-\eta^\rho\big)^{(\alpha-\alpha^*)-1}\big\}\,d\eta
\end{aligned}$$

$$
\begin{aligned}
&+\frac{b\rho^{1+\alpha^*}}{2T^\rho\Gamma(\alpha-\alpha^*)}\int_0^{\mu_1}\left\{\left(\mu_1^\rho-\eta^\rho\right)^{(\alpha-\alpha^*)-1}\eta^{\rho-1}-\eta^{\rho-1}\left(\mu_2^\rho-\eta^\rho\right)^{(\alpha-\alpha^*)-1}\right\}U(\eta)\,d\eta\\
&+\frac{\rho^{1+\alpha^*}a_3K}{\Gamma(\alpha-\alpha^*)}\int_{\mu_1}^{\mu_2}\eta^{\rho-1}\left(\mu_2^\rho-\eta^\rho\right)^{(\alpha-\alpha^*)-1}d\eta\\
&+\frac{b\rho^{1+\alpha^*}}{2T^\rho\Gamma(\alpha-\alpha^*)}\int_{\mu_1}^{\mu_2}\eta^{\rho-1}\left(\mu_2^\rho-\eta^\rho\right)^{(\alpha-\alpha^*)-1}U(\eta)\,d\eta\\
&+\frac{\rho^{\alpha^*}|\phi_T-\phi_0|\{(T^\rho-\mu_1^\rho)^{1-\alpha^*}-(T^\rho-\mu_2^\rho)^{1-\alpha^*}\}}{4T^\rho\Gamma(2-\alpha^*)}\\
&+\frac{a_3K\rho^{\alpha^*}\{(T^\rho-\mu_1^\rho)^{1-\alpha^*}-(T^\rho-\mu_2^\rho)^{1-\alpha^*}\}}{\Gamma(\alpha)(\alpha-\alpha^*)\Gamma(2-\alpha^*)T^{\rho(1-\alpha+\alpha^*)}}\\
&+\frac{b\rho^{1+\alpha^*}\{(T^\rho-\mu_1^\rho)^{1-\alpha^*}-(T^\rho-\mu_2^\rho)^{1-\alpha^*}\}}{2\Gamma(\alpha)\Gamma(2-\alpha^*)T^{2\rho}}\int_0^T\eta^{\rho(\alpha-\alpha^*)-1}U(\eta)\,d\eta\\
&+\frac{a_3K\rho^{\alpha^*}\{(T^\rho-\mu_1^\rho)^{(\alpha-\alpha^*)}-(T^\rho-\mu_2^\rho)^{(\alpha-\alpha^*)}\}}{\Gamma(\alpha)(\alpha-\alpha^*)\Gamma(2-\alpha^*)}\\
&+\frac{b\rho^{1+\alpha^*}}{2T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_{\mu_2}^{T}\left\{\eta^{\rho-1}\left(\eta^\rho-\mu_1^\rho\right)^{(\alpha-\alpha^*)-1}\right.\\
&\left.-\eta^{\rho-1}\left(\eta^\rho-\mu_2^\rho\right)^{(\alpha-\alpha^*)-1}\right\}U(\eta)\,d\eta\\
&+\frac{b\rho^{1+\alpha^*}}{2T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_{\mu_1}^{\mu_2}\eta^{\rho-1}\left(\eta^\rho-\mu_1^\rho\right)^{(\alpha-\alpha^*)-1}U(\eta)\,d\eta.
\end{aligned}
$$

Since $U(\mu)\in L^1[J,\mathbb{R}_+]$ and the functions

$$\eta^{\rho-1}\left(\mu_2^\rho-\eta^\rho\right)^{(\alpha-\alpha^*)-1},$$

$(\eta^{\rho(\alpha-\alpha^*)-1}U(\eta)$, and

$$\left(\left(\mu_1^\rho-\eta^\rho\right)^{(\alpha-\alpha^*)-1}\eta^{\rho-1}-\eta^{\rho-1}\left(\mu_2^\rho-\eta^\rho\right)^{(\alpha-\alpha^*)-1}U(\eta)\right)$$

are Lebesgue integrable on $[0,T]$, so the right-hand side of the above inequality tends to zero as $\mu_1\to\mu_2$. Hence the set of operators $\tilde{T}A_r$ is equicontinuous. Also $\tilde{T}A_r\subseteq A_r$ implies that $\tilde{T}A_r$ is uniformly bounded. Henceforth, $\tilde{T}$ is completely continuous and thus Schauder's fixed point theorem assures the existence of at least one fixed point of operator (6). Hence, by taking into account Lemma 4.2, the proof is finalized. □

