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Blowup for nonlinearly damped viscoelastic equations with logarithmic source and delay terms

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Abstract

In this work, we investigate blowup phenomena for nonlinearly damped viscoelastic equations with logarithmic source effect and time delay in the velocity. Owing to the nonlinear damping term instead of strong or linear dissipation, we cannot apply the concavity method introduced by Levine. Thus, utilizing the energy method, we show that the solutions with not only non-positive initial energy but also some positive initial energy blow up at a finite point in time.

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Keywords: Blowup; Viscoelastic equation; Logarithmic source; Time delay; Nonlinear dissipation

1 Introduction

We discuss the viscoelastic wave equation with nonlinear damping, logarithmic source, and delay terms

$$
\begin{aligned}
&u_{tt} - \Delta u + k * \Delta u + c_1 \left|u_t(t)\right|^{q-2} u_t(t) + c_2 \left|u_t(t-\tau)\right|^{q-2} u_t(t-\tau) \\
&\quad = |u|^{p-2} u \ln|u| \quad \text{in } \Omega \times (0,T), \qquad (1.1)\\
&u = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad (1.2)\\
&u(0) = u_0, \qquad u_t(0) = u_1 \quad \text{in } \Omega, \qquad (1.3)\\
&u_t(t-\tau) = j_0(t-\tau) \quad \text{for } t \in (0,\tau), \qquad (1.4)
\end{aligned}
$$

here $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $k * \Delta u = \int_0^t k(t-s)\Delta u(s)\,ds$, the kernel function $k : [0,\infty) \to (0,\infty)$ is a C^1-function with

$$
k'(t) \le 0 \quad \text{and} \quad 1 - \int_0^\infty k(s)\,ds := k_l > 0,
$$

$\tau > 0$ is time delay, the coefficients $c_1 > 0$ and $c_2 \in \mathbb{R}$ satisfy $0 < |c_2| < c_1$, and the exponents $q \ge 2$ and $p > 2$ are specified later.

Many researchers have studied parabolic or hyperbolic equations with logarithmic nonlinearity [2, 4, 7, 13, 15]. For the physical application of this nonlinearity, we refer to [1, 6]. In [4], the authors discussed a strongly damped equation,

$$u_{tt} - \Delta u - \Delta u_t = |u|^{p-2} u \ln |u|$$

with Dirichlet boundary condition. They showed that the solutions with subcritical and critical initial energy blow up in a finite point under suitable conditions. Moreover, they estimated bounds of the blowup time. The authors of [7] proved similar results to those of [4] for the equation with memory. Most work dealing with wave equations with logarithmic nonlinearity is associated with a strongly or linearly damped mechanism, and blowup results are investigated by virtue of the potential well method and Levine's concavity technique [12].

On the other hand, time delay effect arises in many natural phenomena depending not only on the present state but also on some past occurrences. Thus, partial differential equations with time delay have become an active area of research in resent years. For the physical application of the time delay, we refer to [3, 18]. Recently, Kafini and Messaoudi [8] considered the wave equation with linear damping and delay terms

$$u_{tt} - \Delta u + c_1 u_t(t) + c_2 u_t(t-\tau) = k|u|^{p-2} u \ln |u|$$

with Dirichlet boundary condition. They established a blowup result of the solution with negative initial energy by adapting the energy method. While there are many studies on the existence and asymptotic stability of the solutions of wave equations with delay, there are relatively few studies on blowup. We refer to [5, 10, 16, 17, 19] and [9] for stability and blowup of equations with delay, respectively. Motivated by this pioneering work [8], in this article, we study blowup phenomena for the nonlinearly damped viscoelastic wave equation (1.1)–(1.4) with logarithmic source effect and time delay in the velocity. Due to the presence of nonlinear dissipation instead of strong or linear damping terms, we cannot apply the concavity method. Thus, by applying the energy method, we establish a blowup result of solutions with not only non-positive initial energy but also some positive initial energy. And, it is worth to mention that there are few works dealing with viscoelastic wave equations with nonlinear damping and logarithmic source terms.

Here is the outline of this paper. In Sect. 2, we present notations, hypotheses, and auxiliary functions and lemmas. In Sect. 3, we establish a blowup criterion of solutions with not only non-positive initial energy but also some positive initial energy.

2 Preliminaries

Throughout this article, $(\cdot,\cdot)$ denotes the scalar product in Hilbert space $L^2(\Omega)$. $\|\cdot\|_r$ represents the norm in the space $L^r(\Omega)$. Moreover, $\|\cdot\|_Y$ denotes the norm of a normed space Y. $C > 0$ represents a generic constant. If there is no ambiguity, we omit the variables t and x.

