

RESEARCH

Open Access



Matrix computational collocation approach based on rational Chebyshev functions for nonlinear differential equations

Mohamed A. Abd El Salam¹, Mohamed A. Ramadan², Mahmoud A. Nassar¹, Praveen Agarwal^{3,4,5,6} and Yu-Ming Chu^{7,8*}

*Correspondence:

chuyuming2005@126.com

⁷Department of Mathematics, Huzhou University, Huzhou, 313000, P.R. China

⁸Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha, 410114, China
Full list of author information is available at the end of the article

Abstract

In this work, a numerical technique for solving general nonlinear ordinary differential equations (ODEs) with variable coefficients and given conditions is introduced. The collocation method is used with rational Chebyshev (RC) functions as a matrix discretization to treat the nonlinear ODEs. Rational Chebyshev collocation (RCC) method is used to transform the problem to a system of nonlinear algebraic equations. The discussion of the order of convergence for RC functions is introduced. The proposed base is specified by its ability to deal with boundary conditions with independent variable that may tend to infinity with easy manner without divergence. The technique is tested and verified by two examples, then applied to four real life and applications models. Also, the comparison of our results with other methods is introduced to study the applicability and accuracy.

Keywords: Nonlinear ordinary differential equations; Collocation method; Rational Chebyshev functions

1 Introduction

As known, the nonlinear ordinary differential equations (ODEs) have an important role because of their applications in many fields of science and real life phenomena modeling for instance the logistic growth model in a population [1], the epidemic model [2], the kinetic model [3], the model of ozone decomposition [4], happiness dynamical models [5], the natural convection of Darcian fluid (NCDF) about a vertical full cone embedded in porous media (PM) with a prescribed wall temperature [6], and the Volterra population model [7]. The approximation and numerical techniques are used to treat the nonlinear ODEs, because getting the analytical exact solution is not easy in many cases. Several numerical methods have been given to solve nonlinear ODEs such as linearization method [8] and a sixth-degree B-spline method [9]. Moreover, the Chebyshev collocation matrix method [10] is presented as a numerical solution of nonlinear ODEs. Also, the semi-analytical techniques, such as the Adomian decomposition method [11, 12], the homotopy perturbation method (HPM) [13, 14], and the decomposition method [15], are used to find the solution of nonlinear ODEs.

© The Author(s) 2021. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Many works considered solving differential equations in the finite [16–19] and infinite domain [20–22]. The rational Chebyshev (RC) functions are used in the domain $[0, L]$, where $0 \leq L < \infty$, which provides great success in dealing with differential equations defined in the open domain $[0, L]$. Many authors studied RC functions for solving different problems of differential, integro-differential equations, and some other physical problems with variable coefficients [23–25]. Saeid Abbasbandy et al. [26] investigated the use of the RC collocation method to get an approximate solution of magneto hydro dynamic flow of an incompressible viscous fluid over a stretching sheet. Parand et al. [27] and Ramadan et al. [28] introduced RC functions for solving natural convection of Darcian fluid about a vertical full cone embedded in porous media with a prescribed wall temperature. In [29] Ramadan et al. investigated the use of the RC collocation technique for approximating bio-mathematical problems of continuous population models for single and interacting species, systematically the logistic growth model in a population, prey-predator model: Lotka–Volterra system, the simple two-species Lotka–Volterra competition model, and the prey-predator model with limit cycle periodic behavior.

In this paper, the RC collocation method (RCC) as matrix discretization is introduced to obtain the solution of a general form of nonlinear ODEs defined on the semi-infinite domain. The order of convergence for RC functions is discussed. The validity and applicability of the method are tested through two examples. In addition, four applications as nonlinear real life models are solved using the proposed technique. On the other hand, the RC functions inherit many properties of Chebyshev polynomials as completeness, orthogonality, and strong convergence. The RC functions [23, 30–35] are defined as follows:

$$R_n(x) = \cos n\phi, \quad \text{where } \cos \phi = \left(\frac{x-1}{x+1} \right). \tag{1.1}$$

The RC functions are orthogonal with respect to the weight function $w(x)$ in the semi-infinite domain, where

$$w(x) = \frac{1}{(x+1)\sqrt{x}}.$$

If the substitution $v = \frac{x-1}{x+1}$ is used in $R_n(x)$, then we have the RC functions as polynomials in v , then $R_n(x)$ takes the matrix form:

$$R(x) = V(x)C^T, \tag{1.2}$$

where $R(x)$ and $V(x)$ are vector matrices of the form

$$R(x) = \begin{bmatrix} R_0(x) & R_1(x) & \dots & R_N(x) \end{bmatrix},$$

$$V(x) = \begin{bmatrix} v^0(x) & v^1(x) & \dots & v^N(x) \end{bmatrix},$$

and C is a matrix given by

$$C = \begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \binom{1}{0} & 0 & 0 & \dots & 0 & 0 \\ -\binom{1}{1} & 0 & 2\binom{2}{0} & 0 & \dots & 0 & 0 \\ 0 & -\frac{3}{2}\binom{2}{1} & 0 & 2^2\binom{3}{0} & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-1)^l \binom{l}{l} & 0 & (-1)^{l-1} 2^{\frac{2l}{l+1}} \binom{l+1}{l-1} & 0 & \dots & 2^{2l-1} \binom{2l}{0} & 0 \\ 0 & (-1)^l 2^{\frac{2l}{l+1}} \binom{l+1}{l} & 0 & (-1)^{l-1} 2^{2\frac{2l+1}{l+2}} \binom{l+2}{l-1} \dots & 0 & 2^{2l} \binom{2l+1}{0} \end{bmatrix}.$$

In the matrix C , if N is an odd number, the last row will be used, where $N = 2l + 1$, while the second row from below is used for the even case of N and $N = 2l$. For more instances of the RC functions and their properties, we provide some suggested references for the reader [23, 30, 31].