Theorem 4.4 *Assume that conditions* (H_1^*) *and* (H_2^*) *hold. Then equation* (3) *comports as a unique solution of Problem* (1).

Proof To prove this theorem, we use the Banach fixed point theorem. For this, we first necessitate to confirm that (6) is a self-mapped operator, and afterwards we show that $\tilde{T}$ satisfies the contraction mapping principle. Since we have shown in Theorem 4.3 that $\tilde{T}\phi(\mu),{}_0^{RC}D_T^{\alpha^*,\rho}(\tilde{T}\phi(\mu))\in A_r$, so the operator $\tilde{T}$ satisfies the self-mappedness property under these conditions. Hence, the only stipulation that we need to verify here is contraction. For this, consider

$$\left|\tilde{T}\phi_1(\mu)-\tilde{T}\phi_2(\mu)\right|$$

$$
\begin{aligned}
&\leq \left|g\left(\eta,\phi_1(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta)\right)-g\left(\eta,\phi_2(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta)\right)\right| \\
&\quad\times\left(\frac{\mu^\rho\rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^T\frac{\eta^{\rho-1}}{(T^\rho-\eta^\rho)^{1-\alpha}}\,d\eta+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu\frac{\eta^{\rho-1}}{(\mu^\rho-\eta^\rho)^{1-\alpha}}\,d\eta\right. \\
&\quad\left.+\frac{(T^\rho-\mu^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)T^\rho}\int_0^T\frac{1}{\eta^{1-\alpha\rho}}\,d\eta+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^T\frac{\eta^{\rho-1}}{(\eta^\rho-\mu^\rho)^{1-\alpha}}\,d\eta\right) \\
&\leq \frac{\lambda_1(|\phi_1(\mu)-\phi_2(\mu)|+|{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta))-{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta))|)}{\rho^\alpha\Gamma(\alpha+1)} \\
&\quad\times\left(T^{\alpha\rho}+\mu^{\alpha\rho}+\left(T^\rho-\mu^\rho\right)^\alpha\right) \\
&\leq \frac{\lambda_1}{K_1}\left\|\phi_1(\mu)-\phi_2(\mu)\right\|,
\end{aligned}
$$

