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Existence of a unique weak solution to a non-autonomous time-fractional diffusion equation with space-dependent variable order

Karel Van Bockstal[1*]

*Correspondence: karel.vanbockstal@ugent.be
[1]Department of Electronics and Information systems, Research Group NaM2, Ghent University, Ghent, Belgium

Abstract

In this contribution, we investigate an initial-boundary value problem for a fractional diffusion equation with Caputo fractional derivative of space-dependent variable order where the coefficients are dependent on spatial and time variables. We consider a bounded Lipschitz domain and a homogeneous Dirichlet boundary condition. Variable-order fractional differential operators originate in anomalous diffusion modelling. Using the strongly positive definiteness of the governing kernel, we establish the existence of a unique weak solution in $u \in \mathrm{L}^\infty((0,T),\mathrm{H}_0^1(\Omega))$ to the problem if the initial data belongs to $\mathrm{H}_0^1(\Omega)$. We show that the solution belongs to $\mathrm{C}([0,T],\mathrm{H}_0^1(\Omega)^*)$ in the case of a Caputo fractional derivative of constant order. We generalise a fundamental identity for integro-differential operators of the form $\frac{\mathrm{d}}{\mathrm{d}t}(k * v)(t)$ to a convolution kernel that is also space-dependent and employ this result when searching for more regular solutions. We also discuss the situation that the domain consists of separated subdomains.

Keywords: Time-fractional diffusion equation; Anomalous diffusion; Non-autonomous; Time discretization; Existence; Uniqueness

1 Introduction

1.1 Mathematical setting and motivation

Let $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. The final time is denoted by T, $Q_T := \Omega \times (0,T]$ and $\Sigma_T := \partial\Omega \times (0,T]$. Consider a general second-order linear differential operator given by

$$L(\mathbf{x},t)u(\mathbf{x},t) = -\nabla \cdot \big(\boldsymbol{A}(\mathbf{x},t)\nabla u(\mathbf{x},t)\big) + c(\mathbf{x},t)u(\mathbf{x},t), \tag{1}$$

where $((\mathbf{x},t) \in Q_T)$

$$\boldsymbol{A}(\mathbf{x},t) = \big(a_{i,j}(\mathbf{x},t)\big)_{i,j=1,\dots,d} \quad \text{with } \boldsymbol{A}^\top = \boldsymbol{A}.$$

The goal of this contribution is to show the existence and uniqueness of u for given f and $\tilde{u}_0$ such that

$$\begin{cases} \partial_t(g * (u - \tilde{u}_0))(\mathbf{x},t) + Lu(\mathbf{x},t) = f(\mathbf{x},t) & (\mathbf{x},t) \in Q_T, \\ u(\mathbf{x},t) = 0 & (\mathbf{x},t) \in \Sigma_T, \\ u(\mathbf{x},0) = \tilde{u}_0(\mathbf{x}) & \mathbf{x} \in \Omega. \end{cases} \tag{2}$$

The symbol '$*$' stands for the convolution product defined by

$$(g * z)(\mathbf{x},t) := \big(g_{1-\beta(\mathbf{x})} * z(\mathbf{x})\big)(t) = \int_0^t g_{1-\beta(\mathbf{x})}(t-s) z(\mathbf{x},s)\,\mathrm{d}s, \tag{3}$$

where the kernel $g_{1-\beta(\mathbf{x})}$ is defined by

$$g_{1-\beta(\mathbf{x})}(t) = \frac{t^{-\beta(\mathbf{x})}}{\Gamma(1-\beta(\mathbf{x}))}, \quad t > 0, \mathbf{x} \in \Omega,$$

where Γ denotes the gamma function, and $\beta \in \mathrm{C}(\Omega)$ satisfies

$$0 < \beta(\mathbf{x}) \leq \beta_1 < 1, \quad \mathbf{x} \in \Omega. \tag{4}$$

The term $D_t^{\beta(\mathbf{x})} u := \partial_t(g * (u - \tilde{u}_0))$ in (2) represents the Caputo fractional derivative of space-dependent variable order $\beta(\mathbf{x})$, which arises in the modelling of anomalous diffusion. Anomalous diffusion is a rapidly growing field of research with applications in physics, chemistry, biology and several other branches of engineering [1]. A typical example is heat conduction and fluid flow in porous media. We refer the reader to [2] for a comprehensive overview on fractional calculus and anomalous diffusion. To model diffusion processes in a homogeneous medium, the constant-order (β is constant over Ω) fractional diffusion model is sufficient. But in complex media, the presence of heterogeneous regions causes variations of permeability in different spatial positions. In this situation, the space-dependent variable-order model is more suitable to describe location-dependent diffusion processes, see e.g. [3–11]. We mention also the existence of time-dependent variable-order models when diffusion behaviour changes with the time evolution [12–16] and some other recent interesting applications of fractional calculus [17–20].

1.2 Literature overview

Existence and uniqueness of a regular solution to autonomous (time-independent elliptic part L) constant-order fractional diffusion equation, i.e. $\beta(\mathbf{x}) = \beta$) is studied in [21–23]. The results in these papers are build on the method of eigenfunction expansion and the solution is expressed in terms of the Mittag-Leffler functions, which implies that the solution is analytic on $(0,T)$. In [21], the authors searched for a solution $u(\cdot,t) \in \mathrm{dom}(L) = \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$ of problem (2) such that the governing partial differential equation (PDE) is satisfied in $\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))$. The authors establish in Theorem 2.1 the unique existence of a solution $u \in \mathrm{L}^2((0,T),\mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega))$ with $g * \partial_t u \in \mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))$ if $\tilde{u}_0 \in \mathrm{H}_0^1(\Omega)$ and $f = 0$. In the article [22], the maximum principle for PDEs of parabolic type is extended to the time-fractional diffusion equation. It is used to show the uniqueness of solution to problem (2). Moreover, the generalised solution in sense of Vladimirov is constructed in the form of a Fourier series with respect to the eigenfunctions of a certain Sturm–Liouville eigenvalue problem. In the one-dimensional case, it is shown that the generalised solution is a solution in the classical sense (a twice differentiable function with respect to the spatial variables and β-differentiable with respect to the time variable) under some additional conditions. In [23], the authors give an interpretation of the Caputo derivative in the fractional Sobolev space $\tilde{\mathrm{H}}^{\beta}(0,T) = \{u \in \mathrm{H}^{\beta}(0,T) : u(0) = 0\}$ (see [24, Chapter VII]) and prove some important norm equivalences. These results are used to show the maximum regularity $u \in \mathrm{L}^2((0,T),\mathrm{H}^2(\Omega)\cap\mathrm{H}^1_0(\Omega))\cap\tilde{\mathrm{H}}^{\beta}((0,T),\mathrm{L}^2(\Omega))$ of the solution to problem (2) when $\tilde{u}_0 = 0$ and $f \in \mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))$.

Existence and uniqueness results for weak solutions (using classical integration by parts formula for L) to the non-autonomous (so time-dependent elliptic part) constant-order fractional diffusion equation are derived in [25, 26] by Galerkin method. In [25, Corollary 4.1] and [26, Theorem 1], the authors showed that there exists a unique weak solution u to (2) satisfying $u \in \mathrm{L}^2((0,T),\mathrm{H}^1_0(\Omega))$ with $\partial_t(g * (u - \tilde{u}_0)) \in \mathrm{L}^2((0,T),\mathrm{H}^1_0(\Omega)^*)$ if $\tilde{u}_0 \in \mathrm{L}^2(\Omega)$ and $f \in \mathrm{L}^2((0,T),\mathrm{H}^1_0(\Omega)^*)$. In particular, in [26, Theorem 2] the authors proved that the solution belongs to $\mathrm{C}([0,T],\mathrm{H}^1_0(\Omega)^*)$ if $\beta > \frac{1}{2}$. Moreover, if $\tilde{u}_0 \in \mathrm{H}^1_0(\Omega)$ and $f \in \mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))$, the solution is more regular and belongs to $\mathrm{L}^2((0,T),\mathrm{H}^2(\Omega))$ under some appropriate conditions on the coefficients in L. Furthermore, u belongs to $\mathrm{C}([0,T],\mathrm{L}^2(\Omega))$ if $\beta > \frac{1}{2}$, cf. [26, Theorem 4].

Moreover, we refer for multi-term constant-order time fractional diffusion equations to [27, 28] and for distributed-order time fractional diffusion problems to [29–31] for the well-posedness and asymptotic behaviour of the solution (again using the method of eigenfunction expansion and Mittag-Leffler analysis). For completeness, we also provide some key literature works related to mild solutions of abstract dynamic equations [32–36].

In the recent works [37, 38], using a classical variational approach, the author established the existence of a unique weak solution to a non-autonomous time-fractional diffusion (respectively, wave) equation of constant and distributed order. The solution is continuous on $[0,T]$ without restriction on the order of the fractional derivative and thus under low regularity assumptions. It is important to note here that weakly singular solutions are included in the class of admissible solutions.

The existence of a solution to non-autonomous variable-order fractional differential equation is proved in [39]. However, to the best of our knowledge there is only one result available in the literature related to the well-posedness of the time-fractional diffusion equation with space-dependent variable order. In [40], the authors investigated the existence and uniqueness result for problem (2) in the case that the elliptic operator L is autonomous. The authors do not use the usual definition of weak solution but they characterise the weak solution to (2) as the original of the solution to the Laplace transform (of tempered distributions) of (2) with respect to the time variable.

1.3 New aspects and outline

We study a more general multi-dimensional case for a governing PDE with time-dependent coefficients (non-autonomous system). This implies that the Mittag-Leffler analysis or the approach by means of the Laplace transform is not appropriate and one has to use other tools to succeed. The approach in this paper follows the standard procedure for classical parabolic problems: first we discretize the problem in time using a convolution quadrature, next we obtain a priori estimates, and on the basis of these estimates we show the existence of a solution. In problem (2), we consider a homogeneous

Dirichlet boundary condition. Throughout the paper we discuss also the situation wherein a homogeneous Neumann boundary condition is considered on the boundary. We assume that β is continuous on $\overline{\Omega}$ as for instance in [4, 7]. However, in Sect. 8, we discuss the situation where Ω consists of separated regions and thus β can be discontinuous, see e.g. [3, 5, 7].

The Caputo fractional time derivative of order $\beta \in (0,1)$ is a special case of the variable-order fractional derivative (i.e. $\beta(\mathbf{x}) = \beta$ in Ω). The authors in [41] noted that the smoothness of $\tilde{u}_0$ and f in (2) does not always imply the smoothness of the exact solution. The solution of such a problem is shown in general to have a weak singularity near the initial time $t = 0$. It is important to take this behaviour of the solution into account when studying the well-posedness of problem (2). This implies that compatibility condition as $L(\cdot,0)\tilde{u}_0(x) = f(\cdot,0)$ should be avoided as $\partial_t(g * (u - \tilde{u}_0))(0)$ is not necessarily equal to zero in general [42]. Such a compatibility condition would restrict the class of admissible solutions.

One of the main points in the analysis is the property that the kernel g in the analysis is strongly positive definite. This and other properties of g are discussed in Sect. 2. The strongly positive definiteness of g is used in Theorem 6.2 to show the uniqueness of a solution to problem (2).

In Sect. 3, we generalise a fundamental identity for integro-differential operators of the form $\frac{\mathrm{d}}{\mathrm{d}t}(k * v)(t)$ by Zacher [25] to a convolution kernel k that depends on both the time- and space-variable, see Sect. 3 and in particular Lemma 3.1. Using this generalisation, we can show Corollary 3.1, which is in particular helpful for showing the uniqueness of a solution if we know that $\partial_t(g * u)(t) \in \mathrm{L}^2(\Omega)$ for a.a. $t \in (0,T)$. Moreover, we explain how to discretize the convolution in time (see Eq. (17)), and we derive a discrete estimate (see Lemma 3.3), which will be useful when establishing the a priori estimates later in the article.

In Sect. 4, the assumptions on the data are discussed and the weak formulation of (2) is derived. Afterwards, in Sect. 5, we propose a time-discrete scheme to approximate the solution at a single timestep. Moreover, a priori estimates are derived in the remainder of this section.

Then, the existence of a solution to the variational problem is discussed in Sect. 6. Under low regularity assumptions on the data (we suppose that $\tilde{u}_0 \in \mathrm{H}^1_0(\Omega)$ and consider natural assumptions on f), we show in Theorem 6.1 the existence of a solution to problem (2) in $\mathrm{L}^2((0,T),\mathrm{H}^1_0(\Omega))$. Then, using the strongly positive definiteness of the kernel, we establish the uniqueness of a solution in Theorem 6.2. However, we are not able to show that $u(0) = \tilde{u}_0$ for the Caputo fractional derivative with variable order. Next, in Sect. 7, we show how to overcome this problem if $\beta(\mathbf{x}) = \beta$ with $\beta \in (0,1)$ by showing that $u \in \mathrm{C}([0,T],\mathrm{H}^1_0(\Omega)^*)$ in this situation independent of the order of the fractional derivative. We slightly improve the results stated in [37], see Remark 1.1 for more details. Moreover, we study under which assumptions we can obtain a solution to problem (2) in $\mathrm{L}^2((0,T),\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega))$ if $\beta(\mathbf{x}) = \beta$ and $\boldsymbol{A} = a$. The advantage here is that we can use Corollary 3.1 when establishing the uniqueness of a solution. Next, we show that if $\tilde{u}_0 \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ the solution can belong to $\mathrm{C}([0,T],\mathrm{L}^2(\Omega))$ under appropriate assumptions. We conclude this section with a discussion of the performance of the proposed scheme in the constant-order case. To improve the computational results, we consider a convolution quadrature on a graded mesh and compare the results with the well-known $L1$-algorithm.

Finally, in Sect. 8, we consider a domain Ω consisting of multiple disjunct regions. The existence results obtained before are still valid in the case that $\tilde{u}_0 \in \mathrm{H}_0^1(\Omega)$ but some interface conditions are needed. We also discuss in this section the $\mathrm{L}^2((0,T),\mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega))$ regularity of the solution in this setting.

Remark 1.1 (Comparison with [37]) In this paper, the positive definiteness of the kernel and the theory on Volterra equations is used to establish the uniqueness of a solution to problem (2). This approach is more careful than using [26, Corollary 16] as is done in [37, Theorem 2.1] (for constant and distributed order of the fractional derivative) since it is not clear that $\partial_t(g*u)(t) \in \mathrm{L}^2(\Omega)$ for $t \in (0,T)$ and that u is absolutely continuous under the assumptions of [37, Theorem 2.1]. This is the reason why we consider $\partial_t(g*(u-\tilde{u}_0))$ in the formulation of problem (2) instead of $g*\partial_t u$ in [37, Eq. (2)]. Moreover, although the constant-order fractional derivative is a special case, the space-dependent variable order considered in this contribution leads to a more complicated analysis in comparison with [37].

We conclude this introduction by stating the function spaces used in the paper.

Remark 1.2 (Additional notations) Consider an abstract Banach space X with norm $\|\cdot\|_X$. Let $p \geq 1$.

- $(\cdot,\cdot)$: standard inner product in $\mathrm{L}^2(\Omega)$ with induced norm $\|\cdot\|$;
- $\mathrm{L}^p((0,T),X)$: space of measurable functions $u:(0,T)\to X$ such that

$$\|u\|_{\mathrm{L}^p((0,T),X)} = \left(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} < \infty;$$

- $\mathrm{L}^2_{\mathrm{loc}}((0,\infty),X)$: space of functions belonging to $\mathrm{L}^2((0,T),X)$ for any $T \in (0,\infty)$;
- $\mathrm{C}([0,T],X)$: space of continuous functions $u:[0,T]\to X$ satisfying

$$\|u\|_{\mathrm{C}([0,T],X)} = \max_{t\in[0,T]} \|u(t)\|_X < \infty;$$

- $\mathrm{L}^\infty((0,T),X)$: space of measurable functions $u:(0,T)\to X$ that are essentially bounded, which is

$$\|u\|_{\mathrm{L}^\infty((0,T),X)} = \inf\left\{B : \|u(t)\|_X \leq B \text{ for almost all } t \in (0,T)\right\} < +\infty.$$

- $\mathrm{H}^1((0,T),X)$: space of functions $u:(0,T)\to X$ such that the weak derivative u' exists and

$$\|u\|_{\mathrm{H}^1((0,T),X)} = \left(\int_0^T \|u(t)\|_X^2 + \|u'(t)\|_X^2 \,\mathrm{d}t\right)^{\frac{1}{2}} < \infty.$$

- The values C, ε and C_ε considered throughout the paper are generic and positive constants (independent of the discretization parameter), where ε is arbitrarily small and C_ε arbitrarily large, i.e. $C_\varepsilon = C(1+\varepsilon+\frac{1}{\varepsilon})$.

2 Properties of the singular kernel *g*

In this section, we give the properties of g that will be used throughout the paper. First, note that

$$0 \leq \frac{\min\{1, t^{-\beta_1}\}}{\Gamma(1-\beta_1)} \leq g(\mathbf{x}, t) = \frac{t^{-\beta(\mathbf{x})}}{\Gamma(1-\beta(\mathbf{x}))} \leq \max\{1, t^{-\beta_1}\}, \quad t > 0, \tag{5}$$

since $\Gamma(z) \geq \Gamma(1) = 1$ for all $z \in (0,1)$. Thus

$$g(\cdot, t) \in \mathrm{L}^\infty(\Omega), \quad t > 0.$$

We also obtain that $g(\mathbf{x}, \cdot) \in \mathrm{L}^1(0,T)$ for all $\mathbf{x} \in \Omega$ as follows:

$$\begin{aligned}\int_0^T \left|g(\mathbf{x}, t)\right| \mathrm{d}t &= \frac{1}{\Gamma(1-\beta(\mathbf{x}))} \int_0^T t^{-\beta(\mathbf{x})} \, \mathrm{d}t \\ &= \frac{T^{1-\beta(\mathbf{x})}}{\Gamma(2-\beta(\mathbf{x}))} \leq 2\max\{1, T\},\end{aligned} \tag{6}$$

since $\Gamma(z) \geq \frac{1}{2}$ for $z \in (1,2)$. It is also clear that $g(\mathbf{x}, \cdot) \in \mathrm{L}^1_{\mathrm{loc}}(0,\infty)$ for all $\mathbf{x} \in \Omega$. Moreover, consider $[t_1, t_2] \subset (0,T)$, then

$$\begin{aligned}\int_{t_1}^{t_2} \left|\partial_t g(\mathbf{x}, t)\right| \mathrm{d}t &= \frac{\beta(\mathbf{x})}{\Gamma(1-\beta(\mathbf{x}))} \int_{t_1}^{t_2} t^{-\beta(\mathbf{x})-1} \, \mathrm{d}t \\ &= \frac{t_1^{-\beta(\mathbf{x})} - t_2^{-\beta(\mathbf{x})}}{\Gamma(1-\beta(\mathbf{x}))} \\ &\leq \max\{1, t_1^{-\beta_1}\} - \frac{\min\{1, t_2^{-\beta_1}\}}{\Gamma(1-\beta_1)},\end{aligned} \tag{7}$$

i.e. $\partial_t g(\mathbf{x}, \cdot) \in \mathrm{L}^1_{\mathrm{loc}}(0,T)$. Moreover, as g is decreasing in time, we have the existence of $\tilde{g}$ such that

$$g(\mathbf{x}, t) \geq \frac{\min\{1, T^{-\beta_1}\}}{\Gamma(1-\beta_1)} =: \tilde{g} > 0, \quad t \in (0,T], \mathbf{x} \in \Omega. \tag{8}$$

We note also that the following shift formula is valid for $s > 0$:

$$\int_0^T \left|g(\mathbf{x}, t+s) - g(\mathbf{x}, t)\right| \mathrm{d}t \leq 4\max\{s^{1-\beta_1}, s\}, \quad \mathbf{x} \in \Omega, \tag{9}$$

since

$$\begin{aligned}\int_0^T \left|g(\mathbf{x}, t+s) - g(\mathbf{x}, t)\right| \mathrm{d}t &= \frac{1}{\Gamma(1-\beta(\mathbf{x}))} \int_0^T \left(t^{-\beta(\mathbf{x})} - (t+s)^{-\beta(\mathbf{x})}\right) \mathrm{d}t \\ &= \frac{1}{\Gamma(2-\beta(\mathbf{x}))} \left[T^{1-\beta(\mathbf{x})} - (T+s)^{1-\beta(\mathbf{x})} + s^{1-\beta(\mathbf{x})}\right] \\ &\leq 4 s^{1-\beta(\mathbf{x})},\end{aligned}$$

using that the function $f(x) = x^\alpha$ with $\alpha \in (0,1]$ is α-Hölder continuous.

