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On partial fractional Sturm–Liouville equation and inclusion

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Abstract

The Sturm–Liouville differential equation is one of interesting problems which has been studied by researchers during recent decades. We study the existence of a solution for partial fractional Sturm–Liouville equation by using the α-ψ-contractive mappings. Also, we give an illustrative example. By using the α-ψ-multifunctions, we prove the existence of solutions for inclusion version of the partial fractional Sturm–Liouville problem. Finally by providing another example and some figures, we try to illustrate the related inclusion result.

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1 Introduction

It can be said that most physical or engineering phenomena can be modeled with some categories such time-dependent (or time-fractional), fractional differential and some variables partial equations. One can find many published papers on delayed time-fractional problems, fractional differential equations [1–30] and some variables partial fractional problems [31–36]. During the history of mathematics, physics and engineering, we can find many equations which have a special role in progress of these sciences. One of the important frameworks of problems is the Sturm–Liouville differential equation (in brief SLDE) have been in the spotlight of the mathematicians of applied mathematics, engineering and scientists of physics, quantum mechanics, classical mechanics (see, [37, 38] and the references therein). In such a manner, it is important that mathematicians and researchers design complicated and more general abstract mathematical models of procedures in the format of applicable fractional SLDE [33, 39–41]. One can find a variety of recent work about this equation, but the aim of this work is studying partial version of the Sturm–Liouville differential equation.

Let $\hat{k} = (\hat{k}_1, \hat{k}_2)$ where $\hat{k}_1, \hat{k}_2 > 0$ and $\mathcal{J}_{a_0} = [0, a_0]$ and $\mathcal{J}_{b_0} = [0, b_0]$ where $a_0, b_0 > 0$. For $\sigma \in \mathcal{L}^1(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0}, \mathbb{R}) = \mathcal{L}^1(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0})$, the partial left-sided mixed Riemann–Liouville integral (of order $\hat{k}$) is defined by (see [42])

$$\mathcal{I}_0^{\hat{k}}\sigma(p^\star,q^\star) = \int_0^{p^\star}\int_0^{q^\star} \frac{(p^\star - s)^{\hat{k}_1-1}(q^\star - t)^{\hat{k}_2-1}}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)}\sigma(s,t)\,dt\,ds.$$

Also the partial derivative in the sense of Caputo (of order $\hat{k}$) is defined by

$$\begin{aligned} D_{c_0}^{\hat{k}}\sigma(p^\star,q^\star) &= \mathcal{I}_0^{1-\hat{k}}\left(\frac{\partial^2}{\partial p^\star \partial q^\star}\sigma(p^\star,q^\star)\right) \\ &= \int_0^{p^\star}\int_0^{q^\star} \frac{(p^\star - s)^{\hat{k}_1-1}(q^\star - t)^{\hat{k}_2-1}}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)}\frac{\partial^2}{\partial s\,\partial t}\sigma(s,t)\,dt\,ds. \end{aligned}$$

Let $(\mathcal{Z}^\star, d_{\mathcal{Z}^\star})$ be a metric space. $\mathcal{P}_{cl}(\mathcal{Z}^\star)$ the set of all closed subsets of $\mathcal{Z}^\star$ and $2^{\mathcal{Z}^\star}$ the set of all nonempty subsets of $\mathcal{Z}^\star$. It is well known that the Pompeiu–Hausdorff metric $PH_{d_{\mathcal{Z}^\star}} : \mathcal{P}_{cl}(\mathcal{Z}^\star) \times \mathcal{P}_{cl}(\mathcal{Z}^\star) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$PH_{d_{\mathcal{Z}^\star}}\left(A_1^{d_{\mathcal{Z}^\star}}, A_2^{d_{\mathcal{Z}^\star}}\right) = \max\left\{\sup_{a_1^\star \in A_1^{d_{\mathcal{Z}^\star}}} d_{\mathcal{Z}^\star}\left(a_1^\star, A_2^{d_{\mathcal{Z}^\star}}\right), \sup_{a_2^\star \in A_2^{d_{\mathcal{Z}^\star}}} d_{\mathcal{Z}^\star}\left(A_1^{d_{\mathcal{Z}^\star}}, a_2^\star\right)\right\}$$

for all $A_1^{d_{\mathcal{Z}^\star}}, A_2^{d_{\mathcal{Z}^\star}} \in \mathcal{P}(\mathcal{Z}^\star)$, where $d_{\mathcal{Z}^\star}(a_1^\star, A_2^{d_{\mathcal{Z}^\star}}) = \inf_{a_1^\star \in A_1^{d_{\mathcal{Z}^\star}}} d_{\mathcal{Z}^\star}(a_1^\star, a_2^\star)$ and $d_{\mathcal{Z}^\star}(A_1^{d_{\mathcal{Z}^\star}}, a_2^\star) = \inf_{a_2^\star \in A_2^{d_{\mathcal{Z}^\star}}} d_{\mathcal{Z}^\star}(a_1^\star, a_2^\star)$ [43]. We say that a set-valued mapping $\Psi : \mathcal{Z}^\star \to \mathcal{P}_{cl}(\mathcal{Z}^\star)$ is called Lipschitzian with Lipschitz constant $k > 0$ whenever $PH_{d_{\mathcal{Z}^\star}}(\Psi(\sigma_1), \Psi(\sigma_1)) \leq k d_{\mathcal{Z}^\star}(\sigma_1, \sigma_2)$ for all $\sigma_1, \sigma_2 \in \mathcal{Z}^\star$. If $0 < k < 1$, then we ay that Ψ is a contraction [43]. An operator $\Psi : [0,1] \to \mathcal{P}_{cl}(\mathcal{R})$ is called measurable whenever the function $t \to d_{\mathcal{Z}^\star}(\omega_0, \Psi(t)) = \inf\{|\omega_0 - y| : y \in \Psi(t)\}$ is measurable for all real constant ω [43, 44]. The following notions were introduced in 2012 [45].

- $\Psi = \{\psi \mid \sum_{n=1}^{\infty} \psi^n(t) < \infty, \forall t > 0\}$ where $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$.
- Assume that $\alpha : \mathcal{Z}^\star \times \mathcal{Z}^\star \to [0,\infty)$ and $T : \mathcal{Z}^\star \to \mathcal{Z}^\star$ are two mappings. Now T is α-admissible whenever for each $\sigma_1, \sigma_2 \in \mathcal{Z}^\star$ with $\alpha(\sigma_1, \sigma_2) \geq 1$, we get $\alpha(T\sigma_1, T\sigma_2) \geq 1$.
- T is α-ψ-contractive mapping whence $\alpha(\sigma_1, \sigma_2) d_{\mathcal{Z}^\star}(T\sigma_1, T\sigma_2) \leq \psi(d_{\mathcal{Z}^\star}(\sigma_1, \sigma_2))$ for all $\sigma_1, \sigma_2 \in \mathcal{Z}^\star$.

Lemma 1 ([45]) *Assume that the metric space $(\mathcal{Z}^\star, d_{\mathcal{Z}^\star})$ is complete, T is α-admissible and α-ψ-contractive mapping and there exists $\sigma_0 \in \mathcal{Z}^\star$ such that $\alpha(\sigma_0, T\sigma_0) \geq 1$. Further, for every convergent sequence $\{\sigma_n\}_{n\geq 1} \subseteq \mathcal{Z}^\star$ with $\sigma_n \to \sigma$ and $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$ for all $n \geq 1$, we have $\alpha(\sigma_n, \sigma) \geq 1$ for all $n \geq 1$. Then T has a fixed point.*

After this, multifunction version of α-ψ-contractive maps introduced in 2013 as follows [46].

- A multifunction $F : \mathcal{Z}^\star \to CB(\mathcal{Z}^\star)$ is α-admissible whenever for each $\sigma_1 \in \mathcal{Z}^\star$ and $\sigma_2 \in F\sigma_1$ with $\alpha(\sigma_1, \sigma_2) \geq 1$, we have $\alpha(\sigma_2, w_0) \geq 1$, for all $w_0 \in F\sigma_2$.
- The metric space $\mathcal{Z}^\star$ possesses the C_α-property if for every convergent sequence $\{\sigma_n\}_{n\geq 1} \subseteq \mathcal{Z}^\star$ with $\sigma_n \to \sigma$ and $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$ for all $n \geq 1$, there exists a subsequence $\{\sigma_{n_j}\}_{j\geq 1}$ of σ_n such that $\alpha(\sigma_{n_j}, \sigma) \geq 1$ for all $j \geq 1$.
- F is α-ψ-contractive multifunction whenever $\alpha(\sigma_1, \sigma_2) PH_{d_{\mathcal{Z}^\star}}(T\sigma_1, T\sigma_2) \leq \psi(d_{\mathcal{Z}^\star}(\sigma_1, \sigma_2))$ for all $\sigma_1, \sigma_2 \in \mathcal{Z}^\star$.