where $K_1=\frac{\rho^\alpha\Gamma(\alpha+1)}{L_1}$. Moreover,

$$
\begin{aligned}
&\left|{}_0^{RC}D_T^{\alpha^*,\rho}\left(\tilde{T}\phi_1(\mu)\right)-{}_0^{RC}D_T^{\alpha^*,\rho}\left(\tilde{T}\phi_2(\mu)\right)\right| \\
&\leq \frac{\rho^{1-(\alpha-\alpha^*)}\mu^{\rho(1-\alpha^*)}}{2T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)} \\
&\quad\times\int_0^T\frac{\eta^{\rho-1}|g(\eta,\phi_1(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta))|}{(T^\rho-\eta^\rho)^{1-\alpha}}\,d\eta \\
&\quad+\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)} \\
&\quad\times\int_0^\mu\frac{\eta^{\rho-1}|g(\eta,\phi_1(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta))|}{(\mu^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}\,d\eta \\
&\quad+\frac{\rho^{1-(\alpha-\alpha^*)}(T^\rho-\mu^\rho)^{1-\alpha^*}}{2T^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)} \\
&\quad\times\int_0^T\frac{|g(\eta,\phi_1(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta))|}{\eta^{1-\alpha\rho}}\,d\eta \\
&\quad+\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)} \\
&\quad\times\int_\mu^T\frac{\eta^{\rho-1}|g(\eta,\phi_1(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta))|}{(\eta^\rho-\mu^\rho)^{1-(\alpha-\alpha^*)}}\,d\eta \\
&\leq \lambda_1\frac{\rho^{(\alpha^*-\alpha)}T^{\rho(\alpha-1)}\mu^{\rho(1-\alpha^*)}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)} \\
&\quad\times\left(\left|\phi_1(\mu)-\phi_2(\mu)\right|+\left|{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta)\right)-{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta))\right|) \\
&\quad+\lambda_1\frac{\rho^{(\alpha^*-\alpha)}\mu^{(\alpha-\alpha^*)}}{2\Gamma(2-\alpha^*+1)} \\
&\quad\times\left(\left|\phi_1(\mu)-\phi_2(\mu)\right|+\left|{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta)\right)-{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta))\right|) \\
&\quad+\lambda_1\frac{\rho^{(\alpha^*-\alpha)}T^{\rho(\alpha-1)}(T^\rho-\mu^\rho)^{1-\alpha^*}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)} \\
&\quad\times\left(\left|\phi_1(\mu)-\phi_2(\mu)\right|+\left|{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta)\right)-{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta))\right|)
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda_1 \frac{\rho^{(\alpha^*-\alpha)}(T^\rho - \mu^\rho)^{\alpha-\alpha^*}}{2\Gamma(\alpha-\alpha^*+1)} \\
&\times \left(\left|\phi_1(\mu) - \phi_2(\mu)\right| + \left|{}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta)\right) - {}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta))\right|) \\
&= \frac{\lambda_1}{K_2}\left\|\phi_1(\mu) - \phi_2(\mu)\right\|,
\end{aligned}
$$

where $K_2 = \frac{2\rho^{(\alpha-\alpha^*)}}{L_2}$. Therefore

$$
\left|\tilde{T}\phi_1(\mu) - \tilde{T}\phi_2(\mu)\right| \leq \frac{2\lambda_1}{M}\left\|\phi_1(\mu) - \phi_2(\mu)\right\|
$$

where $M = \max(K_1, K_2)$. Thus the Banach fixed point theorem assures the existence of a unique fixed point of operator (6). So, in consequence of Lemma 4.2, we concluded that (3) is the unique solution of boundary value problem (1). □

Lemma 4.5 *Assume that* $1 < \alpha < 2$, $0 < \beta^* < 1$ *and* $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function. Furthermore, g satisfies*

$$
\left|g\left(\mu, \phi(\mu), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\mu)\right)\right| \leq a_3 + a_4 \max\left|\phi(\mu)\right| + b_2 \max\left|{}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\mu)\right|
$$

for all $a_3, a_4, b_2 \in \mathbb{R}_+$. *Then the solution* $\phi(\mu)$ *of* (1) *exists in* A_r.

Proof The result follows from Theorem 4.3. □

Lemma 4.6 *Assume that* $1 < \alpha < 2$, $0 < \beta^* < 1$ *and* $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function. Furthermore, g satisfies the following condition*:

$$
\left|g\left(\mu, \phi(\mu), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\mu)\right)\right| \leq \frac{\rho^\alpha}{T^\rho}\left|\phi(\mu)\right|.
$$

Then problem (1) *has at least one solution in* A_r.

Proof Let $a_1 = a_2 = 0$ and $U(\mu) = |\phi|$. Then, taking into account Theorem 4.3, the result holds. □

Example 4.7 Consider the following fractional differential equation:

$$
\begin{cases}
{}_0^{RC}D_\pi^{\frac{7}{4},2}u(\mu) = \frac{|u|}{(\mu+4)^2(1+|u|)}, & \mu \in [0,\pi], \\
u(0) = 0, \qquad u(\pi) = 1,
\end{cases}
$$

where $g(\mu, u) = \frac{|u|}{(\mu+4)^2(1+|u|)}$, $\alpha = \frac{7}{4}$, and $T = \pi$. Also, since $\|g(\mu,u) - g(\mu,v)\| \leq \lambda_1\|u - v\|$ with $\lambda_1 = \frac{1}{16}$, therefore Theorem 4.4 assures that the boundary value problem has a unique solution on $[0,\pi]$.