We let the function y be as in [16]

$$y(x,\sigma,t) = u_t(x, t - \sigma\tau) \quad \text{for } (x,\sigma,t) \in \Omega \times (0,1) \times (0,T).$$

Then problem (1.1)–(1.4) reads

$$u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)\,ds + c_1|u_t|^{q-2}u_t + c_2\big|y(x,1,t)\big|^{q-2}y(x,1,t)$$
$$= |u|^{p-2}u\ln|u| \quad \text{in } \Omega\times(0,T), \tag{2.1}$$
$$\tau y_t(x,\sigma,t) + y_\sigma(x,\sigma,t) = 0 \quad \text{for } (x,\sigma,t)\in\Omega\times(0,1)\times(0,T), \tag{2.2}$$
$$u = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{2.3}$$
$$u(0) = u_0, \qquad u_t(0) = u_1 \quad \text{in } \Omega, \tag{2.4}$$
$$y(0) = j_0(x,-\sigma\tau) := y_0 \quad \text{in } \Omega\times(0,1). \tag{2.5}$$

By the arguments of [7, 9], we can state the well-posedness.

Theorem 2.1 *Let* $(u_0,u_1,y_0)\in H_0^1(\Omega)\times L^2(\Omega)\times L^q(\Omega\times(0,1))$, $q\geq 2$, *and*

$$2 < p < \frac{2(n-1)}{n-2} \quad \text{if } n\geq 3, \qquad 2<p<\infty \quad \text{if } n=1,2.$$

Then problem (2.1)–(2.5) *admits a unique local solution* (u,y) *with* $u\in C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega))$ *and* $y\in L^\infty(0,T;L^q(\Omega\times(0,1)))$.

Our goal is to find a blowup result to problem (2.1)–(2.5). For this, we will often use the embedding

$$H_0^1(\Omega)\hookrightarrow L^r(\Omega), \quad \text{where } 2\leq r\leq\bar{r} := \begin{cases}\infty & \text{if } n=1,2;\\ \frac{2n}{n-2} & \text{if } n\geq 3,\end{cases}$$

and Young's inequality

$$ab \leq \frac{1}{r}a^r + \frac{1}{r^*}b^{r^*}, \tag{2.6}$$

where

$$a,b\geq 0, \qquad r,r^*>1, \qquad \frac{1}{r}+\frac{1}{r^*} = 1.$$

Also, we need the lemmas below, which are proved by Kafini and Messaoudi [8], to estimate logarithmic nonlinearity.

Lemma 2.1 *For* $\phi\in L^p(\Omega)\cap H_0^1(\Omega)$, *we have*

$$\|\phi\|_p^s \leq C\big(\|\phi\|_p^p + \|\nabla\phi\|_2^2\big) \quad \textit{for } 2\leq s\leq p.$$

Lemma 2.2 *For* $\phi\in L^{p+1}(\Omega)\cap H_0^1(\Omega)$ *with* $\int_\Omega|\phi|^p\ln|\phi|\,dx\geq 0$, *we have*

$$\left(\int_\Omega|\phi|^p\ln|\phi|\,dx\right)^{\frac{s}{p}} \leq C\left(\|\nabla\phi\|_2^2 + \int_\Omega|\phi|^p\ln|\phi|\,dx\right) \quad \textit{for } 2\leq s\leq p.$$

Lemma 2.3 *For $\phi \in L^p(\Omega) \cap H_0^1(\Omega)$ with $\int_\Omega |\phi|^p \ln|\phi|\,dx \ge 0$, we have*

$$\|\phi\|_p^p \le C\left(\|\nabla\phi\|_2^2 + \int_\Omega |\phi|^p \ln|\phi|\,dx \right).$$

To establish our desired blowup result, we impose the following assumptions:

(H_n) Let $1 \le n \le 5$.

(H_p) Let p satisfy

$$\begin{cases} 2 < p < \infty & \text{if } n = 1,2; \\ 2 < p < \min\{\frac{2(n-1)}{n-2}, \frac{n+2}{n-2}\} & \text{if } n = 3,4,5. \end{cases}$$

(H_q) Let q verify

$$\max\left\{2, \frac{p^2+2p}{p^2-2p+4}\right\} < q < p. \tag{2.7}$$

(H_k) Let k satisfy

$$\int_0^\infty k(s)\,ds < \frac{p(1-\lambda)-2}{p(1-\lambda)-2+\frac{1}{2\eta}}, \tag{2.8}$$

where

$$0 < \lambda < \frac{p-2}{p}, \qquad 0 < \eta < \frac{p(1-\lambda)}{2}. \tag{2.9}$$

From (H_p), there exists $\mu > 0$ satisfying

$$\begin{cases} 2 < p < p+\mu < \infty & \text{if } n = 1,2; \\ 2 < p < p+\mu < \frac{2n}{n-2} & \text{if } n = 3,4,5. \end{cases} \tag{2.10}$$

This implies

$$H_0^1(\Omega) \hookrightarrow L^{p+\mu}(\Omega).$$

Put D_0 be the embedding constant with

$$\|\phi\|_{p+\mu} \le D_0\|\nabla\phi\|_2, \quad \phi \in H_0^1(\Omega). \tag{2.11}$$