Lemma 2 ([46]) *Assume that the metric space* $(\mathcal{Z}^\star, d_{\mathcal{Z}^\star})$ *is complete,* F *is* α*-admissible and* α-ψ*-contractive multifunction and there exist* $\sigma_0 \in \mathcal{Z}^\star$ *and* $\sigma_1 \in F\sigma_0$ *such that* $\alpha(\sigma_0, \sigma_1) \geq 1$. *If* $\mathcal{Z}^\star$ *possesses the* C_α*-property, then* F *has a fixed point.*

In this paper, first we investigate the partial fractional Sturm–Liouville differential equation

$$\begin{cases} D_{c_0}^{\hat{k}}(l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star)) + o(p^\star,q^\star)\sigma(p^\star,q^\star) = h(p^\star,q^\star)f(\sigma(p^\star,q^\star)), \\ (p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}(l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star))_{q^\star=0} = \theta_1(p^\star), \\ (l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star))_{p^\star=0} = \theta_2(q^\star), \\ \sigma(p^\star,0) = \kappa(p^\star) \quad \text{and} \quad \sigma(0,q^\star) = \omega(q^\star), \end{cases} \tag{1}$$

where $\hat{k}, \hat{\ell} \in (0,1] \times (0,1]$, $D_{c_0}^{\hat{k}}$ and $D_{c_0}^{\hat{\ell}}$ denote the Caputo partial fractional derivatives, l, o, h belong to $\mathcal{C}(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0})$ with $l(p^\star,q^\star) \neq 0$ for all $(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$ and $f : \mathbb{R} \to \mathbb{R}$ is a function. Here, $\theta_1 : \mathcal{J}_{a_0} \to \mathbb{R}$, $\theta_2 : \mathcal{J}_{b_0} \to \mathbb{R}$, $\kappa : \mathcal{J}_{a_0} \to \mathbb{R}$ and $\omega : \mathcal{J}_{b_0} \to \mathbb{R}$ are absolutely continuous with $\theta_1(0) = \theta_2(0) = \kappa(0) = \omega(0)$. Also, we investigate the partial fractional Sturm–Liouville differential inclusion problem

$$\begin{cases} D_{c_0}^{\hat{k}}(l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star)) \in \mathcal{H}(p^\star q^\star, \sigma(p^\star,q^\star)), (p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}, \\ (l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star))_{q^\star=0} = \theta_1(p^\star), \\ (l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star))_{p^\star=0} = \theta_2(q^\star), \\ \sigma(p^\star,0) = \kappa(p^\star) \quad \text{and} \quad \sigma(0,q^\star) = \omega(q^\star), \end{cases} \tag{2}$$

where $\hat{k}, \hat{\ell} \in (0,1] \times (0,1]$, $D_{c_0}^{\hat{k}}$ and $D_{c_0}^{\hat{\ell}}$ denote the Caputo partial fractional derivatives, l, o, h belong to $\mathcal{C}(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0})$ with $l(p^\star,q^\star) \neq 0$ for all $(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$ and $f : \mathbb{R} \to \mathbb{R}$ is a function. Here, $\theta_1 : \mathcal{J}_{a_0} \to \mathbb{R}$, $\theta_2 : \mathcal{J}_{b_0} \to \mathbb{R}$, $\kappa : \mathcal{J}_{a_0} \to \mathbb{R}$ and $\omega : \mathcal{J}_{b_0} \to \mathbb{R}$ are absolutely continuous with $\theta_1(0) = \theta_2(0) = \kappa(0) = \omega(0)$. Also, $\mathcal{H} : \mathcal{J}_{a_0} \times \mathcal{J}_{b_0} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is an integrable bounded multifunction so that $\mathcal{H}(.,.,\sigma)$ is measurable for all $\sigma \in \mathbb{R}$.

2 Main results

Assume that $\mathcal{Z}^\star = \{\sigma | \sigma \in \mathcal{C}(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0})\}$ and $\|\sigma\| = \sup_{(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}} |\sigma(p^\star,q^\star)|$, where $\sigma \in \mathcal{Z}^\star$. Then $(\mathcal{Z}^\star, \|.\|)$ is a Banach space.

Lemma 3 *Let* $\hat{k} = (\hat{k}_1, \hat{k}_2), \hat{\ell} = (\hat{\ell}_1, \hat{\ell}_2) \in (0,1] \times (0,1]$ *and* $g \in \mathcal{L}^1(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0})$. *Consider the problem*

$$\mathcal{D}_{c_0}^{\hat{k}}\big(l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star)\big) = g(p^\star,q^\star), \tag{3}$$

with boundary conditions

$$\begin{cases} (l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star))_{y=0} = \theta_1(p^\star), \\ (l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star))_{x=0} = \theta_2(q^\star), \\ \sigma(p^\star,0) = \kappa(p^\star) \quad \textit{and} \quad \sigma(0,q^\star) = \omega(q^\star). \end{cases} \tag{4}$$

Then the function $\sigma \in \mathcal{C}(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0})$ *is a solution of the problem* (3)–(4) *whenever*

$$\sigma(p^\star, q^\star) = \Theta(p^\star, q^\star) + \int_0^{p^\star} \int_0^{q^\star} \int_0^{s} \int_0^{t} \frac{(p^\star - s)^{\hat{k}_1 - 1}(q^\star - t)^{\hat{k}_2 - 1}(s - \wp_{\bar{1}})^{\hat{\ell}_1 - 1}(t - \zeta_{\bar{2}})^{\hat{\ell}_2 - 1} g(\wp_{\bar{1}}, \zeta_{\bar{2}})}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2) l(s,t)} \, dt\, ds\, d\zeta_{\bar{2}}\, d\wp_{\bar{1}},$$

where

$$\Theta(p^\star, q^\star) = \kappa(p^\star) + \omega(q^\star) - \kappa(0) + \mathcal{I}_0^{\hat{\ell}} \left(\frac{\theta_1(p^\star) + \theta_2(q^\star) - \theta_1(0)}{l(p^\star, q^\star)} \right).$$

Proof Note that Eq. (3) can be written as

$$\mathcal{I}_0^{1-\hat{k}} \left(\frac{\partial^2}{\partial p^\star \partial q^\star} \left(l(p^\star, q^\star) \mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) \right) \right) = g(p^\star, q^\star).$$

Operating by $\mathcal{I}_0^{\hat{k}}$ on both sides we get

$$\mathcal{I}_0^{1} \left(\frac{\partial^2}{\partial p^\star \partial q^\star} \left(l(p^\star, q^\star) \mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) \right) \right) = \mathcal{I}_0^{\hat{k}} g(p^\star, q^\star).$$

Since

$$\begin{aligned} &\mathcal{I}_0^{1} \left(\frac{\partial^2}{\partial p^\star \partial q^\star} \left(l(p^\star, q^\star) \mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) \right) \right) \\ &\quad = l(p^\star, q^\star) \mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) - \left(l(p^\star, q^\star) \mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) \right)_{q^\star = 0} \\ &\qquad - \left(l(p^\star, q^\star) \mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) \right)_{p^\star = 0} + \left(l(p^\star, q^\star) \mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) \right)_{p^\star = 0, q^\star = 0} \\ &\quad = l(p^\star, q^\star) \mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) - \theta_1(p^\star) - \theta_2(q^\star) + \theta_1(0), \end{aligned}$$

we get $l(p^\star, q^\star) \mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) = \theta_1(p^\star) + \theta_2(q^\star) - \theta_1(0) + \mathcal{I}_0^{\hat{k}} g(p^\star, q^\star)$. Hence,

$$\mathcal{D}_{c_0}^{\hat{\ell}} \sigma(p^\star, q^\star) = \frac{\theta_1(p^\star) + \theta_2(q^\star) - \theta_1(0)}{l(p^\star, q^\star)} + \frac{1}{l(p^\star, q^\star)} \mathcal{I}_0^{\hat{k}} g(p^\star, q^\star).$$

This equation can be written as

$$\mathcal{I}_0^{1-\hat{\ell}} \left(\frac{\partial^2}{\partial p^\star \partial q^\star} \sigma(p^\star, q^\star) \right) = \frac{\theta_1(p^\star) + \theta_2(q^\star) - \theta_1(0)}{l(p^\star, q^\star)} + \frac{1}{l(p^\star, q^\star)} \mathcal{I}_0^{\hat{k}} g(p^\star, q^\star).$$

Again operating by $\mathcal{I}_0^{\hat{\ell}}$ on both sides, we obtain

$$\mathcal{I}_0^{1} \left(\frac{\partial^2}{\partial p^\star \partial q^\star} \sigma(p^\star, q^\star) \right) = \mathcal{I}_0^{\hat{\ell}} \left(\frac{\theta_1(p^\star) + \theta_2(q^\star) - \theta_1(0)}{l(p^\star, q^\star)} \right) + \mathcal{I}_0^{\hat{\ell}} \left(\frac{1}{l(p^\star, q^\star)} \mathcal{I}_0^{\hat{k}} g(p^\star, q^\star) \right).$$

Since

$$\mathcal{I}_0^1\left(\frac{\partial^2}{\partial p^\star \partial q^\star}\sigma\left(p^\star,q^\star\right)\right) = \sigma\left(p^\star,q^\star\right) - \sigma\left(p^\star,0\right) - \sigma\left(0,q^\star\right) + \sigma(0,0)$$
$$= \sigma\left(p^\star,q^\star\right) - \kappa\left(p^\star\right) - \omega\left(q^\star\right) + \kappa(0),$$

we get

$$\sigma\left(p^\star,q^\star\right) = \kappa\left(p^\star\right) + \omega\left(q^\star\right) - \kappa(0) + \mathcal{I}_0^{\hat{\ell}}\left(\frac{\theta_1(p^\star)+\theta_2(q^\star)-\theta_1(0)}{l(p^\star,q^\star)}\right)$$
$$+ \mathcal{I}_0^{\hat{\ell}}\left(\frac{1}{l(p^\star,q^\star)}\mathcal{I}_0^{\hat{k}}g\left(p^\star,q^\star\right)\right) = \Theta\left(p^\star,q^\star\right) + \mathcal{I}_0^{\hat{\ell}}\left(\frac{1}{l(p^\star,q^\star)}\mathcal{I}_0^{\hat{k}}g\left(p^\star,q^\star\right)\right).$$

On the other hand,

$$\mathcal{I}_0^{\hat{\ell}}\left(\frac{1}{l(p^\star,q^\star)}\mathcal{I}_0^{\hat{k}}g\left(p^\star,q^\star\right)\right)$$
$$= \int_0^{p^\star}\int_0^{q^\star}\frac{(p^\star-s)^{\hat{k}_1-1}(q^\star-t)^{\hat{k}_2-1}}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)}\frac{1}{l(s,t)}\mathcal{I}_0^{\hat{k}}g(s,t)\,dt\,ds$$
$$= \int_0^{p^\star}\int_0^{q^\star}\frac{1}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)}\left(p^\star-s\right)^{\hat{k}_1-1}\left(q^\star-t\right)^{\hat{k}_2-1}\frac{1}{l(s,t)}$$
$$\times\left(\int_0^s\int_0^t\frac{(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}}{\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)}g(\wp_{\bar{1}},\zeta_{\bar{2}})\,d\wp_{\bar{1}}\,d\zeta_{\bar{2}}\right)dt\,ds$$
$$= \int_0^{p^\star}\int_0^{q^\star}\int_0^s\int_0^t$$
$$\frac{(p^\star-s)^{\hat{k}_1-1}(q^\star-t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}g(\wp_{\bar{1}},\zeta_{\bar{2}})}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}}$$

and so

$$\sigma\left(p^\star,q^\star\right) = \Theta\left(p^\star,q^\star\right)$$
$$+ \int_0^{p^\star}\int_0^{q^\star}\int_0^s\int_0^t$$
$$\frac{(p^\star-s)^{\hat{k}_1-1}(q^\star-t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}g(\wp_{\bar{1}},\zeta_{\bar{2}})}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}}.$$

This completes the proof. □

Now we establish and prove our first main theorem.