4.1 Dependence of solutions on the parameters

The stability analysis of fractional differential equations has been carried out by many mathematicians. For details, one can see [36, 39–42] and the references therein. The solutions satisfy various types of stability, and continuous dependence on the initial data is

one of them. This section demonstrates that the solution of problem (1) depends on the parameters α, ϕ_0, ϕ_T, and g provided that the function g satisfies conditions (H_2^*) and (H_2^*). Continuous dependence of solutions on the parameters indicates the stability of solutions.

Theorem 4.8 *Assume that $\phi_1(\eta)$ is the solution of BVP (1) and $\phi_2(\eta)$ is the solution of the following problem*:

$$\begin{cases} {}_0^{RC}D_T^{\alpha-\epsilon,\rho}\phi(\mu) = g(\mu, \phi(\mu), {}_0^{RC}D_T^{\alpha^*,\rho}\phi(\mu)), \quad \mu \in [0,T], \\ \phi(0) = \phi_0, \qquad \phi(T) = \phi_T, \end{cases}$$

where $1 < \alpha - \epsilon < \alpha \leq 2$, $0 < \alpha^* \leq 1$, and g is continuous. Then $\|\phi_1 - \phi_2\| = \mathrm{O}(\varepsilon)$.

Proof Using equation (3), we have

$$\begin{aligned} &\left|\phi_1(\mu) - \phi_2(\mu)\right| \\ &\quad \leq \left|g\left(\eta, \phi_1(\eta), {}_0^{RC}D_T^{\alpha^*,\rho}\phi_1(\eta)\right) - g\left(\eta, \phi_2(\eta), {}_0^{RC}D_T^{\alpha^*,\rho}\phi_2(\eta)\right)\right| \\ &\qquad \times \left\{ \frac{(T^{\alpha\rho} + \mu^{\alpha\rho} + (T^\rho - \mu^\rho)^\alpha)}{\rho^\alpha \Gamma(\alpha+1)} + \frac{(T^{\rho(\alpha-\varepsilon)} + \mu^{\rho(\alpha-\varepsilon)} + (T^\rho - \mu^\rho)^{\alpha-\varepsilon})}{\rho^{\alpha-\varepsilon}\Gamma(\alpha-\varepsilon+1)} \right\} \\ &\quad \leq \lambda_1 \left\{ \frac{(T^{\alpha\rho} + \mu^{\alpha\rho} + (T^\rho - \mu^\rho)^\alpha)}{\rho^\alpha \Gamma(\alpha+1)} + \frac{(T^{\rho(\alpha-\varepsilon)} + \mu^{\rho(\alpha-\varepsilon)} + (T^\rho - \mu^\rho)^{\alpha-\varepsilon})}{\rho^{\alpha-\varepsilon}\Gamma(\alpha-\varepsilon+1)} \right\} \\ &\qquad \times \|\phi_1 - \phi_2\| = \mathrm{O}(\varepsilon). \end{aligned}$$

Also

$$\begin{aligned} \left|{}_0^{RC}D_T^{\alpha^*,\rho}\left(\phi_1(\mu)\right) - {}_0^{RC}D_T^{\alpha^*,\rho}\left(\phi_2(\mu)\right)\right| &\leq \lambda_1 \|\phi_1 - \phi_2\| \left(H(\mu) + H(\mu,\varepsilon)\right) \\ &= \mathrm{O}(\varepsilon), \end{aligned}$$

where

$$\begin{aligned} H(\mu) = {} & \frac{\rho^{(\alpha^*-\alpha)} T^{\rho(\alpha-1)} \mu^{\rho(1-\alpha^*)}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)} + \frac{\rho^{(\alpha^*-\alpha)} \mu^{(\alpha-\alpha^*)}}{2\Gamma(2-\alpha^*+1)} \\ & + \frac{\rho^{(\alpha^*-\alpha)} T^{\rho(\alpha-1)} (T^\rho - \mu^\rho)^{1-\alpha^*}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)} + \frac{\rho^{(\alpha^*-\alpha)} (T^\rho - \mu^\rho)^{\alpha-\alpha^*}}{2\Gamma(\alpha-\alpha^*+1)} \end{aligned}$$