We let $D = \frac{D_0}{\sqrt{k_l}}$ and define a continuously differentiable function K as

$$K(\xi) = -\frac{D^{p+\mu}}{e\mu p}\xi^{p+\mu} + \frac{1}{2}\xi^2 \tag{2.12}$$

and put

$$\xi_K = \left(\frac{e\mu p}{p+\mu}\right)^{\frac{1}{p+\mu-2}} \left(\frac{1}{D}\right)^{\frac{p+\mu}{p+\mu-2}}, \tag{2.13}$$

$$K_{\max} = \left(\frac{1}{2} - \frac{1}{p+\mu}\right)\left(\frac{e\mu p}{p+\mu}\right)^{\frac{2}{p+\mu-2}} \left(\frac{1}{D}\right)^{\frac{2(p+\mu)}{p+\mu-2}}. \tag{2.14}$$

Lemma 2.4 *For $p > 2$ and $\mu > 0$, the function K satisfies*

(i) $K(0) = 0$,
(ii) $\lim_{\xi\to\infty} K(\xi) = -\infty$,
(iii) $K'(\xi) > 0$ *on* $(0, \xi_K)$, $K'(\xi) < 0$ *on* (ξ_K, ∞),
(iv) *K has the maximum value $K_{\max}$ at ξ_K.*

Proof The results (i) and (ii) are clear. Since

$$K'(\xi) = \xi\left(1 - \frac{(p+\mu)D^{p+\mu}}{e\mu p}\xi^{p+\mu-2}\right), \tag{2.15}$$

we have $K'(\xi_K) = 0$, $K'(\xi) > 0$ on $(0, \xi_K)$, and $K'(\xi) < 0$ on (ξ_K, ∞). Thus, K has the maximum value

$$\begin{aligned} K(\xi_K) &= -\frac{D^{p+\mu}}{e\mu p}\xi_K^{p+\mu} + \frac{1}{2}\xi_K^2 \\ &= \left(\frac{1}{2} - \frac{D^{p+\mu}}{e\mu p}\xi_K^{p+\mu-2}\right)\xi_K^2 \\ &= \left(\frac{1}{2} - \frac{1}{p+\mu}\right)\xi_K^2 \\ &= \left(\frac{1}{2} - \frac{1}{p+\mu}\right)\left(\frac{e\mu p}{p+\mu}\right)^{\frac{2}{p+\mu-2}} \left(\frac{1}{D}\right)^{\frac{2(p+\mu)}{p+\mu-2}}. \end{aligned} \tag{2.16}$$

□

We also need the following auxiliary result in the proof of our main theorem.

Lemma 2.5 *For $p > 2$, $\mu > 0$, and $0 < \lambda < \frac{p-2}{p}$, the ξ_K and K_{max} verify*

$$0 < \frac{\xi_K^2}{2p(1-\lambda)}\left(\frac{p(1-\lambda)}{2} - 1\right) < K_{max}. \tag{2.17}$$

Proof First, we claim

$$A := \left(p^2 - 2p + \mu p + 2\mu\right) - \lambda p(p+\mu-4) > 0. \tag{2.18}$$

Indeed, if $p + \mu \le 4$, it is clear that $A > 0$. If $p + \mu > 4$,

$$\min\left\{\frac{p-2}{p}, \frac{p^2 - 2p + \mu p + 2\mu}{p(p+\mu-4)}\right\} = \frac{p-2}{p}.$$

This means

$$0 < \lambda < \frac{p-2}{p} \le \frac{p^2 - 2p + \mu p + 2\mu}{p(p+\mu-4)}.$$

So, we also have $A > 0$ if $p + \mu > 4$. The result (2.18) implies

$$\begin{aligned}
&\frac{1}{2} - \frac{1}{p+\mu} - \frac{1}{2p(1-\lambda)}\left(\frac{p(1-\lambda)}{2} - 1\right) \\
&\quad = \frac{(p^2 - 2p + \mu p + 2\mu) - \lambda p(p+\mu-4)}{4p(p+\mu)(1-\lambda)} > 0,
\end{aligned}$$

which gives

$$0 < \frac{\xi_K^2}{2p(1-\lambda)}\left(\frac{p(1-\lambda)}{2} - 1\right) < \left(\frac{1}{2} - \frac{1}{p+\mu}\right)\xi_K^2 = K_{max}. \qquad \square$$

3 Blowup results

In this part, we search a blowup result of the solution to (2.1)–(2.5) inspired by the ideas in [8, 14].