2 The order of convergence

Let us assume that

$$L_w^2(\Omega) = \left\{ f : \|f\|_w = \left(\int_0^\infty |f(x)|^2 w(x) dx \right)^{\frac{1}{2}} < \infty \right\}, \tag{2.1}$$

the space functions, and we denote the inner product as $\langle v, u \rangle_w$ such that

$$\langle v, v \rangle_w = (\|v\|_w)^2. \tag{2.2}$$

Therefore, from the orthogonality relation for RC functions

$$\int_0^\infty R_n(x)R_m(x)w(x) dx = \frac{c_m\pi}{2} \delta_{nm}, \tag{2.3}$$

we note that the RC functions form a set of orthogonal bases for $L_w^2(\Omega)$. Also, let us define the normed spaces $H_w^r(\Omega)$ and $H_{w,A}^r(\Omega)$ as follows:

$$H_w^r(\Omega) = \left\{ f : \|f\|_{r,w} = \left(\sum_{k=0}^r \left\| \frac{d^k}{dx^k} f \right\|_w \right)^{\frac{1}{2}} < \infty \right\}, \tag{2.4}$$

$$H_{w,\theta}^r(\Omega) = \left\{ f : \|f\|_{r,w,\theta} = \left(\sum_{k=0}^r \left\| (x+1)^{r+k} \frac{d^k}{dx^k} f \right\|_w \right)^{\frac{1}{2}} < \infty \right\}, \tag{2.5}$$

where $r \geq 0$ and k is a positive integer constant, and θ is the Sturm–Liouville operator in the following form:

$$\theta f = -w^{-1}(w^{-1}f)'. \tag{2.6}$$

Let us define $\mathfrak{R}_N = \text{span}\{R_0, R_1, \dots, R_N\}$, N is a positive integer such that $N < \infty$.

Theorem 2.1 *For any $r \geq 0$ and c a generic positive constant independent of any function, and $\phi \in \mathfrak{R}_N$, then*

$$\|\phi\|_{r,w} \leq cN^{2r} \|\phi\|_w.$$

The proof of Theorem 2.1 is found in [36].

Since the set of RC functions is orthogonal and complete, $f(x)$ is defined over Ω , then it can be expanded in terms of RC functions as follows:

$$f(x) = \sum_{n=0}^{\infty} a_n R_n(x), \tag{2.7}$$

where

$$a_n = \frac{\langle f, R_n \rangle_w}{(\|R_n\|_w)^2} = \frac{2}{c_m \pi} \int_0^{\infty} R_n(x) f(x) w(x) dx. \tag{2.8}$$

Relation (2.7) may represent a spectral approximation in the following form:

$$f_N(x) = \sum_{n=0}^N a_n R_n(x). \tag{2.9}$$

To obtain the order of convergence of RC functions approximation, we need to investigate several orthogonal projections. From (2.9), it is evident that f_N is the orthogonal projection of f upon \mathfrak{R}_N with respect to the weighted inner product (2.2). According to Theorem 2.1, the following theorem contains the order of convergence of RC functions.

Theorem 2.2 *For any $f \in H_{w,\theta}^r(\Omega)$, $r \geq 0$, and c is a positive constant, then*

$$\|f_N - f\|_{r,w} \leq cN^{-r} \|f\|_{r,w,\theta}.$$

The proof of Theorem 2.2 can be found in [26], and for more details see Ref [36]; this theorem shows that the RC approximation has exponential convergence.

3 Formulation of the problem

In this work, m th order nonlinear ODEs containing unknown $y(x)$ and its derivatives with variable coefficients can be written in the following general form:

$$\sum_{k=0}^m \sum_{r=0}^n P_{k,r}(x) y^r(x) y^{(k)}(x) + \sum_{k=1}^m \sum_{r=1}^m Q_{k,r}(x) y^{(r)}(x) y^{(k)}(x) = g(x), \tag{3.1}$$

where $y(x)$ is the unknown function, $y^{(k)}(x) = y(x)$ at $k = 0$, $y(x) \in C^m[0, L]$, and L may tend to infinity. Also, $Q_{k,r}(x), P_{k,r}(x)$, and $g(x)$ are known functions that are defined on $[0, \infty)$ and $m, n \in \mathbb{Z}^+$, where m is the order of the differential equation and n represents the greatest power of unknown function $y(x)$.

Equation (3.1) is subjected to the conditions

$$y^{(k)}(a_k) = \mu_k, \quad k = 0, 1, \dots, m - 1, \tag{3.2}$$

where a_k and μ_k are real-valued constants.

The solution of (3.1) according to (2.9) takes the form

$$y(x) \approx \sum_{n=0}^N a_n R_n(x), \quad (3.3)$$

which is the truncated RC functions series, where N is any positive integer greater than the order of the proposed equation, i.e., $N \geq m$ and a_n are the unknown RC coefficients.

4 Fundamental matrix relations

The solution $y(x)$ of (3.1), according to the forms (1.2) and (3.3), can be written in the matrix form with its derivative $y^{(k)}(x)$ in the following form:

$$y(x) = R(x)A = V(x)C^T A, \quad (4.1)$$

where

$$A = [a_0 \quad a_1 \quad \cdots \quad a_N]^T.$$

And

$$y^{(k)}(x) = R^{(k)}(x)A, \quad k = 0, 1, \dots, m \leq N, \quad (4.2)$$

substituting relation (4.1) into Eq. (4.2), we get

$$y^{(k)}(x) = V^{(k)}(x)C^T A, \quad k = 0, 1, \dots, m \leq N. \quad (4.3)$$

The collocation points take the following form if the problem is defined in a semi-infinite domain:

$$x_s = \left(\frac{1 + \cos\left(\frac{s\pi}{N}\right)}{1 - \cos\left(\frac{s\pi}{N}\right)} \right), \quad s = 1, \dots, N, \quad (4.4a)$$

and at $s = 0, x_0 \rightarrow \infty$.

But if $x \in [0, L]$, where $L < \infty$, the collocation points will take the following form:

$$x_s = \frac{s}{N}L, \quad s = 0, 1, 2, \dots, N. \quad (4.4b)$$

Putting the collocation points x_s ((4.4a) or (4.4b)) in (4.3), we get

$$[y^{(k)}(x_s)] = V^{(k)}(x_s)C^T A,$$

or briefly

$$Y^{(k)} = V^{(k)}C^T A, \quad (4.5)$$

where

$$V^{(k)} = \begin{bmatrix} V^k(x_0) \\ V^k(x_1) \\ \vdots \\ V^k(x_N) \end{bmatrix} = \begin{bmatrix} (v^0(x_0))^k & (v^1(x_0))^k & \dots & (v^N(x_0))^k \\ (v^0(x_1))^k & (v^1(x_1))^k & \dots & (v^N(x_1))^k \\ \vdots & \vdots & \ddots & \vdots \\ (v^0(x_N))^k & (v^1(x_N))^k & \dots & (v^N(x_N))^k \end{bmatrix}.$$

Now, we turn to treat the matrix representation for the nonlinear terms in (3.1).