and

$$\begin{aligned} H(\mu,\varepsilon) = {} & \frac{\rho^{(\alpha^*-\alpha-\varepsilon)} T^{\rho(\alpha-\varepsilon-1)} \mu^{\rho(1-\alpha^*)}}{2\Gamma(\alpha-\varepsilon+1)\Gamma(2-\alpha^*)} + \frac{\rho^{(\alpha^*-\alpha-\varepsilon)} \mu^{(\alpha-\varepsilon-\alpha^*)}}{2\Gamma(2-\alpha^*+1)} \\ & + \frac{\rho^{(\alpha^*-\alpha-\varepsilon)} T^{\rho(\alpha-\varepsilon-1)} (T^\rho - \mu^\rho)^{1-\alpha^*}}{2\Gamma(\alpha-\varepsilon+1)\Gamma(2-\alpha^*)} + \frac{\rho^{(\alpha^*-\alpha-\varepsilon)} (T^\rho - \mu^\rho)^{\alpha-\varepsilon-\alpha^*}}{2\Gamma(\alpha-\alpha^*+1)}. \end{aligned}$$

This completes the proof. □

Theorem 4.9 *Assume that the conditions of Theorem 4.4 hold and if $\phi_1(\eta)$ is the solution of BVP (1) and $\phi_2(\eta)$ is the solution of the following problem*:

$$\begin{cases} {}^{RC}_{0}D_{T}^{\alpha,\rho}\phi(\mu) = g(\mu,\phi(\mu),{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\phi(\mu)), \quad \mu\in[0,T] \\ \phi(0) = \phi_0+\varepsilon_1, \qquad \phi(T) = \phi_T+\varepsilon_2, \end{cases}$$

then $\|\phi_1-\phi_2\| = \mathrm{O}(\max\{\varepsilon_1,\varepsilon_2\})$.

Proof We have

$$\begin{aligned} &\big|\phi_1(\mu)-\phi_2(\mu)\big| \\ &\quad\le \frac{(\varepsilon_1+\varepsilon_2)\mu^\rho}{2}+\frac{(\varepsilon_1+\varepsilon_2)}{2T^\rho} \\ &\qquad + \frac{|g(\eta,\phi_1(\eta),{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\phi_2(\eta))|(T^{\alpha\rho}+\mu^{\alpha\rho}+(T^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)} \\ &\quad\le \frac{(\varepsilon_1+\varepsilon_2)\mu^\rho}{2}+\frac{(\varepsilon_1+\varepsilon_2)}{2T^\rho}+\frac{\lambda_1(T^{\alpha\rho}+\mu^{\alpha\rho}+(T^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)}\big\|\phi_1(\mu)-\phi_2(\mu)\big\| \\ &\quad= \mathrm{O}\big(\{\varepsilon_1,\varepsilon_2\}\big). \end{aligned}$$

This gives the desired result. □

Theorem 4.10 *Assume that $\phi_1(\eta)$ is the solution of BVP (1) and $\phi_2(\eta)$ is the solution of the following problem*:

$$\begin{cases} {}^{RC}_{0}D_{T}^{\alpha,\rho}\phi(\mu) = g(\mu,\phi(\mu),{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\phi(\mu))+\epsilon, \quad \mu\in[0,T] \\ \phi(0) = \phi_0, \qquad \phi(T) = \phi_T, \end{cases}$$

where $1<\alpha-\epsilon<\alpha\le 2$ and $0<\alpha^*\le 1$ and g is continuous. Then $\|\phi_1-\phi_2\| = \mathrm{O}(\varepsilon)$.