We define the energy to problem (2.1)–(2.5) by

$$\begin{aligned}
E(t) = {} & \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t k(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(k \circ \nabla u) + \frac{1}{p^2}\|u\|_p^p \\
& - \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx + \xi\tau\int_0^1 \big\|y(\sigma,t)\big\|_q^q\,d\sigma,
\end{aligned} \tag{3.1}$$

where

$$(k \circ \nabla u)(t) = \int_0^t k(t-s)\big\|\nabla u(t) - \nabla u(s)\big\|_2^2\,ds$$

and

$$\frac{(q-1)|c_2|}{q} < \xi < c_1 - \frac{|c_2|}{q}. \tag{3.2}$$

Lemma 3.1 *Under the conditions of Theorem 2.1, the equation*

$$\frac{d}{dt}E(t) \le -\gamma_1\left(\|u_t\|_q^q + \big\|y(1,t)\big\|_q^q\right) - \frac{k(t)}{2}\|\nabla u\|_2^2 + \frac{1}{2}\left(k' \circ \nabla u\right) \le 0 \tag{3.3}$$

is fulfilled for some $\gamma_1 > 0$.

Proof Taking the scalar product (2.1) by u_t in $L^2(\Omega)$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2\right) - \int_0^t k(t-s)\big(\nabla u(s), \nabla u_t(t)\big)\,ds \\
&\quad = -c_1\|u_t\|_q^q - c_2\int_\Omega u_t(x,t)\big|y(x,1,t)\big|^{q-2}y(x,1,t)\,dx + \int_\Omega u_t|u|^{p-2}u\ln|u|\,dx.
\end{aligned}$$

Using the estimate

$$\int_\Omega u_t|u|^{p-2}u\ln|u|\,dx = \frac{1}{p}\frac{d}{dt}\int_\Omega |u|^p\ln|u|\,dx - \frac{1}{p^2}\frac{d}{dt}\|u\|_p^p$$

and

$$\begin{aligned}
&-\int_0^t k(t-s)\big(\nabla u(s),\nabla u_t(t)\big)\,ds\\
&\quad=\frac{1}{2}\frac{d}{dt}\left\{(k\circ\nabla u)-\left(\int_0^t k(s)\,ds\right)\|\nabla u\|_2^2\right\}+\frac{k(t)}{2}\|\nabla u\|_2^2-\frac{1}{2}\big(k'\circ\nabla u\big),
\end{aligned}$$

we get

$$\begin{aligned}
\frac{d}{dt}E(t) &= -c_1\big\|u_t(t)\big\|_q^q - c_2\int_\Omega u_t(x,t)\big|y(x,1,t)\big|^{q-2}y(x,1,t)\,dx\\
&\quad-\frac{k(t)}{2}\|\nabla u\|_2^2+\frac{1}{2}\big(k'\circ\nabla u\big)+\frac{\partial}{\partial t}\left(\xi\tau\int_0^1\big\|y(\sigma,t)\big\|_q^q\,d\sigma\right).
\end{aligned} \tag{3.4}$$

Using (2.6) with $\frac{q-1}{q}+\frac{1}{q}=1$, we have

$$-c_2\int_\Omega\big|y(x,1,t)\big|^{q-2}y(x,1,t)u_t(x,t)\,dx \le \frac{|c_2|(q-1)}{q}\big\|y(1,t)\big\|_q^q+\frac{|c_2|}{q}\big\|u_t(t)\big\|_q^q. \tag{3.5}$$

From (2.2), we find

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^1\big\|y(\sigma,t)\big\|_q^q\,d\sigma &= \int_\Omega\int_0^1 q\big|y(x,\sigma,t)\big|^{q-2}y(x,\sigma,t)y_t(x,\sigma,t)\,d\sigma\,dx\\
&= -\frac{1}{\tau}\int_\Omega\int_0^1 q\big|y(x,\sigma,t)\big|^{q-2}y(x,\sigma,t)y_\sigma(x,\sigma,t)\,d\sigma\,dx\\
&= -\frac{1}{\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\sigma}\big|y(x,\sigma,t)\big|^q\,d\sigma\,dx\\
&= -\frac{1}{\tau}\big\|y(1,t)\big\|_q^q+\frac{1}{\tau}\big\|y(0,t)\big\|_q^q\\
&= -\frac{1}{\tau}\big\|y(1,t)\big\|_q^q+\frac{1}{\tau}\big\|u_t(t)\big\|_q^q.
\end{aligned} \tag{3.6}$$

From (3.4), (3.5), and (3.6), one sees

$$\begin{aligned}
\frac{d}{dt}E(t) &\le -\left(c_1-\frac{|c_2|}{q}-\xi\right)\|u_t\|_q^q-\left(\xi-\frac{|c_2|(q-1)}{q}\right)\big\|y(1,t)\big\|_q^q\\
&\quad-\frac{k(t)}{2}\|\nabla u\|_2^2+\frac{1}{2}\big(k'\circ\nabla u\big).
\end{aligned}$$