As previous, using the collocation points x_s into $y^r(x)$, we get

$$\begin{bmatrix} y^r(x_0) \\ y^r(x_1) \\ \vdots \\ y^r(x_N) \end{bmatrix} = \begin{bmatrix} y(x_0) & 0 & \dots & 0 \\ 0 & y(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y(x_N) \end{bmatrix}^{r-1} \begin{bmatrix} y(x_0) \\ y(x_1) \\ \vdots \\ y(x_N) \end{bmatrix} = (\tilde{Y})^{r-1} Y$$

and

$$\tilde{Y} = \tilde{V} C^T A, \tag{4.6}$$

where

$$\tilde{V} = \begin{bmatrix} V(x_0) & 0 & \dots & 0 \\ 0 & V(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V(x_N) \end{bmatrix}, \quad C^T = \begin{bmatrix} C^T & 0 & \dots & 0 \\ 0 & C^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C^T \end{bmatrix},$$

$$A = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_k \end{bmatrix}.$$

In addition, substituting x_s into the nonlinear term $y^{(r)}y^{(k)}$, we get

$$\begin{bmatrix} y^{(r)}(x_0)y^{(k)}(x_0) \\ y^{(r)}(x_1)y^{(k)}(x_1) \\ \vdots \\ y^{(r)}(x_N)y^{(k)}(x_N) \end{bmatrix} = \begin{bmatrix} y^{(r)}(x_0) & 0 & \dots & 0 \\ 0 & y^{(r)}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y^{(r)}(x_N) \end{bmatrix} \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix} = \tilde{Y}^{(r)} Y^{(k)},$$

by using (4.5), one gets

$$\tilde{Y}^{(r)} Y^{(k)} = \tilde{V}^{(r)} C^T A V^{(k)} C^T A. \tag{4.7}$$

Substituting x_s into Eq. (3.1), we get

$$\sum_{k=0}^m \sum_{r=0}^n P_{k,r} (\tilde{Y})^r Y^{(k)} + \sum_{k=1}^m \sum_{r=1}^m Q_{k,r} \tilde{Y}^{(r)} Y^{(k)} = G, \tag{4.8}$$

where

$$P_{k,r} = \begin{bmatrix} P_k(x_0) & 0 & \dots & 0 \\ 0 & P_k(x_1) & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & P_k(x_N) \end{bmatrix},$$

$$Q_{k,r} = \begin{bmatrix} Q_{k,r}(x_0) & 0 & \dots & 0 \\ 0 & Q_{k,r}(x_1) & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & Q_{k,r}(x_N) \end{bmatrix}, \quad G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

Using (4.5), (4.6), and (4.7) into (4.8), we get

$$\sum_{k=0}^m \sum_{r=0}^n P_{k,r} (\tilde{V} C^T A)^r V^{(k)} C^T A + \sum_{k=1}^m \sum_{r=1}^m Q_{k,r} \tilde{V}^{(r)} C^T A V^{(k)} C^T A = G. \tag{4.9}$$

Substituting the corresponding matrix $y^{(k)}(a_k)$, which depends on the RC coefficients matrix A , into (3.2) and simplifying the result, we obtain

$$V^{(k)}(a_k) C^T A = \mu_k. \tag{4.10}$$

5 Main results

The fundamental matrix Eq. (4.9) for problem (3.1) transforms to a system of nonlinear algebraic equations for $(N + 1)$ unknown.

Equation (4.9) can be written briefly as follows:

$$\psi A = G, \tag{5.1}$$

where

$$\psi = \sum_{k=0}^m \sum_{r=0}^n P_{k,r} (\tilde{V} C^T A)^r V^{(k)} C^T + \sum_{k=1}^m \sum_{r=1}^m Q_{k,r} \tilde{V}^{(r)} C^T A V^{(k)} C^T,$$

the matrix form for the subjected conditions (3.2), by using (4.10), is in the following form:

$$U_i A = \mu_i; \quad \text{or} \quad [U_i; \mu_i], \quad i = 0, 1, \dots, m - 1, \tag{5.2}$$

where

$$U_i = V^{(k)}(a_k).$$

Then the solution of (3.1) under the subjected conditions (3.2) can be found by replacing the produced Eqs. (5.2) with the last equations of system (5.1). Therefore, we get $(N + 1)$ nonlinear algebraic equations in terms of RC coefficient. This nonlinear system can be solved with the aid of a suitable solver. The well-known Newton iterative method can be used here with 100 iterations. Therefore, the corresponding approximate solution $y(x)$ can be obtained.

6 Test examples

Two examples are considered to illustrate the effectiveness and accuracy properties of the proposed method. The calculations are carried out on the PC using our written codes using Mathematica 7.0.

Example 6.1 Consider the nonlinear equation

$$f^{(2)}(x) + f^2(x)f^{(1)}(x) + (x + 1)f^5(x) = \frac{2}{(1 + x)^3}, \quad x \in [0, 4], \tag{6.1}$$

with the following subjected conditions:

$$f(0) = 1, \quad f'(4) = -\frac{1}{25}.$$

The collocation points at $N = 2$ are $x_0 = 0, x_1 = 1.\bar{3}, x_2 = 2.\bar{6}, x_3 = 4$.

The fundamental matrix Eq. (4.9) corresponding to (6.1) becomes

$$\{V^{(2)}C^T + (\tilde{V}C^TA)^2V^{(1)}C^T + P_{0,5}(\tilde{V}C^TA)^4VC^T\}A = G,$$

where $\tilde{V}, C^T, A, V^{(2)}, C^T, V^{(1)}, V, G,$ and $P_{0,5}$ are matrices given as follows:

$$\begin{aligned} \tilde{V} &= \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{9}{25} \end{bmatrix}, & V^{(2)} &= \begin{bmatrix} 0 & -4 & 16 \\ 0 & \frac{-4}{27} & 0 \\ 0 & \frac{-4}{125} & \frac{-16}{625} \end{bmatrix}, \\ C^T &= \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, & A &= \begin{bmatrix} a_0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & a_0 & 0 \\ 0 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_0 \\ 0 & 0 & a_1 \\ 0 & 0 & a_2 \end{bmatrix}, \\ G &= \begin{bmatrix} 2 \\ \frac{2}{27} \\ \frac{2}{127} \end{bmatrix}, & P_{0,5} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, & C^T &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \\ V^{(1)} &= \begin{bmatrix} 0 & 2 & 4 \\ 0 & \frac{2}{9} & \frac{4}{27} \\ 0 & \frac{2}{25} & \frac{12}{125} \end{bmatrix}, & V &= \begin{bmatrix} 1 & -1 & 1 \\ 1 & \frac{1}{3} & \frac{1}{9} \\ 1 & \frac{3}{5} & \frac{9}{25} \end{bmatrix}. \end{aligned}$$