Another important property of g is contained in the following lemma.

Lemma 2.1 (Strongly positive definiteness) *For all* $v \in \mathrm{L}^2_{\mathrm{loc}}((0,\infty),\mathrm{L}^2(\Omega))$, *the kernel* g *satisfies*

$$\int_0^t \big((g * v)(s), v(s)\big)\,\mathrm{d}s \geq \int_\Omega \gamma(\mathbf{x}) \int_0^t (e * v)^2(\mathbf{x},s)\,\mathrm{d}s\,\mathrm{d}\mathbf{x}$$

with

$$e(t) = \mathrm{e}^{-t} \quad \textit{and} \quad \gamma(\mathbf{x}) = \cos\left(\left[1-\beta(\mathbf{x})\right]\frac{\pi}{2}\right).$$

Proof First, we note that we can interchange the order of integration as

$$\begin{aligned}\left|\int_\Omega \int_0^t (g * v)(\mathbf{x},s) v(\mathbf{x},s)\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\right| &\leq \int_\Omega \left\|(g * v)(\mathbf{x})\right\|_{\mathrm{L}^2(0,T)} \left\|v(\mathbf{x})\right\|_{\mathrm{L}^2(0,T)}\,\mathrm{d}\mathbf{x} \\ &\overset{(\star)}{\leq} \int_\Omega \left\|g(\mathbf{x})\right\|_{\mathrm{L}^1(0,T)} \left\|v(\mathbf{x})\right\|^2_{\mathrm{L}^2(0,T)}\,\mathrm{d}\mathbf{x} \\ &\overset{(6)}{\leq} 2\max\{1,T\}\|v\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))},\end{aligned}$$

where we used Young's inequality for convolutions at position $(\star)$:

$$\|f_1 * f_2\|_{\mathrm{L}^r(0,T)} \leq \|f_1\|_{\mathrm{L}^p(0,T)}\|f_2\|_{\mathrm{L}^q(0,T)} \quad \text{for } \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1 \text{ with } 1 \leq p,q \leq r \leq \infty.$$

The function g is a strongly positive definite kernel for every $\mathbf{x} \in \Omega$ since it satisfies

$$g(\mathbf{x},t) \geq 0, \qquad \partial_t g(\mathbf{x},t) \leq 0, \qquad \partial_{tt} g(\mathbf{x},t) \geq 0, \quad \forall t > 0; \qquad \partial_t g(\mathbf{x},t) \not\equiv 0, \tag{10}$$

see [43, Corollary 2.2] or [44], i.e. for any $\mathbf{x} \in \Omega$, there exists a constant $\gamma > 0$ (varying with $\mathbf{x}$) such that $g(\mathbf{x},t) - \gamma\exp(-t)$ is of positive type and thus

$$\int_\Omega \int_0^t \left(\int_0^s \left[g(\mathbf{x},t-\xi) - \gamma(\mathbf{x})\exp\big(-(t-\xi)\big)\right] v(\mathbf{x},\xi)\,\mathrm{d}\xi\right) v(\mathbf{x},s)\,\mathrm{d}s\,\mathrm{d}\mathbf{x} \geq 0,$$

which is valid for all $v \in \mathrm{L}^2_{\mathrm{loc}}((0,\infty),\mathrm{L}^2(\Omega))$. From [43, Corollary 2.1], it follows that $\gamma(\mathbf{x}) = \cos([1-\beta(\mathbf{x})]\frac{\pi}{2})$ since for any $y \in (-\infty,\infty)$ and a.a. $\mathbf{x} \in \Omega$ we have that

$$\begin{aligned}\liminf_{x \searrow 0} \Re\left[\mathcal{L}\{g(\mathbf{x},t)\}(x+iy)\right] &= \Re\left[(iy)^{\beta(\mathbf{x})-1}\right] \\ &= |y|^{\beta(\mathbf{x})-1}\cos\left(\left[1-\beta(\mathbf{x})\right]\frac{\pi}{2}\right) \\ &\geq \frac{1}{1+y^2}\cos\left(\left[1-\beta(\mathbf{x})\right]\frac{\pi}{2}\right),\end{aligned}$$

where we used that

$$\ln(iy) = \begin{cases} \ln(y) + \frac{i\pi}{2} & y > 0, \\ \ln(-y) - \frac{i\pi}{2} & y < 0. \end{cases}$$

We conclude the proof by noting that for a.a. $\mathbf{x} \in \Omega$ it holds that

$$\int_0^t (e * v)(\mathbf{x}, s) v(\mathbf{x}, s) \, ds = \frac{1}{2}(e * v)^2(\mathbf{x}, t) + \int_0^t (e * v)^2(\mathbf{x}, s) \, ds \geq 0. \qquad \square$$

Remark 2.1 If $\beta(\mathbf{x}) = \beta \in (0,1)$, then we have for all $v \in L^2_{\text{loc}}((0,\infty), L^2(\Omega))$ that

$$\int_0^t \big((g * v)(s), v(s)\big) \, ds \geq \tilde{\gamma} \int_0^t \big((e^{-t} * v)(s), v(s)\big) \, ds, \quad \tilde{\gamma} := \cos\left([1-\beta]\frac{\pi}{2}\right). \tag{11}$$

3 Technical lemmas

First, we generalise a fundamental identity for integro-differential operators of the form $\frac{d}{dt}(k * v)(t)$ from Zacher (see [25, Lemma 2.1] or [45, Lemma 2.3.2]) to a convolution kernel that depends on the time- and space-variable.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $T > 0$. Then, for any $v \in L^2((0,T), L^2(\Omega))$ and any $k : \Omega \times [0,T] \to \mathbb{R}$ satisfying*[1]

$$\big|k(\mathbf{x}, 0)\big| \leq C_1, \qquad \big|\partial_t k(\mathbf{x}, t)\big| \leq C_2, \quad \text{for a.a. } \mathbf{x} \in \Omega \text{ and for a.a. } t \in [0,T],$$

it holds for all $t \in (0,T)$ that

$$\begin{aligned}\big(\partial_t(k * v)(t), v(t)\big) &= \frac{1}{2}\partial_t \int_\Omega \big(k * v^2\big)(\mathbf{x}, t) \, d\mathbf{x} + \frac{1}{2}\big(k(t)v(t), v(t)\big) \\ &\quad - \frac{1}{2}\int_0^t \big(\partial_t k(t)\big[v(t) - v(t-s)\big], v(t) - v(t-s)\big) \, ds.\end{aligned}$$

Proof First note that from

$$k(\mathbf{x}, t) = k(\mathbf{x}, 0) + \int_0^t \partial_t k(\mathbf{x}, s) \, ds,$$

it follows for a.a. $\mathbf{x} \in \Omega$ and all $t \in [0,T]$ that

$$\big|k(\mathbf{x}, t)\big| \leq C_1 + C_2 T.$$

From Leibniz's rule for differentiation under the integral sign, we get that

$$\big(\partial_t(k * v)(t), v(t)\big) = \big(k(0)v(t), v(t)\big) + \left(\int_0^t (\partial_t k)(s) v(t-s) \, ds, v(t)\right),$$

which is valid and well-defined for all $t \in (0,T)$. The second term on the right-hand side (RHS) can be handled as follows:

$$\begin{aligned}&\int_0^t \big((\partial_t k)(s) v(t-s), v(t)\big) \, ds \\ &\quad = \int_0^t \big(\big[-(\partial_t k)(s)\big]\big[v(t) - v(t-s)\big], v(t)\big) \, ds + \int_0^t \big((\partial_t k)(s) v(t), v(t)\big) \, ds\end{aligned}$$

[1] Only the right-hand and the left-hand derivatives need to exist at the boundary points $t = 0$ and $t = T$, respectively.

$$= \int_0^t \big(\big[-(\partial_t k)(s)\big]\big[v(t) - v(t-s)\big], v(t) - v(t-s)\big)\,\mathrm{d}s$$
$$+ \big(\big[k(t) - k(0)\big]v(t), v(t)\big) + \int_0^t \big(\big[-(\partial_t k)(s)\big]\big[v(t) - v(t-s)\big], v(t-s)\big)\,\mathrm{d}s,$$

where we switched the order of integration twice, which is allowed since $\partial_t k$ is uniformly bounded. The last integral on the RHS of the previous equation can be rewritten as follows:

$$- \int_0^t \big(v(t) - v(t-s), (\partial_t k)(s) v(t-s)\big)\,\mathrm{d}s$$
$$= -\big(v(t), (\partial_t k * v)(t)\big) \pm \int_\Omega k(\mathbf{x}, 0) v^2(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} + \int_0^t \int_\Omega (\partial_t k)(\mathbf{x}, s) v^2(\mathbf{x}, t-s)\,\mathrm{d}s$$
$$= -\big(v(t), \partial_t (k * v)(t)\big) + \int_\Omega \partial_t \big(k * v^2\big)(\mathbf{x}, t)\,\mathrm{d}\mathbf{x},$$

where we again exchanged the order of integration two times and used Leibniz's rule two times. Combining the previous results concludes the proof. □

The function g does not satisfy the conditions in the previous lemma, and therefore the lemma cannot be applied directly. It is the following lemma that will lead to the useful Corollary 3.1. Note that a similar result for a solely time-dependent kernel can be found in [46, Lemma 2.1].

Lemma 3.2 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $T > 0$. Then, for any $v : [0, T] \to \mathrm{L}^2(\Omega)$ satisfying $v \in \mathrm{H}^1((0, T), \mathrm{L}^2(\Omega))$ and any $k : \Omega \times (0, T] \to \mathbb{R}$ with*

$$\big\|k(\mathbf{x}, \cdot)\big\|_{\mathrm{L}^1(0,T)} \le C_1, \quad \textit{for a.a. } \mathbf{x} \in \Omega,$$
$$\partial_t k(\mathbf{x}, \cdot) \in \mathrm{L}^1_{\mathrm{loc}}(0, T), \quad \textit{for a.a. } \mathbf{x} \in \Omega,$$

and

$$k(\mathbf{x}, t) \ge 0, \quad \textit{and} \quad \partial_t k(\mathbf{x}, t) \le 0, \quad \textit{for all } (\mathbf{x}, t) \in \Omega \times (0, T),$$

it holds for all $\eta \in [0, T]$ that

$$\int_0^\eta \big(\partial_t (k * v)(t), v(t)\big)\,\mathrm{d}t \ge \frac{1}{2} \int_\Omega \big(k * v^2\big)(\mathbf{x}, \eta)\,\mathrm{d}\mathbf{x} + \frac{1}{2} \int_0^\eta \big\|\sqrt{k(t)}\, v(t)\big\|^2\,\mathrm{d}t$$
$$\ge \frac{1}{2} \int_\Omega \big(k * v^2\big)(\mathbf{x}, \eta)\,\mathrm{d}\mathbf{x} + \frac{1}{2} \int_0^\eta \big\|\sqrt{k(\eta)}\, v(t)\big\|^2\,\mathrm{d}t.$$

Proof From $\partial_t k(\mathbf{x}, \cdot) \in \mathrm{L}^1_{\mathrm{loc}}(0, T)$ it follows that $k(\mathbf{x}, \cdot)$ is continuous on $(0, T)$. The only singularity of $k(\mathbf{x}, \cdot)$ is in $t = 0$ since it is a decreasing function in the time variable, i.e.

$$\lim_{t \searrow 0} k(\mathbf{x}, t) = +\infty \quad \text{for all } \mathbf{x} \in \Omega.$$

Hence, we can define the sequence $(k_n)_{n \in \mathbb{N}}$ by

$$k_n(\mathbf{x}, t) = \min\big\{n, k(\mathbf{x}, t)\big\}, \quad (\mathbf{x}, t) \in \Omega \times [0, T], n \in \mathbb{N}.$$

We obtain from the properties of k that

$$0 \le k_n(\mathbf{x},t) \le n, \quad -C(n) \le \partial_t k_n(\mathbf{x},t) \le 0 \quad \text{for a.a. } (\mathbf{x},t) \in \Omega \times [0,T]$$

and

$$k_n(\mathbf{x},t) \to k(\mathbf{x},t) \quad \text{for all } (\mathbf{x},t) \in \Omega \times (0,T) \text{ as } n \to \infty.$$

Hence, the function k_n satisfies the conditions of Lemma 3.1. Moreover, we also have the uniform boundedness in $\mathrm{L}^2(\Omega)$ and the continuity of v in the time variable, i.e. $v \in \mathrm{C}([0,T],\mathrm{L}^2(\Omega))$. Indeed, from $v(\cdot,t) = v(\cdot,0) + \int_0^t \partial_s v(\cdot,s)\,\mathrm{d}s$, we see that

$$\begin{aligned}\|v(t)\| &\le \|v(0)\| + \int_0^t \|\partial_s v(s)\|\,\mathrm{d}s \\ &\le \|v(0)\| + \sqrt{t}\sqrt{\int_0^t \|\partial_s v(s)\|^2\,\mathrm{d}s} \le \tilde{C} \quad \text{for a.a. } t \in [0,T]\end{aligned}$$

and

$$\|v(t) - v(s)\| \le C|t-s|^{\frac{1}{2}}, \quad \forall t,s \in [0,T].$$

Hence, we have that

$$\lim_{t \searrow 0} \int_\Omega (k_n * v^2)(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = 0, \quad \forall n \in \mathbb{N}.$$

Therefore, integrating the result of Lemma 3.1 over $(0,\eta) \subset (0,T)$ gives that

$$\int_0^\eta \big(\partial_t(k_n * v)(t), v(t)\big)\,\mathrm{d}t \ge \frac{1}{2}\int_\Omega (k_n * v^2)(\mathbf{x},\eta)\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_0^\eta \big\|\sqrt{k_n(t)}v(t)\big\|^2\,\mathrm{d}t. \tag{12}$$

Next, we want to pass to the limit $n \to \infty$ in (12). First, note that

$$\int_0^\eta \big(\partial_t(k_n * v)(t), v(t)\big)\,\mathrm{d}t = \int_0^\eta \big((k_n * \partial_t v)(t) + k_n(t)v(0), v(t)\big)\,\mathrm{d}t. \tag{13}$$

The sequence $f_n : Q_T \to \mathbb{R}$ defined by

$$f_n(\mathbf{x},t) = \left(\int_0^t k_n(\mathbf{x},t-s)\partial_t v(\mathbf{x},s)\,\mathrm{d}s\right) v(\mathbf{x},t)$$

converges almost everywhere pointwise to $(k * \partial_t v)(\mathbf{x},t)v(\mathbf{x},t)$, and it is dominated by

$$\tilde{f} : Q_T \to \mathbb{R} : (\mathbf{x},t) \mapsto \tilde{f}(\mathbf{x},t) = \left(\int_0^t k(\mathbf{x},t-s)\big|\partial_t v(\mathbf{x},s)\big|\,\mathrm{d}s\right)\big|v(\mathbf{x},t)\big|.$$

The function $\tilde{f}$ is absolutely integrable on $(0,\eta) \times \Omega$ since

$$0 \le \int_\Omega \int_0^\eta \left(\int_0^t k(\mathbf{x},t-s)\big|\partial_t v(\mathbf{x},s)\big|\,\mathrm{d}s\right)\big|v(\mathbf{x},t)\big|\,\mathrm{d}t\,\mathrm{d}\mathbf{x}$$

$$\begin{aligned}
&\leq \int_\Omega \sqrt{\int_0^\eta \left(k * |\partial_t v|\right)^2(\mathbf{x},t)\,\mathrm{d}t}\sqrt{\int_0^\eta v^2(\mathbf{x},t)\,\mathrm{d}t}\,\mathrm{d}\mathbf{x} \\
&\overset{(\star)}{\leq} \int_\Omega \left\|k(\mathbf{x})\right\|_{\mathrm{L}^1(0,\eta)} \left\|\partial_t v(\mathbf{x})\right\|_{\mathrm{L}^2(0,\eta)} \left\|v(\mathbf{x})\right\|_{\mathrm{L}^2(0,\eta)}\,\mathrm{d}\mathbf{x} \\
&\leq C_1 \|\partial_t v\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))} \|v\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))},
\end{aligned}$$

where we have used Young's inequality for convolutions (in the time variable) at position $(\star)$. Therefore, we can apply the Lebesgue dominated theorem to obtain for the first term on RHS of (13) that

$$\lim_{n\to\infty} \int_0^\eta \left((k_n * \partial_t v)(t), v(t)\right)\mathrm{d}t = \lim_{n\to\infty} \int_0^\eta \int_\Omega f_n(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_0^\eta \left((k * \partial_t v)(t), v(t)\right)\mathrm{d}t.$$

We perform integration by parts in the time variable on the second term in the RHS of (13) and then pass to the limit $n \to \infty$ in this term, i.e. we consider

$$\begin{aligned}
&\int_\Omega v(\mathbf{x},0) \int_0^\eta k_n(\mathbf{x},t) v(\mathbf{x},t)\,\mathrm{d}t\,\mathrm{d}\mathbf{x} \\
&\quad = \int_\Omega v(\mathbf{x},0)\left[\left(\int_0^\eta k_n(\mathbf{x},s)\,\mathrm{d}s\right) v(\mathbf{x},\eta) \right. \\
&\qquad \left. - \int_0^\eta \left(\int_0^t k_n(\mathbf{x},s)\,\mathrm{d}s\right)\partial_t v(\mathbf{x},t)\,\mathrm{d}t\right]\mathrm{d}\mathbf{x}. \qquad (14)
\end{aligned}$$

The dominating functions are absolutely integrable since

$$0 \leq \int_\Omega \left|v(\mathbf{x},0)\right|\left(\int_0^\eta k(\mathbf{x},s)\,\mathrm{d}s\right)\left|v(\mathbf{x},\eta)\right|\mathrm{d}\mathbf{x} \leq C_1 \left\|v(0)\right\|\left\|v(\eta)\right\| \leq C_1 \tilde{C}\left\|v(0)\right\|$$

and

$$\begin{aligned}
0 &\leq \int_\Omega \left|v(\mathbf{x},0)\right| \int_0^\eta \left(\int_0^t k(\mathbf{x},s)\,\mathrm{d}s\right)\left|\partial_t v(\mathbf{x},t)\right|\mathrm{d}t\,\mathrm{d}\mathbf{x} \\
&\leq C_1\sqrt{T}\left\|v(0)\right\| \|\partial_t v\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))}.
\end{aligned}$$

Hence, again by the Lebesgue dominated theorem, we can take the limit $n \to \infty$ in the RHS of (14). Therefore, we obtain for the second term on the RHS of (13) that

$$\lim_{n\to\infty} \int_0^\eta \left(k_n(t)v(0), v(t)\right)\mathrm{d}t = \int_0^\eta \left(k(t)v(0), v(t)\right).$$

Finally, we need to pass to the limit $n \to \infty$ in the RHS of (12). We apply integration by parts on both terms to obtain that

$$\begin{aligned}
\int_\Omega (k_n * v^2)(\mathbf{x},\eta)\,\mathrm{d}\mathbf{x} &= \int_\Omega \left(\int_0^\eta k_n(\mathbf{x},\eta - s)\,\mathrm{d}s\right) v^2(\mathbf{x},\eta)\,\mathrm{d}\mathbf{x} \\
&\quad - 2\int_\Omega \int_0^\eta \left(\int_0^t k_n(\mathbf{x},\eta - s)\,\mathrm{d}s\right) v(\mathbf{x},t)\partial_t v(\mathbf{x},t)\,\mathrm{d}t\,\mathrm{d}\mathbf{x}
\end{aligned}$$

and

$$\int_\Omega \int_0^\eta k_n(\mathbf{x},t)v^2(\mathbf{x},t)\,\mathrm{d}t\,\mathrm{d}\mathbf{x} = \int_\Omega \left(\int_0^\eta k_n(\mathbf{x},s)\,\mathrm{d}s\right) v^2(\mathbf{x},\eta)\,\mathrm{d}\mathbf{x} \\ - 2\int_\Omega \int_0^\eta \left(\int_0^t k_n(\mathbf{x},s)\,\mathrm{d}s\right) v(\mathbf{x},t)\partial_t v(\mathbf{x},t)\,\mathrm{d}t\,\mathrm{d}\mathbf{x}.$$

We only point out the limit transition for the first term by checking that the dominating functions are absolutely integrable as the second term can be handled similarly. Indeed, we get that

$$0 \le \int_\Omega \left(\int_0^\eta k(\mathbf{x},\eta - s)\,\mathrm{d}s\right) v^2(\mathbf{x},\eta)\,\mathrm{d}\mathbf{x} \le C_1 \left\| v(\eta)\right\|^2 \le C_1 \tilde{C}^2$$

and

$$0 \le \int_\Omega \int_0^\eta \left(\int_0^t k(\mathbf{x},s)\,\mathrm{d}s\right) \left|v(\mathbf{x},t)\right| \left|\partial_t v(\mathbf{x},t)\right|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ \le C_1 \|v\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))} \|\partial_t v\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))}.$$

Using the results above, we can pass to the limit $n \to \infty$ in (12), which concludes the proof. □

The function g defined in (3) satisfies the properties of the kernel in the previous lemma, see Eq. ((6)–(10)). However, the solution u does not satisfy $\partial_t u \in \mathrm{L}^2(0,T)$. Fortunately, we can use [23, Definition 3.1] to obtain the following result, which will be helpful later in showing the uniqueness of a solution to problem (2).

