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# Unified multivalued interpolative Reich–Rus–Ćirić-type contractions

Monairah Alansari<sup>1</sup> and Muhammad Usman Ali<sup>2\*</sup>

\*Correspondence:

muh\_usman\_ali@yahoo.com;  
musman.ali@ciit-attock.edu.pk

<sup>2</sup>Department of Mathematics,  
COMSATS University Islamabad,  
Attock Campus, Attock, Pakistan  
Full list of author information is  
available at the end of the article

## Abstract

This article examines new multivalued interpolative Reich–Rus–Ćirić-type contraction conditions and fixed point results for multivalued maps that fulfill these conditions. Earlier defined interpolative contraction type conditions cannot be particularized to any contraction type condition. This slackness of the interpolative contraction type condition is addressed through new multivalued interpolative Reich–Rus–Ćirić-type contraction conditions.

**MSC:** 47H10; 54H25

**Keywords:** Fixed points;  $b$ -metric spaces; Interpolative Kannan contraction; Interpolative contraction type conditions

## 1 Introduction and preliminaries

A fixed point to a self-mapping  $L$  defined on a non-void abstract set  $B$  is a solution to an equation  $Lb = b$ . Banach's fixed point result [1] is the initial result in the metric fixed point theory which deals with the existence of a solution to the aforementioned equation for a self-map  $L$  of a metric space  $(B, d_B)$ . This result requires the following two conditions to ensure the existence and uniqueness of a solution to an equation  $Lb = b$ , equivalently, fixed point of  $L$ :

- (1) The metric space should be complete;
- (2)  $L$  should be contraction map, that is,  $d_B(Lb, Lz) \leq \Omega d_B(b, z)$  for each  $b, z \in B$ , where  $\Omega \in [0, 1)$ .

Above conditions have a pivotal role in the development of the metric fixed point theory. Several generalizations have been concluded by modifying these conditions. For instance, some modified types of metric spaces are known as partial metric spaces [2],  $b$ -metric spaces [3, 4], and extended  $b$ -metric spaces [5]. Meanwhile, the classical and the earliest modifications in contraction map are provided by Kannan [6], and Chatterjea [7], as follows:

A map  $L : (B, d_B) \rightarrow (B, d_B)$  is called a Kannan contraction, if

$$d_B(Lb, Lz) \leq \Omega [d_B(b, Lb) + d_B(z, Lz)]$$

for all  $b, z \in B$ , where  $\Omega \in [0, 1/2)$ .

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A map  $L : (B, d_B) \rightarrow (B, d_B)$  is called a Chatterjea contraction, if

$$d_B(Lb, Lz) \leq \Omega [d_B(b, Lz) + d_B(z, Lb)]$$

for all  $b, z \in B$ , where  $\Omega \in [0, 1/2)$ .

An interpolative Kannan contraction seems like a modified form of Kannan contraction. This notion is derived by Karapınar [8] and further improved by Karapınar, Agarwal and Aydi [9]. Since the introduction of an interpolative Kannan contraction by Karapınar [8] many of the existing contraction type conditions have been modified utilizing the pattern of interpolative Kannan contraction. Details can be found in [10–18]. A few existing interpolative contraction type conditions are as follows:

A map  $L : (B, d_B) \rightarrow (B, d_B)$  is an interpolative Kannan contraction, if

$$d_B(Lb, Lz) \leq \Omega [d_B(b, Lb)]^{\tau_1} [d_B(z, Lz)]^{1-\tau_1}$$

for all  $b, z \in B$  with  $b \neq Lb$ , where  $\Omega \in [0, 1)$  and  $\tau_1 \in (0, 1)$ .

A map  $L : (B, d_B) \rightarrow (B, d_B)$  is an improved interpolative Kannan contraction, if

$$d_B(Lb, Lz) \leq \Omega [d_B(b, Lb)]^{\tau_1} [d_B(z, Lz)]^{1-\tau_1}$$

for all  $b, z \in B \setminus \text{Fix}(L)$ , where  $\Omega \in [0, 1)$ ,  $\tau_1 \in (0, 1)$  and  $\text{Fix}(L) = \{b \in B : Lb = b\}$ .

A map  $L : (B, d_B) \rightarrow (B, d_B)$  is an  $(\Omega, \tau_1, \tau_2)$ -interpolative Kannan contraction, if

$$d_B(Lb, Lz) \leq \Omega [d_B(b, Lb)]^{\tau_1} [d_B(z, Lz)]^{\tau_2}$$

for all  $b, z \in B \setminus \text{Fix}(L)$ , where  $\Omega \in [0, 1)$ ,  $\tau_1, \tau_2 \in (0, 1)$  with  $\tau_1 + \tau_2 < 1$ .

A map  $L : (B, d_B) \rightarrow (B, d_B)$  is an interpolative Reich–Rus–Ćirić-type contraction, if

$$d_B(Lb, Lz) \leq \Omega [[d_B(b, z)]^{\tau_1} [d_B(b, Lb)]^{\tau_2} [d_B(z, Lz)]^{1-\tau_1-\tau_2}]$$

for each  $b, z \in B \setminus \text{Fix}(L)$ , where  $\Omega \in [0, 1)$  and  $\tau_1, \tau_2 \in (0, 1)$  with  $\tau_1 + \tau_2 < 1$ .