We get the RC coefficient as

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}^T,$$

Table 1 Absolute errors for the RCC method at different N

x_j	e_5	e_8	e_{10}
0.0	0	0	0
0.1	8.09945×10^{-6}	2.10677×10^{-7}	7.75121×10^{-8}
0.2	7.19997×10^{-5}	1.36228×10^{-6}	4.47100×10^{-7}
0.3	2.13045×10^{-4}	3.45067×10^{-6}	1.09107×10^{-6}
0.4	4.16020×10^{-4}	6.35635×10^{-6}	1.97909×10^{-6}
0.5	6.67314×10^{-4}	9.99044×10^{-6}	3.08286×10^{-6}
0.6	9.70708×10^{-4}	1.42644×10^{-5}	4.37683×10^{-6}
0.7	1.34619×10^{-3}	1.91020×10^{-5}	5.83775×10^{-6}
0.8	1.82307×10^{-3}	2.44326×10^{-5}	7.44486×10^{-6}
0.9	2.43308×10^{-3}	3.02070×10^{-5}	9.16450×10^{-6}
1.0	3.20542×10^{-3}	3.64697×10^{-5}	1.08977×10^{-5}

Table 2 Comparison of the L_2, L_∞ error norms

	L_2		L_∞	
	Proposed method	Method in [37]	Proposed method	Method in [37]
$N = 5$	3.24915×10^{-5}	7.04723×10^{-6}	3.20542×10^{-3}	1.19252×10^{-4}
$N = 8$	4.20596×10^{-9}	1.58113×10^{-7}	3.64697×10^{-5}	4.45095×10^{-6}
$N = 10$	3.75552×10^{-10}	4.24679×10^{-12}	1.08977×10^{-5}	5.24856×10^{-12}

therefore, the solution is

$$f(x) = \sum_{n=0}^5 a_n R_n(x) = \frac{1}{2}R_0 - \frac{1}{2}R_1,$$

after simplifying

$$f(x) = (x + 1)^{-1},$$

which represents the exact solution of (6.1).

Example 6.2 Consider the nonlinear ODE [37]

$$f'''(x) + f''(x) + f'(x)f(x) = -e^{-2x}, \quad x \in [0, 1], \tag{6.2}$$

subject to the initial conditions

$$f(0) = 1, \quad f'(0) = -1, \quad f''(0) = 1.$$

The exact solution of the present problem (6.2) is $f(x) = e^{-x}$. In Table 1 the comparison of absolute errors e_N of the presented method at $N = 5, 8,$ and 10 is listed. Also, Table 2 contains a comparison of the error norms (L_2 and L_∞ [29]) for the RCC method and the shifted Chebyshev collocation method [37].

7 Applications

In this section four applications are introduced as nonlinear real life models, solved using the proposed technique. The introduced applications are defined on finite and infinite domains as nonlinear differential equations. The proposed method deals with the boundary value problems defined on the semi-infinite domain without divergence.

Table 3 A comparison between the RCC method and other existing methods with the values for $f''(0)$

λ	Runge–Kutta	HAM [6]	RCT [27]	Method in [38]	RCC method $N = 26$
0	-0.76854	-0.77363	-0.76600	-0.769	-0.768659
1/4	-0.88498	-0.88800	-0.88830	-	-0.885073
1/3	-0.92101	-0.92433	-0.92100	-0.921	-0.921085
1/2	-0.98956	-0.99382	-0.98581	-0.992	-0.989664
3/4	-1.08518	-1.08840	-1.08598	-	-1.08526
1	-1.17372	-1.17686	-1.17040	-	-1.17375

Table 4 Comparison between the RCC method and other existing methods for $f'(0)$ with $\lambda = 0$

η	Runge–Kutta	RCT [27]	HAM [6]	Proposed method	
				$N = 26$	$N = 30$
0	1.0000	1.0000	0.9999	1.0000	1.0000
0.2	0.8478	0.8477	0.8474	0.8477	0.8478
0.4	0.7036	0.7036	0.7028	0.7036	0.7036
0.6	0.5733	0.5732	0.5720	0.5732	0.5732
0.8	0.4599	0.4598	0.4583	0.4598	0.4598
1	0.3643	0.3641	0.3623	0.3641	0.3641
1.2	0.2855	0.2853	0.2833	0.2854	0.2854
1.4	0.2218	0.2218	0.2200	0.2218	0.2218
1.6	0.1713	0.1713	0.1698	0.1713	0.1713
1.8	0.1315	0.1316	0.1301	0.1316	0.1316
2	0.1007	0.1007	0.0991	0.1007	0.1007

Application 7.1 In this application the proposed method is introduced to solve NCDF about a vertical full cone embedded in PM with a prescribed wall temperature [6, 27]. Consider the governing equation of fluid flow and heat transfer of full cone embedded in PM that can be written by the following third-order nonlinear ODE, with boundary conditions that naturally have infinity:

$$f''' + \left(\frac{\lambda + 3}{2}\right)ff'' - \lambda(f')^2 = 0, \tag{7.1}$$

for wall temperature the boundary conditions are

$$f(0) = 0, \quad f'(0) = 1 \quad \text{and} \quad f'(\eta) = 0 \quad \text{at} \quad \eta \rightarrow \infty.$$

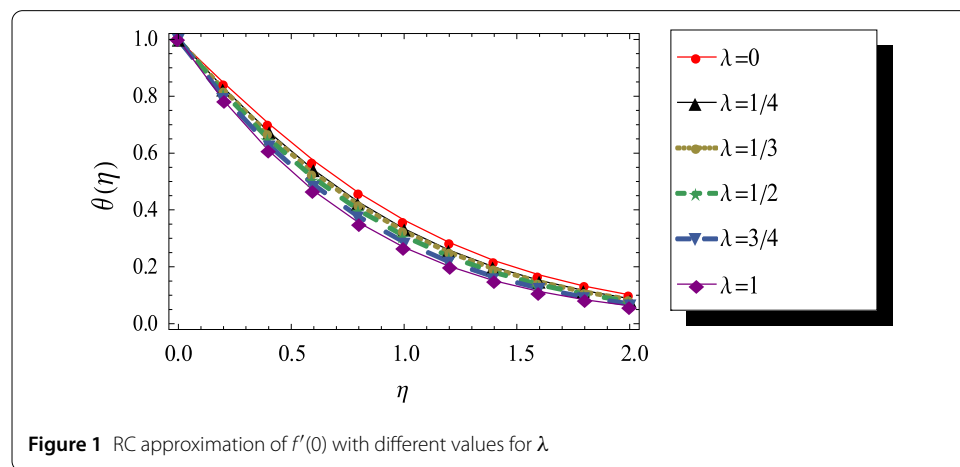
When the RCC method is applied for $N = 26$, the approximate solution for $f(\eta)$ is given. Table 3 shows the comparison of the approximate values for $f''(0)$ of the RCC method at $N = 26$ with previous results obtained from different methods. Our obtained results are close to the ones of the following methods: fourth-order Runge–Kutta method, homotopy analysis method [6], and RCT method [27] along the given domain. Also, the results for $f'(0)$ are shown in Tables 4 and 5 with two selected λ as 0 and 1/2, and the comparison between results of Runge–Kutta, [6], and [27] methods is given. Figure 1 shows the RCC approximation of $f'(0)$ with different values for λ .