Proof From Lemma 4.2, we have

$$\begin{aligned} &\big|\phi_1(\mu)-\phi_2(\mu)\big| \\ &\quad\le \big|g\big(\eta,\phi_1(\eta),{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\phi_1(\eta)\big)-g\big(\eta,\phi_2(\eta),{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\phi_2(\eta)\big)\big| \\ &\qquad\times\left(\frac{(T^{\alpha\rho}+\mu^{\alpha\rho}+(T^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)}\right)+\frac{\varepsilon(T^{\alpha\rho}+\mu^{\alpha\rho}+(T^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)} \\ &\quad\le \big\{\lambda_1\big(\big|\phi_1(\mu)-\phi_2(\mu)\big|+\big|{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\phi_1(\eta)\big)-{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\phi_2(\eta))\big|+\varepsilon)\big\} \\ &\qquad\times\frac{(T^{\alpha\rho}+\mu^{\alpha\rho}+(T^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)} \\ &\quad\le \frac{(T^{\alpha\rho}+\mu^{\alpha\rho}+(T^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)}\big\{\lambda_1\|\phi_1-\phi_2\|+\varepsilon\big\} = \mathrm{O}(\varepsilon). \end{aligned}$$

Moreover,

$$\big|{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\big(\tilde{T}\phi_1(\mu)\big)-{}^{RC}_{0}D_{T}^{\alpha^*,\rho}\big(\tilde{T}\phi_2(\mu)\big)\big|\le H(\mu)\big\{\lambda_1\|\phi_1-\phi_2\|+\varepsilon\big\} = \mathrm{O}(\varepsilon),$$

where

$$H(\mu) = \frac{\rho^{(\alpha^*-\alpha)}T^{\rho(\alpha-1)}\mu^{\rho(1-\alpha^*)}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)} + \frac{\rho^{(\alpha^*-\alpha)}\mu^{(\alpha-\alpha^*)}}{2\Gamma(2-\alpha^*+1)} + \frac{\rho^{(\alpha^*-\alpha)}T^{\rho(\alpha-1)}(T^\rho-\mu^\rho)^{1-\alpha^*}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)} + \frac{\rho^{(\alpha^*-\alpha)}(T^\rho-\mu^\rho)^{\alpha-\alpha^*}}{2\Gamma(\alpha-\alpha^*+1)}.$$

This completes the proof. □

5 Concluding remarks

We presented a generalization of the Riesz fractional operator in this work. We provided some results and inequalities for the new generalized Riesz fractional operators. Furthermore, we proved some equivalence results for the nonlinear fractional differential equation involving the generalized Riesz derivative operator. By using suitable fixed point theorems, we provided the uniqueness of solution of the problem and some several mathematical techniques. Also, we discussed the stability of solutions and showed continuous dependence onto given parameters. An instructive comparison with literature shows that these results present the generalization of various old theorems in the related areas.

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References

1. Katugampola, U.N.: A new approach to generalized fractional derivatives. Bull. Math. Anal. Appl. **6**(4), 1–15 (2014)
2. Alqahtani, B., Aydi, H., Karapinar, E., Rakocevic, V.: A solution for Volterra fractional integral equations by hybrid contractions. Mathematics **7**(8), 694 (2019). https://doi.org/10.3390/math7080694

