

RESEARCH

Open Access



Some new Opial type dynamic inequalities via convex functions and applications

S.H. Saker¹, J. Alzabut^{2,3*} , A.G. Sayed⁴ and D. O'Regan⁵

*Correspondence:
jalzabut@psu.edu.sa

²Department of Mathematics and General Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia

³Group of Mathematics, Faculty of Engineering, OSTIM Technical University, 06374 Ankara, Turkey
Full list of author information is available at the end of the article

Abstract

In this paper, we prove some new Opial-type dynamic inequalities on time scales. Our results are obtained in frame of convexity property and by using the chain rule and Jensen and Hölder inequalities. For illustration purpose, we obtain some particular Opial-type inequalities reported in the literature.

MSC: Primary 26A15; 26A51; 26D10; secondary 26D15; 39A13; 34A40

Keywords: Opial's inequality; Hölder's inequality; Jensen's inequality; Time scales

1 Introduction

The theory of convex analysis has become one of the most significant fields of mathematics due to its widespread usefulness in diverse fields of pure and applied sciences. The concept of convexity has been utilized in several directions using innovative techniques to study and unify different problems. Consequently, many new inequalities associated with convex functions have been derived by many researchers [6–8, 10].

On other direction, integral inequalities on time scale have been a topic of debate amongst interested researchers. Due to their numerous application potentials, several variants have been established by many authors; see for instance [1, 2]. One of the most attractive inequalities that engaged many researchers is the Opial inequality [11]. During the last years, it has been realized that the Opial inequality and its generalizations play a fundamental role in establishing the existence-uniqueness and stability of initial and boundary value problems for various types of differential equations [12, 14]. For the sake of completeness, we review some relevant results of Opial inequalities in the context of time scales calculus.

In [3], the authors proved some dynamic inequalities of Opial type on time scales. One of the results states that: If $\Pi : [0, a] \cap \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable with $\Pi(0) = 0$, then

$$\int_0^a |\Pi(t) + \Pi^\sigma(t)| |\Pi^\Delta(t)| \Delta t \leq a \int_0^a |\Pi^\Delta(t)|^2 \Delta t. \quad (1)$$

© The Author(s) 2021. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Further, it was proved that, if η and ζ are positive rd-continuous functions on $[0, b]_{\mathbb{T}}$, ζ is non-increasing and $\Pi : [0, b] \cap \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable with $\Pi(0) = 0$, then

$$\int_0^b \zeta^\sigma(t) |(\Pi(t) + \Pi^\sigma(t)) \Pi^\Delta(t)| \Delta t \leq \int_0^b \frac{\Delta t}{\eta(t)} \int_0^b \eta(t) \zeta(t) |\Pi^\Delta(t)|^2 \Delta t. \tag{2}$$

In [9], the authors replaced ζ^σ with ζ and proved an inequality similar to (2) of the form

$$\int_a^b \zeta(t) |(\Pi(t) + \Pi^\sigma(t)) \Pi^\Delta(t)| \Delta t \leq K_\zeta(a, b) \int_a^b |\Pi^\Delta(t)|^2 \Delta t, \tag{3}$$

where ζ is a positive function on $[a, b]_{\mathbb{T}}$, $\Pi : [a, b] \cap \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable with $\Pi(a) = 0$, and

$$K_\zeta(a, b) = \left(2 \int_a^b \zeta^2(\eta) (\sigma(\eta) - a) \Delta \eta \right)^{1/2}. \tag{4}$$

On the other hand, the authors in [15, 16] proved that, if ζ is a positive and non-increasing function on $[a, b] \cap \mathbb{T}$, then

$$\int_a^b \zeta(t) |\Pi(t)|^v |\Pi^\Delta(t)|^\varrho \Delta t \leq \frac{\varrho}{v + \varrho} (b - a)^v \int_a^b \zeta(t) |\Pi^\Delta(t)|^{v+\varrho} \Delta t, \tag{5}$$

where $\Pi : [a, b] \cap \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable with $\Pi(a) = 0$.

In [13], the author generalized (5) and proved some new dynamic inequalities with two weight functions η and ζ . In particular, it was proved that, if η and ζ are non-negative functions on $[a, b]_{\mathbb{T}}$ such that

$$\int_a^b \eta^{-1/(v+\varrho-1)}(t) \Delta t < \infty$$

and $\Pi : [a, b] \cap \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable with $\Pi(a) = 0$, then

$$\int_a^b \zeta(t) |\Pi(t)|^v |\Pi^\Delta(t)|^\varrho \Delta t \leq K_1(a, b, v, \varrho) \int_a^b \eta(t) |\Pi^\Delta(t)|^{v+\varrho} \Delta t, \tag{6}$$

where

$$K_1(a, b, v, \varrho) = \left(\frac{\varrho}{v + \varrho} \right)^{\varrho/(v+\varrho)} \times \left(\int_a^b \frac{(\zeta(t))^{(v+\varrho)/v}}{(\eta(t))^{\varrho/v}} \left(\int_a^t \eta^{\frac{-1}{(v+\varrho-1)}}(s) \Delta s \right)^{(v+\varrho-1)} \Delta t \right)^{v/(v+\varrho)}. \tag{7}$$

The objective of this paper is to prove some new dynamic inequalities of Opial type on time scales by using the convexity property and Jensen inequality. Our results in particular cases yield some of the recent results reported on Opial-type inequalities.

The paper adheres to the following plan. In Sect. 2, we present some essential preliminaries on time scales as well as some fundamental inequalities. In Sect. 3, we prove the main results of the paper. For illustration, we derive some particular cases of the main results in Sect. 4.

2 Preliminaries on time scales

In this section, we assemble some definitions and concepts on the theory of time scales calculus. Further, some basic inequalities in the context of time scales are stated. For more details, we refer the reader to the two pioneering monographs [4, 5].

A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} . We assume throughout that \mathbb{T} has the topology that it inherits from the standard topology on the real numbers \mathbb{R} . We define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(\theta) := \inf\{s \in \mathbb{T} : s > \theta\}$, for $\theta \in \mathbb{T}$. The mapping $\mu : \mathbb{T} \rightarrow [0, \infty)$ defined by $\mu(\theta) := \sigma(\theta) - \theta$ is called the graininess of \mathbb{T} .

A point $\theta \in \mathbb{T}$ is said to be right-dense and right-scattered, if $\sigma(\theta) = \theta$ and $\sigma(\theta) > \theta$, respectively. A function $\eta : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd*-continuous provided it is continuous at all right-dense points in \mathbb{T} and its left-sided limits exist (finite) at all left-dense points in \mathbb{T} . The set of all such *rd*-continuous functions is denoted by $C_{rd}(\mathbb{T})$. We define $\eta^\sigma := \eta \circ \sigma$ and define the time scale interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$.