Application 7.2 The Riccati equation [39]

$$f'(x) - 2f(x) + f^2(x) = 1, \quad x \in [0, 1], \tag{7.2}$$

Table 5 Comparison between RCC and other methods for $f'(0)$ with $\lambda = 1/2$

η	Runge–Kutta	RCT [27]	HAM [6]	Proposed method	
				$N = 26$	$N = 30$
0	1.0000	0.9999	1.0004	1.0000	1.0000
0.2	0.8130	0.8130	0.8125	0.8129	0.8129
0.4	0.6500	0.6500	0.6491	0.6499	0.6499
0.6	0.5124	0.5125	0.5111	0.5124	0.5124
0.8	0.3994	0.3994	0.3977	0.3993	0.3993
1	0.3084	0.3084	0.3063	0.3084	0.3084
1.2	0.2364	0.2364	0.2347	0.2364	0.2364
1.4	0.1802	0.1802	0.1785	0.1802	0.1802
1.6	0.1367	0.1367	0.1352	0.1368	0.1367
1.8	0.1034	0.1034	0.1020	0.1034	0.1034
2	0.0780	0.0779	0.0763	0.0780	0.0780



with the subjected initial condition $f(0) = 0$, and the exact analytical solution is

$$f(x) = 1 + \sqrt{2} \tanh \left[\sqrt{2}x + \frac{1}{2} \log \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \right].$$

By the RCC method, the solution of the Riccati equation is presented for deferent values of N . In Table 6 the comparison of the L_2 error norm of the present technique for deferent values of N ($N = 5, 8,$ and 12) with the other methods is given. We substituted by the Chebyshev coefficient [40] and the relation, which give us Adomian and homotopy perturbation solutions [39]. The RCC method is compared with the exact solution at $N = 5, 8,$ and 12 in Fig. 2. Also, the RCC solution at $N = 5$ is compared with the exact solution, shifted Chebyshev collection method [40], Adomian and homotopy perturbation solutions [39] in Fig. 3. It is clear that present numeric results are more convergent to the exact solution, especially, when $x > 1$, while the other methods are divergent.

We have to observe that the shifted Chebyshev domain is defined in the interval $[0, 1]$, therefore, we find it is divergent out from its domain [40]. Although, Adomian and homotopy perturbation methods defend in whole numbers, from Fig. 3 we see that the Adomian method is divergent when $x > 1$. Also, the homotopy perturbation method diverges when $x > 2$, while the RCC method is convergent along the interval.

Table 6 Comparison of the L_2 , error norm

	L_2 proposed method	L_2 shift Chebyshev [40]	L_2 Adomian [39]	L_2 homotopy perturbation [39]
$N=5$	1.13349×10^{-4}	9.41458×10^{-5}	3.86133×10^{-2}	–
$N=8$	2.01918×10^{-5}	–	–	2.93357×10^{-1}
$N=10$	3.43867×10^{-7}	–	–	–

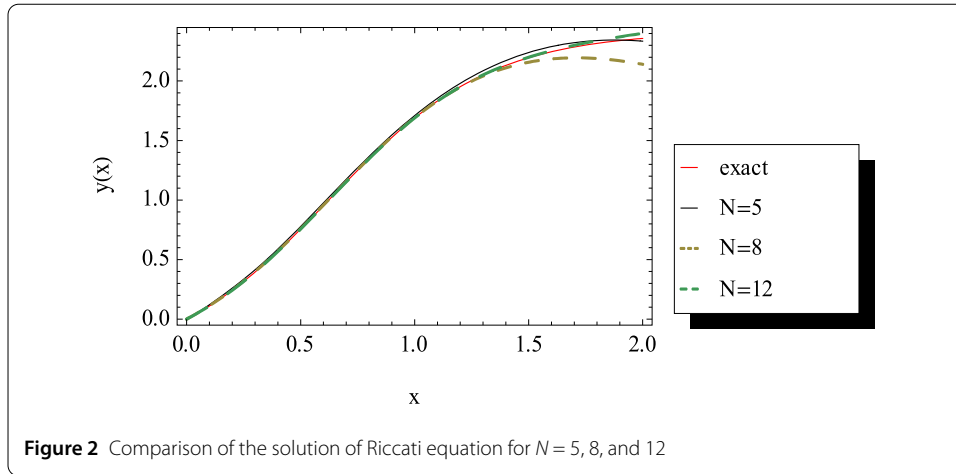


Figure 2 Comparison of the solution of Riccati equation for $N = 5, 8,$ and 12

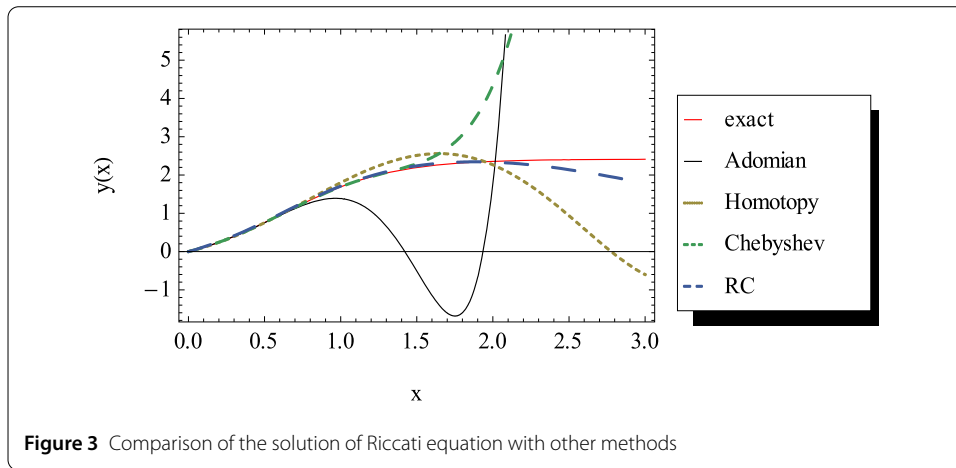


Figure 3 Comparison of the solution of Riccati equation with other methods

Application 7.3 An application of the RCC method is approximating bio-mathematical problems of continuous population models for single and interacting species(CPM). The logistic growth model in a population of CPM [41] is nonlinear ODE of the following form:

$$\begin{cases} \frac{df}{d\tau} = f(1-f), \\ f(0) = 2, \end{cases} \tag{7.3}$$

with $\frac{N_0}{K} = 2$, and the analytical solution is $f(\tau) = \frac{2}{2-e^{-\tau}}$.