The set-valued/multivalued interpolative Reich–Rus–Ćirić-type contraction map was introduced by Debnath and Sen [16] in a  $b$ -metric space.

This article examines new multivalued interpolative Reich–Rus–Ćirić-type contraction maps and fixed point results for such maps. The new multivalued interpolative Reich–Rus–Ćirić-type contraction conditions which are being examined in this article cannot only be particularized to Nadler's type contraction condition but also to some other types of interpolative contraction conditions. Debnath and Sen [16] discussed the existence of fixed points for multivalued interpolative Reich–Rus–Ćirić-type contraction map in  $b$ -metric space, by assuming that all bounded and closed subsets of the  $b$ -metric space are compact. Readers can see that the restriction of compactness is not required in the presented results of this article.

Before moving towards the main results, we discuss the notion of  $b$ -metric spaces, presented by Bakhtin [3] and Czerwik [4], with a few essential concepts.

**Definition 1.1** ([3, 4]) A function  $d_B : B \times B \rightarrow [0, \infty)$  is called a  $b$ -metric on  $B \neq \emptyset$ , if for all  $b, z, c \in B$  and for some  $\lambda \geq 1$ , we get

- (1)  $d_B(b, z) = 0 \Leftrightarrow b = z;$
- (2)  $d_B(b, z) = d_B(z, b);$
- (3)  $d_B(b, c) \leq \lambda[d_B(b, z) + d_B(z, c)].$

Then  $(B, d_B, \lambda)$  denotes  $b$ -metric space along coefficient  $\lambda \geq 1$ .

The concept of  $b$ -metric space is considered as the strongest generalization of metric space and it is reflected by the work of several researchers. The reader may refer to [19–27].

**Definition 1.2** ([4]) Let  $(B, d_B, \lambda)$  be a  $b$ -metric space along coefficient  $\lambda \geq 1$ . Then:

- a sequence  $\{b_n\}$  is Cauchy in  $B$ , if  $\lim_{n,m \rightarrow \infty} d_B(b_n, b_m) = 0;$
- a sequence  $\{b_n\}$  is convergent to  $b_*$  in  $B$ , if  $\lim_{n \rightarrow \infty} d_B(b_n, b_*) = 0$  and  $b_* \in B;$
- $(B, d_B, \lambda)$  is called complete if each Cauchy sequence  $\{b_n\}$  in  $B$  is convergent in  $B$ .

Now on,  $(B, d_B, \lambda)$  denotes the  $b$ -metric space along coefficient  $\lambda \geq 1$  and  $CB(B)$  represents the collection of all non-empty bounded and closed subsets of  $B$ . The functional  $H_B : CB(B) \times CB(B) \rightarrow [0, \infty)$  defined by

$$H_B(D, E) = \max \{ \sup \{ d_B(\omega, E) : \omega \in D \}, \sup \{ d_B(\eta, D) : \eta \in E \} \}$$

is the Pompeiu–Hausdorff  $b$ -metric on  $CB(B)$ , where  $d_B(\omega, E) = \inf \{ d_B(\omega, \eta), \eta \in E \}.$

The following theorem has an important role in the results presented by this article.

**Theorem 1.3** ([19]) Let  $(B, d_B, \lambda)$  be a  $b$ -metric space. Let  $D, E \in CB(B)$  and  $\omega \in D$ . Then, for each  $\Omega > 1$ , there is  $\eta \in E$  with

$$d_B(\omega, \eta) \leq \Omega H_B(D, E).$$

## 2 Main results

This section begins with the following definition.

**Definition 2.1** Assume a  $b$ -metric space  $(B, d_B, \lambda)$  and maps  $L : B \rightarrow CB(B), \gamma : B \times B \rightarrow \mathbb{R} - \{0\}.$  The map  $L$  is called a  $\gamma$ -interpolative Reich–Rus–Ćirić-I-contraction, if

$$[H_B(Lb, Lz)]^{\gamma(b,z)} \leq \Omega [[d_B(b, z)]^{\tau_1} [d_B(b, Lb)]^{\tau_2} [d_B(z, Lz)]^{\tau_3}] \tag{2.1}$$

for each  $b, z \in B$  with

$$\min \{ d_B(b, z), d_B(b, Lb), d_B(z, Lz) \} > 0,$$

where  $\Omega \in (0, \frac{1}{\lambda^2})$  and  $\tau_1, \tau_2, \tau_3 \in [0, 1]$  with  $\tau_1 + \tau_2 + \tau_3 = 1.$

The existence of fixed points for the defined notion is discussed as follows.

**Theorem 2.2** Assume a complete  $b$ -metric space  $(B, d_B, \lambda)$  and  $\gamma$ -interpolative Reich–Rus–Ćirić-I-contraction map  $L$ . Also, assume that:

- (1) there exist  $b_0 \in B$  and  $b_1 \in Lb_0$  with  $\gamma(b_0, b_1) = 1;$

- (2) for each  $b, z \in B$  with  $\gamma(b, z) = 1$ , we have  $\gamma(c, d) = 1 \forall c \in Lb, d \in Lz$ ;
- (3) for each  $\{b_m\}$  in  $B$  with  $b_m \rightarrow b$  and  $\gamma(b_m, b_{m+1}) = 1 \forall m \in \mathbb{N}$ , we have  $\gamma(b_m, b) = 1 \forall m \in \mathbb{N}$ .