Theorem 4 *Assume that* $\upsilon:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is a function and* $\Phi:\mathcal{J}_{a_0}\times\mathcal{J}_{b_0}\to[0,\infty)$ *is a bounded function such that*

$$\left|f\left(\sigma_1\left(p^\star,q^\star\right)\right) - f\left(\sigma_2\left(p^\star,q^\star\right)\right)\right| \le \Phi\left(p^\star,q^\star\right)\left|\sigma_1\left(p^\star,q^\star\right) - \sigma_2\left(p^\star,q^\star\right)\right|$$

for all $\sigma_1, \sigma_2 \in \mathcal{Z}^\star$ *with* $\upsilon(\sigma_1(p^\star, q^\star), \sigma_2(p^\star, q^\star)) \geq 0$, *where* $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. *Suppose that, for all* $\sigma_1, \sigma_2 \in \mathcal{Z}^\star$ *with* $\upsilon(\sigma_1(p^\star, q^\star), \sigma_2(p^\star, q^\star)) \geq 0$, *we have* $\upsilon(\mathcal{Q}_{\bar{0}}^\star \sigma_1(p^\star, q^\star), \mathcal{Q}_{\bar{0}}^\star \sigma_2(p^\star, q^\star)) \geq 0$, *where*

$$\begin{aligned}
&\mathcal{Q}_{\bar{0}}^\star \sigma(p^\star, q^\star) \\
&\quad = \Theta(p^\star, q^\star) \\
&\qquad + \int_0^{p^\star} \int_0^{q^\star} \int_0^s \int_0^t \\
&\qquad \frac{(p^\star - s)^{\hat{k}_1 - 1}(q^\star - t)^{\hat{k}_2 - 1}(s - \wp_{\bar{1}})^{\hat{\ell}_1 - 1}(t - \zeta_{\bar{2}})^{\hat{\ell}_2 - 1}\mathcal{H}(\wp_{\bar{1}}, \zeta_{\bar{2}}, \sigma(\wp_{\bar{1}}, \zeta_{\bar{2}}))}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}},
\end{aligned}$$

$\Theta(p^\star, q^\star) = \kappa(p^\star) + \omega(q^\star) - \kappa(0) + \mathcal{I}_0^{\hat{\ell}}(\frac{\theta_1(p^\star) + \theta_2(q^\star) - \theta_1(0)}{l(p^\star, q^\star)})$,

$$\mathcal{H}(\wp_{\bar{1}}, \zeta_{\bar{2}}, \sigma(\wp_{\bar{1}}, \zeta_{\bar{2}})) = h(\wp_{\bar{1}}, \zeta_{\bar{2}}) f(\sigma(\wp_{\bar{1}}, \zeta_{\bar{2}})) - o(\wp_{\bar{1}}, \zeta_{\bar{2}})\sigma(\wp_{\bar{1}}, \zeta_{\bar{2}})$$

and there exists σ_0 *so that* $\upsilon(\sigma_0(p^\star, q^\star), \mathcal{Q}_{\bar{0}}^\star \sigma_0(p^\star, q^\star)) \geq 0$ *whenever* $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. *Assume that, for every sequence* $\{\sigma_n\}_{n \geq 1} \subseteq \mathcal{Z}^\star$ *with* $\sigma_n \to \sigma$ *and* $\upsilon(\sigma_n(p^\star, q^\star), \sigma_{n+1}(p^\star, q^\star)) \geq 0$ *for all* $n \geq 1$ *and* $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$, *we have* $\upsilon(\sigma_n(p^\star, q^\star), \sigma(p^\star, q^\star)) \geq 0$ *for all* $n \geq 1$ *and* $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. *If* $\frac{a_0^{\hat{k}_1 + \hat{\ell}_1} b_0^{\hat{k}_2 + \hat{\ell}_2}(\|h\|\Phi^* + \|o\|)}{l\Gamma(\hat{k}_1 + \hat{\ell}_1 + 1)\Gamma(\hat{k}_2 + \hat{\ell}_2 + 1)} < 1$, *then the fractional Sturm–Liouville problem* (1) *has a solution, where* $\Phi^* = \sup_{(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}} \Phi(p^\star, q^\star)$ *and*

$$l = \inf_{(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}} l(p^\star, q^\star).$$

Proof By using Lemma 3, σ_0 is a solution of the partial fractional Sturm–Liouville problem (1) if and only if $\sigma_0 = \mathcal{Q}_{\bar{0}}^\star \sigma_0$. Let $\sigma_1, \sigma_2 \in \mathcal{Z}^\star$ and $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$ with

$$\upsilon(\sigma_1(p^\star, q^\star), \sigma_2(p^\star, q^\star)) \geq 0.$$

Hence, we get

$$\begin{aligned}
&\left|\mathcal{Q}_{\bar{0}}^\star \sigma_1(p^\star, q^\star) - \mathcal{Q}_{\bar{0}}^\star \sigma_2(p^\star, q^\star)\right| \\
&\quad = \Bigg|\Theta(p^\star, q^\star) + \int_0^{p^\star} \int_0^{q^\star} \int_0^s \int_0^t \\
&\qquad \frac{(p^\star - s)^{\hat{k}_1 - 1}(q^\star - t)^{\hat{k}_2 - 1}(s - \wp_{\bar{1}})^{\hat{\ell}_1 - 1}(t - \zeta_{\bar{2}})^{\hat{\ell}_2 - 1}\mathcal{H}(\wp_{\bar{1}}, \zeta_{\bar{2}}, \sigma_1(\wp_{\bar{1}}, \zeta_{\bar{2}}))}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \\
&\qquad - \Theta(p^\star, q^\star) - \int_0^{p^\star} \int_0^{q^\star} \int_0^s \int_0^t \\
&\qquad \frac{(p^\star - s)^{\hat{k}_1 - 1}(q^\star - t)^{\hat{k}_2 - 1}(s - \wp_{\bar{1}})^{\hat{\ell}_1 - 1}(t - \zeta_{\bar{2}})^{\hat{\ell}_2 - 1}\mathcal{H}(\wp_{\bar{1}}, \zeta_{\bar{2}}, \sigma_2(\wp_{\bar{1}}, \zeta_{\bar{2}}))}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}}\Bigg| \\
&\quad \leq \int_0^{p^\star} \int_0^{q^\star} \int_0^s \int_0^t \\
&\qquad \frac{(p^\star - s)^{\hat{k}_1 - 1}(q^\star - t)^{\hat{k}_2 - 1}(s - \wp_{\bar{1}})^{\hat{\ell}_1 - 1}(t - \zeta_{\bar{2}})^{\hat{\ell}_2 - 1}|\mathcal{N}(\wp_{\bar{1}}, \zeta_{\bar{2}})|}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}},
\end{aligned}$$

where $\mathcal{N}(\wp_{\bar{1}},\zeta_{\bar{2}}) = \mathcal{H}(\wp_{\bar{1}},\zeta_{\bar{2}},\sigma_1(\wp_{\bar{1}},\zeta_{\bar{2}})) - \mathcal{H}(\wp_{\bar{1}},\zeta_{\bar{2}},\sigma_2(\wp_{\bar{1}},\zeta_{\bar{2}}))$. Since

$$\begin{aligned}
&\left|\mathcal{H}\big(\wp_{\bar{1}},\zeta_{\bar{2}},\sigma_1(\wp_{\bar{1}},\zeta_{\bar{2}})\big) - \mathcal{H}\big(\wp_{\bar{1}},\zeta_{\bar{2}},\sigma_2(\wp_{\bar{1}},\zeta_{\bar{2}})\big)\right| \\
&\quad\leq \left|h(\wp_{\bar{1}},\zeta_{\bar{2}})\big(f\big(\sigma_1(\wp_{\bar{1}},\zeta_{\bar{2}})\big) - f\big(\sigma_2(\wp_{\bar{1}},\zeta_{\bar{2}})\big)\big) - o(\wp_{\bar{1}},\zeta_{\bar{2}})\big(\sigma_1(\wp_{\bar{1}},\zeta_{\bar{2}}) - \sigma_2(\wp_{\bar{1}},\zeta_{\bar{2}})\big)\right| \\
&\quad\leq \left|h(\wp_{\bar{1}},\zeta_{\bar{2}})\right|\left|f\big(\sigma_1(\wp_{\bar{1}},\zeta_{\bar{2}})\big) - f\big(\sigma_2(\wp_{\bar{1}},\zeta_{\bar{2}})\big)\right| + \left|o(\wp_{\bar{1}},\zeta_{\bar{2}})\right|\left|\sigma_1(\wp_{\bar{1}},\zeta_{\bar{2}}) - \sigma_2(\wp_{\bar{1}},\zeta_{\bar{2}})\right| \\
&\quad\leq \left|h(\wp_{\bar{1}},\zeta_{\bar{2}})\right|\Phi(\wp_{\bar{1}},\zeta_{\bar{2}})\left|\sigma_1(\wp_{\bar{1}},\zeta_{\bar{2}}) - \sigma_2(\wp_{\bar{1}},\zeta_{\bar{2}})\right| + \left|o(\wp_{\bar{1}},\zeta_{\bar{2}})\right|\left|\sigma_1(\wp_{\bar{1}},\zeta_{\bar{2}}) - \sigma_2(\wp_{\bar{1}},\zeta_{\bar{2}})\right| \\
&\quad\leq \big(\|h\|\Phi^* + \|o\|\big)\|\sigma_1 - \sigma_2\|,
\end{aligned}$$