The approximate solution for $f(\tau)$ is obtained by applying the presented method at $N = 10$ by the truncated RC series. Table 7 shows the comparison of the absolute errors for the presented method at $N = 10$ and other methods [42–45], where the comparison shows that our method obtains best accuracy along the interval $[0, 1]$. The error functions for

Table 7 Comparing absolute errors for pervious work

	Bessel collocation method [42]	Shifted Chebyshev collocation method [43]	Adomian decomposition method [44]	He's homotopy perturbation method [45]	Present method
τ_i	e_{10}	e_{10}	e_8	e_8	e_{10}
0.0	0.0	0.0	0.0	0.0	2.21096×10^{-8}
0.2	6.5844×10^{-7}	4.75702×10^{-8}	1.5521×10^{-5}	1.8643×10^{-4}	1.44351×10^{-8}
0.4	4.4035×10^{-7}	6.02884×10^{-7}	6.4928×10^{-3}	1.9497×10^{-2}	9.92748×10^{-9}
0.6	2.9727×10^{-7}	7.12326×10^{-7}	2.1100×10^{-1}	2.8160×10^{-1}	7.10665×10^{-9}
0.8	1.4041×10^{-7}	3.90601×10^{-7}	2.433800	1.827000	3.45884×10^{-9}
1.0	2.3094×10^{-5}	1.46765×10^{-5}	15.99100	7.682900	2.21096×10^{-8}

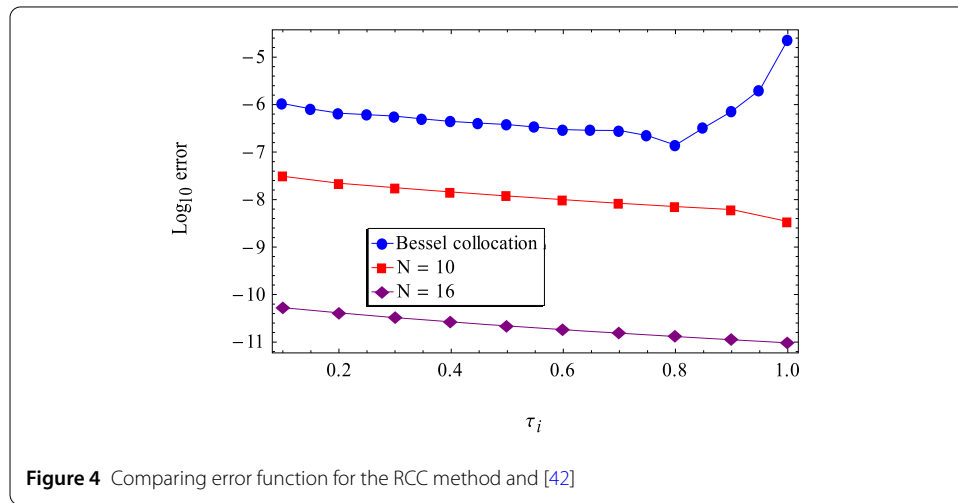


Figure 4 Comparing error function for the RCC method and [42]

the RCC method at $N = 10$ and 16 are compared with the Bessel collocation method [42] in Fig. 4; it is clear that our results are more accurate.

Application 7.4 We apply the RCC method for the Lane–Emden equation (LEE) which has the general form

$$y''(x) + \frac{\alpha}{x}y(x) + q(x)p(y) = g(x), \quad x \geq 0, \tag{7.4}$$

with the initial conditions $y(0) = \lambda, y'(0) = \beta$, where α, β , and λ are appropriate constants and $q(x), p(y)$, and $g(x)$ are given functions. The forms of $p(y)$ give us various types of LEE which model several phenomena. In this application we study deferent cases of LEE types.

The standard LEE when $q(x) = 1, p(y) = y^m, \lambda = 1$, and $\beta = 0$, we get the standard LEE that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [46–51]

$$y''(x) + \frac{2}{x}y'(x) + y^m = 0, \quad x \geq 0, \tag{7.5}$$

Table 8 Comparing numerical results of the RCC method with Runge–Kutta of fourth order at $m = 5$

x	Exact values	Runge–Kutta	RCC method	
			N = 20	N = 25
0.0	1	1	1	1
0.2	0.9933992678	0.9933992543126849	0.9933992939	0.9933992684
0.4	0.9743547036	0.9743546989480394	0.9743547291	0.9743547043
0.6	0.9449111825	0.9449111812575188	0.9449112047	0.9449111830
0.8	0.9078412990	0.9078412996319103	0.9078413170	0.9078412994
1.0	0.8660254037	0.8660254054870272	0.8660254037	0.8660254041
1.2	0.8219949365	0.8219949387903671	0.8219949365	0.8219949367
1.4	0.7777137711	0.7777137735378965	0.7777137710	0.7777137716
1.6	0.7345531603	0.7345531628136189	0.7345531609	0.7345531712
1.8	0.6933752452	0.6933752476639488	0.6933752245	0.6933753566
2.0	0.6546536707	0.6546536728956738	0.6546535327	0.6546543517

Table 9 Comparing absolute errors at $m = 5$

x	Runge–Kutta	RCC method	
		e_{20}	e_{25}
0.0	0	0	0
0.2	1.3486×10^{-8}	2.60697×10^{-8}	6.07152×10^{-10}
0.4	4.7444×10^{-9}	2.53963×10^{-8}	5.69494×10^{-10}
0.6	1.2655×10^{-9}	2.21944×10^{-8}	4.90154×10^{-10}
0.8	6.287×10^{-10}	1.79881×10^{-8}	3.92929×10^{-10}
1.0	1.7025×10^{-9}	1.35314×10^{-8}	2.92358×10^{-10}
1.2	2.2635×10^{-9}	9.29598×10^{-9}	2.02582×10^{-10}
1.4	2.49×10^{-9}	5.48337×10^{-9}	5.82489×10^{-10}
1.6	2.502×10^{-9}	6.28875×10^{-10}	1.08966×10^{-8}
1.8	2.3824×10^{-9}	2.06909×10^{-8}	1.11352×10^{-7}
2.0	2.1877×10^{-9}	1.37968×10^{-7}	6.81077×10^{-7}

with the initial conditions $y(0) = 1, y'(0) = 0$, where m is a real constant $m \geq 0$, if $m = 0, 1$, and 5, we get the analytical solution

$$y(x) = 1 - \frac{1}{3!}x^2, \quad y(x) = \frac{\sin(x)}{x} \quad \text{and} \quad y(x) = \left(1 + \frac{x^2}{3}\right)^{-1/2},$$

respectively, the other values of m have no analytical solution. Now, we apply the RCC method to deal with the standard LEE for $m = 3, 4$, and 5. Table 8 shows the comparing of numerical results of the RCC method, the exact solution, and Runge–Kutta of fourth order at $m = 5$, while Table 9 compares absolute errors. Tables 10 and 11 compare the approximate solution of the RCC method with Runge–Kutta of fourth order at $m = 4$ and 3 respectively.