3. Rezapour, S., Ben Chikh, S., Amara, A., Ntouyas, S.K., Tariboon, J., Etemad, S.: Existence results for Caputo–Hadamard nonlocal fractional multi-order boundary value problems. Mathematics **9**(7), 719 (2021). https://doi.org/10.3390/math9070719
4. Amara, A., Etemad, S., Rezapour, S.: Approximate solutions for a fractional hybrid initial value problem via the Caputo conformable derivative. Adv. Differ. Equ. **2020**, 608 (2020). https://doi.org/10.1186/s13662-020-03072-3
5. Baleanu, D., Etemad, S., Rezapour, S.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. **2020**, 64 (2020). https://doi.org/10.1186/s13661-020-01361-0
6. Baleanu, D., Etemad, S., Rezapour, S.: On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators. Alex. Eng. J. **59**(5), 3019–3027 (2020). https://doi.org/10.1016/j.aej.2020.04.053
7. Herrmann, R.: Fractional Calculus: An Introduction for Physicists. World Scientific, Singapore (2011)
8. Karapinar, E., Fulga, A., Rashid, M., Shahid, L., Aydi, H.: Large contractions on quasi-metric spaces with an application to nonlinear fractional differential equations. Mathematics **7**(5), 444 (2019). https://doi.org/10.3390/math7050444
9. Matar, M.M., Lubbad, A.A., Alzabut, J.: On *p*-Laplacian boundary value problems involving Caputo–Katugampula fractional derivatives. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6534
10. Pratap, A., Raja, R., Alzabut, J., Dianavinnarasi, J., Cao, J., Rajchakit, G.: Finite-time Mittag-Leffler stability of fractional order quaternion-valued memristive neural networks with impulses. Neural Process. Lett. **51**, 1485–1526 (2020). https://doi.org/10.1007/s11063-019-10154-1
11. Seemab, A., Rehman, M.U., Alzabut, J., Hamdi, A.: On the existence of positive solutions for generalized fractional boundary value problems. Bound. Value Probl. **2019**, 186 (2019). https://doi.org/10.1186/s13661-019-01300-8
12. Shukla, A.K., Prajapati, J.C.: On a generalization of Mittag-Leffler function and its properties. J. Math. Anal. Appl. **336**(2), 797–811 (2007). https://doi.org/10.1016/j.jmaa.2007.03.018
13. Jajarmi, A., Baleanu, D.: On the fractional optimal control problems with a general derivative operator. Asian J. Control **23**(2), 1062–1071 (2021). https://doi.org/10.1002/asjc.2282
14. Sajjadi, S.S., Baleanu, D., Jajarmi, A., Pirouz, H.M.: A new adaptive synchronization and hyperchaos control of a biological snap oscillator. Chaos Solitons Fractals **138**, 109919 (2020). https://doi.org/10.1016/j.chaos.2020.109919
15. Patil, J., Chaudhari, A., Abdo, M., Hardan, B.: Upper and lower solution method for positive solution of generalized Caputo fractional differential equations. Adv. Theory of Nonlinear Anal. Appl. **4**(4), 279–291 (2020). https://doi.org/10.31197/atnaa.709442
16. Rezapour, S., Ntouyas, S.K., Amara, A., Etemad, S., Tariboon, J.: Some existence and dependence criteria of solutions to a fractional integro-differential boundary value problem via the generalized Gronwall inequality. Mathematics **9**(11), 1165 (2021) https://doi.org/10.3390/math9111165
17. Jarad, F., Abdeljawad, T.: A modified Laplace transform for certain generalized fractional operators. Results Nonlinear Anal. **1**(2), 88–98 (2019)
18. Marino, G., Scardamglia, B., Karapinar, E.: Strong convergence theorem for strict pseudo-contractions in Hilbert spaces. J. Inequal. Appl. **2016**, 134 (2016). https://doi.org/10.1186/s13660-016-1072-6
19. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
20. Almeida, R.: Fractional variational problems with the Riesz–Caputo derivative. Appl. Math. Lett. **25**(2), 142–148 (2012). https://doi.org/10.1016/j.aml.2011.08.003
21. Mandelbrot, B.B.: The Fractal Geometry of Nature. Freeman, San Francisco (1982)
22. Muslih, S.I., Agrawal, O.P.: Riesz fractional derivatives and fractional dimensional space. Int. J. Theor. Phys. **49**, 270–275 (2010). https://doi.org/10.1007/s10773-009-0200-1
23. Ding, H.F., Li, C.P.: High-order algorithms for Riesz derivative and their applications (III). Fract. Calc. Appl. Anal. **19**(1), 19–55 (2016). https://doi.org/10.1515/fca-2016-0003
24. Ding, H.F., Li, C.P.: High-order algorithms for Riesz derivative and their applications (V). Numer. Methods Partial Differ. Equ. **33**(5), 1754–1794 (2017). https://doi.org/10.1002/num.22169
25. Ding, H.F., Li, C.P., Chen, Y.Q.: High-order algorithms for Riesz derivative and their applications (II). J. Comput. Phys. **293**, 218–237 (2016). https://doi.org/10.1016/j.jcp.2014.06.007
26. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, S.: A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative. Chaos Solitons Fractals **134**, 109705 (2020). https://doi.org/10.1016/j.chaos.2020.109705
27. Mohammadi, H., Kumar, S., Rezapour, S., Etemad, S.: A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. Chaos Solitons Fractals **144**, 110668 (2021). https://doi.org/10.1016/j.chaos.2021.110668
28. Saha Ray, S.: A new numerical approach for single rational soliton solution of Chen–Lee–Liu equation with Riesz fractional derivative in optical fibers. Math. Methods Appl. Sci. **42**(1), 99–114 (2019). https://doi.org/10.1002/mma.5326
29. Wu, G.C., Baleanu, D., Deng, Z.G., Zeng, S.D.: Lattice fractional diffusion equation in terms of a Riesz–Caputo difference. Phys. A, Stat. Mech. Appl. **438**(2), 335–339 (2007). https://doi.org/10.1016/j.physa.2015.06.024
30. Chen, F., Baleanu, D., Wu, G.: Existence results of fractional differential equations with Riesz–Caputo derivative. Eur. Phys. J. Spec. Top. **226**, 3411–3425 (2017). https://doi.org/10.1140/epjst/e2018-00030-6
31. Zhang, W., Liu, W.: Existence of solutions for fractional differential equations with infinite point boundary conditions at resonance. Bound. Value Probl. **2018**, Article ID 36 (2018). https://doi.org/10.1186/s13661-018-0954-6
32. Agrawal, O.P.: Fractional variational calculus in terms of Riesz fractional derivatives. J. Phys. A, Math. Theor. **40**(24), 6287 (2007). https://doi.org/10.1088/1751-8113/40/24/003
33. Agrawal, O.P.: Generalized variational problems and Euler–Lagrange equations. Comput. Math. Appl. **59**(5), 1852–1864 (2010). https://doi.org/10.1016/j.camwa.2009.08.029
34. Katugampola, U.N.: New approach to a generalized fractional integral. Appl. Math. Comput. **218**(3), 860–865 (2011). https://doi.org/10.1016/j.amc.2011.03.062
35. Oliveira, D.S., Capelas de Oliveira, E.: On a Caputo-type fractional derivative. Adv. Pure Appl. Math. **10**(2), 81–91 (2018). https://doi.org/10.1515/apam-2017-0068