we have

$$\begin{aligned}
&\left|\mathcal{Q}_{\bar{0}}^{\star}\sigma_1\big(p^\star,q^\star\big) - \mathcal{Q}_{\bar{0}}^{\star}\sigma_2\big(p^\star,q^\star\big)\right| \\
&\quad\leq \frac{(\|h\|\Phi^* + \|o\|)\|\sigma_1 - \sigma_2\|}{l\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)} \\
&\qquad\times \int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t}\big(p^\star - s\big)^{\hat{k}_1-1}\big(q^\star - t\big)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1} \\
&\qquad\times (t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}},
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
&\int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t}\big(p^\star - s\big)^{\hat{k}_1-1}\big(q^\star - t\big)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \\
&\quad= \int_0^{p^\star}\int_0^{q^\star}\frac{s^{\hat{\ell}_1}(p^\star - s)^{\hat{k}_1-1}}{\hat{\ell}_1}\frac{t^{\hat{\ell}_2}(q^\star - t)^{\hat{k}_2-1}}{\hat{\ell}_2}\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \\
&\quad= \int_0^{p^\star}\frac{s^{\hat{\ell}_1}(p^\star - s)^{\hat{k}_1-1}}{\hat{\ell}_1}\,d\wp_{\bar{1}} \times \int_0^{q^\star}\frac{t^{\hat{\ell}_2}(q^\star - t)^{\hat{k}_2-1}}{\hat{\ell}_2}\,d\zeta_{\bar{2}} \\
&\quad\leq \frac{1}{\hat{\ell}_1\hat{\ell}_2}\int_0^{a_0}s^{\hat{\ell}_1}(a_0 - s)^{\hat{k}_1-1}\,ds \times \int_0^{b_0}t^{\hat{\ell}_2}(b_0 - t)^{\hat{k}_2-1}\,dt.
\end{aligned}$$

Put $s = a_0u$ and $t = b_0v$. Thus, we obtain

$$\begin{aligned}
&\int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t}\big(p^\star - s\big)^{\hat{k}_1-1}\big(q^\star - t\big)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \\
&\quad\leq \frac{a_0^{\hat{k}_1+\hat{\ell}_1}b_0^{\hat{k}_2+\hat{\ell}_2}}{\hat{\ell}_1\hat{\ell}_2}\int_0^1 u^{\hat{\ell}_1}(1-u)^{\hat{k}_1-1}\,du \times \int_0^1 v^{\hat{\ell}_2}(1-v)^{\hat{k}_2-1}\,dv.
\end{aligned}$$

On the other hand,

$$\mathbf{B}(\hat{\ell}_1 + 1,\hat{k}_1) = \int_0^1 u^{\hat{\ell}_1}(1-u)^{\hat{k}_1-1}\,du = \frac{\Gamma(\hat{\ell}_1+1)\Gamma(\hat{k}_1)}{\Gamma(\hat{k}_1+\hat{\ell}_1+1)}$$

and

$$\mathbf{B}(\hat{\ell}_2 + 1,\hat{k}_2) = \int_0^1 v^{\hat{\ell}_2}(1-v)^{\hat{k}_2-1}\,dv = \frac{\Gamma(\hat{\ell}_2+1)\Gamma(\hat{k}_2)}{\Gamma(\hat{k}_2+\hat{\ell}_2+1)}.$$

Hence,

$$\int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t}(p^\star-s)^{\hat{k}_1-1}(q^\star-t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}}$$
$$\leq \frac{a_0^{\hat{k}_1+\hat{\ell}_1}b_0^{\hat{k}_2+\hat{\ell}_2}\Gamma(\hat{\ell}_1+1)\Gamma(\hat{\ell}_2+1)\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)}{\hat{\ell}_1\hat{\ell}_2\Gamma(\hat{k}_1+\hat{\ell}_1+1)\Gamma(\hat{k}_2+\hat{\ell}_2+1)}.$$

By using (5), we derive

$$\begin{aligned}
&\left|\mathcal{Q}_{\bar{0}}^\star\sigma_1(p^\star,q^\star)-\mathcal{Q}_{\bar{0}}^\star\sigma_2(p^\star,q^\star)\right| \\
&\quad\leq \frac{(\|h\|\Phi^*+\|o\|)\|\sigma_1-\sigma_2\|}{l\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)} \\
&\qquad\times\frac{a_0^{\hat{k}_1+\hat{\ell}_1}b_0^{\hat{k}_2+\hat{\ell}_2}\Gamma(\hat{\ell}_1+1)\Gamma(\hat{\ell}_2+1)\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)}{\hat{\ell}_1\hat{\ell}_2\Gamma(\hat{k}_1+\hat{\ell}_1+1)\Gamma(\hat{k}_2+\hat{\ell}_2+1)} \\
&\quad= \frac{a_0^{\hat{k}_1+\hat{\ell}_1}b_0^{\hat{k}_2+\hat{\ell}_2}(\|h\|\Phi^*+\|o\|)}{l\Gamma(\hat{k}_1+\hat{\ell}_1+1)\Gamma(\hat{k}_2+\hat{\ell}_2+1)}\|\sigma_1-\sigma_2\|,
\end{aligned}$$

which means $\|\mathcal{Q}_{\bar{0}}^\star\sigma_1-\mathcal{Q}_{\bar{0}}^\star\sigma_2\|\leq\frac{a_0^{\hat{k}_1+\hat{\ell}_1}b_0^{\hat{k}_2+\hat{\ell}_2}(\|h\|\Phi^*+\|o\|)}{l\Gamma(\hat{k}_1+\hat{\ell}_1+1)\Gamma(\hat{k}_2+\hat{\ell}_2+1)}\|\sigma_1-\sigma_2\|$. Define $\psi:[0,\infty)\to[0,\infty)$ and $\alpha:\mathcal{Z}^\star\times\mathcal{Z}^\star\to[0,\infty)$ by $\psi(t)=\frac{a_0^{\hat{k}_1+\hat{\ell}_1}b_0^{\hat{k}_2+\hat{\ell}_2}(\|h\|\Phi^*+\|o\|)}{l\Gamma(\hat{k}_1+\hat{\ell}_1+1)\Gamma(\hat{k}_2+\hat{\ell}_2+1)}t$ and

$$\alpha(\sigma_1,\sigma_2)=\begin{cases}1, & \upsilon(\sigma_1(p^\star,q^\star),\sigma_2(p^\star,q^\star))\geq 0 \text{ with } (p^\star,q^\star)\in\mathcal{J}_{a_0}\times\mathcal{J}_{b_0},\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that $\psi\in\Psi$. If $\alpha(\sigma_1,\sigma_2)\geq 1$, then $\upsilon(\sigma_1(p^\star,q^\star),\sigma_2(p^\star,q^\star))\geq 0$. From the hypotheses, $\upsilon(\mathcal{Q}_{\bar{0}}^\star\sigma_1(p^\star,q^\star),\mathcal{Q}_{\bar{0}}^\star\sigma_2(p^\star,q^\star))\geq 0$ and so $\alpha(\mathcal{Q}_{\bar{0}}^\star\sigma_1,\mathcal{Q}_{\bar{0}}^\star\sigma_2)\geq 1$. Thus, $\mathcal{Q}_{\bar{0}}^\star$ is an α-admissible mapping. Also, there exists $\sigma_0\in\mathcal{Z}^\star$ such that $\alpha(\sigma_0,\mathcal{Q}_{\bar{0}}^\star\sigma_0)\geq 1$. For every sequence $\{\sigma_n\}_{n\geq 1}\subseteq\mathcal{Z}^\star$ with $\sigma_n\to\sigma$ and $\alpha(\sigma_n,\sigma_{n+1})\geq 1$ for all $n\geq 1$, we have $\alpha(\sigma_n,\sigma)\geq 1$ for all $n\geq 1$. Assume that $\alpha(\sigma_1,\sigma_2)=0$. Then $\alpha(\sigma_1,\sigma_2)\|\mathcal{Q}_{\bar{0}}^\star\sigma_1-\mathcal{Q}_{\bar{0}}^\star\sigma_2\|=0\leq\psi(\|\sigma_1-\sigma_2\|)$ and so

$$\alpha(\sigma_1,\sigma_2)\left\|\mathcal{Q}_{\bar{0}}^\star\sigma_1-\mathcal{Q}_{\bar{0}}^\star\sigma_2\right\|\leq\psi\left(\|\sigma_1-\sigma_2\|\right)$$

for all $\sigma_1,\sigma_2\in\mathcal{Z}^\star$. Thus all conditions of Lemma 1 hold and so $\mathcal{Q}_{\bar{0}}^\star$ has a fixed point which is a solution for the partial fractional Sturm–Liouville problem (1). □

Example 1 Consider the partial fractional Sturm–Liouville equation

$$\begin{cases}D_{c_0}^{(\frac{999}{1000},\frac{1999}{2000})}(100e^{-\sqrt[4]{p^\star}}\cosh q^\star D_{c_0}^{(\frac{89}{90},\frac{79}{80})}\sigma(p^\star,q^\star))+\frac{e^{-p^\star-q^{\star 3}}}{300(1+p^{\star 2}+p^{\star 2})}\sigma(p^\star,q^\star)\\ \quad=\frac{1}{600}e^{\frac{p^{\star 2}}{1+p^{\star 2}}}\sigma(p^\star,q^\star),\quad(p^\star,q^\star)\in[0,1]\times[0,1],\\ (100e^{-\sqrt[4]{p^\star}}\cosh q^\star\mathcal{D}_{c_0}^{(\frac{89}{90},\frac{79}{80})}\sigma(p^\star,q^\star))_{q^\star=0}=\frac{1}{100}p^{\star 2},\\ (100e^{-\sqrt[4]{p^\star}}\cosh q^\star\mathcal{D}_{c_0}^{(\frac{89}{90},\frac{79}{80})}\sigma(p^\star,q^\star))_{q^\star=0}=\frac{1}{400}q^{\star 3},\\ \sigma(p^\star,0)=\frac{1}{500}p^\star\quad\text{and}\quad\sigma(0,q^\star)=\frac{1}{350}q^{\star 2}.\end{cases}\tag{6}$$