Then  $L$  has a fixed point in  $B$ .

*Proof* By (1), there exist  $b_0 \in B$  and  $b_1 \in Lb_0$  with  $\gamma(b_0, b_1) = 1$ . If

$$\min\{d_B(b_0, b_1), d_B(b_0, Lb_0), d_B(b_1, Lb_1)\} = 0$$

then fixed point of  $L$  possesses in  $B$ . Suppose that

$$\min\{d_B(b_0, b_1), d_B(b_0, Lb_0), d_B(b_1, Lb_1)\} > 0.$$

By (2.1), we obtain

$$\begin{aligned} H_B(Lb_0, Lb_1) &= [H_B(Lb_0, Lb_1)]^{\gamma(b_0, b_1)} \\ &\leq \Omega [[d_B(b_0, b_1)]^{\tau_1} [d_B(b_0, Lb_0)]^{\tau_2} [d_B(b_1, Lb_1)]^{\tau_3}]. \end{aligned} \tag{2.2}$$

From (2.2), we obtain

$$\begin{aligned} \frac{1}{\sqrt{\Omega}} d_B(b_1, Lb_1) &\leq \frac{1}{\sqrt{\Omega}} H_B(Lb_0, Lb_1) \\ &\leq \sqrt{\Omega} [[d_B(b_0, b_1)]^{\tau_1} [d_B(b_0, Lb_0)]^{\tau_2} [d_B(b_1, Lb_1)]^{\tau_3}]. \end{aligned} \tag{2.3}$$

As  $\frac{1}{\sqrt{\Omega}} > 1$ , from Theorem 1.3, there should be  $b_2 \in Lb_1$  satisfying

$$d_B(b_1, b_2) \leq \frac{1}{\sqrt{\Omega}} d_B(b_1, Lb_1).$$

By (2.3) and the above fact, we conclude

$$d_B(b_1, b_2) \leq \sqrt{\Omega} [[d_B(b_0, b_1)]^{\tau_1} [d_B(b_0, Lb_0)]^{\tau_2} [d_B(b_1, b_2)]^{\tau_3}]. \tag{2.4}$$

Now, we discuss the proof for the following three choices of  $\tau_3$ :

If  $\tau_3 = 0$  in (2.4), then  $\tau_1 + \tau_2 = 1$ , thus  $d_B(b_1, b_2) \leq \sqrt{\Omega} d_B(b_0, b_1)$ .

If  $\tau_3 = 1$  in (2.4) then  $d_B(b_1, b_2) = 0$ , that is,  $b_1$  is a fixed point of  $L$  and it is not possible under the assumption.

If  $\tau_3 \in (0, 1)$  in (2.4) then we have the following:

$$[d_B(b_1, b_2)]^{1-\tau_3} \leq \sqrt{\Omega} [[d_B(b_0, b_1)]^{\tau_1+\tau_2}] \tag{2.5}$$

since  $1 - \tau_3 = \tau_1 + \tau_2$ , thus, by the above inequality, we get

$$d_B(b_1, b_2) \leq (\sqrt{\Omega})^{\frac{1}{1-\tau_3}} d_B(b_0, b_1) < \sqrt{\Omega} d_B(b_0, b_1).$$

Hence, we arrive at

$$d_B(b_1, b_2) \leq \sqrt{\Omega} d_B(b_0, b_1). \tag{2.6}$$

As  $b_1 \in Lb_0$ ,  $b_2 \in Lb_1$  and  $\gamma(b_0, b_1) = 1$ , then, by (2), we obtain  $\gamma(b_1, b_2) = 1$ . Again, we assume that

$$\min\{d_B(b_1, b_2), d_B(b_1, Lb_1), d_B(b_2, Lb_2)\} > 0$$

then by (2.1) we get

$$\begin{aligned} \frac{1}{\sqrt{\Omega}}d_B(b_2, Lb_2) &\leq \frac{1}{\sqrt{\Omega}}H_B(Lb_1, Lb_2) \\ &= \frac{1}{\sqrt{\Omega}}[H_B(Lb_1, Lb_2)]^{\gamma(b_1, b_2)} \\ &\leq \sqrt{\Omega}[[d_B(b_1, b_2)]^{\tau_1} [d_B(b_1, Lb_1)]^{\tau_2} [d_B(b_2, Lb_2)]^{\tau_3}]. \end{aligned} \tag{2.7}$$

As  $\frac{1}{\sqrt{\Omega}} > 1$ , there should be  $b_3 \in Lb_2$  satisfying

$$d_B(b_2, b_3) \leq \frac{1}{\sqrt{\Omega}}d_B(b_2, Lb_2).$$

Thus, by (2.7) and the above inequality, we get

$$d_B(b_2, b_3) \leq \sqrt{\Omega}[[d_B(b_1, b_2)]^{\tau_1} [d_B(b_1, Lb_1)]^{\tau_2} [d_B(b_2, b_3)]^{\tau_3}]. \tag{2.8}$$

Again, we discuss the proof of the following three choices of  $\tau_3$ :

If  $\tau_3 = 0$  in (2.8), then  $\tau_1 + \tau_2 = 1$ , thus  $d_B(b_2, b_3) \leq \sqrt{\Omega}d_B(b_1, b_2)$ .