36. Diethelm, K.A.: The Analysis of Fractional Differential Equations. Springer, Berlin (2010)
37. Almeida, R., Malinowska, A.B., Odzijewicz, T.: Fractional differential equations with dependence on the Caputo–Katugampola derivative. J. Comput. Nonlinear Dyn. **11**(6), 061017 (2016). https://doi.org/10.1115/1.4034432
38. Almeida, R.: A Gronwall inequality for a general Caputo fractional operator. Math. Inequal. Appl. **20**(4), 1089–1105 (2017). https://doi.org/10.7153/mia-2017-20-70
39. Agarwal, R., Hristova, S., O'Regan, D.: Some stability properties related to initial time difference for Caputo fractional differential equations. Fract. Calc. Appl. Anal. **21**(1), 72–93 (2018). https://doi.org/10.1515/fca-2018-0005
40. Ahmad, B., Sivasundaram, S.: Dynamics and stability of impulsive hybrid set-valued integro-differential equations. Nonlinear Anal., Theory Methods Appl. **65**(11), 2082–2093 (2006). https://doi.org/10.1016/j.na.2005.11.055
41. Ben-Chikh, S., Amara, A., Etemad, S., Rezapour, S.: On Hyers–Ulam stability of a multi-order boundary value problems via Riemann–Liouville derivatives and integrals. Adv. Differ. Equ. **2020**, 547 (2020). https://doi.org/10.1186/s13662-020-03012-1
42. Odibat, Z.M.: Analytic study on linear systems of fractional differential equations. Comput. Math. Appl. **59**(3), 1171–1183 (2010). https://doi.org/10.1016/j.camwa.2009.06.035