Fix $\theta \in \mathbb{T}$ and let $\eta : \mathbb{T} \rightarrow \mathbb{R}$. Define $\eta^\Delta(\theta)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood s of θ with

$$|[\eta^\sigma(\theta) - \eta(s)] - \eta^\Delta(\theta)[\sigma(\theta) - s]| \leq \epsilon |\sigma(\theta) - s|, \quad \text{for } s \in \mathbb{T}.$$

In this case, we say $\eta^\Delta(\theta)$ is the (delta) derivative of η at θ and that η is (delta) differentiable at θ . We will make use of the following product and quotient rules for the derivative of the product $\eta\zeta$ and the quotient η/ζ (where $\zeta\zeta^\sigma \neq 0$, here $\zeta^\sigma := \zeta \circ \sigma$) of two differentiable function η and ζ :

$$(\eta\zeta)^\Delta = \eta^\Delta\zeta + \eta^\sigma\zeta^\Delta = \eta\zeta^\Delta + \eta^\Delta\zeta^\sigma, \quad \left(\frac{\eta}{\zeta}\right)^\Delta = \frac{\eta^\Delta\zeta - \eta\zeta^\Delta}{\zeta\zeta^\sigma}.$$

For $a, b \in \mathbb{T}$ and a delta differentiable function η , the Cauchy integral of η^Δ is defined by $\int_a^b \eta^\Delta(\theta)\Delta\theta = \eta(b) - \eta(a)$. The integration by parts formula on time scales is given by

$$\int_a^b \eta(\theta)\zeta^\Delta(\theta)\Delta\theta = \eta(\theta)\zeta(\theta)|_a^b - \int_a^b \eta^\Delta(\theta)\zeta^\sigma(\theta)\Delta\theta.$$

The chain rule formula (see [4, Theorem 1.87]) for appropriate functions η and ζ is given as

$$(\eta \circ \zeta)^\Delta(\theta) = \eta'(\zeta(d))\zeta^\Delta(\theta), \quad \text{where } d \in [\theta, \sigma(\theta)]. \tag{8}$$

However, we may define another chain rule by

$$(\eta \circ \zeta)^\Delta(\theta) = \left\{ \int_0^1 \eta'(h\zeta^\sigma(\theta) + (1-h)\zeta(\theta)) dh \right\} \zeta^\Delta(\theta), \tag{9}$$

which is a simple consequence of Keller’s chain rule ([4, Theorem 1.90]). Hölder’s inequality (see [4, Theorem 6.13]) on time scales is given by

$$\int_a^b |\eta(\theta)\zeta(\theta)|\Delta\theta \leq \left[\int_a^b |\eta(\theta)|^p \Delta\theta \right]^{\frac{1}{p}} \left[\int_a^b |\zeta(\theta)|^q \Delta\theta \right]^{\frac{1}{q}}, \tag{10}$$

where $a, b \in \mathbb{T}$ and $\eta, \zeta \in C_{rd}(\mathbb{I}, \mathbb{R})$, $p > 1$ and $1/p + 1/q = 1$. The particular case when $p = q = 2$ yields the Cauchy–Schwarz inequality,

$$\int_a^b |\eta(\theta)\zeta(\theta)| \Delta\theta \leq \left[\int_a^b |\eta(\theta)|^2 \Delta\theta \right]^{\frac{1}{2}} \left[\int_a^b |\zeta(\theta)|^2 \Delta\theta \right]^{\frac{1}{2}}. \tag{11}$$

On the other hand, Jensen’s inequality on time scales [4, Theorem 6.17] is given by

$$\phi\left(\frac{\int_a^b \zeta(\theta)\Phi(\theta)\Delta\theta}{\int_a^b \Phi(\theta)\Delta\theta}\right) \leq \frac{\int_a^b \phi(\zeta(\theta))\Phi(\theta)\Delta\theta}{\int_a^b \Phi(\theta)\Delta\theta}, \tag{12}$$

where $a, b \in \mathbb{T}$, $c, d \in \mathbb{R}$, $\zeta, \Phi \in C_{rd}([a, b]_{\mathbb{T}}, (c, d))$ and $\phi \in C((c, d), \mathbb{R})$ is a convex function.

3 Main results

In this section, we state and prove the main results. Throughout this paper we assume that the appropriate functions are delta differentiable and the integrals in the statements of the theorems are assumed to exist.

Theorem 3.1 *Let \mathbb{T} be a time scale with $d, \tau \in \mathbb{T}$ and assume F is non-negative and increasing on $[0, \infty)$. If F is convex and $\zeta : [d, \tau]_{\mathbb{T}} \rightarrow \mathbb{R}$ is such that $\zeta(d) = 0$, then*

$$\int_d^\tau F^\Delta(|\zeta(t)|) \Delta t \leq F\left(\int_d^\tau |\zeta^\Delta(t)| \Delta t\right). \tag{13}$$

Proof Let $\eta(\theta) = \int_d^\theta |\zeta^\Delta(s)| \Delta s$, for $\theta \in [d, \tau]_{\mathbb{T}}$. Then $\eta^\Delta(\theta) = |\zeta^\Delta(\theta)|$, and

$$\eta(\theta) = \int_d^\theta |\zeta^\Delta(s)| \Delta s \geq \left| \int_d^\theta \zeta^\Delta(s) \Delta s \right| = |\zeta(\theta)|, \quad \text{for } \theta \in [d, \tau]_{\mathbb{T}}.$$

This implies that

$$h\eta^\sigma(\theta) + (1-h)\eta(\theta) \geq h|\zeta^\sigma(\theta)| + (1-h)|\zeta(\theta)|, \quad \text{for } h \in (0, 1), \tag{14}$$

and

$$\left(|\zeta(\theta)|\right)^\Delta = \left(\left|\int_d^\theta \zeta^\Delta(s) \Delta s\right|\right)^\Delta \leq \left(\int_d^\theta |\zeta^\Delta(s)| \Delta s\right)^\Delta = |\zeta^\Delta(\theta)|. \tag{15}$$

Applying the chain rule (9), we see that

$$\int_d^\tau F^\Delta(w(\theta)) \Delta\theta = \int_d^\tau \left\{ \int_0^1 F'(hw^\sigma(\theta) + (1-h)w(\theta)) dh \right\} (w(\theta))^\Delta \Delta\theta.$$

Replacing $w(\theta)$ with $|\zeta(\theta)|$ and using (15), we have

$$\begin{aligned} \int_d^\tau F^\Delta(|\zeta(\theta)|) \Delta\theta &= \int_d^\tau \left\{ \int_0^1 F'(h|\zeta^\sigma(\theta)| + (1-h)|\zeta(\theta)|) dh \right\} (|\zeta(\theta)|)^\Delta \Delta\theta \\ &\leq \int_d^\tau \left\{ \int_0^1 F'(h|\zeta^\sigma(\theta)| + (1-h)|\zeta(\theta)|) dh \right\} |\zeta^\Delta(\theta)| \Delta\theta. \end{aligned}$$

Now using (14), we get (note F is a convex function)

$$\begin{aligned} \int_d^\tau F^\Delta(|\zeta(\theta)|)\Delta\theta &\leq \int_d^\tau \left\{ \int_0^1 F'(h|\zeta^\sigma(\theta)| + (1-h)|\zeta(\theta)|) dh \right\} |\zeta^\Delta(\theta)|\Delta\theta \\ &\leq \int_d^\tau \left\{ \int_0^1 F'(h\eta^\sigma(\theta) + (1-h)\eta(\theta)) dh \right\} \eta^\Delta(\theta)\Delta\theta \\ &= \int_d^\tau F^\Delta(\eta(\theta))\Delta\theta = F(\eta(\tau)) - F(\eta(d)) \\ &= F(\eta(\tau)) - F(0) \leq F\left(\int_d^\tau |\zeta^\Delta(\theta)|\Delta\theta\right), \end{aligned}$$

which is (13). The proof is complete. □

Theorem 3.2 *Let \mathbb{T} be a time scale with $\tau, \ell \in \mathbb{T}$ and assume that F is non-negative and increasing on $[0, \infty)$. If F is convex and $\zeta : [\tau, \ell]_{\mathbb{T}} \rightarrow \mathbb{R}$ is such that $\zeta(\ell) = 0$, then*

$$\int_\tau^\ell F^\Delta(|\zeta(t)|)\Delta t \leq F\left(\int_\tau^\ell |\zeta^\Delta(t)|\Delta t\right). \tag{16}$$