If  $\tau_3 = 1$  in (2.8) then  $d_B(b_2, b_3) = 0$ , that is,  $b_2$  is a fixed point of  $L$  and it is not possible under the assumption.

If  $\tau_3 \in (0, 1)$  in (2.8) then we have the following:

$$[d_B(b_2, b_3)]^{1-\tau_3} \leq \sqrt{\Omega}[[d_B(b_1, b_2)]^{\tau_1+\tau_2}]. \tag{2.9}$$

Thus, we arrive at

$$d_B(b_2, b_3) \leq \sqrt{\Omega}d_B(b_1, b_2). \tag{2.10}$$

By (2.10) and (2.6) we obtain

$$d_B(b_2, b_3) \leq (\sqrt{\Omega})^2 d_B(b_0, b_1).$$

Induction yields a sequence  $\{b_m\}$  in  $B$  with  $b_m \in Lb_{m-1}$ ,  $\gamma(b_m, b_{m+1}) = 1 \forall m \in \mathbb{N}$  and

$$d_B(b_m, b_{m+1}) \leq (\sqrt{\Omega})^m d_B(b_0, b_1) \quad \forall m \in \mathbb{N}.$$

Also, we get

$$\min\{d_B(b_m, b_{m+1}), d_B(b_m, Lb_m), d_B(b_{m+1}, Lb_{m+1})\} > 0 \quad \forall m \in \mathbb{N}.$$

By the triangle inequality, for  $n > m$ , we get

$$\begin{aligned} d_B(b_n, b_m) &\leq \sum_{j=m}^{n-1} \lambda^j d_B(b_j, b_{j+1}) \\ &\leq \sum_{j=m}^{n-1} \lambda^j (\sqrt{\Omega})^j d_B(b_0, b_1). \end{aligned}$$

Since  $\sum_{j=1}^{\infty} \lambda^j (\sqrt{\Omega})^j < \infty$ , thus,  $\{b_m\}$  is a Cauchy in  $B$ . For  $\{b_m\}$  the completeness of  $B$  shall give  $b_*$  in  $B$  with  $b_m \rightarrow b_*$ . By considering (3), we obtain  $\gamma(b_m, b_*) = 1 \forall m \in \mathbb{N}$ . Here, we claim that  $b_* \in Lb_*$ . Let us suppose that if the claim is wrong then

$$\min\{d_B(b_m, b_*), d_B(b_m, Lb_m), d_B(b_*, Lb_*)\} > 0 \quad \forall m \geq n_0$$

for some natural number  $n_0$ . By (2.1) we get

$$\begin{aligned} d_B(b_{m+1}, Lb_*) &\leq H_B(Lb_m, Lb_*) \\ &= [H_B(Lb_m, Lb_*)]^{\gamma(b_m, b_*)} \\ &\leq \Omega [ [d_B(b_m, b_*)]^{\tau_1} [d_B(b_m, Lb_m)]^{\tau_2} [d_B(b_*, Lb_*)]^{\tau_3} ] \\ &\leq \Omega [ [d_B(b_m, b_*)]^{\tau_1} [d_B(b_m, b_{m+1})]^{\tau_2} [d_B(b_*, Lb_*)]^{\tau_3} ] \quad \forall m \geq n_0. \end{aligned} \tag{2.11}$$

By the triangle inequality and (2.11), we get

$$\begin{aligned} d_B(b_*, Lb_*) &\leq \lambda [d_B(b_*, b_{m+1}) + d_B(b_{m+1}, Lb_*)] \\ &\leq \lambda d_B(b_*, b_{m+1}) \\ &\quad + \lambda \Omega [ [d_B(b_m, b_*)]^{\tau_1} [d_B(b_m, b_{m+1})]^{\tau_2} [d_B(b_*, Lb_*)]^{\tau_3} ] \quad \forall m \geq n_0. \end{aligned}$$

Suppose that  $\tau_3 \neq 1$  and  $m \rightarrow \infty$  in the above inequality, then we get  $d_B(b_*, Lb_*) = 0$ , that is,  $b_* \in Lb_*$ . Suppose that  $\tau_3 = 1$  and  $m \rightarrow \infty$  in the above inequality, then we get  $d_B(b_*, Lb_*) \leq \lambda \Omega d_B(b_*, Lb_*)$ , which is not possible if  $d_B(b_*, Lb_*) \neq 0$ . Hence, our claim is true,  $b_* \in Lb_*$ . □

*Example 2.3* Consider  $B$  as a set of all integers and define  $d_B(b, b') = |b - b'|^2 \forall b, b' \in B$ . Define  $L : B \rightarrow CB(B)$  by

$$L(b) = \begin{cases} \{0, 1\}, & b \in \{0, 1, 2, 3, \dots\}, \\ \{b, -(b-2)^2\}, & b \in \{-1, -2, -3, \dots\}, \end{cases}$$

and  $\gamma : B \times B \rightarrow \mathbb{R} - \{0\}$  by

$$\gamma(b, b') = \begin{cases} 1, & b, b' \in \{0, 1, 2, 3, \dots\}, \\ -[|b| + |b'| + 8], & \text{otherwise.} \end{cases}$$

Now, one can calculate the following cases.