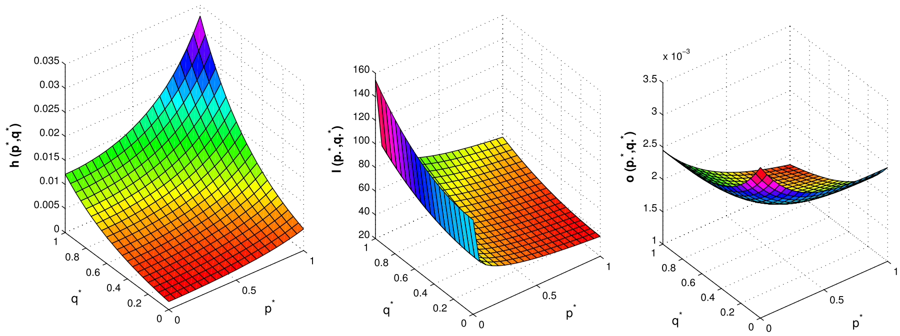

Figure 1 Plots of functions $h(p^*,q^*), l(p^*,q^*), o(p^*,q^*)$ in [0, 1]

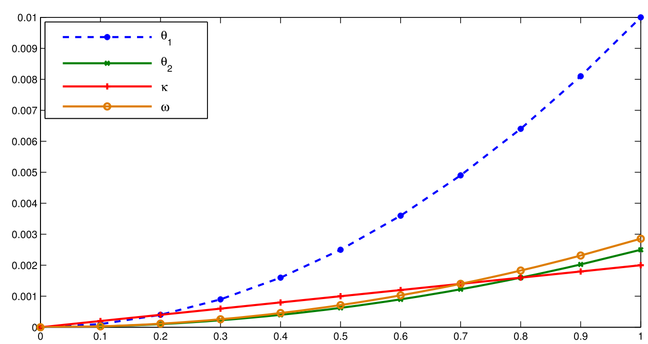

Figure 2 Plots of functions $\theta_1,\theta_2,\kappa,\omega$ in [0, 1]

Put $\hat{k}=(\hat{k}_1,\hat{k}_2)=(\frac{999}{1000},\frac{1999}{2000})$, $\hat{\ell}=(\hat{\ell}_1,\hat{\ell}_2)=(\frac{89}{99},\frac{79}{80})$, $a_0=1$, $b_0=1$, $\theta_1(p^\star)=\frac{1}{100}{p^\star}^2$, $\theta_2(q^\star)=\frac{1}{400}{q^\star}^3$, $\kappa(p^\star)=\frac{1}{500}p^\star$, $\omega(q^\star)=\frac{1}{350}{q^\star}^2$, $l(p^\star,q^\star)=100e^{-\sqrt[4]{p^\star}}\cosh q^\star$, $o(p^\star,q^\star)=\frac{e^{-p^\star-{q^\star}^3}}{300(1+{p^\star}^2+{q^\star}^2)}$, $h(p^\star,q^\star)=\frac{1}{600}e^{\frac{{p^\star}^2}{1+{q^\star}^2}}$. The diagrams are plotted in Figs. 1 and 2, and obviously they satisfy the conditions of the partial Sturm–Liouville differential problem. Put $f(r)=r$, $\upsilon(r_1,r_2)=3$ whenever $|r_1|\leq 1$ and $|r_2|\leq 1$ and $\upsilon(r_1,r_2)=-1$ otherwise, and

$$\begin{aligned}
&\mathcal{Q}_{\bar{0}}^{\star}\sigma\big(p^\star,q^\star\big)\\
&\quad=\Theta\big(p^\star,q^\star\big)+\int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t}\\
&\qquad\frac{(p^\star-s)^{\frac{-1}{1000}}(q^\star-t)^{\frac{-1}{2000}}(s-\wp_{\bar{1}})^{\frac{-1}{90}}(t-\zeta_{\bar{2}})^{\frac{-1}{80}}\mathcal{H}(\wp_{\bar{1}},\zeta_{\bar{2}},\sigma(\wp_{\bar{1}},\zeta_{\bar{2}}))}{\Gamma(\frac{999}{1000})\Gamma(\frac{1999}{2000})\Gamma(\frac{89}{99})\Gamma(\frac{79}{80})100e^{-\sqrt[4]{p^\star}}\frac{\cosh q^\star}{50}}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}},
\end{aligned}$$

where

$$\begin{aligned}
\Theta\big(p^\star,q^\star\big)&=\frac{1}{500}p^\star+\frac{1}{350}{p^\star}^2\\
&\quad+\int_0^{p^\star}\int_0^{q^\star}\frac{(p^\star-s)^{-\frac{1}{90}}(q^\star-t)^{-\frac{1}{80}}(\frac{1}{100}s^2+\frac{1}{400}t^3)}{100e^{-\sqrt[4]{p^\star}}\cosh q^\star\,\Gamma(\frac{89}{90})\Gamma(\frac{79}{80})}\,dt\,ds.
\end{aligned}$$

Note that $\mathcal{H}(\wp_{\bar{1}}, \zeta_{\bar{2}}, \sigma(\wp_{\bar{1}}, \zeta_{\bar{2}})) = h(\wp_{\bar{1}}, \zeta_{\bar{2}})f(\sigma(\wp_{\bar{1}}, \zeta_{\bar{2}})) - o(\wp_{\bar{1}}, \zeta_{\bar{2}})\sigma(\wp_{\bar{1}}, \zeta_{\bar{2}})$. Now, assume that $\upsilon(\sigma_1(p^\star, q^\star), \sigma_2(p^\star, q^\star)) \geq 0$. Then we have $|\sigma_1(p^\star, q^\star)| \leq 1$ and $|\sigma_2(p^\star, q^\star)| \leq 1$. Suppose that $|\sigma_1(p^\star, q^\star)| \leq 1$. Then we get

$$\begin{aligned}
&\left|\mathcal{H}\big(\wp_{\bar{1}}, \zeta_{\bar{2}}, \sigma(\wp_{\bar{1}}, \zeta_{\bar{2}})\big)\right| \\
&= \left| \frac{1}{600} e^{\frac{\wp_{\bar{1}}^2}{1+\zeta_{\bar{2}}^2}} f\big(\sigma(\wp_{\bar{1}}, \zeta_{\bar{2}})\big) - \frac{e^{-\wp_{\bar{1}}-\zeta_{\bar{2}}^3}}{300(1+\wp_{\bar{1}}^2+\zeta_{\bar{2}}^2)} \sigma_1(\wp_{\bar{1}}, \zeta_{\bar{2}}) \right| \\
&\leq \left| \frac{1}{600} e^{\frac{\wp_{\bar{1}}^2}{1+\zeta_{\bar{2}}^2}} \right| \left|\sigma_1(\wp_{\bar{1}}, \zeta_{\bar{2}})\right| + \left| \frac{e^{-p^\star - {p^\star}^3}}{300(1+{p^\star}^2+{p^\star}^2)} \right| \left|\sigma_1(\wp_{\bar{1}}, \zeta_{\bar{2}})\right| \\
&\leq \frac{e}{600} + \frac{1}{300} = \frac{e+2}{600}
\end{aligned}$$

and so

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_0^s \int_0^t \\
&\quad \frac{(p^\star - s)^{\frac{-1}{1000}} (q^\star - t)^{\frac{-1}{2000}} (s - \wp_{\bar{1}})^{\frac{-1}{90}} (t - \zeta_{\bar{2}})^{\frac{-1}{80}} |\mathcal{H}(\wp_{\bar{1}}, \zeta_{\bar{2}}, \sigma_1(\wp_{\bar{1}}, \zeta_{\bar{2}}))|}{\Gamma(\frac{999}{1000})\Gamma(\frac{1999}{2000})\Gamma(\frac{89}{90})\Gamma(\frac{79}{80}) 100 e^{-\sqrt[4]{p^\star}} \cosh q^\star} \, dt \, ds \, d\zeta_{\bar{2}} \, d\wp_{\bar{1}} \\
&\quad \leq 0.0000285064 \int_0^{p^\star} \int_0^{q^\star} \int_0^s \int_0^t (p^\star - s)^{\frac{-1}{1000}} (q^\star - t)^{\frac{-1}{2000}} (s - \wp_{\bar{1}})^{\frac{-1}{90}} \\
&\qquad \times (t - \zeta_{\bar{2}})^{\frac{-1}{80}} \, dt \, ds \, d\zeta_{\bar{2}} \, d\wp_{\bar{1}} \\
&\quad \leq 0.0000285064 \times \frac{\Gamma(\frac{89}{90}+1)\Gamma(\frac{79}{80}+1)\Gamma(\frac{999}{1000})\Gamma(\frac{1999}{2000})}{\frac{89}{90}\frac{79}{80}\Gamma(\frac{999}{1000}+\frac{89}{90}+1)\Gamma(\frac{1999}{2000}+\frac{79}{80}+1)} = 0.0000296489.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|\Theta\big(p^\star, q^\star\big)\right| \\
&\leq \left| \frac{1}{500} p^\star + \frac{1}{350} {q^\star}^2 \right| + \left| \int_0^{p^\star} \int_0^{q^\star} \frac{(p^\star - s)^{-\frac{1}{90}} (q^\star - t)^{-\frac{1}{80}} (\frac{1}{100} {p^\star}^2 + \frac{1}{400} {p^\star}^3)}{100 e^{-\sqrt[4]{p^\star}} \cosh q^\star \Gamma(\frac{89}{90})\Gamma(\frac{79}{80})} \, dt \, ds \right| \\
&\leq \frac{1}{500} + \frac{1}{350} + 0.0000453519 \int_0^1 \int_0^1 (1-s)^{-\frac{1}{90}} (1-t)^{-\frac{1}{80}} \, dt \, ds \\
&= \frac{1}{500} + \frac{1}{350} + 0.0000453519 \times 1.0240364102 = 0.0053797753.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\mathcal{Q}_0^\star \sigma_1\big(p^\star, q^\star\big)\right| \\
&\leq \left|\Theta\big(p^\star, q^\star\big)\right| + \int_0^{p^\star} \int_0^{q^\star} \int_0^s \int_0^t \\
&\quad \frac{(p^\star - s)^{\frac{-1}{1000}} (q^\star - t)^{\frac{-1}{2000}} (s - \wp_{\bar{1}})^{\frac{-1}{90}} (t - \zeta_{\bar{2}})^{\frac{-1}{80}} |\mathcal{H}(\wp_{\bar{1}}, \zeta_{\bar{2}}, \sigma_1(\wp_{\bar{1}}, \zeta_{\bar{2}}))|}{\Gamma(\frac{999}{1000})\Gamma(\frac{1999}{2000})\Gamma(\frac{89}{99})\Gamma(\frac{79}{80}) 100 e^{-\sqrt[4]{p^\star}} \frac{\cosh q^\star}{50}} \, dt \, ds \, d\zeta_{\bar{2}} \, d\wp_{\bar{1}} \\
&\leq 0.0053797753 + 0.0000296489 = 0.0054094242 \leq 1.
\end{aligned}$$