Letting

$$\gamma_1=\min\left\{c_1-\frac{|c_2|}{q}-\xi,\ \xi-\frac{|c_2|(q-1)}{q}\right\},$$

we obtain (3.3) from (3.2). □

Lemma 3.2 *Let (u, y) be the solution of (2.1)–(2.5). If the initial datum (u_0, y_0) satisfies*

$$E(0) < K_{\max} \quad \textit{and} \quad \sqrt{k_l}\|\nabla u_0\|_2 \geq \xi_K, \tag{3.7}$$

there exists $\xi_1 > \xi_K$ such that

$$\sqrt{k_l}\big\|\nabla u(t)\big\|_2 \geq \xi_1, \quad t \geq 0. \tag{3.8}$$

Proof Set

$$\Omega_1 = \big\{x \in \Omega \mid \big|u(x,t)\big| \geq 1\big\} \quad \text{and} \quad \Omega_2 = \big\{x \in \Omega \mid \big|u(x,t)\big| < 1\big\}.$$

Using (3.1), (2.11), (2.12), and the relation (Lemma 2.1, [11])

$$0 \leq \ln a \leq \frac{a^r}{er} \quad \text{for } r > 0, a \geq 1,$$

we derive

$$\begin{aligned}
E(t) &\geq \frac{1}{2}\left(1 - \int_0^t k(s)\,ds\right)\|\nabla u\|_2^2 - \frac{1}{p}\int_{\Omega_1}|u|^p \ln|u|\,dx - \frac{1}{p}\int_{\Omega_2}|u|^p \ln|u|\,dx \\
&\geq \frac{k_l}{2}\|\nabla u\|_2^2 - \frac{1}{e\mu p}\int_{\Omega_1}|u|^{p+\mu}\,dx \\
&\geq \frac{k_l}{2}\|\nabla u\|_2^2 - \frac{1}{e\mu p}\|u\|_{p+\mu}^{p+\mu} \\
&\geq \frac{k_l}{2}\|\nabla u\|_2^2 - \frac{D_0^{p+\mu}}{e\mu p}\|\nabla u\|_2^{p+\mu} \\
&= K\big(\sqrt{k_l}\|\nabla u\|_2\big).
\end{aligned} \tag{3.9}$$

Since $E(0) < K_{\max}$, there exists $\xi_1 > \xi_K$ with $K(\xi_1) = E(0)$. From (3.9), we have

$$K(\xi_1) = E(0) \geq K\big(\sqrt{k_l}\|\nabla u_0\|_2\big). \tag{3.10}$$

Since K is decreasing on (ξ_K, ∞), (3.10) gives

$$\sqrt{k_l}\|\nabla u_0\|_2 \geq \xi_1.$$

To show (3.8), we use a contradiction. Suppose there exists $t_1 > 0$ such that $\sqrt{k_l}\|\nabla u(t_1)\|_2 < \xi_1$. The continuity of u corresponding to t gives the existence of $t_0 > 0$ with

$$\xi_K < \sqrt{k_l}\big\|\nabla u(t_0)\big\|_2 < \xi_1.$$

From this and (3.9), we get

$$E(t_0) \geq K\big(\sqrt{k_l}\big\|\nabla u(t_0)\big\|_2\big) > K(\xi_1) = E(0),$$

but this contradicts (3.3). □

Now, we are ready to state our main theorem.

Definition 3.1 We say that the solution (u,y) of problem (2.1)–(2.5) blows up in a finite time if there exists a time T^*, $0 < T^* < \infty$, such that

$$\lim_{t\to T^*} \|\nabla u(t)\|_2 = \infty.$$

Theorem 3.1 *Let (H_n), (H_p), (H_q), (H_k), and the assumptions of Lemma 3.2 hold. Furthermore, we assume*

$$E(0) < \frac{\xi_K^2}{2k_l p(1-\lambda)}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)\right\}.$$

Then the solution (u,y) to problem (2.1)–(2.5) blows up after finite time.

Proof Let $\overline{E}$ with

$$E(0) < \overline{E} < \frac{\xi_K^2}{2k_l p(1-\lambda)}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)\right\}. \tag{3.11}$$

From this and (2.17), we see

$$E(0) < \overline{E} < \frac{\xi_K^2}{2p(1-\lambda)}\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{\xi_K^2(1-k_l)}{8k_l\eta p(1-\lambda)} < K_{max}. \tag{3.12}$$

Define

$$R(t) = \overline{E} - E(t), \tag{3.13}$$

then

$$R'(t) = -E'(t) \ge \gamma_1\left(\|u_t\|_q^q + \|y(1,t)\|_q^q\right) + \frac{k(t)}{2}\|\nabla u\|_2^2 - \frac{1}{2}\left(k'\circ\nabla u\right) \ge 0. \tag{3.14}$$