If  $b, b' \in \{2, 3, 4, \dots\}$  with  $b \neq b'$ , we obtain  $H_B(Lb, Lb')^{\gamma(b,b')} = 0$ .

If  $b, b' < 0$  with  $b \neq b'$ , we obtain  $H_B(Lb, Lb')^{\gamma(b,b')} = \frac{1}{[|-(b-2)^2+(b'-2)^2|^{2|b|+|b'|+8}]}$ .

If  $b < 0$  and  $b' \geq 2$ , we obtain  $H_B(Lb, Lb')^{\gamma(b,b')} = \frac{1}{[|-(b-2)^2|^{2|b|+|b'|+8}]}$ .

These calculations verify the validity of (2.1). The remaining axioms of Theorem 2.2 are also valid. Hence,  $L$  has a fixed point.

By assuming  $\tau_1 = 1$  and  $\tau_2 = \tau_3 = 0$  in the above result, we arrive at the following results.

**Corollary 2.4** *Assume we have a complete  $b$ -metric space  $(B, d_B, \lambda)$  and maps  $L : B \rightarrow CB(B)$ ,  $\gamma : B \times B \rightarrow \mathbb{R} - \{0\}$  such that*

$$[H_B(Lb, Lz)]^{\gamma(b,z)} \leq \Omega d_B(b, z) \tag{2.12}$$

for each  $b, z \in B$  with

$$\min\{d_B(b, z), d_B(b, Lb), d_B(z, Lz)\} > 0,$$

where  $\Omega \in [0, \frac{1}{\lambda^2})$ . Also, assume that:

- (1) there exist  $b_0 \in B$  and  $b_1 \in Lb_0$  with  $\gamma(b_0, b_1) = 1$ ;
- (2) for each  $b, z \in B$  with  $\gamma(b, z) = 1$ , we have  $\gamma(c, d) = 1 \forall c \in Lb, d \in Lz$ ;
- (3) for each  $\{b_m\}$  in  $B$  with  $b_m \rightarrow b$  and  $\gamma(b_m, b_{m+1}) = 1 \forall m \in \mathbb{N}$ , we have  $\gamma(b_m, b) = 1 \forall m \in \mathbb{N}$ .

Then  $L$  has a fixed point in  $B$ .

By assuming  $\gamma(b, z) = 1$  for all  $b, z \in B$  in the above corollary, we obtain the following result which can be considered as an extended form of Nadler’s fixed point theorem.

**Corollary 2.5** *Assume a complete  $b$ -metric space  $(B, d_B, \lambda)$  and a map  $L : B \rightarrow CB(B)$  satisfying the following inequality:*

$$H_B(Lb, Lz) \leq \Omega d_B(b, z) \tag{2.13}$$

for each  $b, z \in B$  with

$$\min\{d_B(b, z), d_B(b, Lb), d_B(z, Lz)\} > 0,$$

where  $\Omega \in [0, \frac{1}{\lambda^2})$ . Then  $L$  has a fixed point in  $B$ .

*Remark 2.6* By considering (2.13) one can say that  $\gamma$ -interpolative Reich–Rus–Ćirić-I-contraction can be particularized to Nadler’s type contraction.

The right side of (2.14) is more analogous to interpolative Reich–Rus–Ćirić-contraction.

**Definition 2.7** *Assume a  $b$ -metric space  $(B, d_B, \lambda)$  and maps  $L : B \rightarrow CB(B)$ ,  $\gamma : B \times B \rightarrow \mathbb{R} - \{0\}$ . The map  $L$  is called a reduced  $\gamma$ -interpolative Reich–Rus–Ćirić-I-contraction, if*

$$[H_B(Lb, Lz)]^{\gamma(b,z)} \leq \Omega [[d_B(b, z)]^{\tau_1} [d_B(b, Lb)]^{\tau_2} [d_B(z, Lz)]^{1-\tau_1-\tau_2}] \tag{2.14}$$

for each  $b, z \in B$  with

$$\min\{d_B(b, z), d_B(b, Lb), d_B(z, Lz)\} > 0,$$

where  $\Omega \in (0, \frac{1}{\lambda^2})$  and  $\tau_1, \tau_2 \in [0, 1)$  with  $0 < \tau_1 + \tau_2 < 1$ .

*Remark 2.8* Consider  $\varsigma_1, \varsigma_2 \in [0, 1)$  with  $0 < \varsigma_1 + \varsigma_2 < 1$ . Define  $\tau_1 = \varsigma_1, \tau_2 = \varsigma_2$  and  $\tau_3 = 1 - \varsigma_1 - \varsigma_2$ , then  $\tau_1 + \tau_2 + \tau_3 = \varsigma_1 + \varsigma_2 + (1 - \varsigma_1 - \varsigma_2) = 1$ . Thus, (2.1) of Definition 2.1 gives (2.14) of Definition 2.7.