Similarly, we obtain $|\mathcal{Q}_{\bar{0}}^{\star}\sigma_1(p^\star,q^\star)| \leq 1$ and $\upsilon(\mathcal{Q}_{\bar{0}}^{\star}\sigma_1(p^\star,q^\star), \mathcal{Q}_{\bar{0}}^{\star}\sigma_2(p^\star,q^\star)) \geq 0$. Assume that $\{\sigma_n\}_{n\geq 1} \subseteq \mathcal{Z}^\star$ is a sequence such that $\sigma_n \to \sigma$ and

$$\upsilon\big(\sigma_n\big(p^\star,q^\star\big),\sigma_{n+1}\big(p^\star,q^\star\big)\big) \geq 0$$

for all $n \geq 1$ and $(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. Then we have $|\sigma_n(p^\star,q^\star)| \leq 1$ and $|\sigma_{n+1}(p^\star,q^\star)| \leq 1$ for all $n \geq 1$ and $(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. Since $\sigma_n \to \sigma$, $|\sigma(p^\star,q^\star)| = \lim_{n\to\infty} |\sigma_n(p^\star,q^\star)| \leq 1$, we obtain $\upsilon(\mathcal{Q}_{\bar{0}}^{\star}\sigma_n(p^\star,q^\star), \mathcal{Q}_{\bar{0}}^{\star}\sigma(p^\star,q^\star)) \geq 0$ for all $n \geq 1$ and $(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. Since $|0| \leq 1$ and $|\mathcal{Q}_{\bar{0}}^{\star}0| \leq 1$, we get $\upsilon(0, \mathcal{Q}_{\bar{0}}^{\star}0) \geq 1$. Note that $\Phi^* = 1$,

$$l = \inf_{(p^\star,q^\star)\in\mathcal{J}_{a_0}\times\mathcal{J}_{b_0}} l\big(p^\star,q^\star\big) = \inf_{(p^\star,q^\star)\in\mathcal{J}_{a_0}\times\mathcal{J}_{b_0}} 100e^{-\sqrt[4]{p^\star}}\cosh q^\star = 100e,$$

$$\|o\| = \sup_{(p^\star,q^\star)\in\mathcal{J}_{a_0}\times\mathcal{J}_{b_0}} o\big(p^\star,q^\star\big) = \sup_{(p^\star,q^\star)\in\mathcal{J}_{a_0}\times\mathcal{J}_{b_0}} \frac{e^{-p^\star-{q^\star}^3}}{300(1+{p^\star}^2+{q^\star}^2)} = \frac{1}{300},$$

and $\|h\| = \sup_{(p^\star,q^\star)\in\mathcal{J}_{a_0}\times\mathcal{J}_{b_0}} \frac{1}{600}e^{\frac{{p^\star}^2}{1+{q^\star}^2}} = \frac{e}{600}$. Hence,

$$\begin{aligned}\frac{a_0^{\hat{k}_1+\hat{\ell}_1}b_0^{\hat{k}_2+\hat{\ell}_2}(\|h\|\Phi^*+\|o\|)}{l\Gamma(\hat{k}_1+\hat{\ell}_1+1)\Gamma(\hat{k}_2+\hat{\ell}_2+1)} &= \frac{\frac{e}{600}+\frac{1}{300}}{100e\Gamma(\frac{999}{1000}+\frac{89}{90}+1)\Gamma(\frac{1999}{2000}+\frac{79}{80}+1)} \\ &= 0.0000074014 \leq 1.\end{aligned}$$

Now by using Theorem 4, the problem (6) has a solution.

Definition 5 We say that a function $\sigma \in \mathcal{C}(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0})$ is a solution for the partial fractional Sturm–Liouville differential inclusion problem problem (2) whenever there is a function ν in $\mathcal{L}^1(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0}, \mathbb{R})$ such that $\nu(p^\star,q^\star) \in \mathcal{H}(p^\star,q^\star,\sigma(p^\star,q^\star))$ for almost all $(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$,

$$\begin{cases}(l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star))_{q^\star=0} = \theta_1(p^\star),\\ (l(p^\star,q^\star)\mathcal{D}_{c_0}^{\hat{\ell}}\sigma(p^\star,q^\star))_{p^\star=0} = \theta_2(q^\star),\\ \sigma(p^\star,0) = \kappa(p^\star) \quad \text{and} \quad \sigma(0,q^\star) = \omega(q^\star),\end{cases}$$

and

$$\begin{aligned}&\sigma\big(p^\star,q^\star\big)\\ &\quad = \Theta\big(p^\star,q^\star\big)\\ &\qquad + \int_0^{p^\star}\int_0^{q^\star}\int_0^s\int_0^t\\ &\qquad \frac{(p^\star-s)^{\hat{k}_1-1}(q^\star-t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}\nu(\wp_{\bar{1}},\zeta_{\bar{2}})}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}},\end{aligned}$$

for all $(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$.

For given $\sigma \in \mathcal{Z}^\star$, define the set

$$\mathcal{S}_{\mathcal{H},\sigma} = \{ \nu \in \mathcal{L}^1(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0}) | \nu \in \mathcal{H}(p^\star, q^\star, \sigma(p^\star, q^\star)) \text{ on } \mathcal{J}_{a_0} \times \mathcal{J}_{b_0} \}.$$

Assume that

($H1$) $\mathcal{H} : \mathcal{J}_{a_0} \times \mathcal{J}_{b_0} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is an integrable bounded multifunction so that $\mathcal{H}(\cdot,\cdot,\sigma)$ is measurable for all $\sigma \in \mathbb{R}$.

($H2$) There exists $\rho \in \mathcal{C}(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0}, \mathbb{R}^+)$ so that

$$PH_{d_{\mathcal{Z}^\star}}(\mathcal{H}(p^\star, q^\star, r_1), \mathcal{H}(p^\star, q^\star, r_2)) \le \rho(p^\star, q^\star)\psi(|r_1 - r_2|)$$

for all $r_1, r_2 \in \mathbb{R}$, where $\psi \in \Psi$, $\frac{a_0^{\hat{k}_1+\hat{\ell}_1} b_0^{\hat{k}_2+\hat{\ell}_2} \|\rho\|_\infty}{l\Gamma(\hat{k}_1+\hat{\ell}_1+1)\Gamma(\hat{k}_2+\hat{\ell}_2+1)} \le 1$ and

$$l = \inf_{(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}} l(p^\star, q^\star).$$

($H3$) Define $N : \mathcal{Z}^\star \to 2^{\mathcal{Z}^\star}$ by

$$N(\sigma) = \{ h \in \mathcal{Z}^\star | \text{ there exists } \nu \in S_{\mathcal{H},\sigma} \text{ so that } h(p^\star, q^\star) = w(p^\star, q^\star) \\ \text{for all } (p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0} \},$$

where

$$\begin{aligned} & w(p^\star, q^\star) \\ & = \Theta(p^\star, q^\star) + \int_0^{p^\star} \int_0^{q^\star} \int_0^s \int_0^t \\ & \frac{(p^\star - s)^{\hat{k}_1-1}(q^\star - t)^{\hat{k}_2-1}(s - \wp_{\bar{1}})^{\hat{\ell}_1-1}(t - \zeta_{\bar{2}})^{\hat{\ell}_2-1}\nu(\wp_{\bar{1}}, \zeta_{\bar{2}})}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)} \, dt \, ds \, d\zeta_{\bar{2}} \, d\wp_{\bar{1}}. \end{aligned}$$

($H4$) Suppose that $\upsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a real valued function and for every convergent sequence $\{\sigma_n\}_{n\ge1} \subseteq \mathcal{Z}^\star$ with $\sigma_n \to \sigma$ and $\upsilon(\sigma_n(p^\star, q^\star), \sigma(p^\star, q^\star)) \ge 0$ for all n and $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$, there exists a subsequence $\{\sigma_{n_j}\}_{j\ge1}$ of σ_n so that $\upsilon(\sigma_{n_j}(p^\star, q^\star), \sigma(p^\star, q^\star)) \ge 0$ for all n and $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. Assume that, for every $\sigma \in \mathcal{Z}^\star$ and $h \in N(\sigma)$ with $\upsilon(\sigma(p^\star, q^\star), h(p^\star, q^\star)) \ge 0$ for each $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$, there exists $w \in N(\sigma)$ so that $\upsilon(h(p^\star, q^\star), w(p^\star, q^\star)) \ge 0$ for all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. Suppose that there exists $\sigma_0 \in \mathcal{Z}^\star$ and $h \in N(\sigma_0)$ so that $\upsilon(\sigma_0(p^\star, q^\star), h(p^\star, q^\star)) \ge 0$ for all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$.