Proof Let $\eta(\theta) = \int_\theta^\ell |\zeta^\Delta(s)|\Delta s$, for $\theta \in [\tau, \ell]_{\mathbb{T}}$. Then $\eta^\Delta(\theta) = -|\zeta^\Delta(\theta)|$, and

$$\eta(\theta) = \int_\theta^\ell |\zeta^\Delta(s)|\Delta s \geq \left| \int_\theta^\ell \zeta^\Delta(s)\Delta s \right| = |\zeta(\theta)|, \quad \text{for } \theta \in [\tau, \ell]_{\mathbb{T}}.$$

This implies that

$$h\eta^\sigma(\theta) + (1-h)\eta(\theta) \geq h|\zeta^\sigma(\theta)| + (1-h)|\zeta(\theta)|, \quad \text{for } h \in (0, 1), \tag{17}$$

and

$$\left(|\zeta(\theta)|\right)^\Delta = \left(\left|\int_\theta^\ell \zeta^\Delta(s)\Delta s\right|\right)^\Delta \leq \left(\int_\theta^\ell |\zeta^\Delta(s)|\Delta s\right)^\Delta = |\zeta^\Delta(\theta)|. \tag{18}$$

Applying the chain rule (9), we see that

$$\int_\tau^\ell F^\Delta(w(\theta))\Delta\theta = \int_\tau^\ell \left\{ \int_0^1 F'(hw^\sigma(\theta) + (1-h)w(\theta)) dh \right\} (w(\theta))^\Delta \Delta\theta.$$

Replacing $w(\theta)$ with $|\zeta(\theta)|$ and using (18), we have

$$\begin{aligned} \int_\tau^\ell F^\Delta(|\zeta(\theta)|)\Delta\theta &= \int_\tau^\ell \left\{ \int_0^1 F'(h|\zeta^\sigma(\theta)| + (1-h)|\zeta(\theta)|) dh \right\} \left(|\zeta(\theta)|\right)^\Delta \Delta\theta \\ &\leq \int_\tau^\ell \left\{ \int_0^1 F'(h|\zeta^\sigma(\theta)| + (1-h)|\zeta(\theta)|) dh \right\} |\zeta^\Delta(\theta)|\Delta\theta. \end{aligned}$$

Now using (17), we get (note F is non-negative and convex)

$$\int_\tau^\ell F^\Delta(|\zeta(\theta)|)\Delta\theta \leq \int_\tau^\ell \left\{ \int_0^1 F'(h|\zeta^\sigma(\theta)| + (1-h)|\zeta(\theta)|) dh \right\} |\zeta^\Delta(\theta)|\Delta\theta$$

$$\begin{aligned} &\leq - \int_{\tau}^{\ell} \left\{ \int_0^1 F'(h\eta^{\sigma}(\theta) + (1-h)\eta(\theta)) dh \right\} \eta^{\Delta}(\theta) \Delta\theta \\ &= - \int_{\tau}^{\ell} F^{\Delta}(\eta(\theta)) \Delta\theta = F(\eta(\tau)) - F(\eta(\ell)) \\ &= F(\eta(\tau)) - F(0) \leq F\left(\int_{\tau}^{\ell} |\zeta^{\Delta}(\theta)| \Delta\theta\right), \end{aligned}$$

which is (16). The proof is complete. □

If we assume that there exists $\tau \in (d, \ell)$ so that

$$\int_d^{\tau} |\zeta^{\Delta}(\theta)| \Delta\theta = \int_{\tau}^{\ell} |\zeta^{\Delta}(\theta)| \Delta\theta = \frac{1}{2} \int_d^{\ell} |\zeta^{\Delta}(\theta)| \Delta\theta, \tag{19}$$

then we have the following two results when $\zeta(d) = 0 = \zeta(\ell)$.

Theorem 3.3 *Let (19) be satisfied and \mathbb{T} be a time scale with $d, \ell \in \mathbb{T}$. Assume that F is non-negative and increasing on $[0, \infty)$. If F is convex and $\zeta : [d, \ell]_{\mathbb{T}} \rightarrow \mathbb{R}$ is such that $\zeta(d) = 0 = \zeta(\ell)$, then*

$$\int_d^{\ell} F^{\Delta}(|\zeta(\theta)|) \Delta\theta \leq 2F\left(\frac{1}{2} \int_d^{\ell} |\zeta^{\Delta}(\theta)| \Delta\theta\right). \tag{20}$$

Proof It follows that

$$\int_d^{\ell} F^{\Delta}(|\zeta(\theta)|) \Delta\theta = \int_d^{\tau} F^{\Delta}(|\zeta(\theta)|) \Delta\theta + \int_{\tau}^{\ell} F^{\Delta}(|\zeta(\theta)|) \Delta\theta.$$

Thus, using Theorems 3.1, 3.2 and (19), we obtain

$$\begin{aligned} \int_d^{\ell} F^{\Delta}(|\zeta(\theta)|) \Delta\theta &= \int_d^{\tau} F^{\Delta}(|\zeta(\theta)|) \Delta\theta + \int_{\tau}^{\ell} F^{\Delta}(|\zeta(\theta)|) \Delta\theta \\ &\leq F\left(\int_d^{\tau} |\zeta^{\Delta}(\theta)| \Delta\theta\right) + F\left(\int_{\tau}^{\ell} |\zeta^{\Delta}(\theta)| \Delta\theta\right) \\ &= F\left(\frac{1}{2} \int_d^{\ell} |\zeta^{\Delta}(\theta)| \Delta\theta\right) + F\left(\frac{1}{2} \int_d^{\ell} |\zeta^{\Delta}(\theta)| \Delta\theta\right) \\ &= 2F\left(\frac{1}{2} \int_d^{\ell} |\zeta^{\Delta}(\theta)| \Delta\theta\right), \end{aligned}$$

which is the desired inequality (20). The proof is complete. □

Theorem 3.4 *Let (19) be satisfied and \mathbb{T} be a time scale with $d, \ell \in \mathbb{T}$. Assume that F is non-negative convex and increasing on $[0, \infty)$, and h is convex and increasing on $[0, \infty)$. Suppose η is positive on $[d, \ell]_{\mathbb{T}}$ and $\int_d^{\ell} \eta(t) \Delta t = 1$. If $\zeta : [d, \ell]_{\mathbb{T}} \rightarrow \mathbb{R}$ is such that $\zeta(d) = 0 = \zeta(\ell)$, then*

$$\int_d^{\ell} F^{\Delta}(|\zeta(\theta)|) \Delta\theta \leq 2F\left(h^{-1}\left(\int_d^{\ell} \eta(\theta) h\left(\frac{|\zeta^{\Delta}(\theta)|}{2\eta(\theta)}\right) \Delta\theta\right)\right). \tag{21}$$

Proof Since h is a convex function, then using Jensen’s inequality (12), we have

$$h\left(\frac{\int_d^\ell \eta(\theta) \left(\frac{|\zeta^\Delta(\theta)|}{2\eta(\theta)}\right) \Delta\theta}{\int_d^\ell \eta(\theta) \Delta\theta}\right) \leq \frac{\int_d^\ell \eta(\theta) h\left(\frac{|\zeta^\Delta(\theta)|}{2\eta(\theta)}\right) \Delta\theta}{\int_d^\ell \eta(\theta) \Delta\theta},$$

and since $\int_d^\ell \eta(\theta) \Delta\theta = 1$ and h is increasing we obtain

$$\frac{1}{2} \int_d^\ell |\zeta^\Delta(\theta)| \Delta\theta \leq h^{-1}\left(\int_d^\ell \eta(\theta) h\left(\frac{|\zeta^\Delta(\theta)|}{2\eta(\theta)}\right) \Delta\theta\right). \tag{22}$$

Finally, using the increasing behavior of F and substituting (22) into (20), we have

$$\begin{aligned} \int_d^\ell F^\Delta(|\zeta(\theta)|) \Delta\theta &\leq 2F\left(\frac{1}{2} \int_d^\ell |\zeta^\Delta(\theta)| \Delta\theta\right) \\ &\leq 2F\left(h^{-1}\left(\int_d^\ell \eta(\theta) h\left(\frac{|\zeta^\Delta(\theta)|}{2\eta(\theta)}\right) \Delta\theta\right)\right), \end{aligned}$$

which is (21). The proof is complete. □

4 Some applications

In this section, we use the main results of Sect. 3 to obtain Opial-type inequalities.