From this, (3.1), (3.12), Lemma 3.2, and the definition of ξ_K and $K_{\max}$, we obtain

$$\begin{aligned} 0 < R(0) \le R(t) &\le \overline{E} - \frac{k_l}{2}\|\nabla u\|_2^2 + \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx \\ &< K_{\max} - \frac{\xi_1^2}{2} + \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx \\ &< K_{\max} - \frac{\xi_K^2}{2} + \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx \\ &= -\frac{D^{p+\mu}}{e\mu p}\xi_K^{p+\mu} + \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx \\ &< \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx, \end{aligned} \tag{3.15}$$

which also ensures

$$\int_\Omega |u|^p \ln|u|\,dx > 0. \tag{3.16}$$

Now, we put

$$L(t) = R^{1-\beta}(t) + \epsilon\big(u(t), u_t(t)\big), \tag{3.17}$$

where $\epsilon > 0$ and

$$\frac{2(p-q)}{p^2(q-1)} \le \beta \le \min\left\{\frac{p-q}{p(q-1)}, \frac{p-2}{2p}\right\}. \tag{3.18}$$

From (2.1)–(2.5), we get

$$\begin{aligned} L'(t) &= (1-\beta)R^{-\beta}(t)R'(t) + \epsilon\|u_t\|_2^2 - \epsilon\left(1 - \int_0^t k(s)\,ds\right)\|\nabla u\|_2^2 \\ &\quad + \epsilon\int_\Omega |u|^p \ln|u|\,dx + \epsilon\int_0^t k(t-s)\big(\nabla u(t), \nabla u(s) - \nabla u(t)\big)\,ds \\ &\quad - \epsilon c_1\int_\Omega u|u_t|^{q-2}u_t\,dx - \epsilon c_2\int_\Omega u(x,t)\big|y(x,1,t)\big|^{q-2}y(x,1,t)\,dx. \end{aligned} \tag{3.19}$$

Using (2.6) with $\frac{q-1}{q} + \frac{1}{q} = 1$, we have

$$c_1\int_\Omega u|u_t|^{q-2}u_t\,dx \le \frac{c_1(q-1)}{q}\chi^{-\frac{q}{q-1}}\|u_t\|_q^q + \frac{c_1\chi^q}{q}\|u\|_q^q$$

and

$$c_2\int_\Omega u(x,t)\big|y(x,1,t)\big|^{q-2}y(x,1,t)\,dx \le \frac{|c_2|\chi^q}{q}\|u\|_q^q + \frac{|c_2|(q-1)}{q}\chi^{-\frac{q}{q-1}}\big\|y(1,t)\big\|_q^q$$

for any $\chi > 0$. From these and the relation $0 < |c_2| < c_1$, we get

$$\begin{aligned} &c_1\int_\Omega u|u_t|^{q-2}u_t\,dx + c_2\int_\Omega u(x,t)\big|y(x,1,t)\big|^{q-2}y(x,1,t)\,dx \\ &\quad\le \frac{c_1(q-1)}{q}\chi^{-\frac{q}{q-1}}\left(\|u_t\|_q^q + \big\|y(1,t)\big\|_q^q\right) + \frac{(c_1+|c_2|)\chi^q}{q}\|u\|_q^q. \end{aligned}$$

Taking $\chi = (\theta R^{-\beta}(t))^{-\frac{q-1}{q}}$, $\theta > 0$, and applying (3.14), we derive

$$\begin{aligned} &c_1\int_\Omega u|u_t|^{q-2}u_t\,dx + c_2\int_\Omega u\big|y(1,t)\big|^{q-2}y(1,t)\,dx \\ &\quad\le \frac{c_1(q-1)\theta}{q}\big(R(t)\big)^{-\beta}\left(\|u_t\|_q^q + \big\|y(1,t)\big\|_q^q\right) + \frac{c_1+|c_2|}{q\theta^{q-1}}\big(R(t)\big)^{\beta(q-1)}\|u\|_q^q \\ &\quad\le \frac{c_1(q-1)\theta}{q\gamma_1}\big(R(t)\big)^{-\beta}R'(t) + \frac{C}{q\theta^{q-1}}\big(R(t)\big)^{\beta(q-1)}\|u\|_p^q. \end{aligned} \tag{3.20}$$

Using (3.16), (3.15), Lemma 2.3, and (2.6) with $\frac{q}{p} + \frac{p-q}{p} = 1$, we find

$$\begin{aligned} &\big(R(t)\big)^{\beta(q-1)}\|u\|_p^q \\ &\quad\le C\left(\int_\Omega |u|^p \ln|u|\,dx\right)^{\beta(q-1)}\left(\int_\Omega |u|^p \ln|u|\,dx + \|\nabla u\|_2^2\right)^{\frac{q}{p}} \end{aligned}$$