We note that the results at $m = 5$ with the exact solution for the proposed method are better than the given results by Runge–Kutta at $N = 25$ as Tables 8 and 9 show, so we expected that the results are also stable and better than those of fourth-order Runge–Kutta in other cases.

8 Conclusion

In this work, the rational Chebyshev collection method is applied to solve general non-linear ordinary differential equations (ODEs) with variable coefficients and given conditions. The proposed method is applied as a matrix discretization to treat the nonlinear

Table 10 Comparing approximate solution of the RCC method with Runge–Kutta of fourth order at $m = 4$

x	Runge–Kutta	RCC method	
		$N = 20$	$N = 25$
0.0	1	1	1
0.2	0.9933861998	0.9933861701	0.9933862093
0.4	0.9741584037	0.9741583659	0.9741584049
0.6	0.944011289	0.9440112521	0.9440112873
0.8	0.9053459238	0.9053458911	0.9053459209
1.0	0.8608138133	0.8608137861	0.8608138098
1.2	0.8129249433	0.8129249221	0.8129249398
1.4	0.7637988358	0.7637988205	0.7637988312
1.6	0.7150639513	0.7150639437	0.7150639123
1.8	0.6678666657	0.6678666855	0.6678662631
2.0	0.6229407738	0.62294092500	0.6229382065

Table 11 Comparing approximate solution of the RCC method with Runge–Kutta of fourth order at $m = 3$

x	Runge–Kutta	RCC method	
		$N = 20$	$N = 25$
0.0	1	1	1
0.2	0.99337307946	0.9933731281	0.9933730891
0.4	0.97395825015	0.9739582909	0.9739582516
0.6	0.94307317012	0.9430732051	0.9430731688
0.8	0.90267208835	0.9026721177	0.9026720857
1.0	0.85505756895	0.8550575927	0.8550575657
1.2	0.80259192167	0.8025919339	0.8025919183
1.4	0.74746481001	0.7474648231	0.7474648054
1.6	0.69154411316	0.6915439103	0.6915440763
1.8	0.63630944306	0.6363094342	0.6363090646
2.0	0.58285051702	0.5828504530	0.5828481005

ODEs which take a block matrix form that is transformed to a nonlinear algebraic system of equations. The order of convergence for RC functions has been discussed. The technique has been tested and verified by two examples, then applied to natural convection of Darcian fluid (NCDF) about a vertical full cone embedded in porous media (PM) with a prescribed wall temperature which is a third-order nonlinear ODE with condition naturally tending to infinity. The second application is the Riccati equation, and the third one is continuous population models for single and interacting species (CPM). The standard Lane–Emden equation (LEE) is the last application, and we compare our results with other existing methods to study the applicability and accuracy.

Acknowledgements

This work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11701176, 11626101, 11601485). Praveen Agarwal is very thankful to the SERB (project TAR/2018/000001), DST(project DST/INT/DAAD/P-21/2019 and INT/RUS/RFBR/308), and NBHM (DAE)(project 02011/12/2020 NBHM(R.P)/R\&D II/7867) for their necessary support.

Funding

This work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11701176, 11626101, 11601485).

Availability of data and materials

All authors declare that no data were used to support this study.

Conflicts of interest

All authors declare that they have no conflicts of interest.

Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authors' contributions

The authors contributed equally to the writing of this paper. They read and approved the final manuscript.

Author details

¹Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt. ²Mathematics Department, Faculty of Science, Menoufia University, Shebein El-Kom, Egypt. ³Department of Mathematics, Anand International College of Engineering, Jaipur, 303012, India. ⁴International Center for Basic and Applied Sciences, Jaipur, 302029, India. ⁵Department of Mathematics, Harish-Chandra Research Institute, Allahabad, 211 019, India. ⁶Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE. ⁷Department of Mathematics, Huzhou University, Huzhou, 313000, P.R. China. ⁸Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha, 410114, China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 11 September 2020 Accepted: 26 June 2021 Published online: 13 July 2021