Theorem 4.1 *Let \mathbb{T} be a time scale with $d, \tau \in \mathbb{T}$ and $\varrho > 1$. If $\zeta : [d, \tau]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(d) = 0$, then*

$$\int_d^\tau |\zeta(\theta)|^v |\zeta^\Delta(\theta)|^\varrho \Delta\theta \leq \frac{\varrho(\tau - d)^v}{v + \varrho} \int_d^\tau |\zeta^\Delta(\theta)|^{v+\varrho} \Delta\theta, \quad \text{for } v \geq 0. \tag{23}$$

Proof Let $\eta(\theta) = \int_d^\theta |\zeta^\Delta(s)|^\varrho \Delta s$, for $\theta \in [d, \tau]_{\mathbb{T}}$. Then $\eta(d) = 0$ and

$$\eta^\Delta(\theta) = |\zeta^\Delta(\theta)|^\varrho > 0. \tag{24}$$

By Hölder’s inequality (10) with indices ϱ and $\varrho/(\varrho - 1)$, we get

$$\begin{aligned} |\zeta(\theta)| &= \left| \int_d^\theta \zeta^\Delta(s) \Delta s \right| \leq \int_d^\theta |\zeta^\Delta(s)| \Delta s \\ &\leq \left(\int_d^\theta \Delta s\right)^{\frac{\varrho-1}{\varrho}} \left(\int_d^\theta |\zeta^\Delta(s)|^\varrho \Delta s\right)^{\frac{1}{\varrho}} \leq (\theta - d)^{\frac{\varrho-1}{\varrho}} \eta^{\frac{1}{\varrho}}(\theta), \end{aligned}$$

which yields

$$|\zeta(\theta)|^v \leq (\theta - d)^{\frac{v(\varrho-1)}{\varrho}} \eta^{\frac{v}{\varrho}}(\theta). \tag{25}$$

Applying the chain rule (8), we obtain

$$\left(\eta^{\frac{v+\varrho}{\varrho}}(\theta)\right)^\Delta = \frac{v + \varrho}{\varrho} \eta^{\frac{v}{\varrho}}(d) \eta^\Delta(\theta), \quad \text{where } d \in [\theta, \sigma(\theta)].$$

Since $\eta^\Delta(\theta) > 0$, and $d \geq \theta$, we see that

$$\left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta)\right)^\Delta \geq \frac{\nu+\varrho}{\varrho} \eta^{\frac{\nu}{\varrho}}(\theta) \eta^\Delta(\theta). \tag{26}$$

By virtue of (24), (25) and (26) we have

$$\begin{aligned} \int_d^\tau |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta &\leq \int_d^\tau (\theta-d)^{\frac{\nu(\varrho-1)}{\varrho}} \eta^{\frac{\nu}{\varrho}}(\theta) \eta^\Delta(\theta) \Delta\theta \\ &\leq (\tau-d)^{\frac{\nu(\varrho-1)}{\varrho}} \int_d^\tau \eta^{\frac{\nu}{\varrho}}(\theta) \eta^\Delta(\theta) \Delta\theta \\ &\leq \frac{\varrho(\tau-d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu+\varrho} \int_d^\tau \left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta)\right)^\Delta \Delta\theta. \end{aligned}$$

Applying Theorem 3.1, by setting $F(\eta) = \eta^{\frac{\nu+\varrho}{\varrho}}$, we obtain

$$\begin{aligned} \int_d^\tau |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta &\leq \frac{\varrho(\tau-d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu+\varrho} \int_d^\tau \left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta)\right)^\Delta \Delta\theta \\ &\leq \frac{\varrho(\tau-d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu+\varrho} \left(\int_d^\tau \eta^\Delta(\theta) \Delta\theta\right)^{\frac{\nu+\varrho}{\varrho}} \\ &= \frac{\varrho(\tau-d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu+\varrho} \left(\int_d^\tau |\zeta^\Delta(\theta)|^\varrho \Delta\theta\right)^{\frac{\nu+\varrho}{\varrho}}. \end{aligned}$$

Now applying Hölder’s inequality (10) with indices $(\nu+\varrho)/\nu$ and $(\nu+\varrho)/\varrho$, we have

$$\int_d^\tau |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta \leq \frac{\varrho(\tau-d)^\nu}{\nu+\varrho} \int_d^\tau |\zeta^\Delta(\theta)|^{\nu+\varrho} \Delta\theta,$$

which is the desired inequality (23). The proof is complete. □

Theorem 4.2 *Let \mathbb{T} be a time scale with $d, \tau \in \mathbb{T}$. If $\zeta : [d, \tau]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(d) = 0$ then*

$$\int_d^\tau |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{(\tau-d)^\nu}{\nu+1} \int_d^\tau |\zeta^\Delta(\theta)|^{\nu+1} \Delta\theta, \quad \text{for } \nu \geq 0. \tag{27}$$

Proof Let $\eta(\theta) = \int_d^\theta |\zeta^\Delta(s)| \Delta s$, for $\theta \in [d, \tau]_{\mathbb{T}}$. Then $\eta(d) = 0$ and

$$|\zeta(\theta)| = \left| \int_d^\theta \zeta^\Delta(s) \Delta s \right| \leq \int_d^\theta |\zeta^\Delta(s)| \Delta s = \eta(\theta). \tag{28}$$

Applying the chain rule (8), we obtain

$$\left(\eta^{\nu+1}(\theta)\right)^\Delta = (\nu+1) \eta^\nu(d) \eta^\Delta(\theta), \quad \text{where } d \in [\theta, \sigma(\theta)].$$

Since $\eta^\Delta(\theta) > 0$, and $d \geq \theta$, we see that

$$\left(\eta^{\nu+1}(\theta)\right)^\Delta = (\nu+1) \eta^\nu(d) \eta^\Delta(\theta) \geq (\nu+1) \eta^\nu(\theta) \eta^\Delta(\theta). \tag{29}$$

Now, from (28) and (29) we have

$$\begin{aligned} \int_d^\tau |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta &\leq \int_d^\tau \eta^\nu(\theta) \eta^\Delta(\theta) \Delta\theta \\ &\leq \int_d^\tau \eta^\nu(d) \eta^\Delta(\theta) \Delta\theta \leq \frac{1}{\nu+1} \int_d^\tau (\eta^{\nu+1}(\theta))^\Delta \Delta\theta. \end{aligned}$$

Applying Theorem 3.1, by setting $F(\eta) = \eta^{\nu+1}$, we obtain

$$\begin{aligned} \int_d^\tau |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta &\leq \frac{1}{\nu+1} \int_d^\tau (\eta^{\nu+1}(\theta))^\Delta \Delta\theta \\ &\leq \frac{1}{\nu+1} \left(\int_d^\tau \eta^\Delta(\theta) \Delta\theta \right)^{\nu+1} = \frac{1}{\nu+1} \left(\int_d^\tau |\zeta^\Delta(\theta)| \Delta\theta \right)^{\nu+1}. \end{aligned}$$

Now applying Hölder’s inequality (10) with indices $(\nu + 1)$ and $(\nu + 1)/\nu$, we have

$$\int_d^\tau |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{(\tau - d)^\nu}{\nu + 1} \int_d^\tau |\zeta^\Delta(\theta)|^{\nu+1} \Delta\theta,$$

which is (27). The proof is complete. □

Remark 4.1 If $\nu = 1$, then inequality (27) becomes the Olech inequality

$$\int_d^\tau |\zeta(\theta)| |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{(\tau - d)}{2} \int_d^\tau |\zeta^\Delta(\theta)|^2 \Delta\theta.$$

Following the same arguments of the proofs of Theorem 4.1 and Theorem 4.2, one can prove the following theorems.