$$\begin{aligned}
&\leq C\left(\int_\Omega |u|^p \ln|u|\,dx\right)^{\beta(q-1)}\left\{\left(\int_\Omega |u|^p \ln|u|\,dx\right)^{\frac{q}{p}} + \|\nabla u\|_2^{\frac{2q}{p}}\right\} \\
&\leq C\left(\int_\Omega |u|^p \ln|u|\,dx\right)^{\beta(q-1)+\frac{q}{p}} + C\left(\int_\Omega |u|^p \ln|u|\,dx\right)^{\beta(q-1)}\|\nabla u\|_2^{\frac{2q}{p}} \\
&\leq C\left(\int_\Omega |u|^p \ln|u|\,dx\right)^{\frac{\beta p(q-1)+q}{p}} + C\left(\int_\Omega |u|^p \ln|u|\,dx\right)^{\frac{\beta(q-1)p}{p-q}} + \|\nabla u\|_2^2. \qquad (3.21)
\end{aligned}$$

From (3.18), we see $0 < \beta \leq \frac{p-q}{p(q-1)}$, which gives

$$2 \leq \beta p(q-1) + q \leq p.$$

Thanks to (H_p), we also note the solution u to (2.1)–(2.5) belongs to $L^{p+1}(\Omega)$. So, we can apply Lemma 2.2 to get

$$\left(\int_\Omega |u|^p \ln|u|\,dx\right)^{\frac{\beta p(q-1)+q}{p}} \leq C\left(\|\nabla u\|_2^2 + \int_\Omega |u|^p \ln|u|\,dx\right). \qquad (3.22)$$

Similarly, from (3.18), we see $\frac{2(p-q)}{p^2(q-1)} \leq \beta \leq \frac{p-q}{p(q-1)}$, which implies

$$2 \leq \frac{\beta(q-1)p^2}{p-q} \leq p.$$

So, we have

$$\left(\int_\Omega |u|^p \ln|u|\,dx\right)^{\frac{\beta(q-1)p}{p-q}} \leq C\left(\|\nabla u\|_2^2 + \int_\Omega |u|^p \ln|u|\,dx\right). \qquad (3.23)$$

Inserting (3.22) and (3.23) to (3.21), we obtain

$$\big(R(t)\big)^{\beta(q-1)}\|u\|_p^q \leq C\left(\|\nabla u\|_2^2 + \int_\Omega |u|^p \ln|u|\,dx\right).$$

From this and (3.20),

$$\begin{aligned}
&c_1 \int_\Omega |u_t|^{q-2}u_t\,dx + c_2\int_\Omega u\big|y(1,t)\big|^{q-2}y(1,t)\,dx \\
&\quad\leq \frac{c_1(q-1)\theta}{q\gamma_1}\big(R(t)\big)^{-\beta}R'(t) + \frac{C}{q\theta^{q-1}}\left(\int_\Omega |u|^p \ln|u|\,dx + \|\nabla u\|_2^2\right).
\end{aligned}$$

Applying this and the estimate

$$\int_0^t k(t-s)\big(\nabla u(t), \nabla u(s) - \nabla u(t)\big)\,ds \geq -\eta(k\circ\nabla u) - \frac{1}{4\eta}\left(\int_0^t k(s)\,ds\right)\|\nabla u\|_2^2$$

to (3.19), we get

$$L'(t) \geq \left((1-\beta) - \frac{\epsilon c_1(q-1)\theta}{q\gamma_1}\right)R^{-\beta}(t)R'(t) + \epsilon\|u_t\|_2^2$$

$$
\begin{aligned}
&+\epsilon\int_\Omega |u|^p\ln|u|\,dx-\epsilon\eta(k\circ\nabla u)-\frac{\epsilon C}{q\theta^{q-1}}\int_\Omega |u|^p\ln|u|\,dx\\
&-\epsilon\left(1-\int_0^t k(s)\,ds+\frac{1}{4\eta}\int_0^t k(s)\,ds+\frac{C}{q\theta^{q-1}}\right)\|\nabla u\|_2^2. \qquad (3.24)
\end{aligned}
$$

Subtracting and adding the term $\epsilon\lambda\int_\Omega |u|^p\ln|u|\,dx$, where λ is given in (2.9), and using (3.1) and (3.13), we get

$$
\begin{aligned}
L'(t)\geq{}&\left((1-\beta)-\frac{\epsilon c_1(q-1)\theta}{q\gamma_1}\right)R^{-\beta}(t)R'(t)+\epsilon\left(1+\frac{p(1-\lambda)}{2}\right)\|u_t\|_2^2\\
&+\epsilon\left(\lambda-\frac{C}{q\theta^{q-1}}\right)\int_\Omega |u|^p\ln|u|\,dx+\frac{\epsilon(1-\lambda)}{p}\|u\|_p^p+\epsilon p(1-\lambda)R(t)\\
&+\epsilon\xi\tau p(1-\lambda)\int_0^1\|y(\sigma,t)\|_q^q\,d\sigma+\epsilon\left(\frac{p(1-\lambda)}{2}-\eta\right)(k\circ\nabla u)-\epsilon p(1-\lambda)\overline{E}\\
&+\frac{\epsilon}{2}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)-\frac{C}{q\theta^{q-1}}\right\}\|\nabla u\|_2^2\\
&+\frac{\epsilon}{2}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)-\frac{C}{q\theta^{q-1}}\right\}\|\nabla u\|_2^2. \qquad (3.25)
\end{aligned}
$$