Now one can easily understand that Theorem 2.9 is a simple consequence of Theorem 2.2.

**Theorem 2.9** Assume a complete  $b$ -metric space  $(B, d_B, \lambda)$  and reduced  $\gamma$ -interpolative Reich–Rus–Ćirić-I-contraction map  $L$ . Also, assume that:

- (1) there exist  $b_0 \in B$  and  $b_1 \in Lb_0$  with  $\gamma(b_0, b_1) = 1$ ;
- (2) for each  $b, z \in B$  with  $\gamma(b, z) = 1$ , we have  $\gamma(c, d) = 1 \forall c \in Lb, d \in Lz$ ;
- (3) for each  $\{b_m\}$  in  $B$  with  $b_m \rightarrow b$  and  $\gamma(b_m, b_{m+1}) = 1 \forall m \in \mathbb{N}$ , we have  $\gamma(b_m, b) = 1 \forall m \in \mathbb{N}$ .

Then  $L$  has a fixed point in  $B$ .

By assuming  $\tau_1 = 0$  and  $\tau_2 = \tau \in (0, 1)$  in the above result we reach the following result.

**Corollary 2.10** Assume a complete  $b$ -metric space  $(B, d_B, \lambda)$  and maps  $L : B \rightarrow CB(B), \gamma : B \times B \rightarrow \mathbb{R} - \{0\}$  such that

$$[H_B(Lb, Lz)]^{\gamma(b,z)} \leq \Omega [[d_B(b, Lb)]^\tau [d_B(z, Lz)]^{1-\tau}] \tag{2.15}$$

for each  $b, z \in B$  with

$$\min\{d_B(b, z), d_B(b, Lb), d_B(z, Lz)\} > 0,$$

where  $\Omega \in [0, \frac{1}{\lambda^2})$  and  $\tau \in (0, 1)$ . Also, assume that:

- (1) there exist  $b_0 \in B$  and  $b_1 \in Lb_0$  with  $\gamma(b_0, b_1) = 1$ ;
- (2) for each  $b, z \in B$  with  $\gamma(b, z) = 1$ , we have  $\gamma(c, d) = 1 \forall c \in Lb, d \in Lz$ ;
- (3) for each  $\{b_m\}$  in  $B$  with  $b_m \rightarrow b$  and  $\gamma(b_m, b_{m+1}) = 1 \forall m \in \mathbb{N}$ , we have  $\gamma(b_m, b) = 1 \forall m \in \mathbb{N}$ .

Then  $L$  has a fixed point in  $B$ .

*Remark 2.11* Inequality (2.15) is a generalized form of improved interpolative Kannan contraction.

The following definition provides another way to generalize interpolative Reich–Rus–Ćirić-contraction maps.

**Definition 2.12** Assume a  $b$ -metric space  $(B, d_B, \lambda)$  and maps  $L : B \rightarrow CB(B), \gamma : B \times B \rightarrow [0, \infty)$ . The map  $L$  is called a  $\gamma$ -interpolative Reich–Rus–Ćirić-II-contraction, if

$$\gamma(b, z)H_B(Lb, Lz) \leq \Omega [[d_B(b, z)]^{\tau_1} [d_B(b, Lb)]^{\tau_2} [d_B(z, Lz)]^{\tau_3}] \tag{2.16}$$



for each  $b, z \in B$  with

$$\min\{d_B(b, z), d_B(b, Lb), d_B(z, Lz)\} > 0,$$

where  $\Omega \in (0, \frac{1}{\lambda^2})$  and  $\tau_1, \tau_2, \tau_3 \in [0, 1]$  with  $\tau_1 + \tau_2 + \tau_3 = 1$ .

The existence of fixed points for the above defined notion are verified through the following result.

**Theorem 2.13** *Assume a complete  $b$ -metric space  $(B, d_B, \lambda)$  and  $\gamma$ -interpolative Reich–Rus–Ćirić-II-contraction map  $L$ . Also, assume that:*

- (1) *there exist  $b_0 \in B$  and  $b_1 \in Lb_0$  with  $\gamma(b_0, b_1) \geq 1$ ;*
- (2) *for each  $b, z \in B$  with  $\gamma(b, z) \geq 1$ , we have  $\gamma(c, d) \geq 1 \forall c \in Lb, d \in Lz$ ;*
- (3) *for each  $\{b_m\}$  in  $B$  with  $b_m \rightarrow b$  and  $\gamma(b_m, b_{m+1}) \geq 1 \forall m \in \mathbb{N}$ , we have  $\gamma(b_m, b) \geq 1 \forall m \in \mathbb{N}$ .*

*Then  $L$  has a fixed point in  $B$ .*

*Proof* Axiom (1) says that there are elements  $b_0 \in B$  and  $b_1 \in Lb_0$  with  $\gamma(b_0, b_1) \geq 1$ . Assume that

$$\min\{d_B(b_0, b_1), d_B(b_0, Lb_0), d_B(b_1, Lb_1)\} > 0;$$

otherwise a fixed point of  $L$  occurs in  $B$ . Then, by (2.16), we arrive at

$$\begin{aligned} H_B(Lb_0, Lb_1) &\leq \gamma(b_0, b_1)H_B(Lb_0, Lb_1) \\ &\leq \Omega[[d_B(b_0, b_1)]^{\tau_1} [d_B(b_0, Lb_0)]^{\tau_2} [d_B(b_1, Lb_1)]^{\tau_3}]. \end{aligned} \tag{2.17}$$