Theorem 4.3 *Let \mathbb{T} be a time scale with $\tau, \ell \in \mathbb{T}$ and $\varrho > 1$. If $\zeta : [\tau, \ell]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(\ell) = 0$ then*

$$\int_\tau^\ell |\zeta^\sigma(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta \leq \frac{\varrho(\ell - \tau)^\nu}{\nu + \varrho} \int_\tau^\ell |\zeta^\Delta(\theta)|^{\nu+\varrho} \Delta\theta, \quad \text{for } \nu \geq 0. \tag{30}$$

Theorem 4.4 *Let \mathbb{T} be a time scale with $\tau, \ell \in \mathbb{T}$. If $\zeta : [\tau, \ell]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(\ell) = 0$ then*

$$\int_\tau^\ell |\zeta^\sigma(t)|^\nu |\zeta^\Delta(t)| \Delta t \leq \frac{(\ell - \tau)^\nu}{\nu + 1} \int_\tau^\ell |\zeta^\Delta(t)|^{\nu+1} \Delta t, \quad \text{for } \nu \geq 0. \tag{31}$$

Remark 4.2 If $\nu = 1$, then inequality (31) becomes

$$\int_\tau^\ell |\zeta^\sigma(\theta)| |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{(\ell - \tau)}{2} \int_\tau^\ell |\zeta^\Delta(\theta)|^2 \Delta\theta.$$

Theorem 4.5 *Let \mathbb{T} be a time scale with $d, \tau \in \mathbb{T}$, $\varrho > 1$, and Φ be non-negative and non-increasing on $[d, \tau]_{\mathbb{T}}$. If $\zeta : [d, \tau]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(d) = 0$ then for $\nu \geq 0$*

$$\int_d^\tau \Phi(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta \leq \frac{\varrho(\tau - d)^\nu}{\nu + \varrho} \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+\varrho} \Delta\theta. \tag{32}$$

Proof Let $\eta(\theta) = \int_d^\theta \Phi^{\frac{\varrho}{\nu+\varrho}}(s) |\zeta^\Delta(s)|^\varrho \Delta s$, for $\theta \in [d, \tau]_{\mathbb{T}}$. Then $\eta(d) = 0$ and

$$\eta^\Delta(\theta) = \Phi^{\frac{\varrho}{\nu+\varrho}}(\theta) |\zeta^\Delta(\theta)|^\varrho > 0. \tag{33}$$

Applying Hölder’s inequality (10) with indices ϱ and $\varrho/(\varrho - 1)$, we get

$$\begin{aligned} |\zeta(\theta)| &= \left| \int_d^\theta \zeta^\Delta(s) \Delta s \right| \leq \int_d^\theta |\zeta^\Delta(s)| \Delta s \\ &= \int_d^\theta \Phi^{\frac{-1}{\nu+\varrho}}(s) \Phi^{\frac{1}{\nu+\varrho}}(s) |\zeta^\Delta(s)| \Delta s \\ &\leq \left(\int_d^\theta (\Phi^{\frac{-1}{\nu+\varrho}}(s))^{\frac{\varrho}{\varrho-1}} \Delta s \right)^{\frac{\varrho-1}{\varrho}} \left(\int_d^\theta \Phi^{\frac{\varrho}{\nu+\varrho}}(s) |\zeta^\Delta(s)|^\varrho \Delta s \right)^{\frac{1}{\varrho}} \\ &\leq \Phi^{\frac{-1}{\nu+\varrho}}(\theta) (\theta - d)^{\frac{\varrho-1}{\varrho}} \eta^{\frac{1}{\varrho}}(\theta), \end{aligned}$$

which yields

$$\Phi^{\frac{\nu}{\nu+\varrho}}(s) |\zeta(\theta)|^\nu \leq (\theta - d)^{\frac{\nu(\varrho-1)}{\varrho}} \eta^{\frac{\nu}{\varrho}}(\theta). \tag{34}$$

By applying the chain rule (8), we obtain

$$\left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta) \right)^\Delta = \frac{\nu + \varrho}{\varrho} \eta^{\frac{\nu}{\varrho}}(d) \eta^\Delta(\theta), \quad \text{where } d \in [\theta, \sigma(\theta)].$$

Since $\eta^\Delta(\theta) = \Phi^{\frac{\varrho}{\nu+\varrho}}(\theta) |\zeta^\Delta(\theta)|^\varrho > 0$, and $d \geq \theta$, we see that

$$\left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta) \right)^\Delta \geq \frac{\nu + \varrho}{\varrho} \eta^{\frac{\nu}{\varrho}}(\theta) \eta^\Delta(\theta). \tag{35}$$

From (33), (34) and (35), we have

$$\begin{aligned} \int_d^\tau \Phi(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta \theta &= \int_d^\tau \Phi^{\frac{\nu}{\nu+\varrho}}(\theta) |\zeta(\theta)|^\nu \Phi^{\frac{\varrho}{\nu+\varrho}}(\theta) |\zeta^\Delta(\theta)|^\varrho \\ &\leq \int_d^\tau (\theta - d)^{\frac{\nu(\varrho-1)}{\varrho}} \eta^{\frac{\nu}{\varrho}}(\theta) \eta^\Delta(\theta) \Delta \theta \\ &\leq (\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}} \int_d^\tau \eta^{\frac{\nu}{\varrho}}(\theta) \eta^\Delta(\theta) \Delta \theta \\ &\leq \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \int_d^\tau \left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta) \right)^\Delta \Delta \theta. \end{aligned}$$

Applying Theorem 3.1, with $F(\eta) = \eta^{\frac{\nu+\varrho}{\varrho}}$, we obtain

$$\begin{aligned} \int_d^\tau \Phi(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta \theta &\leq \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \int_d^\tau \left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta) \right)^\Delta \Delta \theta \\ &\leq \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \left(\int_d^\tau \eta^\Delta(\theta) \Delta \theta \right)^{\frac{\nu+\varrho}{\varrho}} \end{aligned}$$

$$= \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \left(\int_d^\tau \Phi^{\frac{\varrho}{\nu+\varrho}}(\theta) |\zeta^\Delta(\theta)|^\varrho \Delta\theta \right)^{\frac{\nu+\varrho}{\varrho}}.$$

Now applying Hölder’s inequality (10) with indices $(\nu + \varrho)/\nu$ and $(\nu + \varrho)/\varrho$, we get

$$\int_d^\tau \Phi(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta \leq \frac{\varrho(\tau - d)^\nu}{\nu + \varrho} \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+\varrho} \Delta\theta,$$

which is the desired inequality (32). The proof is complete. □

Theorem 4.6 *Let \mathbb{T} be a time scale with $d, \tau \in \mathbb{T}$ and Φ be non-negative and non-increasing on $[d, \tau]_{\mathbb{T}}$. If $\zeta : [d, \tau]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(d) = 0$ then for $\nu \geq 0$*

$$\int_d^\tau \Phi(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{(\tau - d)^\nu}{\nu + 1} \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+1} \Delta\theta. \tag{36}$$