Theorem 6 *Assume that* ($H1$)–($H4$) *hold. Then the partial fractional Sturm–Liouville problem* (2) *has a solution.*

Proof We prove that the multifunction $N : \mathcal{Z}^\star \to 2^{\mathcal{Z}^\star}$ has a fixed point which provides a solution for the partial fractional Sturm–Liouville problem (2). Note that the multifunction $(p^\star, q^\star) \to \mathcal{H}(p^\star, q^\star, \sigma(p^\star, q^\star))$ has a measurable selection. Since it has closed and has measurable values for aa $\sigma \in \mathcal{Z}^\star$, $S_{\mathcal{H},\sigma}$ is nonempty for every $\sigma \in \mathcal{Z}^\star$. We prove that $N(\sigma)$

is closed subset of $\mathcal{Z}^\star$. For this aim assume that $\sigma \in \mathcal{Z}^*$ and $\{h_n\} \subset N(\sigma)$ is a sequence with $h_n \to h$. For each n, choose $v_n \in S_{\mathcal{H},\sigma}$ such that

$$\begin{aligned} &h_n(p^\star, q^\star) \\ &\quad = \Theta(p^\star, q^\star) + \int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t} \\ &\qquad \frac{(p^\star - s)^{\hat{k}_1-1}(q^\star - t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}v_n(\wp_{\bar{1}},\zeta_{\bar{2}})}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}}, \end{aligned}$$

for all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. On the other hand, $\mathcal{H}$ has compact values. Thus, we may assume that $\{v_n\}$ converges to some $v \in \mathcal{L}^1(\mathcal{J}_{a_0} \times \mathcal{J}_{b_0})$. Hence,

$$\begin{aligned} &h(p^\star, q^\star) \\ &\quad = \Theta(p^\star, q^\star) + \int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t} \\ &\qquad \frac{(p^\star - s)^{\hat{k}_1-1}(q^\star - t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}v(\wp_{\bar{1}},\zeta_{\bar{2}})}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \end{aligned}$$

and so $h \in N(\sigma)$. Since $\mathcal{H}$ is a compact map, $N(\sigma)$ is a bounded set for al $\sigma \in \mathcal{Z}^\star$. Now, define the function $\alpha : \mathcal{Z}^\star \times \mathcal{Z}^\star \to \mathbb{R}^+$ by $\alpha(\sigma_1, \sigma_2) \geq 1$ whenever $\upsilon(\sigma_1(p^\star, q^\star), \sigma_2(p^\star, q^\star)) \geq 0$ for almost all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$ and $\alpha(\sigma_1, \sigma_2) = 0$ otherwise. We show that N is α-ψ-contractive. Let $\sigma_1, \sigma_2 \in \mathcal{Z}^\star$ and $h_1 \in N(\sigma_2)$. Choose $v_1 \in S_{\mathcal{H},\sigma_2}$ such that

$$\begin{aligned} &h_1(p^\star, q^\star) \\ &\quad = \Theta(p^\star, q^\star) + \int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t} \\ &\qquad \frac{(p^\star - s)^{\hat{k}_1-1}(q^\star - t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}v_1(\wp_{\bar{1}},\zeta_{\bar{2}})}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \end{aligned}$$

for almost all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. Then we get

$$\begin{aligned} &PH_{d_{\mathcal{Z}^\star}}\big(\mathcal{H}(p^\star, q^\star, \sigma_1(p^\star, q^\star)), \mathcal{H}(p^\star, q^\star, \sigma_2(p^\star, q^\star))\big) \\ &\quad \leq \rho(p^\star, q^\star)\psi\big(|\sigma_1(p^\star, q^\star) - \sigma_2(p^\star, q^\star)|\big), \end{aligned}$$

for almost all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$ with $\upsilon(\sigma_1(p^\star, q^\star), \sigma_2(p^\star, q^\star)) \geq 0$. Now, choose $w \in \mathcal{H}(p^\star, q^\star, \sigma_1(p^\star, q^\star))$ so that $|v_1(t) - w| \leq \rho(p^\star, q^\star)\psi(|\sigma_1(p^\star, q^\star) - \sigma_2(p^\star, q^\star)|)$ for all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. Now, define $\mathcal{U} : \mathcal{J}_{a_0} \times \mathcal{J}_{b_0} \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{U}(p^\star, q^\star) = \big\{w \in \mathbb{R} \,|\, |v_1(p^\star, q^\star) - w| \leq \rho(p^\star, q^\star)\psi\big(|\sigma_1(p^\star, q^\star) - \sigma_2(p^\star, q^\star)|\big)\big\}.$$

Since v_1 and $\omega_0^* = \rho(p^\star, q^\star)\psi(|\sigma_1(p^\star, q^\star) - \sigma_2(p^\star, q^\star)|)$ are measurable, the multifunction $\mathcal{U}(\cdot,\cdot) \cap \mathcal{H}(\cdot,\cdot,\sigma(\cdot,\cdot))$ is measurable. Choose $v_2 \in \mathcal{H}(p^\star, q^\star, \sigma_1(p^\star, q^\star))$ such that

$$\begin{aligned} \big|v_1(p^\star, q^\star) - v_2(p^\star, q^\star)\big| &\leq \rho(p^\star, q^\star)\psi\big(|\sigma_1(p^\star, q^\star) - \sigma_2(p^\star, q^\star)|\big) \\ &\leq \|\rho\|_\infty \psi\big(\|\sigma_1 - \sigma_2\|\big) \end{aligned} \tag{7}$$

for all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. Now, choose the element $h_2 \in N(\sigma_1)$ defined by

$$\begin{aligned} &h_2(p^\star, q^\star) \\ &= \Theta(p^\star, q^\star) + \int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t} \\ &\quad \frac{(p^\star - s)^{\hat{k}_1-1}(q^\star - t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}\nu_2(\wp_{\bar{1}}, \zeta_{\bar{2}})}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \end{aligned}$$

for all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. By using (7), we get

$$\begin{aligned} &\left|h_2(p^\star, q^\star) - h_1(p^\star, q^\star)\right| \\ &\leq \int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t} \\ &\quad \frac{(p^\star - s)^{\hat{k}_1-1}(q^\star - t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}|\nu_2(\wp_{\bar{1}}, \zeta_{\bar{2}}) - \nu_1(\wp_{\bar{1}}, \zeta_{\bar{2}})|}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \\ &\leq \|\rho\|_\infty \int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t} \\ &\quad \frac{(p^\star - s)^{\hat{k}_1-1}(q^\star - t)^{\hat{k}_2-1}(s-\wp_{\bar{1}})^{\hat{\ell}_1-1}(t-\zeta_{\bar{2}})^{\hat{\ell}_2-1}}{\Gamma(\hat{k}_1)\Gamma(\hat{k}_2)\Gamma(\hat{\ell}_1)\Gamma(\hat{\ell}_2)l(s,t)}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}}\,\psi\left(\|\sigma_1 - \sigma_2\|\right) \\ &\leq \frac{a_0^{\hat{k}_1+\hat{\ell}_1} b_0^{\hat{k}_2+\hat{\ell}_2}\|\rho\|_\infty}{l\Gamma(\hat{k}_1+\hat{\ell}_1+1)\Gamma(\hat{k}_2+\hat{\ell}_2+1)}\psi\left(\|\sigma_1 - \sigma_2\|\right) \leq \psi\left(\|\sigma_1 - \sigma_2\|\right) \end{aligned}$$

and so $\|h_1 - h_2\| \leq \psi(\|\sigma_1 - \sigma_2\|)$. Note that

$$\alpha(\sigma_1, \sigma_2) PH_{d_{\mathcal{Z}^\star}}\left(N(\sigma_1), N(\sigma_2)\right) \leq \psi\left(\|\sigma_1 - \sigma_2\|\right) \quad \text{for all } \sigma_1, \sigma_2 \in \mathcal{Z}^\star.$$

Thus, N is α-ψ-contraction. Let $\sigma_1 \in \mathcal{Z}^\star$ and $\sigma_2 \in N(\sigma_1)$ be such that $\alpha(\sigma_1, \sigma_2) \geq 1$. Then $\upsilon(\sigma_1(p^\star, q^\star), \sigma_2(p^\star, q^\star)) \geq 0$ for all $(p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$. Hence, there exists $w \in N(\sigma_2)$ such that $\upsilon(\sigma_1(p^\star, q^\star), w(p^\star, q^\star)) \geq 0$. This implies that $\alpha(\sigma_1, w) \geq 1$. Thus, N is α-admissible. Now by using Lemma 2, N has a fixed point which is a solution of the partial fractional Sturm–Liouville problem (2). □

Example 2 Consider the partial fractional Sturm–Liouville inclusion problem

$$\begin{cases} D_{c_0}^{(\frac{1}{2},\frac{1}{3})}(13e^{p^\star q^\star} D_{c_0}^{(\frac{1}{7},\frac{1}{8})}\sigma(p^\star, q^\star)) \in [\sigma(p^\star, q^\star), \sigma(p^\star, q^\star) + \frac{e^{-p^{\star 2} - q^{\star 2}}|\sigma(p^\star, q^\star)|}{4(1+|\sigma(p^\star, p^{\star 2})|)}], \\ (p^\star, q^\star) \in [0,1] \times [0,1], \\ (13e^{p^\star q^\star}\mathcal{D}_{c_0}^{(\frac{1}{7},\frac{1}{8})}\sigma(p^\star, q^\star))_{q^\star=0} = \frac{3}{40}p^{\star 2}, \\ (13e^{p^\star q^\star}\mathcal{D}_{c_0}^{(\frac{1}{7},\frac{1}{8})}\sigma(p^\star, q^\star))_{p^\star=0} = \frac{1}{25}q^{\star 3}, \\ \sigma(p^\star, 0) = \frac{7}{50}p^{\star 2} \quad \text{and} \quad \sigma(0, q^\star) = \frac{8}{35}q^{\star 3}. \end{cases} \tag{8}$$