Corollary 3.1 *For any $v : [0,T] \to \mathrm{L}^2(\Omega)$ satisfying*

$$v \in \mathrm{L}^2\big((0,T),\mathrm{L}^2(\Omega)\big) \quad \textit{with } g * v \in \mathrm{H}^1\big((0,T),\mathrm{L}^2(\Omega)\big),$$

it holds for all $\eta \in [0,T]$ that

$$\int_0^\eta \big(\partial_t (g * v)(t), v(t)\big)\,\mathrm{d}t \ge \frac{\tilde{g}}{2}\int_0^\eta \left\|v(t)\right\|^2\,\mathrm{d}t.$$

Proof For any $\mathbf{x} \in \Omega$, by [23, Definition 3.1], we have the existence of a sequence $\{v_n(\mathbf{x},\cdot)\} \subset \mathrm{H}^2(0,T)$ such that $v_n(\mathbf{x},0) = 0$, $v_n(\mathbf{x},\cdot) \to v(\mathbf{x},\cdot)$ in $\mathrm{L}^2(0,T)$ and $\partial_t(g*v_n)(\mathbf{x},\cdot) \to \partial_t(g*v)(\mathbf{x},\cdot)$ in $\mathrm{L}^2(0,T)$. We consider $v = v_n$ and $k = g$ in Lemma 3.2, i.e.

$$\int_\Omega \int_0^\eta \partial_t (g * v_n)(\mathbf{x},t) v_n(\mathbf{x},t)\,\mathrm{d}t\,\mathrm{d}\mathbf{x} \overset{(8)}{\ge} \frac{\tilde{g}}{2}\int_\Omega \int_0^\eta v_n(\mathbf{x},t)^2\,\mathrm{d}t\,\mathrm{d}\mathbf{x}.$$

Next, we can pass to the limit $n \to \infty$ and obtain the result. □

In order to be able to show the well-posedness of problem (2), we prove a discrete version of Lemma 3.2, which is crucial to establishing a priori estimates in Sect. 5. First, we discretize the time interval $[0,T]$ into $n \in \mathbb{N}$ equidistant subintervals $[t_{i-1},t_i]$ with length

$\tau = \frac{T}{n}$. The approximation of a function z at time $t = t_i$, $0 \le i \le n$, is denoted by z_i. The same notation is also used for a given function. Moreover, we approximate $\partial_t z(\mathbf{x}, t_i)$, $1 \le i \le n$, by the backward Euler difference $\delta z_i(\mathbf{x}) = \frac{z_i(\mathbf{x})-z_{i-1}(\mathbf{x})}{\tau}$. Finally, for $k : \Omega \times (0, T] \to \mathbb{R}$ and $v : \Omega \times [0, T] \to \mathbb{R}$ satisfying

$$(k * v)(\mathbf{x}, 0) = \lim_{t \searrow 0} (k * v)(\mathbf{x}, t) = 0, \quad \text{a.a. } \mathbf{x} \in \Omega, \tag{15}$$

and

$$\int_\Omega \left(k * v^2\right)(\mathbf{x}, 0) \,\mathrm{d}\mathbf{x} = \lim_{t \searrow 0} \int_\Omega \left(k * v^2\right)(\mathbf{x}, t) \,\mathrm{d}\mathbf{x} = 0, \tag{16}$$

the time-discrete convolution is defined as follows (a.a. $\mathbf{x} \in \Omega$):

$$(k * v)(\mathbf{x}, t_i) \approx (k * v)_i(\mathbf{x}) := \sum_{l=1}^{i} k_{i+1-l}(\mathbf{x}) v_l(\mathbf{x}) \tau, \tag{17}$$

with

$$(k * v)_0(\mathbf{x}) := 0 \quad \text{and} \quad \int_\Omega \left(k * v^2\right)_0(\mathbf{x}) \,\mathrm{d}\mathbf{x} := 0.$$

In (17), a possible singularity of k at $t = 0$ is avoided. The proof of the following lemma follows the same lines as the proof of [46, Lemma 3.2] for a solely time-dependent kernel.

Lemma 3.3 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $(v_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$ be sequences of real-valued functions defined on Ω. Suppose that $(v_i)_{i\in\mathbb{N}}$ in $\mathrm{L}^2(\Omega)$ and that the sequence $(k_i)_{i\in\mathbb{N}}$ is positive, uniformly bounded and decreasing, i.e. for a.a. $\mathbf{x} \in \Omega$ it holds that*

$$0 \le k_1(\mathbf{x}) \le C, \quad k_{i+1}(\mathbf{x}) \le k_i(\mathbf{x}), \quad \forall i \in \mathbb{N}.$$

Then

$$\sum_{i=1}^{j} \left(\delta (k * v)_i, v_i\right) \tau \ge \frac{1}{2} \int_\Omega \left(k * v^2\right)_j(\mathbf{x}) \,\mathrm{d}\mathbf{x} + \frac{1}{2} \sum_{i=1}^{j} \|\sqrt{k_i} v_i\|^2 \tau, \quad j \in \mathbb{N},$$

where $(k * v)_i$ is the time-discrete convolution defined in (17).

Proof The result follows from multiplying the following inequality with τ, integrating it over Ω and summing up the result over $i = 1, \dots, j$ (note that $\int_\Omega (k * v^2)_0(\mathbf{x}) \,\mathrm{d}\mathbf{x} = 0$):

$$2\delta (k * v)_i(\mathbf{x}) v_i(\mathbf{x}) \ge \delta \left(k * v^2\right)_i(\mathbf{x}) + k_i(\mathbf{x}) v_i^2(\mathbf{x}), \quad i \in \mathbb{N}, \mathbf{x} \in \Omega.$$

We prove that this inequality is satisfied. First, notice that for a.a. $\mathbf{x} \in \Omega$ and $i \ge 1$, it holds that

$$\begin{aligned} &\delta (k * v)_i(\mathbf{x}) \\ &\quad = k_1(\mathbf{x}) v_i(\mathbf{x}) + \sum_{l=1}^{i-1} \delta k_{i+1-l}(\mathbf{x}) v_l(\mathbf{x}) \tau \quad \text{as } (k * v)_0 = 0 \text{ in } \Omega, \end{aligned} \tag{18}$$

$$= k_i(\mathbf{x})v_0(\mathbf{x}) + \sum_{l=1}^{i} k_{i+1-l}(\mathbf{x})\delta v_l(\mathbf{x})\tau = k_i(\mathbf{x})v_0(\mathbf{x}) + (k * \delta v)_i(\mathbf{x}). \tag{19}$$

From the properties of the sequence $(k_i)_{i\in\mathbb{N}}$, it follows for a.a. $\mathbf{x} \in \Omega$ that

$$\begin{aligned}
&\delta\big(k * v^2\big)_i(\mathbf{x}) + k_i(\mathbf{x})v_i^2(\mathbf{x}) \\
&\le \delta\big(k * v^2\big)_i(\mathbf{x}) + k_i(\mathbf{x})v_i^2(\mathbf{x}) - \sum_{l=1}^{i-1} \underbrace{\delta k_{i+1-l}(\mathbf{x})}_{\le 0}\big(v_i(\mathbf{x}) - v_l(\mathbf{x})\big)^2\tau \\
&\overset{(18)}{=} \big(k_1(\mathbf{x}) + k_i(\mathbf{x})\big)v_i^2(\mathbf{x}) + \sum_{l=1}^{i-1} \delta k_{i+1-l}(\mathbf{x})\big[v_l^2(\mathbf{x}) - \big(v_i(\mathbf{x}) - v_l(\mathbf{x})\big)^2\big]\tau \\
&= \big(k_1(\mathbf{x}) + k_i(\mathbf{x})\big)v_i^2(\mathbf{x}) + 2v_i(\mathbf{x})\sum_{l=1}^{i-1} \delta k_{i+1-l}(\mathbf{x})v_l(\mathbf{x})\tau - v_i^2(\mathbf{x})\underbrace{\sum_{l=1}^{i-1}\delta k_{i+1-l}(\mathbf{x})\tau}_{=k_i(\mathbf{x})-k_1(\mathbf{x})} \\
&\overset{(18)}{=} 2\delta(k * v)_i(\mathbf{x})v_i(\mathbf{x}).
\end{aligned}$$

□

4 Assumptions, weak formulation and uniqueness of a solution

In this section, we first summarise all assumptions that are necessary to obtain the well-posedness result in Theorem 6.1. We assume that

- **AS-1:** $\boldsymbol{A} = (a_{ij}(\mathbf{x}, t))$ is a $d \times d$ matrix-valued function such that

$$\boldsymbol{A} \in \big(\mathrm{L}^\infty(Q_T)\big)^{d\times d} \quad \text{and} \quad \boldsymbol{A}^T = \boldsymbol{A};$$

- **AS-2:** The matrix $\boldsymbol{A}$ is uniformly elliptic, i.e. there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(\mathbf{x}, t)\xi_i\xi_j \ge \alpha|\boldsymbol{\xi}|^2 \quad \text{for a.a. } (\mathbf{x}, t) \in Q_T \text{ and for all } \boldsymbol{\xi} \in \mathbb{R}^d;$$

- **AS-3:** $c \in \mathrm{L}^\infty(Q_T)$ and

$$c(\mathbf{x}, t) \ge 0, \quad (\mathbf{x}, t) \in Q_T;$$

- **AS-4:** $\partial_t\boldsymbol{A} \in (\mathrm{L}^\infty(Q_T))^{d\times d}$ and $\partial_t c \in \mathrm{L}^\infty(Q_T)$;
- **AS-5:** $f \in \mathrm{H}^1((0, T), \mathrm{L}^2(\Omega))$ or

$$\left\|\frac{\partial^\ell}{\partial t^\ell}f(t)\right\| \le C\big(1 + t^{\gamma-\ell}\big) \quad \text{for } \ell = 0, 1 \text{ and } \gamma \in (0, 1);$$

- **AS-6:** $\tilde{u}_0 \in \mathrm{H}_0^1(\Omega)$.

Further, we associate a bilinear form $\mathcal{L}$ with the differential operator L defined in (1) as follows:

$$\mathcal{L}(t)\big(u(t), \varphi\big) := (Lu, \varphi) = \big(\boldsymbol{A}(t)\nabla u(t), \nabla\varphi\big) + \big(c(t)u(t), \varphi\big),$$

with $u(t), \varphi \in \mathrm{H}_0^1(\Omega)$. Using the properties above, we obtain that

$$\mathcal{L}(t)(u,\varphi) \le C\|u\|_{\mathrm{H}^1(\Omega)}\|\varphi\|_{\mathrm{H}^1(\Omega)}, \quad \forall u,\varphi \in \mathrm{H}_0^1(\Omega),$$

and

$$\mathcal{L}(t)(\varphi,\varphi) \ge \alpha\|\nabla\varphi\|^2, \quad \forall \varphi \in \mathrm{H}_0^1(\Omega), \tag{20}$$

i.e. the bilinear form $\mathcal{L}$ is continuous, and $\mathrm{H}_0^1(\Omega)$-elliptic due to Friedrichs's inequality.

Now, we define the variational formulation of problem (2).

Definition 4.1 (Weak formulation) Let assumptions AS-(1–6) be fulfilled. Search $u \in \mathrm{L}^\infty((0,T),\mathrm{H}_0^1(\Omega))$ with $\partial_t(g*(u-\tilde{u}_0)) \in \mathrm{L}^2((0,T),\mathrm{H}_0^1(\Omega)^*)$ such that for a.a. $t \in (0,T)$ it holds that

$$\bigl\langle \partial_t\bigl(g*(u-\tilde{u}_0)\bigr)(t),\varphi\bigr\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)} + \mathcal{L}(t)\bigl(u(t),\varphi\bigr) = \bigl(f(t),\varphi\bigr), \quad \forall\varphi \in \mathrm{H}_0^1(\Omega). \tag{21}$$

Next, we show that conditions (15) and (16) with $k = g$ are satisfied for $v = u$ when the solution u satisfies the regularity assumptions in Definition 4.1. First, we have by Hölder's inequality that

$$\begin{aligned}
&\lim_{t\searrow 0}\bigl\|(g*u)(t)\bigr\|^2 \\
&\quad= \lim_{t\searrow 0}\int_\Omega\left|\int_0^t g(\mathbf{x},t-s)u(\mathbf{x},s)\,\mathrm{d}s\right|^2\mathrm{d}\mathbf{x} \\
&\quad\overset{(5)}{\le} \lim_{t\searrow 0}\frac{1}{\Gamma(1-\beta_1)^2}\int_\Omega\left(\int_0^t (t-s)^{-\beta_1}\bigl|u(\mathbf{x},s)\bigr|\,\mathrm{d}s\right)^2\mathrm{d}\mathbf{x} \\
&\quad\le \lim_{t\searrow 0}\int_\Omega\left(\int_0^t (t-s)^{-\beta_1}\,\mathrm{d}s\right)\left(\int_0^t (t-s)^{-\beta_1}\bigl|u(\mathbf{x},s)\bigr|^2\,\mathrm{d}s\right)\mathrm{d}\mathbf{x} \\
&\quad\le \lim_{t\searrow 0}\left(\int_0^t (t-s)^{-\beta_1}\,\mathrm{d}s\right)\left(\int_0^t (t-s)^{-\beta_1}\left(\int_\Omega\bigl|u(\mathbf{x},s)\bigr|^2\,\mathrm{d}\mathbf{x}\right)\mathrm{d}s\right) \\
&\quad\le \|u\|_{\mathrm{L}^\infty((0,T),\mathrm{L}^2(\Omega))}\lim_{t\searrow 0}\left(\int_0^t r^{-\beta_1}\,\mathrm{d}r\right)^2 = 0.
\end{aligned} \tag{22}$$

Secondly, we see that

$$\begin{aligned}
\lim_{t\searrow 0}\left|\int_\Omega (g*u^2)(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\right| &\overset{(5)}{\le} \lim_{t\searrow 0}\frac{1}{\Gamma(1-\beta_1)}\int_\Omega\int_0^t (t-s)^{-\beta_1}\bigl|u(\mathbf{x},s)\bigr|^2\,\mathrm{d}s\,\mathrm{d}\mathbf{x} \\
&\le \frac{1}{\Gamma(1-\beta_1)}\|u\|_{\mathrm{L}^\infty((0,T),\mathrm{L}^2(\Omega))}\lim_{t\searrow 0}\int_0^t r^{-\beta_1}\,\mathrm{d}r = 0.
\end{aligned} \tag{23}$$

Thus, conditions (22) and (23) are satisfied for $v = u$.

5 Time discretization

In this section, a time-discrete numerical scheme to solve problem (21) is presented. The time interval $[0,T]$ is discretized into $n \in \mathbb{N}$ equidistant subintervals $[t_{i-1},t_i]$ with length

$\tau = \frac{T}{n} < 1$. The approximation of u at time $t = t_i$ $(0 \le i \le n)$ is denoted by u_i. Moreover, the time derivative at time $t = t_i$ is approximated by the backward Euler finite-difference formula

$$\partial_t u(t_i) \approx \delta u_i = (u_i - u_{i-1})/\tau, \quad 1 \le i \le n.$$

These notations are also used for any function $z \neq u$. We propose the following time-discrete variational problem:

$$\begin{aligned}&\text{Find } u_i \in \mathrm{H}^1_0(\Omega), i = 1,2,\dots,n, \text{ such that}\\ &\quad \big((g * \delta u)_i, \varphi\big) + \mathcal{L}_i(u_i,\varphi) = (f_i,\varphi), \quad \forall \varphi \in \mathrm{H}^1_0(\Omega).\end{aligned} \tag{24}$$

Using the time-discrete convolution (17), the discrete problem can be equivalently written as

$$a_i(u_i,\varphi) = \langle F_i,\varphi\rangle, \quad \forall \varphi \in \mathrm{H}^1_0(\Omega), \tag{25}$$

with

$$a_i(u_i,\varphi) := \big(g(\tau)u_i,\varphi\big) + \mathcal{L}_i(u_i,\varphi)$$

and

$$\langle F_i,\varphi\rangle := (f_i,\varphi) + \big(g(\tau)u_{i-1},\varphi\big) - \sum_{k=1}^{i-1}(g_{i+1-k}\delta u_k,\varphi)\tau.$$

The summation occurring in F_i is not contributing for $i = 1$. The well-posedness of this problem under appropriate assumptions on the data is stated in the following lemma.

Lemma 5.1 *Suppose that AS-1, AS-2 and AS-3 are satisfied. Moreover, assume that* $\tilde{u}_0 \in \mathrm{L}^2(\Omega)$ *and* $f \in \mathrm{L}^2([0,T],\mathrm{L}^2(\Omega))$. *Then, for any* $i = 1,2,\dots,n$, *there exists unique* $u_i \in \mathrm{H}^1_0(\Omega)$ *solving* (24).

Proof From the properties of $\mathcal{L}$ and g, it follows that the bilinear form a_i is $\mathrm{H}^1_0(\Omega)$-elliptic and continuous. If $\tilde{u}_0,\dots,u_{i-1} \in \mathrm{L}^2(\Omega)$, then the linear functional F_i is bounded since

$$\begin{aligned}\big|\langle F_i,\varphi\rangle\big| &\overset{(5)}{\le} \|f_i\|\|\varphi\| + \tau^{-\beta_1}\|u_{i-1}\|\|\varphi\| + \tau^{-\beta_1}\|\varphi\|\sum_{k=1}^{i-1}\|u_k - u_{k-1}\|\\ &\le C\big(\tau^{-\beta_1}\big)\|\varphi\|_{\mathrm{H}^1(\Omega)}.\end{aligned}$$

The existence and uniqueness of $u_i \in \mathrm{H}^1_0(\Omega)$ to problem (25) follows from the Lax–Milgram lemma. □

We prove some a priori estimates in the following two lemmas. These are required to ensure the existence of a solution to (21) and to prove the convergence of approximations towards this solution.