Proof Let $\eta(\theta) = \int_d^\theta \Phi^{\frac{1}{\nu+1}}(s) |\zeta^\Delta(s)| \Delta s$, for $\theta \in [d, \tau]_{\mathbb{T}}$. Then $\eta(d) = 0$ and

$$\begin{aligned} |\zeta(\theta)| &\leq \int_d^\theta |\zeta^\Delta(s)| \Delta s \leq \int_d^\theta \Phi^{\frac{-1}{\nu+1}}(s) \Phi^{\frac{1}{\nu+1}}(s) |\zeta^\Delta(s)| \Delta s \\ &\leq \Phi^{\frac{-1}{\nu+1}}(\theta) \int_d^\theta \Phi^{\frac{1}{\nu+1}}(s) |\zeta^\Delta(s)| \Delta s \leq \Phi^{\frac{-1}{\nu+1}}(\theta) \eta(\theta). \end{aligned} \tag{37}$$

Applying the chain rule (8), we obtain

$$(\eta^{\nu+1}(\theta))^\Delta = (\nu + 1) \eta^\nu(d) \eta^\Delta(\theta), \quad \text{where } d \in [\theta, \sigma(\theta)].$$

Since $\eta^\Delta(\theta) = \Phi^{\frac{1}{\nu+1}}(\theta) |\zeta^\Delta(\theta)| > 0$, and $d \geq \theta$, we see that

$$(\eta^{\nu+1}(\theta))^\Delta = (\nu + 1) \eta^\nu(d) \eta^\Delta(\theta) \geq (\nu + 1) \eta^\nu(\theta) \eta^\Delta(\theta). \tag{38}$$

From (37) and (38) we have

$$\begin{aligned} &\int_d^\tau \Phi(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta \\ &\leq \int_d^\tau \Phi^{\frac{\nu}{\nu+1}}(\theta) |\zeta(\theta)|^\nu \Phi^{\frac{1}{\nu+1}}(\theta) |\zeta^\Delta(\theta)| \Delta\theta \leq \int_d^\tau \eta^\nu(\theta) \eta^\Delta(\theta) \Delta\theta \\ &\leq \int_d^\tau \eta^\nu(d) \eta^\Delta(\theta) \Delta\theta \leq \frac{1}{\nu + 1} \int_d^\tau (\eta^{\nu+1}(\theta))^\Delta \Delta\theta. \end{aligned}$$

Applying Theorem 3.1, with $F(\eta) = \eta^{\nu+1}$, we obtain

$$\begin{aligned} \int_d^\tau \Phi(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta &\leq \frac{1}{\nu + 1} \int_d^\tau (\eta^{\nu+1}(\theta))^\Delta \Delta\theta. \\ &\leq \frac{1}{\nu + 1} \left(\int_d^\tau \eta^\Delta(\theta) \Delta\theta \right)^{\nu+1} \\ &= \frac{1}{\nu + 1} \left(\int_d^\tau \Phi^{\frac{1}{\nu+1}}(\theta) |\zeta^\Delta(\theta)| \Delta\theta \right)^{\nu+1}. \end{aligned}$$

By applying Hölder’s inequality (10) with indices $(\nu + 1)$ and $(\nu + 1)/\nu$, we have

$$\int_d^\tau \Phi(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{(\tau - d)^\nu}{\nu + 1} \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+1} \Delta\theta,$$

which is (36). The proof is complete. □

Remark 4.3 If $\nu = 1$, then inequality (36) becomes

$$\int_d^\tau \Phi(\theta) |\zeta(\theta)| |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{(\tau - d)}{2} \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^2 \Delta\theta.$$

Similar to Theorem 4.5 and Theorem 4.6, one can prove the following theorems.

Theorem 4.7 *Let \mathbb{T} be a time scale with $\tau, \ell \in \mathbb{T}$, $\varrho > 1$ and Φ be non-negative and nondecreasing on $[\tau, \ell]_{\mathbb{T}}$. If $\zeta : [\tau, \ell]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(\ell) = 0$ then for $\nu \geq 0$*

$$\int_\tau^\ell \Phi(\theta) |\zeta^\sigma(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta \leq \frac{\varrho(\ell - \tau)^\nu}{\nu + \varrho} \int_\tau^\ell \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+\varrho} \Delta\theta. \tag{39}$$

Theorem 4.8 *Let \mathbb{T} be a time scale with $\tau, \ell \in \mathbb{T}$ and Φ be non-negative and nondecreasing on $[\tau, \ell]_{\mathbb{T}}$. If $\zeta : [\tau, \ell]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(\ell) = 0$ then for $\nu \geq 0$*

$$\int_\tau^\ell \Phi(\theta) |\zeta^\sigma(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{(\ell - \tau)^\nu}{\nu + 1} \int_\tau^\ell \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+1} \Delta\theta. \tag{40}$$

Remark 4.4 If $\nu = 1$, then inequality (40) becomes

$$\int_\tau^\ell \Phi(t) |\zeta^\sigma(t)| |\zeta^\Delta(t)| \Delta t \leq \frac{(\ell - \tau)}{2} \int_\tau^\ell \Phi(t) |\zeta^\Delta(t)|^2 \Delta t.$$

Theorem 4.9 *Let \mathbb{T} be a time scale with $d, \tau \in \mathbb{T}$, $\varrho > 1$, s be non-negative and non-increasing on $[d, \tau]_{\mathbb{T}}$, and Φ be a non-negative function on $[d, \tau]_{\mathbb{T}}$. If $\zeta : [d, \tau]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(d) = 0$ then for $\nu \geq 0$*

$$\int_d^\tau s(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta \leq K_1(d, \tau, \nu, \varrho) \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+\varrho} \Delta\theta, \tag{41}$$

where

$$K_1(d, \tau, \nu, \varrho) = \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \left(\int_d^\tau \left(\frac{s(\theta)}{\Phi(\theta)} \right)^{\varrho/\nu} \Delta\theta \right)^{\nu/\varrho}.$$

Proof Let $\eta(\theta) = \int_d^\theta s^{\frac{\varrho}{\nu+\varrho}}(x) |\zeta^\Delta(x)|^\varrho \Delta x$, for $\theta \in [d, \tau]_\theta$. Then $\eta(d) = 0$ and

$$\eta^\Delta(\theta) = s^{\frac{\varrho}{\nu+\varrho}}(\theta) |\zeta^\Delta(\theta)|^\varrho > 0. \tag{42}$$

By Hölder’s inequality (10) with indices ϱ and $\varrho/(\varrho - 1)$ we get

$$|\zeta(\theta)| = \left| \int_d^\theta \zeta^\Delta(x) \Delta x \right| \leq \int_d^\theta |\zeta^\Delta(x)| \Delta x$$

$$\begin{aligned}
 &= \int_d^\theta s^{\frac{-1}{\nu+\varrho}}(x) s^{\frac{1}{\nu+\varrho}}(x) |\zeta^\Delta(x)| \Delta x \\
 &\leq \left(\int_d^\theta (s^{\frac{-1}{\nu+\varrho}}(x))^{\frac{\varrho}{\varrho-1}} \Delta x \right)^{\frac{\varrho-1}{\varrho}} \left(\int_d^\theta s^{\frac{\varrho}{\nu+\varrho}}(x) |\zeta^\Delta(x)|^\varrho \Delta x \right)^{\frac{1}{\varrho}} \\
 &\leq s^{\frac{-1}{\nu+\varrho}}(\theta) (\theta - d)^{\frac{\varrho-1}{\varrho}} \eta^{\frac{1}{\varrho}}(\theta),
 \end{aligned}$$

which yields

$$s^{\frac{\nu}{\nu+\varrho}}(\theta) |\zeta(\theta)|^\nu \leq (\theta - d)^{\frac{\nu(\varrho-1)}{\varrho}} \eta^{\frac{\nu}{\varrho}}(\theta). \tag{43}$$