From (2.17), we obtain

$$\begin{aligned} \frac{1}{\sqrt{\Omega}}d_B(b_1, Lb_1) &\leq \frac{1}{\sqrt{\Omega}}H_B(Lb_0, Lb_1) \\ &\leq \sqrt{\Omega}[[d_B(b_0, b_1)]^{\tau_1} [d_B(b_0, Lb_0)]^{\tau_2} [d_B(b_1, Lb_1)]^{\tau_3}]. \end{aligned} \tag{2.18}$$

As  $\frac{1}{\sqrt{\Omega}} > 1$ , there should be  $b_2 \in Lb_1$  satisfying

$$d_B(b_1, b_2) \leq \frac{1}{\sqrt{\Omega}}d_B(b_1, Lb_1).$$

By (2.18) and the above inequality, we get

$$d_B(b_1, b_2) \leq \sqrt{\Omega}[[d_B(b_0, b_1)]^{\tau_1} [d_B(b_0, b_1)]^{\tau_2} [d_B(b_1, b_2)]^{\tau_3}]. \tag{2.19}$$

Now, we discuss the proof for the following three choices of  $\tau_3$ :

If  $\tau_3 = 0$  in (2.19), then  $\tau_1 + \tau_2 = 1$ , thus  $d_B(b_1, b_2) \leq \sqrt{\Omega}d_B(b_0, b_1)$ .

If  $\tau_3 = 1$  in (2.19) then  $d_B(b_1, b_2) = 0$ , that is,  $b_1$  is a fixed point of  $L$  and it is not possible under the assumption.

If  $\tau_3 \in (0, 1)$  in (2.19), then we get the following:

$$[d_B(b_1, b_2)]^{1-\tau_3} \leq \sqrt{\Omega} [[d_B(b_0, b_1)]^{\tau_1+\tau_2}] \tag{2.20}$$

since  $1 - \tau_3 = \tau_1 + \tau_2$ , thus, by the above inequality, we get

$$d_B(b_1, b_2) \leq (\sqrt{\Omega})^{\frac{1}{1-\tau_3}} d_B(b_0, b_1).$$

Hence, we arrive at

$$d_B(b_1, b_2) \leq \sqrt{\Omega} d_B(b_0, b_1). \tag{2.21}$$

As  $b_1 \in Lb_0, b_2 \in Lb_1$  and  $\gamma(b_0, b_1) \geq 1$ , then by axiom (2), we arrive at  $\gamma(b_1, b_2) \geq 1$ . By the repetition of (2.16) and axiom (2), we arrive at a sequence  $\{b_m\}$  in  $B$  with  $b_m \in Lb_{m-1}, \gamma(b_m, b_{m+1}) \geq 1 \forall m \in \mathbb{N}$  and

$$d_B(b_m, b_{m+1}) \leq (\sqrt{\Omega})^m d_B(b_0, b_1) \quad \forall m \in \mathbb{N}.$$

Also,

$$\min\{d_B(b_m, b_{m+1}), d_B(b_m, Lb_m), d_B(b_{m+1}, Lb_{m+1})\} > 0 \quad \forall m \in \mathbb{N}.$$

From the proof of Theorem 2.2, we can see that  $\{b_m\}$  is a Cauchy in  $B$  and there should be  $b_*$  in  $B$  with  $b_m \rightarrow b_*$ . Also, by (3),  $\gamma(b_m, b_*) \geq 1 \forall m \in \mathbb{N}$ . Now we can claim that  $b_* \in Lb_*$ . If our claim is wrong, then  $\min\{d_B(b_m, b_*), d_B(b_m, Lb_m), d_B(b_*, Lb_*)\} > 0$  for each  $m \geq n_0$  (for some natural number  $n_0$ ). By (2.16), we arrive at

$$\begin{aligned} d_B(b_{m+1}, Lb_*) &\leq H_B(Lb_m, Lb_*) \\ &\leq \gamma(b_m, b_*) H_B(Lb_m, Lb_*) \\ &\leq \Omega [[d_B(b_m, b_*)]^{\tau_1} [d_B(b_m, Lb_m)]^{\tau_2} [d_B(b_*, Lb_*)]^{\tau_3}] \\ &\leq \Omega [[d_B(b_m, b_*)]^{\tau_1} [d_B(b_m, b_{m+1})]^{\tau_2} [d_B(b_*, Lb_*)]^{\tau_3}] \quad \forall m \geq n_0. \end{aligned} \tag{2.22}$$

By (2.22) and the triangle inequality, we arrive at

$$\begin{aligned} d_B(b_*, Lb_*) &\leq \lambda [d_B(b_*, b_{m+1}) + d_B(b_{m+1}, Lb_*)] \\ &\leq \lambda d_B(b_*, b_{m+1}) \\ &\quad + \lambda \Omega [[d_B(b_m, b_*)]^{\tau_1} [d_B(b_m, b_{m+1})]^{\tau_2} [d_B(b_*, Lb_*)]^{\tau_3}] \quad \forall m \geq n_0. \end{aligned}$$