Lemma 5.2 *Let the assumptions of Lemma 5.1 be fulfilled. Then a positive constant C exists such that, for every $j = 1, 2, \dots, n$, the following relation holds*:

$$\int_\Omega (g * u^2)_j(\mathbf{x})\,\mathrm{d}\mathbf{x} + \sum_{i=1}^{j} \|\sqrt{g_i}u_i\|^2\tau + \sum_{i=1}^{j} \|\nabla u_i\|^2\tau \le C.$$

Proof We set $\varphi = u_i\tau$ in (24) and sum it up for $i = 1, \dots, j$ with $1 \le j \le n$. Using relation (19), we obtain that

$$\sum_{i=1}^{j} \big(\delta(g * u)_i, u_i\big)\tau + \sum_{i=1}^{j} \mathcal{L}_i(u_i, u_i)\tau = \sum_{i=1}^{j} (f_i, u_i)\tau + \sum_{i=1}^{j} (g_i\tilde{u}_0, u_i)\tau.$$

Using the ε-Young inequality, we estimate the second term in the RHS as follows:

$$\begin{aligned}\left|\sum_{i=1}^{j} (g_i\tilde{u}_0, u_i)\tau\right| &\le C_{\varepsilon_1} \int_\Omega \tilde{u}_0^2(\mathbf{x}) \left(\sum_{i=1}^{j} g_i(\mathbf{x})\tau\right) \mathrm{d}\mathbf{x} + \varepsilon_1 \sum_{i=1}^{j} \tau \int_\Omega g_i(\mathbf{x}) u_i^2(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &\overset{(6)}{\le} C_{\varepsilon_1}\|\tilde{u}_0\|^2 + \varepsilon_1 \sum_{i=1}^{j} \|\sqrt{g_i}u_i\|^2\tau.\end{aligned}$$

Using Friedrichs's inequality, we obtain that

$$\left|\sum_{i=1}^{j} (f_i, u_i)\tau\right| \le C_{\varepsilon_2} + \varepsilon_2 \sum_{i=1}^{j} \|\nabla u_i\|^2\tau.$$

Employing Lemma 3.3 and Eq. (20) implies

$$\begin{aligned}&\frac{1}{2}\int_\Omega (g * u^2)_j(\mathbf{x})\,\mathrm{d}\mathbf{x} + \left(\frac{1}{2} - \varepsilon_1\right) \sum_{i=1}^{j} \|\sqrt{g_i}u_i\|^2\tau + \left(\frac{\alpha}{2} - \varepsilon_2\right) \sum_{i=1}^{j} \|\nabla u_i\|^2\tau \\ &\quad\le C_{\varepsilon_1}\|\tilde{u}_0\|^2 + C_{\varepsilon_2}.\end{aligned}$$

Fixing $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ sufficiently small gives the result. □

Lemma 5.3 *Let assumptions AS-*(1–6) *be fulfilled. Then positive constants C and τ_0 exist such that, for any $\tau < \tau_0$ and for every $j = 1, 2, \dots, n$, the following relation holds*:

$$\|\nabla u_j\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \le C.$$

Proof We set $\varphi = \delta u_i\tau$ (here $\tilde{u}_0 \in \mathrm{H}_0^1(\Omega)$ is needed) in (24) and sum the result up for $i = 1, \dots, j$ with $1 \le j \le n$. We obtain that

$$\sum_{i=1}^{j} \big((g * \delta u)_i, \delta u_i\big)\tau + \sum_{i=1}^{j} \mathcal{L}_i(u_i, \delta u_i)\tau = \sum_{i=1}^{j} (f_i, \delta u_i)\tau. \tag{26}$$

The first term on the left-hand side is positive since we can apply [47, Eq. 3.2]. Next, we recall the following per partes formula for a symmetric bilinear form [48, Eq. 3.16]:

$$\begin{aligned}&\sum_{i=1}^{j} r(t_i;z_i,z_i-z_{i-1})\\&\quad=\frac{1}{2}r(t_j;z_j,z_j)-\frac{1}{2}r(0;z_0,z_0)+\frac{1}{2}\sum_{i=1}^{j}\bigl(r(t_i;\delta z_i,\delta z_i)\tau^2-\delta r(t_i;z_{i-1},z_{i-1})\tau\bigr).\end{aligned}\tag{27}$$

Hence, due to the symmetry of $\boldsymbol{A}$, we have that

$$\begin{aligned}&\sum_{i=1}^{j}(\boldsymbol{A}_i\nabla u_i,\nabla\delta u_i)\tau\\&\quad=\frac{1}{2}(\boldsymbol{A}_j\nabla u_j,\nabla u_j)-\frac{1}{2}(\boldsymbol{A}_0\nabla\tilde{u}_0,\nabla\tilde{u}_0)\\&\qquad-\frac{1}{2}\sum_{i=1}^{j}(\delta\boldsymbol{A}_i\nabla u_{i-1},\nabla u_{i-1})\tau+\frac{1}{2}\sum_{i=1}^{j}\bigl(\boldsymbol{A}_i(\nabla u_i-\nabla u_{i-1}),\nabla u_i-\nabla u_{i-1}\bigr),\end{aligned}$$

and thus by Lemma 5.2 we get that

$$\sum_{i=1}^{j}(\boldsymbol{A}_i\nabla u_i,\nabla\delta u_i)\tau\geq\frac{\alpha}{2}\|\nabla u_j\|^2-C+\frac{\alpha}{2}\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2.$$

Similarly, using the conditions on c and Lemma 5.2, we get that

$$\sum_{i=1}^{j}(c_iu_i,\delta u_i)\tau\geq -C.$$

The term on the RHS of (26) can be estimated by using the partial summation rule

$$\sum_{i=1}^{j}(f_i,\delta u_i)\tau=(f_j,u_j)-(f_0,\tilde{u}_0)-\sum_{i=1}^{j}(\delta f_i,u_{i-1})\tau.\tag{28}$$

If $f\in \mathrm{H}^1((0,T),\mathrm{L}^2(\Omega))\subset \mathrm{C}([0,T],\mathrm{L}^2(\Omega))$, then by the ε-Young inequality, Friedrichs's inequality and Lemma 5.2, we get that

$$\left|\sum_{i=1}^{j}(f_i,\delta u_i)\tau\right|\leq C_\varepsilon+\varepsilon\|\nabla u_j\|^2.$$

The result of the lemma follows by fixing ε sufficiently small.

If $\|\frac{\partial^\ell}{\partial t^\ell}f(t)\|\leq C(1+t^{\gamma-\ell})$ for $\ell=0,1$ and $\gamma\in(0,1)$, then

$$\|f(t)\|\leq\|f(0)\|+\int_0^T\|\partial_t f(s)\|\,\mathrm{d}s\leq C\quad\text{for all } t\in[0,T].$$

Moreover, we have that

$$\|\delta f_1\| = \frac{1}{\tau}\|f_1 - f_0\| \leq C\tau^{-1},$$

and by the mean value theorem

$$\|\delta f_i\| = \frac{1}{\tau}\|f_i - f_{i-1}\| \leq C\big(1 + t_{i-1}^{\gamma-1}\big), \quad i \geq 2.$$

Hence, we obtain by (28), the ε-Young inequality, Friedrichs's inequality and $t^{\gamma-1} \in \mathrm{L}^1(0,T)$ that

$$\begin{aligned}
\left|\sum_{i=1}^{j}(f_i, \delta u_i)\tau\right| &\overset{(28)}{\leq} C_\varepsilon + \varepsilon\|\nabla u_j\|^2 + \|\delta f_1\|\|\tilde{u}_0\|\tau + \sum_{i=2}^{j}\|\delta f_i\|\|u_{i-1}\|\tau \\
&\leq C_\varepsilon + \varepsilon\|\nabla u_j\|^2 + C\sum_{i=2}^{j}\big(1 + t_{i-1}^{\gamma-1}\big)\|u_{i-1}\|\tau \\
&\leq C_\varepsilon + \varepsilon\|\nabla u_j\|^2 + C\left(1 + \sum_{i=2}^{j} t_{i-1}^{\gamma-1}\|\nabla u_{i-1}\|^2\tau\right) \\
&\leq C_\varepsilon + \varepsilon\|\nabla u_j\|^2 + C\sum_{i=1}^{j-1} t_{i}^{\gamma-1}\|\nabla u_{i}\|^2\tau.
\end{aligned}$$

Now, the estimate follows from the discrete Grönwall lemma [49, Corollary 15.5]. □

Corollary 5.1 *Let the assumptions of Lemma 5.2 be fulfilled. Then positive constants C and τ_0 exist such that, for any $\tau < \tau_0$ and for every $j = 1, 2, \ldots, n$, the following relation holds*:

$$\big\|(g * \delta u)_j\big\|_{\mathrm{H}_0^1(\Omega)^*} \leq C.$$

Proof The estimate follows from

$$\begin{aligned}
\big\|(g * \delta u)_i\big\|_{\mathrm{H}_0^1(\Omega)^*} &= \sup_{\|\varphi\|_{\mathrm{H}_0^1(\Omega)}=1}\big|\big\langle(g * \delta u)_i, \varphi\big\rangle_{\mathrm{H}_0^1(\Omega)^* \times \mathrm{H}_0^1(\Omega)}\big| \\
&= \sup_{\|\varphi\|_{\mathrm{H}_0^1(\Omega)}=1}\big|\big((g * \delta u)_i, \varphi\big)\big| \\
&\overset{(24)}{=} \sup_{\|\varphi\|_{\mathrm{H}_0^1(\Omega)}=1}\big|(f_i, \varphi) - \mathcal{L}_i(u_i, \varphi)\big| \\
&\leq \|f_i\| + C\|\nabla u_i\|,
\end{aligned}$$

and the result of Lemma 5.2. □

6 Existence

Before showing the existence of a solution in Theorem 6.1, we introduce Rothe's functions u_n and $\overline{u}_n$:

$$u_n : [0,T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} \tilde{u}_0 & t = 0, \\ u_{i-1} + (t - t_{i-1})\delta u_i & t \in (t_{i-1}, t_i], \end{cases} \quad 1 \leq i \leq n,$$

$$\overline{u}_n : [0,T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} \tilde{u}_0 & t = 0, \\ u_i & t \in (t_{i-1}, t_i], \end{cases} \quad 1 \leq i \leq n.$$

Similarly, we define $\overline{g}_n$, $\overline{\mathcal{L}}_n$ and $\overline{f}_n$. Moreover, we define

$$(g * u)_n : [0,T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} 0 & t = 0, \\ (g * u)_{i-1} + (t - t_{i-1})\delta(g * u)_i & t \in (t_{i-1}, t_i], \end{cases}$$

$$\overline{(g * u)}_n : [0,T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} 0 & t = 0, \\ (g * u)_i & t \in (t_{i-1}, t_i]. \end{cases}$$

Further, we use also the notation $\lceil t \rceil_\tau = t_i$ when $t \in (t_{i-1}, t_i]$. Now, using Eq. (19), we can rewrite Eq. (24) on the whole time frame as follows:

$$\big(\partial_t (g * u)_n(t) - \overline{g}_n(t)\tilde{u}_0, \varphi\big) + \overline{\mathcal{L}}_n(t)\big(\overline{u}_n(t), \varphi\big) = \big(\overline{f}_n(t), \varphi\big), \quad \forall \varphi \in \mathrm{H}_0^1(\Omega), \tag{29}$$

where

$$\overline{\mathcal{L}}_n(t)\big(\overline{u}_n(t), \varphi\big) = \big(\overline{\boldsymbol{A}}_n(t)\nabla\overline{u}_n(t), \nabla\varphi\big) + \big(\overline{c}_n(t)\overline{u}_n(t), \varphi\big).$$

Theorem 6.1 (Existence) *Let assumptions AS-(1–6) be fulfilled. Then there exists a weak solution u to problem (21) with $u \in \mathrm{L}^\infty((0,T), \mathrm{H}_0^1(\Omega))$ and $\partial_t(g * (u - \tilde{u}_0)) \in \mathrm{L}^\infty((0,T), \mathrm{H}_0^1(\Omega)^*)$.*

Proof From Lemmas 5.2 and 5.3, we get that the sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ is uniformly bounded and 2-mean equicontinuous in $\mathrm{L}^2((0,T), Q_T) = \mathrm{L}^2((0,T), \mathrm{L}^2(\Omega))$. Therefore, from the Riesz–Frechét–Kolmogorov [50, Theorem 2.13.1] and the reflexivity of the space $\mathrm{L}^2((0,T), \mathrm{H}_0^1(\Omega))$, we have the existence of an element u in $\mathrm{L}^2((0,T), \mathrm{L}^2(\Omega))$ and a subsequence $(\overline{u}_{n_l})_{l\in\mathbb{N}}$ of $(\overline{u}_n)_{n\in\mathbb{N}}$ such that

$$\overline{u}_{n_l} \to u \quad \text{in } \mathrm{L}^2\big((0,T), \mathrm{L}^2(\Omega)\big) \text{ as } l \to \infty \tag{30}$$

and

$$\overline{u}_{n_l} \rightharpoonup u \quad \text{in } \mathrm{L}^2\big((0,T), \mathrm{H}_0^1(\Omega)\big) \text{ as } l \to \infty.$$

Moreover, Lemma 5.3 gives that

$$\overline{u}_{n_l}(t) \rightharpoonup u(t) \quad \text{in } \mathrm{H}_0^1(\Omega) \text{ as } l \to \infty \text{ for all } t \in (0,T).$$

Therefore, $u \in \mathrm{L}^\infty((0,T),\mathrm{H}_0^1(\Omega))$. Now, we integrate Eq. (29) in time over $(0,\eta) \subset (0,T)$ for the resulting subsequence to get that

$$\begin{aligned} &\big((g*u)_{n_l}(\eta),\varphi\big) - \int_0^\eta \big(\overline{g}_{n_l}(t)\tilde{u}_0,\varphi\big)\,\mathrm{d}t \\ &\quad + \int_0^\eta \overline{\mathcal{L}}_{n_l}(t)\big(\overline{u}_{n_l}(t),\varphi\big)\,\mathrm{d}t = \int_0^\eta \big(\overline{f}_{n_l}(t),\varphi\big)\,\mathrm{d}t. \end{aligned} \tag{31}$$

If $\partial_t f \in \mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))$, then for $t \in (t_{i-1},t_i]$, we have that

$$\big\|f_{n_l}(t) - f(t)\big\|_{\mathrm{L}^2(\Omega)} \le C\sqrt{\tau}.$$

If $\|\partial_t f(t)\| \le C(1+t^{\gamma-1})$ with $\gamma \in (0,1)$, then we have that

$$\big\|f_{n_l}(t) - f(t)\big\|_{\mathrm{L}^2(\Omega)} \le C\tau + C\int_t^{t_i} s^{\gamma-1}\,\mathrm{d}s \le C\tau^\gamma$$

by the α-Hölder continuity of $f(x) = x^\alpha$ with $\alpha \in (0,1)$. Therefore,

$$\lim_{l\to\infty} \big\|\overline{f}_{n_l}(t) - f(t)\big\| = 0 \quad \text{for all } t \in (0,T).$$

Similarly, we have that $\|\overline{\boldsymbol{A}}_{n_l}(t) - \boldsymbol{A}(t)\|_{(\mathrm{L}^\infty(\Omega))^{d\times d}} \to 0$ and $\|\overline{c}_{n_l}(t) - c(t)\|_{\mathrm{L}^\infty(\Omega)} \to 0$ for all $t \in [0,T]$ as $l \to \infty$. Hence, employing the results above, for $\eta \in (0,T)$, we see that

$$\left|\int_0^\eta \overline{\mathcal{L}}_{n_l}(t)\big(\overline{u}_{n_l}(t),\varphi\big)\,\mathrm{d}t - \int_0^\eta \mathcal{L}(t)\big(u(t),\varphi\big)\,\mathrm{d}t\right| \to 0 \quad \text{as } l \to \infty,$$

and

$$\left|\int_0^\eta \big(\overline{f}_{n_l}(t),\varphi\big)\,\mathrm{d}t - \int_0^\eta \big(f(t),\varphi\big)\,\mathrm{d}t\right| \to 0 \quad \text{as } l \to \infty.$$

At this moment we are not able to perform the limit transition in the first term of (31). Hence, we integrate Eq. (31) once more in time over $\eta \in (0,\xi) \subset (0,T)$. We get that

$$\begin{aligned} &\int_0^\xi \big((g*u)_{n_l}(\eta),\varphi\big)\,\mathrm{d}\eta - \int_0^\xi \left(\tilde{u}_0 \int_0^\eta \overline{g}_{n_l}(t)\,\mathrm{d}t,\varphi\right)\mathrm{d}\eta \\ &\quad + \int_0^\xi \int_0^\eta \overline{\mathcal{L}}_{n_l}(t)\big(\overline{u}_{n_l}(t),\varphi\big)\,\mathrm{d}t\,\mathrm{d}\eta = \int_0^\xi \int_0^\eta \big(\overline{f}_{n_l}(t),\varphi\big)\,\mathrm{d}t\,\mathrm{d}\eta. \end{aligned} \tag{32}$$

We make the limit transition in three steps:

(i) $\displaystyle \lim_{l\to\infty}\left|\int_0^T \big((g*u)_{n_l}(\eta),\varphi\big)\,\mathrm{d}\eta - \int_0^T \big(\overline{(g*u)}_{n_l}(\eta),\varphi\big)\,\mathrm{d}\eta\right| = 0;$

(ii) $\displaystyle \lim_{l\to\infty}\left|\int_0^T \big(\overline{(g*u)}_{n_l}(\eta),\varphi\big)\,\mathrm{d}\eta - \int_0^T \big((g*\overline{u}_{n_l})(\eta),\varphi\big)\,\mathrm{d}\eta\right| = 0;$

(iii) $\displaystyle \lim_{l\to\infty}\left|\int_0^T \big((g*\overline{u}_{n_l})(\eta),\varphi\big)\,\mathrm{d}\eta - \int_0^T \big((g*u)(\eta),\varphi\big)\,\mathrm{d}\eta\right| = 0.$

We employ Corollary 5.1 to obtain that

$$
\begin{aligned}
&\left|\int_0^T \big((g*u)_{n_l}(\eta),\varphi\big)\,\mathrm{d}\eta - \int_0^T \big(\overline{(g*u)}_{n_l}(\eta),\varphi\big)\,\mathrm{d}\eta\right| \\
&\quad= \left|\sum_{i=1}^{n_l}\int_{t_{i-1}}^{t_i} \big((t-t_i)\delta(g*u)_i,\varphi\big)\,\mathrm{d}t\right| \\
&\quad\overset{(19)}{=} \left|\sum_{i=1}^{n_l}\int_{t_{i-1}}^{t_i} (t-t_i)\times\big((g*\delta u)_i - g_i\tilde{u}_0,\varphi\big)\,\mathrm{d}t\right| \\
&\quad\overset{\varphi\in \mathrm{H}_0^1(\Omega)}{\leq} \sum_{i=1}^{n_l}\tau_{n_l}^2\left|\langle (g*\delta u)_i,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right| + \sum_{i=1}^{n_l}\tau_{n_l}^2\int_\Omega \left|g_i(\mathbf{x},t)\right|\left|\tilde{u}_0(\mathbf{x})\right|\left|\varphi(\mathbf{x})\right|\,\mathrm{d}t \\
&\quad\overset{(5)}{\leq} C\tau_{n_l} + C\tau_{n_l}^{1-\beta_1} \to 0 \quad \text{as } l\to\infty.
\end{aligned}
$$

So the first limit transition is satisfied. For the second limit transition, we first note that $\overline{(g*u)}_{n_l}(t) = (\overline{g}_{n_l} * \overline{u}_{n_l})(\lceil t\rceil_{\tau_{n_l}})$ for any $t\in(0,T)$. Moreover, we have that

$$
\begin{aligned}
&\big((\overline{g}_{n_l} * \overline{u}_{n_l})\big(\lceil t\rceil_{\tau_{n_l}}\big),\varphi\big) - \big((g*\overline{u}_{n_l})(t),\varphi\big) \\
&\quad= \left(\int_t^{\lceil t\rceil_\tau} \overline{g}_{n_l}\big(\lceil t\rceil_\tau - s\big)\overline{u}_{n_l}(s)\,\mathrm{d}s,\varphi\right) \\
&\qquad+ \left(\int_0^t \big[\overline{g}_{n_l}\big(\lceil t\rceil_\tau - s\big) - g(t-s)\big]\overline{u}_{n_l}(s)\,\mathrm{d}s,\varphi\right).
\end{aligned} \tag{33}
$$