By applying the chain rule (8), we obtain

$$\left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta)\right)^\Delta = \frac{\nu + \varrho}{\varrho} \eta^{\frac{\nu}{\varrho}}(d) \eta^\Delta(\theta), \quad \text{where } d \in [\theta, \sigma(\theta)].$$

Since $\eta^\Delta(\theta) = s^{\frac{\varrho}{\nu+\varrho}}(\theta) |\zeta^\Delta(\theta)|^\varrho > 0$, and $d \geq \theta$, we see that

$$\left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta)\right)^\Delta \geq \frac{\nu + \varrho}{\varrho} \eta^{\frac{\nu}{\varrho}}(\theta) \eta^\Delta(\theta). \tag{44}$$

Now, from (42), (43) and (44) we have

$$\begin{aligned}
 \int_d^\tau s(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta &= \int_d^\tau s^{\frac{\nu}{\nu+\varrho}}(\theta) |\zeta(\theta)|^\nu s^{\frac{\varrho}{\nu+\varrho}}(\theta) |\zeta^\Delta(\theta)|^\varrho \Delta\theta \\
 &\leq \int_d^\tau (\theta - d)^{\frac{\nu(\varrho-1)}{\varrho}} \eta^{\frac{\nu}{\varrho}}(\theta) \eta^\Delta(\theta) \Delta\theta \\
 &\leq (\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}} \int_d^\tau \eta^{\frac{\nu}{\varrho}}(\theta) \eta^\Delta(\theta) \Delta\theta \\
 &\leq \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \int_d^\tau \left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta)\right)^\Delta \Delta\theta.
 \end{aligned}$$

Applying Theorem 3.1, with $F(\eta) = \eta^{\frac{\nu+\varrho}{\varrho}}$, we obtain

$$\begin{aligned}
 \int_d^\tau s(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta &\leq \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \int_d^\tau \left(\eta^{\frac{\nu+\varrho}{\varrho}}(\theta)\right)^\Delta \Delta\theta \\
 &\leq \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \left(\int_d^\tau \eta^\Delta(\theta) \Delta\theta \right)^{\frac{\nu+\varrho}{\varrho}} \\
 &= \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \left(\int_d^\tau s^{\frac{\varrho}{\nu+\varrho}}(\theta) |\zeta^\Delta(\theta)|^\varrho \Delta\theta \right)^{\frac{\nu+\varrho}{\varrho}}.
 \end{aligned}$$

Now applying Hölder’s inequality (10) with indices $(\nu + \varrho)/\nu$ and $(\nu + \varrho)/\varrho$ we have

$$\begin{aligned}
 &\int_d^\tau s(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta \\
 &= \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \left(\int_d^\tau s^{\frac{\varrho}{\nu+\varrho}}(\theta) \Phi^{\frac{-\varrho}{\nu+\varrho}}(\theta) \Phi^{\frac{\varrho}{\nu+\varrho}}(\theta) |\zeta^\Delta(\theta)|^\varrho \Delta\theta \right)^{\frac{\nu+\varrho}{\varrho}}
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \left(\int_d^\tau \left(\frac{s(\theta)}{\Phi(\theta)} \right)^{\varrho/\nu} \Delta\theta \right)^{\nu/\varrho} \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+\varrho} \Delta\theta \\ &= K_1(d, \tau, \nu, \varrho) \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+\varrho} \Delta\theta, \end{aligned}$$

where

$$K_1(d, \tau, \nu, \varrho) = \frac{\varrho(\tau - d)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \left(\int_d^\tau \left(\frac{s(\theta)}{\Phi(\theta)} \right)^{\varrho/\nu} \Delta\theta \right)^{\nu/\varrho},$$

which is the desired inequality (41). The proof is complete. □

Theorem 4.10 *Let \mathbb{T} be a time scale with $d, \tau \in \mathbb{T}$, s be non-negative and non-increasing on $[d, \tau]_{\mathbb{T}}$, and Φ be a non-negative function on $[d, \tau]_{\mathbb{T}}$. If $\zeta : [d, \tau]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(d) = 0$ then for $\nu \geq 0$*

$$\int_d^\tau s(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta \leq K_1(d, \tau, \nu) \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+1} \Delta\theta, \tag{45}$$

where

$$K_1(d, \tau, \nu) = \frac{1}{\nu + 1} \left(\int_d^\tau \left(\frac{s(\theta)}{\Phi(\theta)} \right)^{1/\nu} \Delta\theta \right)^\nu.$$

Proof Let $\eta(\theta) = \int_d^\theta s^{\frac{1}{\nu+1}}(x) |\zeta^\Delta(x)| \Delta x$, for $\theta \in [d, \tau]_{\mathbb{T}}$. Then $\eta(d) = 0$ and

$$\begin{aligned} |\zeta(\theta)| &\leq \int_d^\theta |\zeta^\Delta(x)| \Delta x \leq \int_d^\theta s^{\frac{-1}{\nu+1}}(x) s^{\frac{1}{\nu+1}}(x) |\zeta^\Delta(x)| \Delta x \\ &\leq s^{\frac{-1}{\nu+1}}(\theta) \int_d^\theta s^{\frac{1}{\nu+1}}(x) |\zeta^\Delta(x)| \Delta x \leq s^{\frac{-1}{\nu+1}}(\theta) \eta(\theta). \end{aligned} \tag{46}$$

Applying the chain rule (8), we obtain

$$(\eta^{\nu+1}(\theta))^\Delta = (\nu + 1) \eta^\nu(d) \eta^\Delta(\theta), \quad \text{where } d \in [\theta, \sigma(\theta)].$$

Since $\eta^\Delta(\theta) = s^{\frac{1}{\nu+1}}(\theta) |\zeta^\Delta(\theta)| > 0$, and $d \geq \theta$, we see that

$$(\eta^{\nu+1}(\theta))^\Delta = (\nu + 1) \eta^\nu(d) \eta^\Delta(\theta) \geq (\nu + 1) \eta^\nu(\theta) \eta^\Delta(\theta). \tag{47}$$

Now, from (46) and (47) we have

$$\begin{aligned} \int_d^\tau s(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta &\leq \int_d^\tau s^{\frac{\nu}{\nu+1}}(\theta) |\zeta(\theta)|^\nu s^{\frac{1}{\nu+1}}(\theta) |\zeta^\Delta(\theta)| \Delta\theta \\ &\leq \int_d^\tau \eta^\nu(\theta) \eta^\Delta(\theta) \Delta\theta \leq \int_d^\tau \eta^\nu(d) \eta^\Delta(\theta) \Delta\theta \\ &\leq \frac{1}{\nu + 1} \int_d^\tau (\eta^{\nu+1}(\theta))^\Delta \Delta\theta. \end{aligned}$$