Consider  $\tau_3 \neq 1$  and  $m \rightarrow \infty$  in the above inequality, then we get  $d_B(b_*, Lb_*) = 0$ , that is,  $b_* \in Lb_*$ . Consider  $\tau_3 = 1$  and  $m \rightarrow \infty$  in the above inequality, then we get  $d_B(b_*, Lb_*) \leq \lambda \Omega d_B(b_*, Lb_*)$ , which is not possible if  $d_B(b_*, Lb_*) \neq 0$ . Hence, our claim is true,  $b_* \in Lb_*$ . □

Now we shall discuss the notion of reduced  $\gamma$ -interpolative Reich–Rus–Ćirić-II-contraction map and related fixed point result.

**Definition 2.14** Assume a  $b$ -metric space  $(B, d_B, \lambda)$  and maps  $L : B \rightarrow CB(B)$ ,  $\gamma : B \times B \rightarrow [0, \infty)$ . The map  $L$  is called a reduced  $\gamma$ -interpolative Reich–Rus–Ćirić-II-contraction, if

$$\gamma(b, z)H_B(Lb, Lz) \leq \Omega \left[ [d_B(b, z)]^{\tau_1} [d_B(b, Lb)]^{\tau_2} [d_B(z, Lz)]^{1-\tau_1-\tau_2} \right] \tag{2.23}$$

for each  $b, z \in B$  with

$$\min\{d_B(b, z), d_B(b, Lb), d_B(z, Lz)\} > 0,$$

where  $\Omega \in (0, \frac{1}{\lambda^2})$  and  $\tau_1, \tau_2 \in [0, 1)$  with  $0 < \tau_1 + \tau_2 < 1$ .

The following result is a simple consequence of Theorem 2.13.

**Theorem 2.15** Assume a complete  $b$ -metric space  $(B, d_B, \lambda)$  and reduced  $\gamma$ -interpolative Reich–Rus–Ćirić-II-contraction map  $L$ . Also, assume that:

- (1) there exist  $b_0 \in B$  and  $b_1 \in Lb_0$  with  $\gamma(b_0, b_1) \geq 1$ ;
- (2) for each  $b, z \in B$  with  $\gamma(b, z) \geq 1$ , we have  $\gamma(c, d) \geq 1 \forall c \in Lb, d \in Lz$ ;
- (3) for each  $\{b_m\}$  in  $B$  with  $b_m \rightarrow b$  and  $\gamma(b_m, b_{m+1}) \geq 1 \forall m \in \mathbb{N}$ , we have  $\gamma(b_m, b) \geq 1 \forall m \in \mathbb{N}$ .

Then  $L$  has a fixed point in  $B$ .

*Example 2.16* Consider  $B$  as a set of all real numbers and  $d_B(b, b') = |b - b'|$  for all  $b, b' \in B$ . Define  $L : B \rightarrow CB(B)$  by

$$L(b) = \begin{cases} [0, \frac{b}{8}], & b \geq 0, \\ \{0, 2b\}, & b \leq 0, \end{cases}$$

and  $\xi : B \times B \rightarrow [0, \infty)$  by

$$\xi(b, b') = \begin{cases} 1, & b, b' \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For instance, take  $b = -1$  and  $b' = -3$ , then  $H_B(Tb, Tb') = 4$ ,  $d_B(b, b') = 2$ ,  $d_B(b, Tb) = 1$  and  $d_B(b', Tb') = 3$ . Also

$$[d_B(b, b')]^{\tau_1} [d_B(b, Lb)]^{\tau_2} [d_B(b', Lb')]^{1-\tau_1-\tau_2} < 3 \quad \forall \tau_1, \tau_2 \in (0, 1).$$

Thus, it can be seen that the set-valued versions based on the structure of  $b$ -metric spaces for the interpolative contraction type conditions given in [8–12, 16], with many other existing interpolative contraction type conditions, are not applicable on the above defined function  $L$  with respect to the above  $d_B$ . Meanwhile, all the axioms of Theorem 2.13 are valid on the above defined functions.

### 3 Conclusion

This article presents new multivalued interpolative Reich–Rus–Ćirić-type contraction conditions and fixed point results for multivalued maps which fulfil these conditions in

a complete  $b$ -metric space. Earlier defined interpolative contraction type conditions cannot be particularized to any contraction type condition. This slackness of interpolative contraction type condition is addressed through the introduction of new multivalued interpolative Reich–Rus–Ćirić-type contraction conditions. A few examples are given to support the findings of this article.

#### Acknowledgements

This work was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia, under grant number KEP-5-130-42. The authors, therefore, gratefully acknowledge the DSR for technical and financial support.

#### Funding

The Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, has funded this project under grant number KEP-5-130-42.

#### Availability of data and materials

The data used to support the findings of this study are available from the corresponding author upon request.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

Both authors have contributed equally in writing this article. Both authors have read and approved the final manuscript.

#### Author details

<sup>1</sup>Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia. <sup>2</sup>Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock, Pakistan.

#### Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 10 February 2021 Accepted: 8 June 2021 Published online: 26 June 2021

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