We show that

$$
\left|\int_0^T\left(\int_t^{\lceil t\rceil_\tau} \overline{g}_{n_l}\big(\lceil t\rceil_\tau - s\big)\overline{u}_{n_l}(s)\,\mathrm{d}s,\varphi\right)\mathrm{d}t\right| \to 0 \quad \text{as } l\to\infty
$$

and

$$
\left|\int_0^T\left(\int_0^t \big[\overline{g}_{n_l}\big(\lceil t\rceil_\tau - s\big) - g(t-s)\big]\overline{u}_{n_l}(s)\,\mathrm{d}s,\varphi\right)\mathrm{d}t\right| \to 0 \quad \text{as } l\to\infty
$$

such that limit transition (ii) is satisfied. First, we have for $t\in(t_{i-1},t_i]$ that

$$
\begin{aligned}
&\left|\left(\int_t^{\lceil t\rceil_\tau} \overline{g}_{n_l}\big(\lceil t\rceil_\tau - s\big)\overline{u}_{n_l}(s)\,\mathrm{d}s,\varphi\right)\right| \\
&\quad\leq \|\varphi\|\left\|\int_t^{\lceil t\rceil_\tau} \overline{g}_{n_l}\big(\lceil t\rceil_\tau - s\big)\overline{u}_{n_l}(s)(s)\,\mathrm{d}s\right\| \\
&\quad\leq \|\varphi\|\left[\int_\Omega\left(\int_t^{\lceil t\rceil_\tau} \overline{g}_{n_l}\big(\mathbf{x},\lceil t\rceil_\tau - s\big)\,\mathrm{d}s\right)\right. \\
&\qquad\times\left.\left(\int_t^{\lceil t\rceil_\tau} \overline{g}_{n_l}\big(\mathbf{x},\lceil t\rceil_\tau - s\big)\overline{u}_{n_l}^2(\mathbf{x},s)\,\mathrm{d}s\right)\mathrm{d}\mathbf{x}\right]^{\frac{1}{2}}.
\end{aligned} \tag{34}
$$

For $t\in(t_{i-1},t_i]$ and a.a. $\mathbf{x}\in\Omega$, it holds that

$$
\int_t^{\lceil t\rceil_\tau} \overline{g}_{n_l}\big(\mathbf{x},\lceil t\rceil_\tau - s\big)\,\mathrm{d}s
$$

$$= \int_0^{t_i-t} \overline{g}_{n_l}(\mathbf{x},\xi)\,\mathrm{d}\xi = \int_0^{t_i-t} \frac{\tau_{n_l}^{-\beta(\mathbf{x})}}{\Gamma(1-\beta(\mathbf{x}))}\,\mathrm{d}\xi \le \frac{\tau_{n_l}^{1-\beta(\mathbf{x})}}{\Gamma(1-\beta(\mathbf{x}))} \le \tau_{n_l}^{1-\beta_1}.$$

Then, using Lemma 5.2, we obtain from Eq. (34) for $t \in (t_{i-1}, t_i]$ that

$$\begin{aligned}
&\left|\left(\int_t^{\lceil t\rceil_\tau} \overline{g}_{n_l}(\lceil t\rceil_\tau - s)\overline{u}_{n_l}(s)\,\mathrm{d}s, \varphi\right)\right|^2 \\
&\quad\le \tau_{n_l}^{1-\beta_1}\|\varphi\|^2 \int_\Omega \left(\int_t^{\lceil t\rceil_\tau} \overline{g}_{n_l}(\mathbf{x}, \lceil t\rceil_\tau - s)\overline{u}_{n_l}^2(\mathbf{x},s)\,\mathrm{d}s\right)\mathrm{d}\mathbf{x} \\
&\quad\le \tau_{n_l}^{1-\beta_1}\|\varphi\|^2 \int_\Omega \int_0^{t_i} \overline{g}_{n_l}(\mathbf{x}, t_i - s)\overline{u}_{n_l}^2(\mathbf{x},s)\,\mathrm{d}s\,\mathrm{d}\mathbf{x} \\
&\quad\le \tau_{n_l}^{1-\beta_1}\|\varphi\|^2 \int_\Omega \sum_{j=1}^{i} g_{i-j+1}(\mathbf{x})u_j^2(\mathbf{x})\tau\,\mathrm{d}\mathbf{x} \\
&\quad= \tau_{n_l}^{1-\beta_1}\|\varphi\|^2 \int_\Omega \left(g * u^2\right)_i(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\quad\le C\tau_{n_l}^{1-\beta_1},
\end{aligned}$$

which is valid for $i = 1, \ldots, n$. Secondly, we deduce for the second term on the RHS of (33) that

$$\begin{aligned}
&\left|\int_0^T \left(\int_0^t \left[\overline{g}_{n_l}(\lceil t\rceil_\tau - s) - g(t-s)\right]\overline{u}_{n_l}(s)\,\mathrm{d}s, \varphi\right)\mathrm{d}t\right| \\
&\quad\le \int_\Omega \left[\int_0^T \left(\int_0^t \left|\overline{g}_{n_l}(\mathbf{x}, \lceil t\rceil_\tau - s) - g(\mathbf{x}, t-s)\right|\left|\overline{u}_{n_l}(\mathbf{x},s)\right|\mathrm{d}s\right)\mathrm{d}t\right]\left|\varphi(\mathbf{x})\right|\mathrm{d}\mathbf{x} \\
&\quad\le \int_\Omega \left|\varphi(\mathbf{x})\right|\left[\int_0^T \left(\int_0^t \left|\overline{g}_{n_l}(\mathbf{x}, \lceil t\rceil_\tau - s) - g(\mathbf{x}, t-s)\right|\mathrm{d}s\right)^{\frac{1}{2}}\right. \\
&\qquad\times \left.\left(\int_0^t \left|\overline{g}_{n_l}(\mathbf{x}, \lceil t\rceil_\tau - s) - g(\mathbf{x}, t-s)\right|\overline{u}_{n_l}^2(\mathbf{x},s)\,\mathrm{d}s\right)^{\frac{1}{2}}\mathrm{d}t\right]\mathrm{d}\mathbf{x} \\
&\quad\le C\tau_{n_l}^{\frac{1}{2}(1-\beta_1)} \\
&\qquad\times \int_\Omega \left|\varphi(\mathbf{x})\right|\left[\int_0^T \left(\int_0^t \left|\overline{g}_{n_l}(\mathbf{x}, \lceil t\rceil_\tau - s) - g(\mathbf{x}, t-s)\right|\overline{u}_{n_l}^2(\mathbf{x},s)\,\mathrm{d}s\right)^{\frac{1}{2}}\mathrm{d}t\right]\mathrm{d}\mathbf{x} \\
&\quad\le C\tau_{n_l}^{\frac{1}{2}(1-\beta_1)}\sqrt{T} \\
&\qquad\times \int_\Omega \left|\varphi(\mathbf{x})\right|\left[\int_0^T \left(\int_0^t \left|\overline{g}_{n_l}(\mathbf{x}, \lceil t\rceil_\tau - s) - g(\mathbf{x}, t-s)\right|\overline{u}_{n_l}^2(\mathbf{x},s)\,\mathrm{d}s\right)\mathrm{d}t\right]^{\frac{1}{2}}\mathrm{d}\mathbf{x}
\end{aligned}$$

since (remember that the function g is decreasing in time and thus $\overline{g}_{n_l} \le g$) we have for $t \in (t_{i-1}, t_i]$ that

$$\begin{aligned}
&\int_0^t \left|\overline{g}_{n_l}(\mathbf{x}, \lceil t\rceil_\tau - s) - g(\mathbf{x}, t-s)\right|\mathrm{d}s \\
&\quad= \int_0^{t_{i-1}} \left(g(\mathbf{x}, t-s) - \overline{g}_{n_l}(\mathbf{x}, t_i - s)\right)\mathrm{d}s + \int_{t_{i-1}}^t \left(g(\mathbf{x}, t-s) - \overline{g}_{n_l}(\mathbf{x}, t_i - s)\right)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^{t_{i-1}} \big(g(\mathbf{x}, t_{i-1}-s) - g(\mathbf{x}, t_{i+1}-s)\big)\,\mathrm{d}s + 2\int_{t_{i-1}}^{t} g(\mathbf{x}, t-s)\,\mathrm{d}s \\
&\le \frac{t_{i-1}^{1-\beta(\mathbf{x})}}{\Gamma(2-\beta(\mathbf{x}))} + \frac{(2\tau_{n_l})^{1-\beta(\mathbf{x})}}{\Gamma(2-\beta(\mathbf{x}))} - \frac{t_{i+1}^{1-\beta(\mathbf{x})}}{\Gamma(2-\beta(\mathbf{x}))} + 2\frac{t^{1-\beta(x)} - t_{i-1}^{1-\beta(x)}}{\Gamma(2-\beta(\mathbf{x}))} \\
&\le C\tau_{n_l}^{1-\beta_1},
\end{aligned}
$$

where we used the α-Hölder continuity of $f(x) = x^\alpha$ with $\alpha \in (0,1]$. Next, we see that

$$
\begin{aligned}
&\int_0^T \left(\int_0^t \left|\overline{g}_{n_l}\big(\mathbf{x}, \lceil t\rceil_\tau - s\big) - g(\mathbf{x}, t-s)\right| \overline{u}_{n_l}^2(\mathbf{x}, s)\,\mathrm{d}s \right) \mathrm{d}t \\
&\quad = \int_0^T \left(\big(g * \overline{u}_{n_l}^2\big)(\mathbf{x}, t) - \int_0^t \overline{g}_{n_l}\big(\mathbf{x}, \lceil t\rceil_\tau - s\big)\overline{u}_{n_l}^2(\mathbf{x}, s)\,\mathrm{d}s \right) \mathrm{d}t \\
&\quad \overset{(\star)}{\le} \left\|\overline{g}_{n_l}(\mathbf{x})\right\|_{\mathrm{L}^1(0,T)} \left\|\overline{u}_{n_l}^2(\mathbf{x})\right\|_{\mathrm{L}^1(0,T)} \\
&\qquad + \int_0^T \left(\int_0^t \overline{g}_{n_l}\big(\mathbf{x}, \lceil t\rceil_\tau - s\big)\overline{u}_{n_l}^2(\mathbf{x}, s)\,\mathrm{d}s \right) \mathrm{d}t \\
&\quad \overset{(6)}{\le} C\left\|\overline{u}_{n_l}^2(\mathbf{x})\right\|_{\mathrm{L}^1(0,T)} + \int_0^T \left(\int_0^t \overline{g}_{n_l}\big(\mathbf{x}, \lceil t\rceil_\tau - s\big)\overline{u}_{n_l}^2(\mathbf{x}, s)\,\mathrm{d}s \right) \mathrm{d}t,
\end{aligned}
$$

where we used Young's inequality for convolutions at position $(\star)$. Therefore, using Lemma 5.2, we finally obtain that

$$
\begin{aligned}
&\left| \int_0^T \left(\int_0^t \left[\overline{g}_{n_l}\big(\lceil t\rceil_\tau - s\big) - \overline{g}_{n_l}(t-s)\right]\overline{u}_{n_l}^2(s)\,\mathrm{d}s, \varphi \right) \mathrm{d}t \right| \\
&\quad \le C\tau_{n_l}^{\frac{1}{2}(1-\beta_1)} \|\varphi\| \left[\int_\Omega \left\|\overline{u}_{n_l}^2(\mathbf{x})\right\|_{\mathrm{L}^1(0,T)} \mathrm{d}\mathbf{x} \right. \\
&\qquad \left. + \int_\Omega \left(\int_0^T \left(\int_0^t \overline{g}_{n_l}\big(\mathbf{x}, \lceil t\rceil_\tau - s\big)\overline{u}_{n_l}^2(\mathbf{x}, s)\,\mathrm{d}s \right) \mathrm{d}t \right) \mathrm{d}\mathbf{x} \right]^{\frac{1}{2}} \\
&\quad \le C\tau_{n_l}^{\frac{1}{2}(1-\beta_1)} \left[\sum_{i=1}^{n_l} \|u_i\|^2\tau + \sum_{i=1}^{n_l} \tau \int_\Omega \big(g * u^2\big)_i(\mathbf{x})\,\mathrm{d}\mathbf{x} \right]^{\frac{1}{2}} \\
&\quad \le C\tau_{n_l}^{\frac{1}{2}(1-\beta_1)}.
\end{aligned}
$$

We conclude that the second limit transmission is valid. Limit transmission (iii) follows from Eq. (30) since

$$
\begin{aligned}
&\left| \int_0^T \big((g * (\overline{u}_{n_l} - u))(t), \varphi \big)\,\mathrm{d}t \right| \\
&\quad \le \int_\Omega |\varphi(\mathbf{x})| \left(\int_0^T \big(g * |(\overline{u}_{n_l} - u)|\big)(\mathbf{x}, t)\,\mathrm{d}t \right) \mathrm{d}\mathbf{x} \\
&\quad \le \sqrt{T} \int_\Omega |\varphi(\mathbf{x})| \left(\int_0^T \big(g * |(\overline{u}_{n_l} - u)|\big)^2(\mathbf{x}, t)\,\mathrm{d}t \right)^{\frac{1}{2}} \mathrm{d}\mathbf{x} \\
&\quad \overset{(\star)}{\le} \sqrt{T} \int_\Omega |\varphi(\mathbf{x})| \left\|g(\mathbf{x})\right\|_{\mathrm{L}^1(0,T)} \left\|(\overline{u}_{n_l} - u)(\mathbf{x})\right\|_{\mathrm{L}^2(0,T)} \mathrm{d}\mathbf{x}
\end{aligned}
$$

$$\stackrel{(6)}{\leq} C \int_\Omega |\varphi(\mathbf{x})| \|(\overline{u}_{n_l} - u)(\mathbf{x})\|_{\mathrm{L}^2(0,T)} \,\mathrm{d}\mathbf{x}$$
$$\leq C\|\varphi\|\|\overline{u}_{n_l} - u\|_{\mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))},$$

using Young's inequality for convolutions at position ($\star$). Now, we are able to pass to the limit $l \to \infty$ in (32), and we obtain that

$$\int_0^\xi \big((g * u)(\eta), \varphi\big)\,\mathrm{d}\eta - \int_0^\xi \left(\tilde{u}_0 \int_0^\eta g(t)\,\mathrm{d}t, \varphi\right)\mathrm{d}\eta + \int_0^\xi \int_0^\eta \mathcal{L}(t)\big(u(t), \varphi\big)\,\mathrm{d}t\,\mathrm{d}\eta$$
$$= \int_0^\xi \int_0^\eta \big(f(t), \varphi\big)\,\mathrm{d}t\,\mathrm{d}\eta.$$

Next, we differentiate the previous relation with respect to ξ, i.e.

$$\big((g * u)(\xi), \varphi\big) - \left(\tilde{u}_0 \int_0^\xi g(t)\,\mathrm{d}t, \varphi\right) + \int_0^\xi \mathcal{L}(t)\big(u(t), \varphi\big)\,\mathrm{d}t = \int_0^\xi \big(f(t), \varphi\big)\,\mathrm{d}t. \tag{35}$$

Since $u \in \mathrm{L}^\infty((0,T), \mathrm{H}^1_0(\Omega))$ and $f \in \mathrm{C}([0,T], \mathrm{L}^2(\Omega))$, we immediately see that

$$\lim_{\xi \searrow 0}\big((g * u)(\xi), \varphi\big) = 0 \quad \Rightarrow \quad (g * u)(0) = 0 \quad \text{in } \mathrm{L}^2(\Omega).$$

This calculation confirms (22). Moreover, from (35), we also get that $(g * (u - \tilde{u}_0))(\xi)$ is absolutely continuous in time with values in $\mathrm{H}^1_0(\Omega)^*$ and with $(g * (u - \tilde{u}_0))(0) = 0$ as

$$\big(g * (u - \tilde{u}_0)\big)(\xi) = \int_0^\xi f(t)\,\mathrm{d}t - \int_0^\xi L(t)u(t)\,\mathrm{d}t \quad \text{in } \mathrm{H}^1_0(\Omega)^*, \tag{36}$$

where

$$\langle f(t) - L(t)u(t), \varphi\rangle = \big(f(t), \varphi\big) - \mathcal{L}(t)\big(u(t), \varphi\big).$$

We differentiate this relation with respect to ξ and replace ξ with t to obtain that

$$\partial_t\big(g * (u - \tilde{u}_0)\big)(t) = f(t) - L(t)u(t) \quad \text{in } \mathrm{H}^1_0(\Omega)^* \text{ for a.a. } t \in (0,T), \tag{37}$$

i.e. u satisfies the weak formulation (21). Moreover, using that $u \in \mathrm{L}^\infty((0,T), \mathrm{H}^1_0(\Omega))$ and $f \in \mathrm{C}([0,T], \mathrm{L}^2(\Omega))$, we get that

$$\partial_t\big(g * (u - \tilde{u}_0)\big) \in \mathrm{L}^\infty\big((0,T), \mathrm{H}^1_0(\Omega)^*\big). \qquad \square$$

Theorem 6.2 (Uniqueness) *Let assumptions AS-*(1–6) *be fulfilled. Then there exists a unique weak solution u to problem* (21) *with $u \in \mathrm{L}^\infty((0,T), \mathrm{H}^1_0(\Omega))$ and $\partial_t(g * (u - \tilde{u}_0)) \in \mathrm{L}^\infty((0,T), \mathrm{H}^1_0(\Omega)^*)$.*

Proof We show the uniqueness of a solution by contradiction. We suppose that two solutions u_1 and u_2 to (21) exist and set $u = u_1 - u_2$. Then u satisfies (21) with $f = 0$ in Q_T

and $u(\mathbf{x},0) = 0$ in Ω. Then, we integrate the result over $t \in (0,\eta) \subset (0,T)$ and put $\varphi = u(\eta)$. Afterwards, we integrate in time over $\eta \in (0,\xi) \subset (0,T)$ and obtain that

$$\int_0^\xi \big((g * u)(\eta), u(\eta)\big)\,\mathrm{d}\eta + \int_0^\xi \int_0^\eta \mathcal{L}(t)\big(u(t), u(\eta)\big)\,\mathrm{d}t\,\mathrm{d}\eta = \int_0^\xi \int_0^\eta \big(f(t), u(\eta)\big)\,\mathrm{d}t\,\mathrm{d}\eta.$$

From Lemma 2.1, it follows that

$$\int_0^\xi \big((g * u)(\eta), u(\eta)\big)\,\mathrm{d}\eta \geq \int_\Omega \gamma(\mathbf{x}) \int_0^\xi (e * u)^2(\mathbf{x}, s)\,\mathrm{d}s\,\mathrm{d}\mathbf{x}.$$

Using the symmetry of $\boldsymbol{A}$ and integration by parts, we see that

$$\begin{aligned}\int_0^\xi \left(\int_0^\eta \boldsymbol{A}(t)\nabla u(t)\,\mathrm{d}t, \nabla u(\eta)\right)\mathrm{d}\eta &= \frac{1}{2}\left(\boldsymbol{A}(\xi)\left[\int_0^\xi \nabla u(s)\,\mathrm{d}s\right], \int_0^\xi \nabla u(t)\,\mathrm{d}t\right) \\ &\quad + \frac{1}{2}\int_0^\xi \left(\int_0^\eta \nabla u(t)\,\mathrm{d}t, \partial_t \boldsymbol{A}(\eta)\left[\int_0^\eta \nabla u(s)\,\mathrm{d}s\right]\right)\mathrm{d}\eta \\ &\quad - \left(\int_0^\xi \partial_t \boldsymbol{A}(t)\left[\int_0^t \nabla u(s)\,\mathrm{d}s\right]\mathrm{d}t, \int_0^\xi \nabla u(t)\,\mathrm{d}t\right).\end{aligned}$$

Hence, using the ε-Young inequality, we see that

$$\begin{aligned}&\int_0^\xi \left(\int_0^\eta \boldsymbol{A}(t)\nabla u(t)\,\mathrm{d}t, \nabla u(\eta)\right)\mathrm{d}\eta \\ &\quad \geq \left(\frac{\alpha}{2} - \varepsilon\right)\left\|\int_0^\xi \nabla u(t)\,\mathrm{d}t\right\|^2 - C_\varepsilon \int_0^\xi \left\|\int_0^\eta \nabla u(t)\,\mathrm{d}t\right\|^2 \mathrm{d}\eta.\end{aligned}$$

Similarly, we have that

$$\left|\int_0^\xi \left(\int_0^\eta c(t)u(t)\,\mathrm{d}t, u(\eta)\right)\mathrm{d}\eta\right| \geq -\varepsilon\left\|\int_0^\xi \nabla u(t)\,\mathrm{d}t\right\|^2 - C_\varepsilon \int_0^\xi \left\|\int_0^\eta \nabla u(t)\,\mathrm{d}t\right\|^2 \mathrm{d}\eta.$$

Therefore, fixing ε sufficiently small and applying the Grönwall lemma imply that

$$\int_\Omega \gamma(\mathbf{x}) \int_0^\xi (e * u)^2(\mathbf{x}, s)\,\mathrm{d}s\,\mathrm{d}\mathbf{x} + \left\|\int_0^\xi \nabla u(t)\,\mathrm{d}t\right\|^2 = 0.$$

Hence, we obtain that $e * u = 0$ a.e. in Q_T. From the theory on Volterra equations, it follows that $u = 0$ a.e. in Q_T, cf. [51, Theorem 3.5]. □

Remark 6.1 (Neumann boundary condition) We made use of Friedrichs's inequality to handle the term

$$\int_0^\xi \left(\int_0^\eta c(t)u(t)\,\mathrm{d}t, u(\eta)\right)\mathrm{d}\eta$$

in Theorem 6.2. This step is violated when a Neumann condition is considered on the whole boundary of the domain ($\boldsymbol{A}(t)\nabla u(t) \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$ for $t > 0$). However, we can assume that $c \geq c_0 > 0$ in Q_T in order to be able to obtain the uniqueness of a solution. In fact, this assumption is also necessary when establishing the a priori estimate in Lemma 5.3 if a Neumann boundary condition is considered.