Applying Theorem 3.1, with $F(\eta) = \eta^{\nu+1}$, we obtain

$$\begin{aligned} \int_d^\tau s(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta &\leq \frac{1}{\nu+1} \int_d^\tau (\eta^{\nu+1}(\theta))^\Delta \Delta\theta \leq \frac{1}{\nu+1} \left(\int_d^\tau \eta^\Delta(\theta) \Delta\theta \right)^{\nu+1} \\ &= \frac{1}{\nu+1} \left(\int_d^\tau s^{\frac{1}{\nu+1}}(\theta) |\zeta^\Delta(\theta)| \Delta\theta \right)^{\nu+1} \\ &= \frac{1}{\nu+1} \left(\int_d^\tau s^{\frac{1}{\nu+1}}(\theta) \Phi^{\frac{-1}{\nu+1}}(\theta) \Phi^{\frac{1}{\nu+1}}(\theta) |\zeta^\Delta(\theta)| \Delta\theta \right)^{\nu+1}. \end{aligned}$$

Now applying Hölder’s inequality (10) with indices $(\nu + 1)$ and $(\nu + 1)/\nu$ we have

$$\int_d^\tau s(\theta) |\zeta(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{1}{\nu+1} \left(\int_d^\tau \left(\frac{s(\theta)}{\Phi(\theta)} \right)^{\frac{1}{\nu}} \Delta\theta \right)^\nu \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+1} \Delta\theta,$$

which is the desired inequality (45). The proof is complete. □

Remark 4.5 If $\nu = 1$, then inequality (45) becomes

$$\int_d^\tau s(\theta) |\zeta(\theta)| |\zeta^\Delta(\theta)| \Delta\theta \leq \frac{1}{2} \int_d^\tau \frac{s(\theta)}{\Phi(\theta)} \Delta\theta \int_d^\tau \Phi(\theta) |\zeta^\Delta(\theta)|^2 \Delta\theta.$$

As in the proofs of Theorem 4.9 and Theorem 4.10, one can prove the following theorems.

Theorem 4.11 *Let \mathbb{T} be a time scale with $\tau, \ell \in \mathbb{T}$, $\varrho > 1$, s be non-negative and nondecreasing on $[\tau, \ell]_{\mathbb{T}}$, and Φ be a non-negative function on $[\tau, \ell]_{\mathbb{T}}$. If $\zeta : [\tau, \ell]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(\ell) = 0$ then for $\nu \geq 0$*

$$\int_\tau^\ell s(\theta) |\zeta^\sigma(\theta)|^\nu |\zeta^\Delta(\theta)|^\varrho \Delta\theta \leq K_2(\tau, \ell, \nu, \varrho) \int_\tau^\ell \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+\varrho} \Delta\theta, \tag{48}$$

where

$$K_2(\tau, \ell, \nu, \varrho) = \frac{\varrho(\ell - \tau)^{\frac{\nu(\varrho-1)}{\varrho}}}{\nu + \varrho} \left(\int_\tau^\ell \left(\frac{s(\theta)}{\Phi(\theta)} \right)^{\varrho/\nu} \Delta\theta \right)^{\nu/\varrho}.$$

Theorem 4.12 *Let \mathbb{T} be a time scale with $\tau, \ell \in \mathbb{T}$, s be non-negative and nondecreasing on $[\tau, \ell]_{\mathbb{T}}$, and Φ be a non-negative function on $[\tau, \ell]_{\mathbb{T}}$. If $\zeta : [\tau, \ell]_{\mathbb{T}} \rightarrow \mathbb{R}$ is delta differentiable with $\zeta(\ell) = 0$ then for $\nu \geq 0$*

$$\int_\tau^\ell s(\theta) |\zeta^\sigma(\theta)|^\nu |\zeta^\Delta(\theta)| \Delta\theta \leq K_2(\tau, \ell, \nu) \int_\tau^\ell \Phi(\theta) |\zeta^\Delta(\theta)|^{\nu+1} \Delta\theta, \tag{49}$$

where

$$K_2(\tau, \ell, \nu) = \frac{1}{\nu+1} \left(\int_\tau^\ell \left(\frac{s(\theta)}{\Phi(\theta)} \right)^{1/\nu} \Delta\theta \right)^\nu.$$

Remark 4.6 If $\nu = 1$, then inequality (49) becomes

$$\int_{\tau}^{\ell} s(\theta) |\zeta^{\sigma}(\theta)| |\zeta^{\Delta}(\theta)| \Delta\theta \leq \frac{1}{2} \int_{\tau}^{\ell} \frac{s(\theta)}{\Phi(\theta)} \Delta\theta \int_{\tau}^{\ell} \Phi(\theta) |\zeta^{\Delta}(\theta)|^2 \Delta\theta.$$

Acknowledgements

Jehad Alzabut is highly thankful to the Prince Sultan University for supporting this work.

Funding

None.

Availability of data and materials

None.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have taken part in this research equally and they read and approved the final manuscript.

Author details

¹Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt. ²Department of Mathematics and General Sciences, Prince Sultan University, 11586 Riyadh, Saudi Arabia. ³Group of Mathematics, Faculty of Engineering, OSTIM Technical University, 06374 Ankara, Turkey. ⁴Department of Mathematics, Faculty of Science, Al-Azhar University, 11884 Nasr City, Egypt. ⁵School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 1 December 2020 Accepted: 11 June 2021 Published online: 23 June 2021

References

1. Agarwal, R.P., Bohner, M.: Basic calculus on time scales and some of its applications. *Results Math.* **35**, 3–22 (1999)
2. Agarwal, R.P., O'Regan, D., Saker, S.H.: *Dynamic Inequalities on Time Scales*. Springer, Heidelberg (2014)
3. Bohner, M., Kaymakçalan, B.: Opial inequalities on time scales. *Ann. Pol. Math.* **77**(1), 11–20 (2001)
4. Bohner, M., Peterson, A.: *Dynamic Equations on Time Scales: An Introduction with Application*. Birkhäuser, Boston (2001)
5. Bohner, M., Peterson, A.: *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston (2003)
6. Cristescu, G.: Improved integral inequalities for products of convex functions. *JIPAM. J. Inequal. Pure Appl. Math.* **6** (2005) 6 pages
7. Cristescu, G., Lupşa, L.: *Non-connected Convexities and Applications*, Applied Optimization. Kluwer Academic, Dordrecht (2002)
8. Kalsoom, H., Latif, M.A., Junjua, M.D., Hussain, S., Shahzadi, G.: Some (p, q) -estimates of Hermite-Hadamard-type inequalities for coordinated convex and quasi-convex functions. *Mathematics* **7**, 683 (2019)
9. Karpuz, B., Kaymakçalan, B., Öclan, Ö.: A generalization of Opial's inequality and applications to second order dynamic equations. *Differ. Equ. Dyn. Syst.* **18**(1–2), 11–18 (2010)
10. Latif, A.M.: On some inequalities for h -convex functions. *Int. J. Math. Anal.* **30**, 1473–1482 (2010)
11. Opial, Z.: Sur une inégalité. *Ann. Pol. Math.* **8**, 29–32 (1960)
12. Pachpatte, B.G.: On Opial-type integral inequalities. *J. Math. Anal. Appl.* **120**, 547–556 (1986)
13. Saker, S.H.: Some Opial-type inequalities on time scales. *Abstr. Appl. Anal.* **2011**, Article ID 265316 (2011)
14. Saker, S.H.: New inequalities of Opial's type on time scales and some of their applications. *Discrete Dyn. Nat. Soc.* **2012**, Article ID 362526 (2012)
15. Srivastava, H.M., Tseng, K.L., Tseng, S.J., Lo, J.C.: Some weighted Opial-type inequalities on time scale. *Taiwan. J. Math.* **14**(1), 107–122 (2010)
16. Wong, F.H., Lin, W.C., Yu, S.L., Yeh, C.C.: Some generalizations of Opial's inequalities on time scales. *Taiwan. J. Math.* **12**(2), 463–471 (2008)