By (H_k), we note

$$\frac{p(1-\lambda)}{2}-\eta>0$$

and

$$k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)>0.$$

Firstly, we take $\theta>0$ appropriately large to guarantee

$$\lambda-\frac{C}{q\theta^{q-1}}>0$$

and

$$k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)-\frac{C}{q\theta^{q-1}}>0.$$

Next, we claim

$$-\epsilon p(1-\lambda)\overline{E}+\frac{\epsilon}{2}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)-\frac{C}{q\theta^{q-1}}\right\}\|\nabla u\|_2^2>0. \qquad (3.26)$$

Indeed, it is seen from Lemma 3.2 that

$$
\begin{aligned}
&-p(1-\lambda)\overline{E}+\frac{1}{2}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)-\frac{C}{q\theta^{q-1}}\right\}\|\nabla u\|_2^2\\
&\quad\geq -p(1-\lambda)\overline{E}+\frac{\xi_1^2}{2k_l}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)-\frac{C}{q\theta^{q-1}}\right\}\\
&\quad> -p(1-\lambda)\overline{E}+\frac{\xi_K^2}{2k_l}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)-\frac{C}{q\theta^{q-1}}\right\}. \qquad (3.27)
\end{aligned}
$$

From (3.11), we have

$$-p(1-\lambda)\overline{E}+\frac{\xi_K^2}{2k_l}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)\right\}>0.$$

Thus, we can fix $\theta>0$ suitably large again to get

$$-p(1-\lambda)\overline{E}+\frac{\xi_K^2}{2k_l}\left\{k_l\left(\frac{p(1-\lambda)}{2}-1\right)-\frac{1}{4\eta}(1-k_l)\right\}-\frac{C}{2k_lq\theta^{q-1}}\xi_K^2>0.$$

This and (3.27) imply (3.26).

Finally, we pick $\epsilon>0$ suitably small to have

$$(1-\beta)-\frac{\epsilon c_1(q-1)\theta}{q\gamma_1}>0$$

and

$$L(0)=R^{1-\beta}(0)+\epsilon(u_0,u_1)>0.$$

Therefore, from (3.25) we arrive at

$$L'(t)\geq C\left(\|u_t\|_2^2+\|u\|_p^p+R(t)+\int_\Omega|u|^p\ln|u|\,dx+\|\nabla u\|_2^2\right). \tag{3.28}$$

Next, from (3.17), we know

$$L^{\frac{1}{1-\beta}}(t)\leq C\bigl(R(t)+\bigl|(u,u_t)\bigr|^{\frac{1}{1-\beta}}\bigr). \tag{3.29}$$

Using (2.6) with $\frac{1}{2(1-\beta)}+\frac{1-2\beta}{2(1-\beta)}=1$, noting $2\leq\frac{2}{1-2\beta}\leq p$ from (3.18), and applying Lemma 2.1, we deduce

$$\begin{aligned}\bigl|(u,u_t)\bigr|^{\frac{1}{1-\beta}}&\leq\|u\|_2^{\frac{1}{1-\beta}}\|u_t\|_2^{\frac{1}{1-\beta}}\\&\leq C\|u\|_p^{\frac{1}{1-\beta}}\|u_t\|_2^{\frac{1}{1-\beta}}\\&\leq C\bigl(\|u_t\|_2^2+\|u\|_p^{\frac{2}{1-2\beta}}\bigr)\\&\leq C\bigl(\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2\bigr).\end{aligned} \tag{3.30}$$

Combining (3.29), (3.30) and noting (3.16), we derive

$$\begin{aligned}L^{\frac{1}{1-\beta}}(t)&\leq C\bigl(R(t)+\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2\bigr)\\&\leq C\left(R(t)+\|u_t\|_2^2+\|u\|_p^p+\|\nabla u\|_2^2+\int_\Omega|u|^p\ln|u|\,dx\right).\end{aligned}$$

From this and (3.29), we conclude

$$L'(t)\geq CL^{\frac{1}{1-\beta}}(t),\quad t\geq0,$$

which shows that the solution u blows up after finite time $T^*\leq\frac{1-\beta}{C\beta L^{\frac{\beta}{1-\beta}}(0)}$. □

4 Conclusion

In this paper, we considered a viscoelastic wave equation with nonlinear damping and time delay terms and logarithmic source effect. Under the conditions (H_n), (H_p), (H_q), and (H_k), we showed that the solutions with not only a non-positive initial energy but also some positive initial energy blow up after a finite time by utilizing the energy method.

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