Remark 6.2 Note that the convergence of Rothe's functions towards the weak solution in Theorem 6.1 is also valid for the entire Rothe's sequence as the solution is unique.

We are not able to show that $u(0) = \tilde{u}_0$ for the space-dependent variable-order fractional derivative under consideration. In the next section, we explain how to obtain this convergence result in the case of a constant-order fractional derivative.

7 Fractional derivative of constant order

In this section, we suppose that $\beta(\mathbf{x}) = \beta$ with $\beta \in (0,1)$. Therefore, the kernel g only depends on the time-variable. We have the existence of the kernel $l(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ such that $(l * g)(t) = 1$ for all $t \in [0,T]$. Next, we apply the convolution operation with kernel l on (37). We obtain that

$$\bigl(l * \bigl[\partial_t\bigl(g * (u - \tilde{u}_0)\bigr)\bigr]\bigr)(t) = \bigl(l * [f - Lu]\bigr)(t) \quad \text{in } \mathrm{H}_0^1(\Omega)^*.$$

From the absolute continuity of $(g * (u - \tilde{u}_0))(t)$, $(g * (u - \tilde{u}_0))(0) = 0$ and $(l * g)(t) = 1$, we have that

$$\bigl(l * \bigl[\partial_t\bigl(g * (u - \tilde{u}_0)\bigr)\bigr]\bigr)(t) = \partial_t\bigl(l * g * (u - \tilde{u}_0)\bigr)(t) = u(t) - \tilde{u}_0.$$

Hence,

$$u(t) - \tilde{u}_0 = \bigl(l * [f - Lu]\bigr)(t) \quad \text{in } \mathrm{H}_0^1(\Omega)^*,$$

For all $\varphi \in \mathrm{H}_0^1(\Omega)$, since $u \in \mathrm{L}^\infty((0,T),\mathrm{H}_0^1(\Omega))$ and $f \in \mathrm{C}([0,T],\mathrm{L}^2(\Omega))$, we obtain that

$$\begin{aligned}\lim_{t\searrow 0}\bigl|\langle u(t) - \tilde{u}_0, \varphi\rangle_{\mathrm{H}_0^1(\Omega)^* \times \mathrm{H}_0^1(\Omega)}\bigr| &= \lim_{t\searrow 0}\left|\int_0^t l(t-s)\bigl[\bigl(f(s),\varphi\bigr) - \mathcal{L}(s)\bigl(u(s),\varphi\bigr)\bigr]\,\mathrm{d}s\right| \\ &\le C\lim_{t\searrow 0}\int_0^t l(t-s)\,\mathrm{d}s = 0.\end{aligned}$$

Thus $u(0) = \tilde{u}_0$ in $\mathrm{H}_0^1(\Omega)^*$ and

$$u \in \mathrm{C}\bigl([0,T],\mathrm{H}_0^1(\Omega)^*\bigr).$$

Finally, we summarise the results in the following theorem, which improves the result obtained in [37, Theorem 3.1] (see also [38, Theorem 1]).

Theorem 7.1 *Let assumptions AS-*(1–6) *be fulfilled and put* $\beta(\mathbf{x}) = \beta$ *with* $\beta \in (0,1)$. *Then there exists a unique weak solution* u *to problem* (21) *with* $u \in \mathrm{C}([0,T],\mathrm{H}_0^1(\Omega)^*) \cap \mathrm{L}^\infty((0,T),\mathrm{H}_0^1(\Omega))$ *and* $\partial_t(g * (u - \tilde{u}_0)) \in \mathrm{L}^\infty((0,T),\mathrm{H}_0^1(\Omega)^*)$.

Remark 7.1 In [26, Proposition 7], the authors proved that $u \in \mathrm{C}([0,T],X)$ and $u(0) = \tilde{u}_0$ for $\beta \in (\frac{1}{p},1]$ if $u \in \mathrm{L}^1((0,T),X)$, $p \in (1,\infty)$, $(g * (u - \tilde{u}_0))(0) = 0$ and $\partial_t(g * (u - \tilde{u}_0)) \in$

$L^p((0,T),X)$. The authors made use of [52, Theorem 3.6], which is also satisfied for $p = \infty$. Therefore, we deduce that [26, Proposition 7] is also satisfied for $p = \infty$, which is in accordance with the result obtained above.

Remark 7.2 (Neumann boundary condition) Theorem 7.1 is also satisfied when considering a homogeneous Neumann boundary condition if $c \geq c_0 > 0$.

Remark 7.3 We use the absolute continuity in time with values in $H_0^1(\Omega)^*$ of $(g * (u - \tilde{u}_0))(t)$ to obtain the continuity in time of $u(t)$ with values in $H_0^1(\Omega)^*$. We note that no information about $\partial_t u$ itself is obtained.

7.1 More regular solution

In this section, we show that the solution u to problem (2) belongs to

$$u \in L^2\big((0,T), H^2(\Omega) \cap H_0^1(\Omega)\big)$$

if the tensor $\boldsymbol{A}$ in the principal part of the differential operator L is a 1×1 matrix, i.e. we suppose that

$$L(\mathbf{x},t)u(\mathbf{x},t) = -\nabla \cdot \big(a(\mathbf{x},t)\nabla u(\mathbf{x},t)\big) + c(\mathbf{x},t)u(\mathbf{x},t).$$

This leads to the following additional assumptions:

- **AS-7:** $\beta \in C(\overline{\Omega})$ with $0 < \beta(\mathbf{x}) \leq \beta_1 < 1$ for all $\mathbf{x} \in \overline{\Omega}$;
- **AS-8:** $\boldsymbol{A} \in (L^\infty(Q_T))^{1\times 1}$, i.e. $\boldsymbol{A} = a$, and $\nabla a \in (L^\infty(Q_T))^d$.

We note that the ellipticity assumption AS-2 reduces to

$$a(\mathbf{x},t) \geq \alpha > 0 \quad \text{for a.a. } (\mathbf{x},t) \in Q_T.$$

The goal is to derive some higher stability results for the discrete solution. We have already from the Lax–Milgram lemma that there exists a unique $u_i \in H_0^1(\Omega)$ solving problem (24) for any $i = 1,\dots,n$, see Lemma 5.1. Moreover, from (24), it follows that

$$-a_i \Delta u_i = f_i + \nabla a_i \cdot \nabla u_i - (g * \delta u)_i - c_i u_i \quad \text{in } H_0^1(\Omega)^*.$$

However, as the RHS is an element of $L^2(\Omega)$ (thanks to assumption AS-8), we also have that $-a_i\Delta u_i$ is an element of $L^2(\Omega)$ for any $i = 1,\dots,n$. Hence, from the ellipticity assumption, we get that $\Delta u_i \in L^2(\Omega)$ for any $i = 1,\dots,n$. Now, we are able to establish the following a priori estimate.

Lemma 7.1 *Let assumptions AS-*(1–8) *be fulfilled. Then a positive constant C exists such that, for every* $j = 1,2,\dots,n$, *the following relation holds*:

$$\big(g * \|\nabla u\|^2\big)_j + \sum_{i=1}^{j} g_i \|\nabla u_i\|^2 \tau + \sum_{i=1}^{j} \|\Delta u_i\|^2 \tau \leq C.$$

Proof We multiply

$$(g * \delta u)_i - a_i \Delta u_i - \nabla a_i \cdot \nabla u_i + c_i u_i = f_i \tag{38}$$

by $-\Delta u_i \tau$, integrate the result over Ω and sum up the result for $i = 1, \dots j$ keeping $1 \le j \le n$. We obtain that

$$\begin{aligned} &-\sum_{i=1}^{j} \bigl((g * \delta u)_i, \Delta u_i\bigr)\tau + \sum_{i=1}^{j} (a_i \Delta u_i, \Delta u_i)\tau + \sum_{i=1}^{j} (\nabla a_i \cdot \nabla u_i, \Delta u_i)\tau - \sum_{i=1}^{j} (c_i u_i, \Delta u_i)\tau \\ &\quad = \sum_{i=1}^{j} (f_i, \Delta u_i)\tau. \end{aligned}$$

We apply Green's theorem on the first term on the LHS. Since g is space-independent here and $u_i|_\Gamma = 0$ for $i = 0, \dots, n$, we get that

$$\begin{aligned} -\sum_{i=1}^{j} \bigl((g * \delta u)_i, \Delta u_i\bigr)\tau &= \sum_{i=1}^{j} \bigl((g * \nabla \delta u)_i, \nabla u_i\bigr)\tau - \sum_{i=1}^{j} \bigl((g * \delta u)_i, \nabla u_i \cdot \boldsymbol{\nu}\bigr)_\Gamma \tau \\ &= \sum_{i=1}^{j} \bigl((g * \delta \nabla u)_i, \nabla u_i\bigr)\tau. \end{aligned}$$

Hence, using relation (19), we obtain that

$$\begin{aligned} &\sum_{i=1}^{j} \bigl(\delta(g * \nabla u)_i, \nabla u_i\bigr)\tau + \sum_{i=1}^{j} (a_i \Delta u_i, \Delta u_i)\tau \\ &\quad = \sum_{i=1}^{j} (f_i, \Delta u_i)\tau + \sum_{i=1}^{j} (g_i \nabla \tilde{u}_0, \nabla u_i)\tau + \sum_{i=1}^{j} (c_i u_i, \Delta u_i)\tau - \sum_{i=1}^{j} (\nabla a_i \cdot \nabla u_i, \Delta u_i)\tau. \end{aligned}$$

From Lemma 3.3, again using that g is solely time-dependent, we obtain that

$$\sum_{i=1}^{j} \bigl(\delta(g * \nabla u)_i, \nabla u_i\bigr)\tau \ge \frac{1}{2}\bigl(g * \|\nabla u\|^2\bigr)_j + \frac{1}{2}\sum_{i=1}^{j} g_i \|\nabla u_i\|^2 \tau.$$

Using the ε-Young inequality, we estimate the second term in the RHS as follows:

$$\left| \sum_{i=1}^{j} (g_i \nabla \tilde{u}_0, \nabla u_i)\tau \right| \le C_{\varepsilon_1} + \varepsilon_1 \sum_{i=1}^{j} g_i \|\nabla u_i\|^2 \tau.$$

Using Lemma 5.2, we easily see that

$$\left| \sum_{i=1}^{j} (f_i, \Delta u_i)\tau + \sum_{i=1}^{j} (c_i u_i, \Delta u_i)\tau - \sum_{i=1}^{j} (\nabla a_i \cdot \nabla u_i, \Delta u_i)\tau \right| \le C_{\varepsilon_2} + \varepsilon_2 \sum_{i=1}^{j} \|\Delta u_i\|^2 \tau.$$

Therefore, we obtain the following estimate:

$$\frac{1}{2}\bigl(g * \|\nabla u\|^2\bigr)_j + \left(\frac{1}{2} - \varepsilon_1\right) \sum_{i=1}^{j} g_i \|\nabla u_i\|^2 \tau + (\alpha - \varepsilon_2) \sum_{i=1}^{j} \|\Delta u_i\|^2 \tau \le C_{\varepsilon_1} + C_{\varepsilon_2}.$$

We conclude the proof by fixing ε_1 and ε_2 sufficiently small. □

In the space $\mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$, the norms $\|\Delta u\|$ and $\|u\|_{\mathrm{H}^2(\Omega)}$ are equivalent [53, Theorem 1]. Hence, Lemma 7.1 implies that the Rothe sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2((0,T),\mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega))$. The reflexivity of this space implies that the weak solution u to problem (21) belongs to this space. Moreover, from (21) it also follows that $\partial_t(g*(u-\tilde{u}_0)) \in \mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))$.

This latter result implies that the conditions of Corollary 3.1 are satisfied. It is here that we can use this result to obtain in a different way (before we used the positive definiteness of the kernel) the uniqueness of solution to problem (21). We consider $u = u_1 - u_2$ with u_1 and u_2 different solutions to (21). Thus u satisfies (21) with $f = 0$ in Q_T and $u(\cdot,0) = 0$ in Ω. Therefore, taking $\varphi = u(t)$ in (21) and integrating with respect to time over $(0,\eta) \subset (0,T)$ gives

$$\int_0^\eta \big(\partial_t(g*u)(t),u(t)\big)\,\mathrm{d}t + \int_0^\eta \mathcal{L}(t)\big(u(t),u(t)\big)\,\mathrm{d}t = 0.$$

From Corollary 3.1, it follows for any $\eta \in (0,T)$ that (note that $\beta_1 = \beta$ in (8))

$$\frac{\tilde{g}}{2}\int_0^\eta \big\|u(t)\big\|^2\,\mathrm{d}t + \alpha\int_0^\eta \big\|\nabla u(t)\big\|^2\,\mathrm{d}t \le 0, \tag{39}$$

i.e. $u = 0$ a.e. in Q_T.

Theorem 7.2 *Let assumptions AS-*(1–8) *be fulfilled and put $\beta(\mathbf{x}) = \beta$ with $\beta \in (0,1)$. Then there exists a unique weak solution u to problem* (21) *with $u \in \mathrm{C}([0,T],\mathrm{H}_0^1(\Omega)^*) \cap \mathrm{L}^2((0,T),\mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega))$ and $\partial_t(g*(u-\tilde{u}_0)) \in \mathrm{L}^2((0,T),\mathrm{L}^2(\Omega))$.*

Remark 7.4 (Neumann boundary condition) We discussed before in Remark 6.1 that the coefficient c needs to satisfy additionally $c \ge c_0 > 0$ (next to assumptions AS-(1–6)) in order to obtain the uniqueness of a solution and Lemma 5.3 when a Neumann condition is considered on the whole boundary of the domain. We note that Eq. (39) is also satisfied without additional assumption on c in the case of a Neumann boundary condition on the complete boundary (as $f \equiv 0$ in the proof of uniqueness). Moreover, from [54, Theorem 2.50], it follows that the norms $\|u\| + \|\Delta u\|$ and $\|u\|_{\mathrm{H}^2(\Omega)}$ are equivalent for $u \in \mathrm{H}^2(\Omega)$ satisfying $\nabla u \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$. This implies that Theorem 7.2 is also satisfied when considering a homogeneous Neumann boundary condition if $c \ge c_0 > 0$.

In the next estimate, we show that under the additional assumptions

- **AS-9:** $\tilde{u}_0 \in \mathrm{H}^2(\Omega)$, i.e. $\tilde{u}_0 \in \mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega)$;
- **AS-10:** $\nabla a(t) = \mathbf{0}$ and $\nabla c(t) = \mathbf{0}$ a.e. in Ω for all $t \in (0,T]$,

we have that the solution u to problem (21) belongs to $\mathrm{L}^\infty((0,T),\mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega))$. Then from

$$u(t) - \tilde{u}_0 = \big(l*[f + a\Delta u - cu]\big)(t) \quad \text{in } \mathrm{L}^2(\Omega),$$

we get that $\lim_{t\searrow 0}\|u(t) - \tilde{u}_0\| = 0$.

Lemma 7.2 *Let assumptions AS-*(1–10) *be fulfilled. Then positive constants C and* τ_0 *exist such that, for any* $\tau < \tau_0$ *and for every* $j = 1, 2, \ldots, n$, *the following relation holds*:

$$\|\Delta u_j\|^2 + \sum_{i=1}^{j} \|\Delta u_i - \Delta u_{i-1}\|^2 \leq C.$$

Proof We start with multiplying (38) by $-\Delta\delta u_i\tau$, integrating the result over Ω and summing it up for $i = 1, \ldots j$ keeping $1 \leq j \leq n$. We get by AS-10 and Green's theorem the following equality:

$$\begin{aligned}&\sum_{i=1}^{j}\big((g * \nabla\delta u)_i, \nabla\delta u_i\big)\tau + \sum_{i=1}^{j}(a_i\Delta u_i, \Delta\delta u_i)\tau + \sum_{i=1}^{j}(c_i\nabla u_i, \nabla\delta u_i)\tau\\&\quad= \sum_{i=1}^{j}(f_i, \Delta\delta u_i)\tau.\end{aligned} \tag{40}$$

Further, we follow the lines of the proof of Lemma 5.3. The first term on the LHS of (40) is positive, and for the second we use (27) in order to obtain that

$$\sum_{i=1}^{j}(a_i\Delta u_i, \Delta\delta u_i)\tau \geq \frac{\alpha}{2}\|\Delta u_j\|^2 - C + \frac{\alpha}{2}\sum_{i=1}^{j}\|\Delta u_i - \Delta u_{i-1}\|^2.$$

Using again (27) and the result of Lemma 5.3 gives us

$$\left|\sum_{i=1}^{j}(c_i\nabla u_i, \nabla\delta u_i)\tau\right| \leq C.$$

If $f \in \mathrm{H}^1((0,T), \mathrm{L}^2(\Omega))$, then the term on the RHS of (40) can be estimated as follows:

$$\left|\sum_{i=1}^{j}(f_i, \delta\Delta u_i)\tau\right| \overset{(28)}{\leq} C_\varepsilon + \varepsilon\|\Delta u_j\|^2.$$

The result of the lemma follows by fixing ε sufficiently small.

If $\|\frac{\partial^\ell}{\partial t^\ell}f(t)\| \leq C(1 + t^{\gamma-\ell})$ for $\ell = 0, 1$ and $\gamma \in (0,1)$, then

$$\left|\sum_{i=1}^{j}(f_i, \delta\Delta u_i)\tau\right| \overset{(28)}{\leq} C_\varepsilon + \varepsilon\|\Delta u_j\|^2 + C\sum_{i=1}^{j-1} t_i^{\gamma-1}\|\Delta u_i\|^2\tau.$$

Now, the estimate follows from the discrete Grönwall lemma [49, Corollary 15.5]. □

We summarise the consequences of Lemma 7.2 in the following theorem.