Put $\hat{k} = (\hat{k}_1, \hat{k}_2) = (\frac{1}{2}, \frac{1}{3})$, $\hat{\ell} = (\hat{\ell}_1, \hat{\ell}_2) = (\frac{1}{7}, \frac{1}{8})$, $a = 1$, $b = 1$, $\theta_1(p^\star) = \frac{3}{40}p^{\star 2}$, $\theta_2(q^\star) = \frac{1}{25}q^{\star 3}$, $\kappa(p^\star) = \frac{7}{50}p^{\star 2}$, $\omega(q^\star) = \frac{8}{35}q^{\star 3}$ and $l(p^\star, q^\star) = 13e^{p^\star q^\star}$. The plotted diagrams in Figs. 3 and

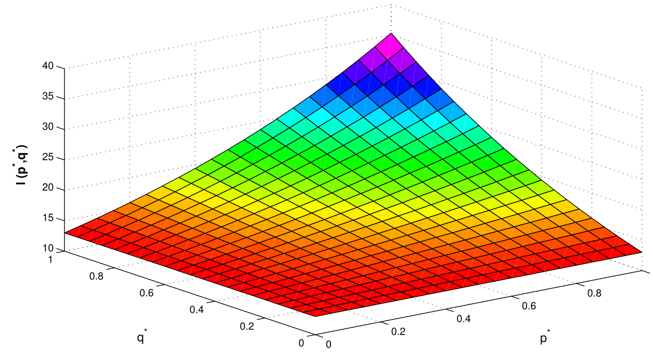

Figure 3 Plot of function $I(p^*, q^*)$ in [0, 1]

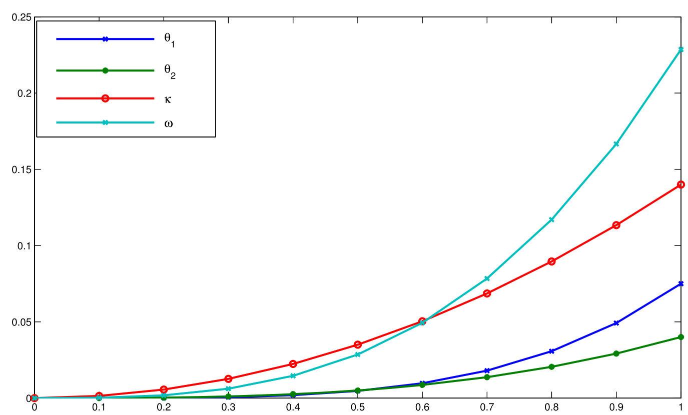

Figure 4 Plots of functions $\theta_1, \theta_2, \kappa, \omega$ in [0, 1]

4 show that the conditions of the partial Sturm–Liouville differential inclusion problem hold. Also, put $\mathcal{H}(p^\star, q^\star, r) = [r, r + \frac{e^{-p^{\star 2} - q^{\star 2}}|r|}{4(1+|r|)}]$, $\upsilon(r_1, r_2) = 1$ whenever $r_1 \geq 0$ and $r_2 \geq 0$ and $\upsilon(r_1, r_2) = -1$ otherwise,

$$\begin{aligned} N(\sigma) = \big\{ & h \in \mathcal{Z}^\star \mid \text{there exists } \nu \in S_{\mathcal{H},\sigma} \text{ so that } h(p^\star, q^\star) = w(p^\star, q^\star) \\ & \text{for all } (p^\star, q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0} \big\} \end{aligned}$$

where

$$\begin{aligned} & w(p^\star, q^\star) \\ & \quad = \Theta(p^\star, q^\star) \\ & \qquad + \int_0^{p^\star}\int_0^{q^\star}\int_0^{s}\int_0^{t} \frac{(p^\star - s)^{-\frac{1}{2}}(q^\star - t)^{-\frac{2}{3}}(s - \wp_{\bar{1}})^{-\frac{6}{7}}(t - \zeta_{\bar{2}})^{-\frac{7}{8}}\nu(\wp_{\bar{1}}, \zeta_{\bar{2}})}{13e^{st}\Gamma(\frac{1}{2})\Gamma(\frac{1}{3})\Gamma(\frac{1}{7})\Gamma(\frac{1}{8})}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \end{aligned}$$

and $\Theta(p^\star, q^\star) = \frac{7}{50}p^{\star 2} + \frac{8}{35}p^{\star 3} + \int_0^{p^\star}\int_0^{q^\star} \frac{(p^\star - s)^{-\frac{1}{90}}(q^\star - t)^{-\frac{1}{80}}(\frac{3}{40}s^2 + \frac{1}{25}t^3)}{13e^{st}\Gamma(\frac{89}{90})\Gamma(\frac{79}{80})}\,dt\,ds$. Assume that $\sigma \in \mathcal{Z}^\star$ and $h \in N(\sigma)$ with $\upsilon(\sigma(p^\star, q^\star), h(p^\star, q^\star)) \geq 0$ for all $(p^\star, q^\star) \in [0, 1] \times [0, 1]$. Then we have

$\sigma(p^\star,q^\star) \geq 0$ and $h(p^\star,q^\star) \geq 0$ for all $(p^\star,q^\star) \in [0,1] \times [0,1]$. Since $\sigma(p^\star,q^\star) \geq 0$, we get

$$\mathcal{H}\big(p^\star,q^\star,\sigma\big(p^\star,q^\star\big)\big) = \left[\sigma\big(p^\star,q^\star\big), \sigma\big(p^\star,q^\star\big) + \frac{e^{-{p^\star}^2-{q^\star}^2}|\sigma(p^\star,q^\star)|}{4(1+|\sigma(p^\star,q^\star)|)}\right] \subseteq [0,\infty).$$

Choose $\nu(p^\star,q^\star) \in \mathcal{H}(p^\star,q^\star,\sigma(p^\star,q^\star))$ so that $\nu(p^\star,q^\star) \geq 0$ for all $(p^\star,q^\star)$ in $[0,1] \times [0,1]$. Since $\Theta(p^\star,q^\star) \geq 0$, we get

$$\begin{aligned} &w\big(p^\star,q^\star\big) \\ &\quad := \Theta\big(p^\star,q^\star\big) + \int_0^{p^\star}\int_0^{q^\star}\int_0^s\int_0^t \\ &\qquad \frac{(p^\star-s)^{-\frac{1}{2}}(q^\star-t)^{-\frac{2}{3}}(s-\wp_{\bar{1}})^{-\frac{6}{7}}(t-\zeta_{\bar{2}})^{-\frac{7}{8}}\nu(\wp_{\bar{1}},\zeta_{\bar{2}})}{13e^{st}\Gamma(\frac{1}{2})\Gamma(\frac{1}{3})\Gamma(\frac{1}{7})\Gamma(\frac{1}{8})}\,dt\,ds\,d\zeta_{\bar{2}}\,d\wp_{\bar{1}} \geq 0 \end{aligned}$$

and so $w(p^\star,q^\star) \geq 0$. Thus, $\upsilon(h(p^\star,q^\star),w(p^\star,q^\star)) \geq 0$. Note that $\upsilon(0,h(p^\star,q^\star)) \geq 0$ for $h \in N(\sigma)$ and also for each convergent sequence $\{\sigma_n\}_{n\geq 1} \subseteq \mathcal{Z}^\star$ with $\sigma_n \to \sigma$ and $\upsilon(\sigma_n(p^\star,q^\star),\sigma(p^\star,q^\star)) \geq 0$ for all n and $(p^\star,q^\star) \in \mathcal{J}_{a_0} \times \mathcal{J}_{b_0}$, there exists a subsequence $\{\sigma_{n_j}\}_{j\geq 1}$ of $\{\sigma_n\}_{n\geq 1}$ such that $\upsilon(\sigma_{n_j}(p^\star,q^\star),\sigma(p^\star,q^\star)) \geq 0$. Thus,

$$\begin{aligned} &PH_{d_{\mathcal{Z}^\star}}\big(\mathcal{H}\big(p^\star,q^\star,r_1\big),\mathcal{H}\big(p^\star,q^\star,r_2\big)\big) \\ &\quad \leq \frac{e^{-{p^\star}^2-{q^\star}^2}}{4}\left|\frac{|r_1|}{1+|r_1|} - \frac{|r_2|}{1+|r_2|}\right| \\ &\quad = \frac{e^{-{p^\star}^2-{q^\star}^2}}{4}\big||r_1|-|r_2|\big| \leq \frac{e^{-{p^\star}^2-{q^\star}^2}}{4}|r_1-r_2|. \end{aligned}$$

If $\rho(p^\star,q^\star) = e^{-{p^\star}^2-{q^\star}^2}$ and $\psi(t) = \frac{1}{4}t$, then

$$PH_{d_{\mathcal{Z}^\star}}\big(\mathcal{H}\big(p^\star,q^\star,r_1\big),\mathcal{H}\big(p^\star,q^\star,r_2\big)\big) \leq \Phi\big(p^\star,q^\star\big)\psi\big(|r_1-r_2|\big)$$

and $\|\rho\|_\infty = 1$. Put $l(p^\star,q^\star) = 13e^{p^\star q^\star}$. Then $l = 13$ and so

$$\frac{a_0^{\hat{k}_1+\hat{\ell}_1}b_0^{\hat{k}_2+\hat{\ell}_2}\|\rho\|_\infty}{l\Gamma(\hat{k}_1+\hat{\ell}_1+1)\Gamma(\hat{k}_2+\hat{\ell}_2+1)} = \frac{1}{13\Gamma(\frac{1}{2}+\frac{1}{7}+1)\Gamma(\frac{1}{3}+\frac{1}{8}+1)} = 0.0966114627 \leq 1.$$

Now by using Theorem 6, the problem (8) has a solution.

3 Conclusion

In this work, we studied a partial fractional version of the Sturm–Liouville differential equation by using the Caputo derivative. Also, we reviewed inclusion version of the problem. First, by using the technique of α-ψ-contractive mappings, we investigated the existence of solutions for the partial fractional Sturm–Liouville equation. We presented an illustrated example to clear more the result. Secondly, we have investigated the partial fractional Sturm–Liouville inclusion problem by using the technique of α-ψ-contractive multifunctions. We provided an illustrated an example for explaining the second result. In this way, we provided some related figures for the examples.

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