Theorem 7.3 *Let assumptions AS-*(1–10) *be fulfilled and put* $\beta(\mathbf{x}) = \beta$ *with* $\beta \in (0,1)$. *Then there exists a unique weak solution u to problem* (21) *with* $u \in \mathrm{C}([0,T], \mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty((0,T), \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega))$ *and* $\partial_t(g * (u - \tilde{u}_0)) \in \mathrm{L}^\infty((0,T), \mathrm{L}^2(\Omega))$.

7.2 Numerical experiments

In this subsection, we first test the performance of the time-discrete scheme (25) (temporal error) for solving (2) for a smooth solution and a typical solution given by

$$u_{\mathrm{ex}}^{1,\beta}(x,t) = t^3 \sin(\pi x) \quad \text{and} \quad u_{\mathrm{ex}}^{2,\beta}(x,t) = t^\beta \sin(\pi x),$$

respectively. We will perform several numerical experiments for different values of the order of the fractional derivative β. We consider $\Omega = (0,1)$ and $T = 1$. We solve problems (25) at each time step by the finite element method (FEM) using the first-order (P1–FEM) Lagrange polynomials for the space discretization, where we take the number of space discretization intervals equal to 128. We use the finite element library DOLFIN [55, 56] from the FEniCS project [57, 58] to solve these problems. The CPU time (in seconds, Intel® Core$^{\mathrm{TM}}$ i7-1065G7 Processor) increases fast when increasing the number of time discretization intervals, as at each time step one has to use the numerical solutions at all preceding time levels. In the numerical examples, we consider the following errors:

$$E_{\max}^n = \max_{1\le i\le n} \max_{x\in\overline{\Omega}} \left|\overline{u}_n(x,t_i) - u_{\mathrm{ex}}(x,t_i)\right|,$$

$$E_{\mathrm{conv}}^n = \max_{1\le i\le n} \left\|(g * u)_n(t_i) - (g * u_{\mathrm{ex}})(t_i)\right\|.$$

The error E_{conv}^n is motivated by considering that we have $\partial_t(g * (u - \tilde{u}_0)$ in the governing PDE instead of $\partial_t u$ in a classical parabolic PDE. In the first experiment, we consider the smooth solution $u_{\mathrm{ex}}^{1,\beta}$ for different values of β, i.e. $\beta = 0.2, 0.4, 0.6, 0.8$. On Figs. 1 and 2, the errors $E_{\max}^n$ and E_{conv}^n are depicted on a logarithmic scale for $\tau = \{2^{-j} : j = 5 \dots 10\}$. We may

Figure 1 Asymptotic convergence rate for $u_{\mathrm{ex}}^{1,\beta}$ on a logarithmic scale using (17): $\log_2 E_{\max}^n$

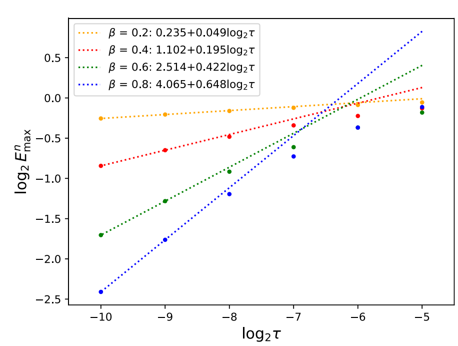


Figure 2 Convergence rate for $u_{\mathrm{ex}}^{1,\beta}$ on a logarithmic scale using (17): $\log_2 E_{\mathrm{conv}}^n$

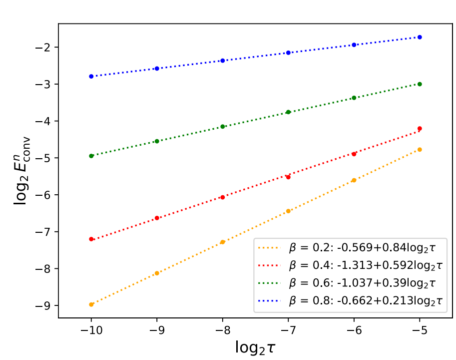


Figure 3 Asymptotic convergence rate for $u_{ex}^{2,\beta}$ on a logarithmic scale using (17): $\log_2 E_{max}^n$

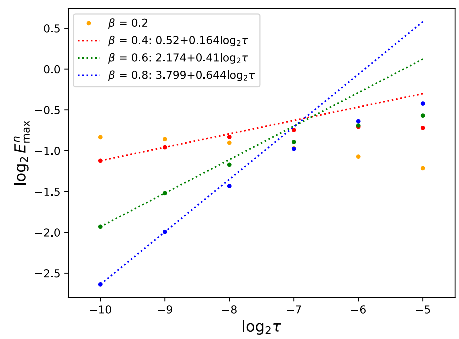
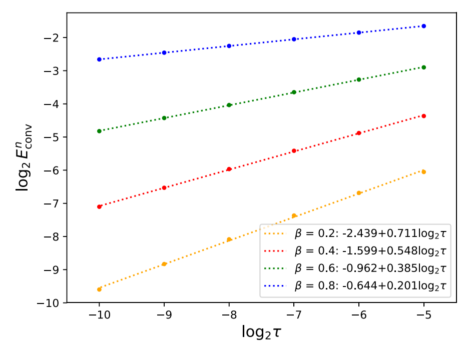


Figure 4 Convergence rate for $u_{ex}^{2,\beta}$ on a logarithmic scale using (17): $\log_2 E_{conv}^n$



conclude that the asymptotic rate of convergence for E_{max}^n is of order τ^β, but a very fine timestep τ will be needed to obtain this convergence rate, which will lead to a huge computational complexity. Moreover, it is clear that we have $\mathcal{O}(\tau^{1-\beta})$ convergence for E_{conv}^n. The same conclusions can be drawn from Figs. 3 and 4, where we investigate the algorithm in case of $u_{ex}^{2,\beta}$ for the different values of $\beta \in \{0.2, 0.4, 0.6, 0.8\}$. For these typical solutions, the timestep τ needs to be smaller in comparison with the smooth solutions (especially for small $\beta = 0.2$). The numerical results obtained for these examples validate the convergence of the proposed algorithms. Important to note is that $E_{max}^n = \max_{x\in\overline{\Omega}} |\overline{u}_n(x,t_1) - u_{ex}(x,t_1)|$ in all these experiments. An interesting direction for future research is to investigate theoretically the order of convergence of E_{max}^n and E_{conv}^n.

As we mentioned before, in order to obtain good numerical approximations, a very small timestep τ is needed, which is very time consuming. We may conclude that the time-discrete scheme (25) mainly has theoretical advantages. For this reason, we try to improve the time-discrete scheme (25) (leading to the theoretical results obtained in this paper) from computational viewpoint by considering a graded mesh and allocating more time-points around $t = 0$ [59–61]. We consider a graded time-partitioning of the form $t_j = T(j/n)^r$ for $j = 0, 1, \ldots, n$, where the constant mesh grading is assumed to satisfy $r \geq 1$. We put $\tau_j := t_j - t_{j-1}$ for $j = 1, \ldots, n$ and consider the following time-discrete convolution on the graded mesh:

$$(g * v)(t_i) \approx (g * v)_i^{\text{graded}} := \sum_{l=1}^{i} g(t_i - t_{l-1}) v_l \tau_l. \tag{41}$$

If $r = 1$, then the mesh is uniform and we easily see that we get the approximation (17) on a uniform mesh. The use of a graded mesh increases the temporal mesh near $t = T$ (it is coarser), and so it is possible that the error around $t = T$ starts to dominate the error around the initial time $t = 0$ (possibly also the space discretization error starts to interfere the results). This is the reason why we also calculate the maximum error on the initial time layer $I := [0, t_1]$ and investigate its behaviour, i.e.

$$E^n_{\max_I} = \max_{x\in\overline{\Omega}} \left| \overline{u}_n(x, t_1) - u_{\mathrm{ex}}(x, t_1) \right|.$$

In case of the graded mesh, the convergence rates are derived as follows:

$$\mathrm{rate}_{E^n_{\max}} := \log_2\left(\frac{E^n_{\max}}{E^{2n}_{\max}}\right), \qquad \mathrm{rate}_{E^n_{\max_I}} := \log_2\left(\frac{E^n_{\max_I}}{E^{2n}_{\max_I}}\right),$$
$$\mathrm{rate}_{E^n_{\mathrm{conv}}} := \log_2\left(\frac{E^n_{\mathrm{conv}}}{E^{2n}_{\mathrm{conv}}}\right).$$

In the experiments, we take $r = \frac{2-\beta}{\beta}$ in accordance with the optimal mesh grading for the classical $L1$-approximation for the Caputo fractional derivative [41, 62]. In Tables 1 and 2, we give the maximum errors and the orders of convergence for $u^{1,\beta}_{\mathrm{ex}}$ and $u^{2,\beta}_{\mathrm{ex}}$, respectively. For $r = \frac{2-\beta}{\beta}$, we see that the asymptotic convergence rate of $E^n_{\max_I}$ is $2 - \beta$, and we again observe that the asymptotic convergence rate of E^n_{conv} is $1 - \beta$. From these experiments, we are not able to make conclusions for the convergence rate of $E^n_{\max}$. However, it is clear that the maximum error is smaller in comparison with the uniform scheme (17). The same order of accuracy can only be obtained for the uniform scheme in the case of a very small timestep τ.

Table 1 Errors and order convergence for $u^{1,\beta}_{\mathrm{ex}}$ using (41) with $r = \frac{2-\beta}{\beta}$

$r = \frac{2-\beta}{\beta}$		$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1028$
$\beta = 0.2$	$E^n_{\max_I}$	2.479E−2	7.233E−3	2.087E−3	6.000E−4	1.724E−4	4.951E−5
	$\mathrm{rate}_{E^n_{\max_I}}$	1.777	1.793	1.798	1.799	1.800	
	$E^n_{\max}$	5.647E−1	3.356E−1	1.823E−1	9.430E−2	4.748E−2	2.354E−2
	$\mathrm{rate}_{E^n_{\max}}$	0.751	0.880	0.951	0.990	1.012	
	$\mathrm{rate}_{E^n_{\mathrm{conv}}}$	0.700	0.746	0.767	0.777	0.783	
$\beta = 0.4$	$E^n_{\max_I}$	6.318E−2	2.163E−2	7.225E−3	2.393E−3	7.906E−4	2.609E−4
	$\mathrm{rate}_{E^n_{\max_I}}$	1.547	1.582	1.594	1.598	1.599	
	$E^n_{\max}$	2.863E−1	1.468E−1	7.060E−2	3.209E−2	1.358E−2	5.086E−3
	$\mathrm{rate}_{E^n_{\max}}$	0.964	1.056	1.138	1.240	1.417	
	$\mathrm{rate}_{E^n_{\mathrm{conv}}}$	0.565	0.581	0.589	0.593	0.596	
$\beta = 0.6$	$E^n_{\max_I}$	1.758E−1	7.328E−2	2.886E−2	1.110E−2	4.231E−3	1.607E−3
	$\mathrm{rate}_{E^n_{\max_I}}$	1.263	1.344	1.378	1.392	1.397	
	$E^n_{\max}$	1.758E−1	7.328E−2	2.886E−2	1.110E−2	1.225E−2	1.296E−2
	$\mathrm{rate}_{E^n_{\max}}$	1.263	1.344	1.378	-0.142	-0.082	
	$\mathrm{rate}_{E^n_{\mathrm{conv}}}$	0.398	0.405	0.407	0.407	0.406	
$\beta = 0.8$	$E^n_{\max_I}$	5.185E−1	2.947E−1	1.479E−1	6.900E−2	3.100E−2	1.369E−2
	$\mathrm{rate}_{E^n_{\max_I}}$	0.815	0.994	1.100	1.154	1.180	
	$E^n_{\max}$	5.185E−1	2.947E−1	1.479E−1	7.440E−2	7.131E−2	6.516E−2
	$\mathrm{rate}_{E^n_{\max}}$	0.815	0.994	0.992	0.061	0.130	
	$\mathrm{rate}_{E^n_{\mathrm{conv}}}$	0.214	0.217	0.218	0.216	0.215	

Table 2 Errors and order convergence for $u_{\mathrm{ex}}^{2,\beta}$ using using (41) with $r = \frac{2-\beta}{\beta}$

$r = \frac{2-\beta}{\beta}$		$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1028$
$\beta = 0.2$	$E^n_{\max_l}$	2.204E–2	6.439E–3	1.858E–3	5.344E–4	1.535E–4	4.410E–5
	$\mathrm{rate}_{E^n_{\max_l}}$	1.775	1.793	1.798	1.799	1.800	
	$E^n_{\max}$	4.902E–2	2.435E–2	1.203E–2	5.917E–3	2.899E–3	1.413E–3
	$\mathrm{rate}_{E^n_{\max}}$	1.009	1.017	1.024	1.030	1.036	
	$\mathrm{rate}_{E^n_{\mathrm{conv}}}$	0.638	0.659	0.681	0.700	0.716	
$\beta = 0.4$	$E^n_{\max_l}$	5.527E–2	1.897E–2	6.342E–3	2.101E–3	6.943E–4	2.291E–4
	$\mathrm{rate}_{E^n_{\max_l}}$	1.543	1.581	1.594	1.598	1.599	
	$E^n_{\max}$	5.527E–2	1.897E–2	8.380E–3	3.611E–3	1.421E–3	1.058E–3
	$\mathrm{rate}_{E^n_{\max}}$	1.543	1.179	1.214	1.346	0.426	
	$\mathrm{rate}_{E^n_{\mathrm{conv}}}$	0.523	0.539	0.553	0.564	0.573	
$\beta = 0.6$	$E^n_{\max_l}$	1.515E–1	6.342E–2	2.502E–2	9.630E–3	3.671E–3	1.394E–3
	$\mathrm{rate}_{E^n_{\max_l}}$	1.256	1.342	1.377	1.391	1.397	
	$E^n_{\max}$	1.515E–1	6.342E–2	2.502E–2	9.630E–3	6.739E–3	5.411E–3
	$\mathrm{rate}_{E^n_{\max}}$	1.256	1.342	1.377	0.515	0.317	
	$\mathrm{rate}_{E^n_{\mathrm{conv}}}$	0.375	0.383	0.389	0.393	0.395	
$\beta = 0.8$	$E^n_{\max_l}$	4.380E–1	2.510E–1	1.265E–1	5.908E–2	2.656E–2	1.173E–2
	$\mathrm{rate}_{E^n_{\max_l}}$	0.803	0.989	1.098	1.153	1.179	
	$E^n_{\max}$	4.380E–1	2.510E–1	1.265E–1	5.908E–2	2.697E–2	2.394E–2
	$\mathrm{rate}_{E^n_{\max}}$	0.803	0.989	1.098	1.131	0.172	
	$\mathrm{rate}_{E^n_{\mathrm{conv}}}$	0.199	0.202	0.203	0.204	0.204	

Finally, we compare these results with the results obtained when discretising the Caputo time fractional derivative using the well-known $L1$-algorithm on uniform [63] and graded meshes [41]. The $L1$-approximation is defined as follows:

$$\begin{aligned}(g * v)(t_i) \approx (g * v)_i^{\mathrm{L1}} &:= \sum_{l=1}^{i} u_l \int_{t_{l-1}}^{t_l} \mathrm{g}_{1-\beta}(t_i - s)\,\mathrm{d}s \\ &= \sum_{l=1}^{i} u_l \big[\mathrm{g}_{2-\beta}(t_i - t_{l-1}) - \mathrm{g}_{2-\beta}(t_i - t_l)\big], \end{aligned} \tag{42}$$

where $\mathrm{g}_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. Hence, the $L1$-approximation to the Caputo fractional derivative of order $\beta \in (0,1)$ at the time point t_i is given by

$$\left.\frac{\partial^\beta u}{\partial t^\beta}\right|_{t=t_i} = \sum_{l=1}^{i} \tilde{a}_{i,l}(u_l - u_{l-1}) \quad \text{with } \tilde{a}_{i,l} = \frac{\mathrm{g}_{2-\beta}(t_i - t_{l-1}) - \mathrm{g}_{2-\beta}(t_i - t_l)}{\tau_l}.$$

Therefore, using this approximation, the time-discrete scheme (25) becomes

$$\tilde{a}_{i,i}(u_i, \varphi) + \mathcal{L}_i(u_i, \varphi) = (f_i, \varphi) + \tilde{a}_{i,i}(u_{i-1}, \varphi) - \sum_{l=1}^{i-1} \tilde{a}_{i,l}(u_l - u_{l-1}, \varphi).$$

Now, we perform the same experiments as before using the $L1$-approximation. The results for a uniform mesh (i.e. (42) with $r = 1$) are plotted in Figs. 5–8. The maximum error is smaller in comparison with the uniform time-discrete convolution (17). The results suggest an asymptotic convergence rate of $\mathcal{O}(\tau^\beta)$ for $E^n_{\max}$ and of $\mathcal{O}(\tau)$ for E^n_{conv} when using

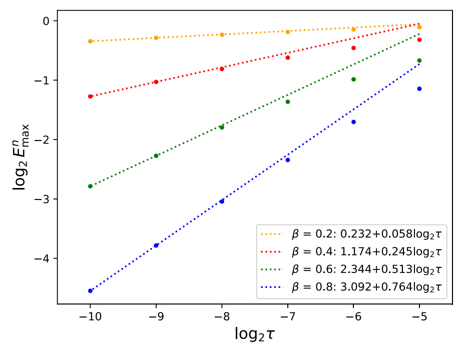


Figure 5 Asymptotic convergence rate for $u_{\text{ex}}^{1,\beta}$ on a logarithmic scale using (41) with $r = 1$: $\log_2 E^n_{\max}$

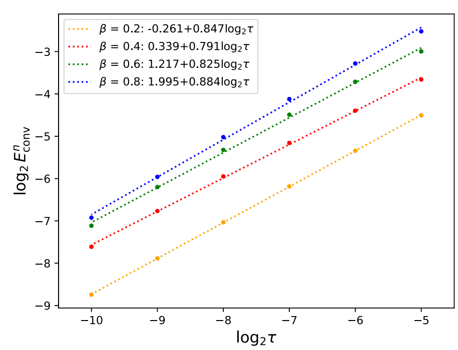


Figure 6 Convergence rate for $u_{\text{ex}}^{1,\beta}$ on a logarithmic scale using (41) with $r = 1$: $\log_2 E^n_{\text{conv}}$

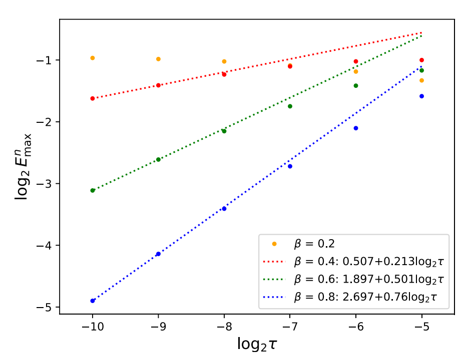


Figure 7 Asymptotic convergence rate for $u_{\text{ex}}^{2,\beta}$ on a logarithmic scale using (41) with $r = 1$: $\log_2 E^n_{\max}$

(42). The results for the graded mesh with $r = \frac{2-\beta}{\beta}$ are given in Tables 3 and 4. We again see that the asymptotic convergence rate of $E^n_{\max_I}$ is $2 - \beta$ and that the asymptotic convergence rate of E^n_{conv} is 1. We observe that the error away from $t = 0$ can dominate the error around the initial time, which gives a possible explanation for not obtaining the convergence rate of $2 - \beta$ for $E^n_{\max}$ as was obtained before in [41]. We note that the maximum error $E^n_{\max_I}$ is smaller in comparison with using the time-discrete convolution quadrature (41) with $r = \frac{2-\beta}{\beta}$ for every experiment. However, we observe that the maximum error $E^n_{\max}$ is (slightly) smaller in comparison with using the time-discrete convolution (41) with $r = \frac{2-\beta}{\beta}$ for $\beta = 0.6$ and $\beta = 0.8$, but it is (slightly) larger for $\beta = 0.2$ and $\beta = 0.4$. Further research should be undertaken to investigate these observations in more detail.

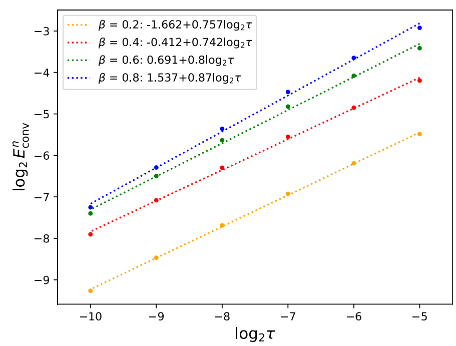


Figure 8 Convergence rate for $u_{\mathrm{ex}}^{2,\beta}$ on a logarithmic scale using (41) with $r = 1$: $\log_2 E_{\mathrm{conv}}^n$

Table 3 Errors and order convergence for $u_{\mathrm{ex}}^{1,\beta}$ using (42) with $r = \frac{2-\beta}{\beta}$

$r = \frac{2-\beta}{\beta}$		$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1028$
$\beta = 0.2$	$E_{\max_I}^n$	1.992E–2	5.794E–3	1.670E–3	4.801E–4	1.379E–4	3.961E–5
	$\mathrm{rate}_{E_{\max_I}^n}$	1.782	1.795	1.798	1.800	1.800	
	$E_{\max}^n$	5.719E–1	3.434E–1	1.894E–1	9.953E–2	5.104E–2	2.584E–2
	$\mathrm{rate}_{E_{\max}^n}$	0.736	0.859	0.928	0.964	0.982	
	$\mathrm{rate}_{E_{\mathrm{conv}}^n}$	0.921	0.981	0.994	1.004	1.005	
$\beta = 0.4$	$E_{\max_I}^n$	3.875E–2	1.307E–2	4.346E–3	1.437E–3	4.745E–4	1.566E–4
	$\mathrm{rate}_{E_{\max_I}^n}$	1.567	1.589	1.596	1.599	1.600	
	$E_{\max}^n$	3.110E–1	1.691E–1	8.825E–2	4.511E–2	2.281E–2	1.147E–2
	$\mathrm{rate}_{E_{\max}^n}$	0.879	0.938	0.968	0.984	0.992	
	$\mathrm{rate}_{E_{\mathrm{conv}}^n}$	0.999	1.020	1.023	1.020	1.016	
$\beta = 0.6$	$E_{\max_I}^n$	7.709E–2	3.042E–2	1.171E–2	4.465E–3	1.696E–3	6.432E–4
	$\mathrm{rate}_{E_{\max_I}^n}$	1.341	1.377	1.391	1.397	1.399	
	$E_{\max}^n$	1.927E–1	1.011E–1	5.184E–2	2.627E–2	1.323E–2	6.639E–3
	$\mathrm{rate}_{E_{\max}^n}$	0.930	0.963	0.981	0.990	0.995	
	$\mathrm{rate}_{E_{\mathrm{conv}}^n}$	1.024	1.037	1.038	1.034	1.030	
$\beta = 0.8$	$E_{\max_I}^n$	1.551E–1	7.262E–2	3.268E–2	1.444E–2	6.326E–3	2.761E–3
	$\mathrm{rate}_{E_{\max_I}^n}$	1.095	1.152	1.179	1.191	1.196	
	$E_{\max}^n$	1.551E–1	7.262E–2	3.303E–2	1.668E–2	8.391E–3	4.210E–3
	$\mathrm{rate}_{E_{\max}^n}$	1.095	1.137	0.985	0.991	0.995	
	$\mathrm{rate}_{E_{\mathrm{conv}}^n}$	1.016	1.031	1.036	1.037	1.037	

8 Multiple regions

In this section, we consider a composite medium consisting out of N separated subdomains (e.g. layers). For ease of exposition, we take $N = 2$ and make the following assumptions:

- **AS-11:** $\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{S}$, where Ω_1 and Ω_2 are non-overlapping Lipschitz subdomains with interface $\mathcal{S} := \overline{\Omega}_1 \cap \overline{\Omega}_2$;
- **AS-12:** The normal flux of u is continuous along $\mathcal{S}$, i.e.

$$[\![\boldsymbol{A}\nabla u(t) \cdot \mathbf{n}]\!]_{\mathcal{S}} = \left(\boldsymbol{A}^2 \nabla u^2(t) - \boldsymbol{A}^1 \nabla u^1(t)\right) \cdot \mathbf{n} = 0, \quad t \in (0, T], \tag{43}$$

where $\mathbf{n}$ denotes the outer normal on Ω_1, and u^1, u^2 are the limiting values of the function u as $\mathcal{S}$ is approached from Ω_1, Ω_2, respectively.

Table 4 Errors and order convergence for $u_{\text{ex}}^{2,\beta}$ using (42) with $r = \frac{2-\beta}{\beta}$

$r = \frac{2-\beta}{\beta}$		$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1028$
$\beta = 0.2$	$E^n_{\max_l}$	1.733E−2	5.046E−3	1.455E−3	4.183E−4	1.202E−4	3.451E−5
	$\text{rate}_{E^n_{\max_l}}$	1.780	1.794	1.798	1.800	1.800	
	$E^n_{\max}$	5.105E−2	2.580E−2	1.279E−2	6.399E−3	3.188E−3	1.598E−3
	$\text{rate}_{E^n_{\max}}$	0.984	1.012	1.000	1.005	0.996	
	$\text{rate}_{E^n_{\text{conv}}}$	1.083	1.058	1.004	1.039	1.052	
$\beta = 0.4$	$E^n_{\max_l}$	3.239E−2	1.096E−2	3.645E−3	1.206E−3	3.982E−4	1.314E−4
	$\text{rate}_{E^n_{\max_l}}$	1.564	1.588	1.596	1.599	1.600	
	$E^n_{\max}$	4.603E−2	2.310E−2	1.157E−2	5.786E−3	2.892E−3	1.444E−3
	$\text{rate}_{E^n_{\max}}$	0.995	0.998	0.999	1.001	1.002	
	$\text{rate}_{E^n_{\text{conv}}}$	1.060	1.043	1.030	1.022	1.018	
$\beta = 0.6$	$E^n_{\max_l}$	6.202E−2	2.460E−2	9.487E−3	3.619E−3	1.375E−3	5.216E−4
	$\text{rate}_{E^n_{\max_l}}$	1.334	1.374	1.390	1.396	1.399	
	$E^n_{\max}$	6.202E−2	2.460E−2	1.028E−2	5.141E−3	2.568E−3	1.282E−3
	$\text{rate}_{E^n_{\max}}$	1.334	1.258	1.000	1.001	1.003	
	$\text{rate}_{E^n_{\text{conv}}}$	1.059	1.051	1.043	1.036	1.031	
$\beta = 0.8$	$E^n_{\max_l}$	1.201E−1	5.673E−2	2.563E−2	1.134E−2	4.974E−3	2.172E−3
	$\text{rate}_{E^n_{\max_l}}$	1.082	1.146	1.176	1.189	1.195	
	$E^n_{\max}$	1.201E−1	5.673E−2	2.563E−2	1.134E−2	4.974E−3	2.172E−3
	$\text{rate}_{E^n_{\max}}$	1.082	1.146	1.176	1.189	1.195	
	$\text{rate}_{E^n_{\text{conv}}}$	1.384	1.444	1.256	1.037	1.038	

The weak formulation (4.1) is also satisfied in this situation as AS-12 implies that for all $\varphi \in \mathrm{H}^1_0(\Omega)$ it holds that

$$\begin{aligned} -\big(\nabla \cdot \big[\boldsymbol{A}(t)\nabla u(t)\big], \varphi\big) &= -\big(\nabla \cdot \big[\boldsymbol{A}(t)\nabla u(t)\big], \varphi\big)_{\Omega_1} - \big(\nabla \cdot \big[\boldsymbol{A}(t)\nabla u(t)\big], \varphi\big)_{\Omega_2} \\ &= \big(\boldsymbol{A}(t)\nabla u(t), \nabla\varphi\big)_{\Omega} + \big(\llbracket \boldsymbol{A}\nabla u(t)\cdot \mathbf{n}\rrbracket_{\mathcal{S}}, \varphi\big)_{\mathcal{S}} \\ &= \big(\boldsymbol{A}(t)\nabla u(t), \nabla\varphi\big). \end{aligned}$$

The analysis in Sect. 6 stays valid. Therefore, we can conclude the following theorem.

Theorem 8.1 (Existence and uniqueness: multiple regions) *Let assumptions AS-*(1–6) *and AS-*(11–12) *be fulfilled. Then there exists a unique weak solution u to problem* (21) *with* $u \in \mathrm{L}^\infty((0,T), \mathrm{H}^1_0(\Omega))$ *and* $\partial_t(g * (u - \tilde{u}_0)) \in \mathrm{L}^\infty((0,T), \mathrm{H}^1_0(\Omega)^*)$.

Next, we show the $\mathrm{L}^2((0,T), \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega))$ regularity of the solution when $\beta(\mathbf{x})$ is a step function. We consider the following assumptions:

- **AS-13:** $\beta(\mathbf{x})$ is a step function, i.e.

$$\beta(\mathbf{x}) = \begin{cases} \overline{\beta}_1 & \mathbf{x} \in \Omega_1; \\ \overline{\beta}_2 & \mathbf{x} \in \Omega_2, \end{cases} \tag{44}$$

  with $0 < \overline{\beta}_1, \overline{\beta}_2 < 1$;

- **AS-14:**

$$\boldsymbol{A}(\mathbf{x},t) = a(\mathbf{x}) = \begin{cases} \overline{a}_1 & \mathbf{x} \in \Omega_1; \\ \overline{a}_2 & \mathbf{x} \in \Omega_2. \end{cases}$$

Next, we are able to obtain a similar estimate as in Lemma 7.1 as $\nabla g = 0$ in Ω_1 and Ω_2.

Lemma 8.1 *Let assumptions AS-*(1–6) *and AS-*(11–14) *be fulfilled. Then a positive constant C exists such that, for every $j = 1, 2, \ldots, n$, the following relation holds*:

$$\int_\Omega \left(g * |\nabla u|^2\right)_j(\mathbf{x})\,\mathrm{d}\mathbf{x} + \sum_{i=1}^j \|\sqrt{g_i}\nabla u_i\|^2\tau + \sum_{i=1}^j \|\Delta u_i\|^2\tau \le C$$

Proof We multiply (38) by $-a\Delta u_i\tau$ and integrate the result over Ω_1 and Ω_2, respectively. In particular for the first term on the LHS, using $\nabla g = 0$ in Ω_1 and Ω_2, we obtain that

$$-\left((g * \delta u)_i, a\Delta u_i\right)_{\Omega_1} = \left((g * \nabla \delta u)_i, a\nabla u_i\right)_{\Omega_1} - \left(\left(g * \delta u^1\right)_i, \overline{a}_1 \nabla u_i^1 \cdot \boldsymbol{\nu}\right)_{\mathcal{S}},$$
$$-\left((g * \delta u)_i, a\Delta u_i\right)_{\Omega_2} = \left((g * \nabla \delta u)_i, a\nabla u_i\right)_{\Omega_2} - \left(\left(g * \delta u^2\right)_i, \overline{a}_2 \nabla u_i^2 \cdot \boldsymbol{\nu}\right)_{\mathcal{S}}.$$

Hence, by $u_i \in \mathrm{H}_0^1(\Omega)$ and AS-12, we get that

$$\begin{aligned} -\left((g * \delta u)_i, a\Delta u_i\right) &= \left((g * \nabla \delta u)_i, a\nabla u_i\right) + \left((g * \delta u)_i, [\![a\nabla u_i \cdot \mathbf{n}]\!]_{\mathcal{S}}\right) \\ &\overset{(43)}{=} \left((g * \nabla \delta u)_i, a\nabla u_i\right). \end{aligned}$$

We add up the resulting equations on Ω_1 and Ω_2 and sum the result over $i = 1, \ldots j$ with $1 \le j \le n$. Using relation (19), we get that

$$\begin{aligned} &\sum_{i=1}^j \left(\delta\left(g * (\sqrt{a}\nabla u)\right)_i, \sqrt{a}\nabla u_i\right)\tau + \sum_{i=1}^j \|a\Delta u_i\|^2\tau \\ &\quad = \sum_{i=1}^j (f_i, a\Delta u_i)\tau + \sum_{i=1}^j (g_i \nabla \tilde{u}_0, a\nabla u_i)\tau + \sum_{i=1}^j (c_i u_i, a\Delta u_i)\tau. \end{aligned} \tag{45}$$

We obtain from Lemma 3.3 that

$$\sum_{i=1}^j \left(\delta\left(g * (\sqrt{a}\nabla u)\right)_i, \sqrt{a}\nabla u_i\right)\tau \ge \frac{1}{2}\int_\Omega \left(g * |\sqrt{a}\nabla u|^2\right)_j(\mathbf{x})\,\mathrm{d}\mathbf{x} + \frac{1}{2}\sum_{i=1}^j \|\sqrt{g_i}\sqrt{a}\nabla u_i\|^2\tau.$$

All the other terms in (45) can be handled as in Lemma 7.1, and so we can conclude the proof. □

From this lemma it follows that $u \in \mathrm{L}^2((0,T), \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega))$ and $\partial_t(g * (u - \tilde{u}_0)) \in \mathrm{L}^2((0,T), \mathrm{L}^2(\Omega))$ with

$$\partial_t\left(g * (u - \tilde{u}_0)\right) = f + a\Delta u - cu \quad \text{a.e. in } Q_T. \tag{46}$$

When discussing the uniqueness of a solution, we get again from Corollary 3.1 inequality (39) where $\beta_1 = \max\{\overline{\beta}_1, \overline{\beta}_2\}$ in (8). We conclude the following theorem.

Theorem 8.2 *Let assumptions AS-*(1–6) *and AS-*(11–14) *be fulfilled. Then there exists a unique weak solution u to problem* (21) *with*

$$u \in \mathrm{L}^\infty\big((0,T),\mathrm{H}_0^1(\Omega)\big) \cap \mathrm{L}^2\big((0,T),\mathrm{H}^2(\Omega)\cap \mathrm{H}_0^1(\Omega)\big) \quad \textit{and}$$
$$\partial_t\big(g*(u-\tilde{u}_0)\big) \in \mathrm{L}^2\big((0,T),\mathrm{L}^2(\Omega)\big).$$

We are not able to show the continuity of the solution in time by the argument used before for the fractional derivative of constant order. Let us consider

$$\partial_t\big(g*(u-\tilde{u}_0)\big)(t) = f(t) - L(t)u(t) \quad \text{in } \mathrm{H}_0^1(\Omega_i)^* \text{ for a.a. } t \in (0,T), i = 1,2;$$

and define $l_i(t) := \frac{t^{\overline{\beta}_i - 1}}{\Gamma(\overline{\beta}_i)}$ for $i = 1,2$. Then, for $\varphi \in \mathrm{H}_0^1(\Omega)$ and $i = 1,2$, it holds that

$$\langle u(t) - \tilde{u}_0, \varphi\rangle_{\mathrm{H}_0^1(\Omega_i)^* \times \mathrm{H}_0^1(\Omega_i)} = \big(l_i * \big[(f,\varphi)_{\Omega_i} - (a\nabla u, \nabla\varphi)_{\Omega_i} + (a\nabla u\cdot \boldsymbol{\nu}, \varphi)_{\mathcal{S}}\big]\big)(t).$$

We do not know how to get control over the boundary term as $l_1 \neq l_2$. However, if $\tilde{u}_0$ in $\mathrm{H}^2(\Omega)$, then we can show the continuity in time by multiplying (38) by $-a\Delta\delta u_i\tau$. Then, as in the proof of Lemma 8.1, we have that

$$-\big((g*\delta u)_i, a\Delta\delta u_i\big) = \big((g*\nabla\delta u)_i, a\nabla\delta u_i\big) + \big((g*\delta u)_i, [\![a\nabla\delta u_i\cdot\mathbf{n}]\!]_{\mathcal{S}}\big).$$

The boundary term cancels out if AS-12 is satisfied at $t = 0$. Together with $\tilde{u}_0 \in \mathrm{H}^2(\Omega)$ we see that necessarily $[\![a]\!]_{\mathcal{S}} = 0$. Hence, we immediately assume that

- **AS-15:** $\boldsymbol{A}(\mathbf{x},t) = \overline{a} \in (0,\infty)$ in Q_T.

Lemma 8.2 *Let assumptions AS-*(1–6), *AS-*9, *AS-*(11–13) *and AS-*15 *be fulfilled. Then positive constants C and* τ_0 *exist such that, for any* $\tau < \tau_0$ *and for every* $j = 1,2,\dots,n$, *the following relation holds*:

$$\|\Delta u_j\|^2 + \sum_{i=1}^{j} \|\Delta u_i - \Delta u_{i-1}\|^2 \le C.$$

Proof We multiply (38) by $-\Delta\delta u_i\tau$ and integrate the result over Ω_1 and Ω_2, respectively. Now, as in the proof of Lemma 8.1, we have that

$$\begin{aligned}-\big((g*\delta u)_i, \Delta\delta u_i\big) &= \big((g*\nabla\delta u)_i, \nabla\delta u_i\big) + \big((g*\delta u)_i, [\![\nabla\delta u_i\cdot\mathbf{n}]\!]_{\mathcal{S}}\big)\\ &= \big((g*\nabla\delta u)_i, \nabla\delta u_i\big)\end{aligned}$$

and

$$-(c_i u_i, \Delta\delta u_i) = (c_i\nabla u_i, \nabla\delta u_i) + \big(c_i u_i, [\![\nabla\delta u_i\cdot\mathbf{n}]\!]_{\mathcal{S}}\big) \overset{(43)}{=} (c_i\nabla u_i, \nabla\delta u_i).$$

Hence, we obtain (40) and we can proceed as in Lemma 7.2. □

From this lemma, we obtain that $u \in \mathrm{L}^\infty((0,T),\mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega))$. Hence, from

$$u(t) - \tilde{u}_0 = \big(l_i * [f + a\Delta u - cu]\big)(t) \quad \text{a.e. in } \Omega_i \text{ for a.a. } t \in (0,T), i = 1,2,$$

we get that

$$\lim_{t\searrow 0}\|u(t) - \tilde{u}_0\| = \sum_{i=1}^{2}\lim_{t\searrow 0}\|u(t) - \tilde{u}_0\|_{\mathrm{L}^2(\Omega_i)} = 0.$$

Theorem 8.3 *Let assumptions AS-*(1–6), *AS*-9, *AS*-(11–13) *and AS*-15 *be fulfilled. Then there exists a unique weak solution u to problem* (21) *with* $u \in \mathrm{C}([0,T],\mathrm{L}^2(\Omega))\cap\mathrm{L}^\infty((0,T),\mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega))$ *and* $\partial_t(g*(u-\tilde{u}_0)) \in \mathrm{L}^\infty((0,T),\mathrm{L}^2(\Omega))$.

9 Conclusion

We have investigated an initial-boundary value problem for a fractional diffusion equation with space-dependent variable order where the coefficients are dependent on spatial and time variables. First, we have studied the properties of the governing kernel in Sect. 2. Afterwards, in Sect. 3, we have generalised a fundamental identity for integro-differential operators of the form $\frac{\mathrm{d}}{\mathrm{d}t}(k * v)(t)$ to a convolution kernel that is also space-dependent. By the aid of a convolution quadrature on a uniform mesh, we have proven the existence and uniqueness of a unique weak solution to the problem by aid of Rothe's method in the next sections. Moreover, we discussed in detail the constant-order case and under which assumptions a more regular solution can be obtained. We considered a time-discrete convolution on a graded mesh in order to improve the computational results. We also investigated a composite medium consisting of a finite number of separated subdomains. An interesting direction for future research is to investigate the well-posedness of the fractional wave equation of variable order.

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