References

1. Murray, J.D.: *Mathematical Biology: I. An Introduction* (Vol. 17). Springer, Berlin (2007)
2. Chinviriyasit, S., Chinviriyasit, W.: Numerical modeling of an SIR epidemic model with diffusion. *Appl. Math. Comput.* **216**(2), 395–409 (2010)
3. Youssef, C.B., Goma, G., Olmos-Dichara, A.: Kinetic modeling of lactobacillus casei ssp. Rhamnosus growth and lactic acid production in batch cultures under various medium conditions. *Biotechnol. Lett.* **27**(22), 1785–1789 (2005)
4. Kim, M.S., Cha, D., Lee, K.M., Lee, H.J., Kim, T., Lee, C.: Modeling of ozone decomposition, oxidant exposures, and the abatement of micro pollutants during ozonation processes. *Water Res.* **169**, 115230 (2020)
5. Sprott, J.C.: Dynamical models of happiness. *Nonlinear Dyn. Psychol. Life Sci.* **9**(1), 23–36 (2005)
6. Sohoul, A.R., Domairry, D., Famouri, M., Mohsenzadeh, A.: Analytical solution of natural convection of Darcian fluid about a vertical full cone embedded in porous media prescribed wall temperature by means of HAM. *Int. Commun. Heat Mass Transf.* **35**(10), 1380–1384 (2008)
7. Marzban, H.R., Hoseini, S.M., Razzaghi, M.: Solution of Volterra's population model via block-pulse functions and Lagrange-interpolating polynomials. *Math. Methods Appl. Sci.* **32**(2), 127–134 (2009)
8. Ramos, J.I.: Linearization techniques for singular initial-value problems of ordinary differential equations. *Appl. Math. Comput.* **161**(2), 525–542 (2005)
9. Caglar, H.N., Caglar, S.H., Twizell, E.H.: The numerical solution of fifth-order boundary value problems with sixth-degree B-spline functions. *Appl. Math. Lett.* **12**(5), 25–30 (1999)
10. Daşcıoğlu, A., Yaslan, H.: The solution of high-order nonlinear ordinary differential equations by Chebyshev polynomials. *Appl. Math. Comput.* **217**(2), 5658–5666 (2011)
11. Wazwaz, A.M.: A new method for solving singular initial value problems in the second-order ordinary differential equations. *Appl. Math. Comput.* **128**(1), 45–57 (2002)
12. Adomian, G.: *Nonlinear Stochastic Operator Equations*. Acad. Press, Can Diego (1986)
13. Aminikhah, H., Hemmatnezhad, M.: An efficient method for quadratic Riccati differential equation. *Commun. Nonlinear Sci. Numer. Simul.* **15**(4), 835–839 (2010)
14. Khan, N.A., Ahmed, S., Hameed, T., Raja, M.A.Z.: Expedite homotopy perturbation method based on metaheuristic technique mimicked by the flashing behavior of fireflies. *AIMS Math.* **4**(4), 1114 (2019)
15. Wazwaz, A.M.: The numerical solution of fifth-order boundary value problems by the decomposition method. *J. Comput. Appl. Math.* **136**(1–2), 259–270 (2001)
16. Aziz, S.H., Rasheed, M., Shihab, S.: New properties of modified second kind Chebyshev polynomials. *J. Southwest Jiaotong Univ.* **55**, 3 (2020)
17. Agarwal, P., Attary, M., Maghasedi, M., Kumam, P.: Solving higher-order boundary and initial value problems via Chebyshev-spectral method: application in elastic foundation. *Symmetry* **12**(6), 987 (2020)
18. Khan, N.A., Razaq, O.A., Hameed, T., Ayaz, M.: Numerical scheme for global optimization of fractional optimal control problem with boundary conditions. *Int. J. Innov. Comput. Inf. Control* **13**(5), 1669–1679 (2017)
19. Khan, N.A., Hameed, T., Razaq, O.A., Ayaz, M.: Intelligent computing for Duffing-harmonic oscillator equation via the bio-evolutionary optimization algorithm. *J. Low Freq. Noise Vib. Act. Control* **38**(3–4), 1327–1337 (2019)
20. Koç, A.B., Kurnaz, A.: A new kind of double Chebyshev polynomial approximation on unbounded domains. *Bound. Value Probl.* **2013**(1), 10 (2013)
21. Ren, Y., Yu, X., Wang, Z.: Diagonalized Chebyshev rational spectral methods for second-order elliptic problems on unbounded domains. *Numer. Math., Theory Methods Appl.* **12**(1), 265–284 (2019)
22. Guo, S., Mei, L., Li, C., Zhang, Z., Li, Y.: Semi-implicit Hermite-Galerkin spectral method for distributed-order fractional-in-space nonlinear reaction-diffusion equations in multidimensional unbounded domains. *J. Sci. Comput.* **85**(1), 1–27 (2020)
23. Deniz, S., Sezer, M.: Rational Chebyshev collocation method for solving nonlinear heat transfer equations. *Int. Commun. Heat Mass Transf.* **114**, 104595 (2020)
24. Malachivskyy, P.S., Pizyur, Y.V., Malachivskyy, R.P.: Chebyshev approximation by a rational expression for functions of many variables. *Cybern. Syst. Anal.* **56**(5), 811–819 (2020)
25. Zhang, X., Boyd, J.P.: Revisiting the Thomas-Fermi equation: accelerating rational Chebyshev series through coordinate transformations. *Appl. Numer. Math.* **135**, 186–205 (2019)

26. Abbasbandy, S., Ghehsareh, H.R., Hashim, I.: An approximate solution of the MHD flow over a non-linear stretching sheet by rational Chebyshev collocation method. *UPB Sci. Bull.* **74**(4), 47–58 (2012)
27. Parand, K., Delafkar, Z., Baharifard, F.: Rational Chebyshev Tau method for solving natural convection of Darcian fluid about a vertical full cone embedded in porous media with a prescribed wall temperature. *World Acad. Sci., Eng. Technol.* **5**(8), 1186–1191 (2011)
28. Ramadan, M., Raslan, K., Nassar, M.: Solving natural convection of Darcian fluid about a vertical full cone embedded in porous media with a prescribed wall temperature is introduced using rational. *Appl. Math. Inf. Sci.* **14**(5), 1–8 (2020)
29. Ramadan, M., Baleanu, D., Nassar, M.: Highly accurate numerical technique for population models via rational Chebyshev collocation method. *Mathematics* **7**(10), 913 (2019)
30. Yalçınbas, S., Özsoy, N., Sezer, M.: Approximate solution of higher order linear differential equations by means of a new rational Chebyshev collocation method. *Math. Comput. Appl.* **15**(1), 45–56 (2010)
31. Sezer, M., Gülsu, M., Tanay, B.: Rational Chebyshev collocation method for solving higher-order linear ordinary differential equations. *Numer. Methods Partial Differ. Equ.* **27**(5), 1130–1142 (2011)
32. Abdrabou, A., Heikal, A.M., Obayya, S.S.A.: Efficient rational Chebyshev pseudo-spectral method with domain decomposition for optical waveguides modal analysis. *Opt. Express* **24**(10), 10495–10511 (2016)
33. Golbabai, A., Samadpour, S.: Rational Chebyshev collocation method for the similarity solution of two dimensional stagnation point flow. *Indian J. Pure Appl. Math.* **49**(3), 505–519 (2018)
34. Boyd, J.P.: Rational Chebyshev spectral methods for unbounded solutions on an infinite interval using polynomial-growth special basis functions. *Comput. Math. Appl.* **41**(10–11), 1293–1315 (2001)
35. Boyd, J.P.: Orthogonal rational functions on a semi-infinite interval. *J. Comput. Phys.* **70**(1), 63–88 (1987)
36. Guo, B.Y., Shen, J., Wang, Z.Q.: Chebyshev rational spectral and pseudospectral methods on a semi-infinite interval. *Int. J. Numer. Methods Eng.* **53**(1), 65–84 (2002)
37. Öztürk, Y., Gülsu, M.: The approximate solution of high-order nonlinear ordinary differential equations by improved collocation method with terms of shifted Chebyshev polynomials. *Int. J. Appl. Comput. Math.* **2**(4), 519–531 (2016)
38. Cheng, P., Le, T.T., Pop, I.: Natural convection of a Darcian fluid about a cone. *Int. Commun. Heat Mass Transf.* **12**(6), 705–717 (1985)
39. Abbasbandy, S.: Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method. *Appl. Math. Comput.* **172**(1), 485–490 (2006)
40. Akyüz-Dağcıoğlu, A., Çerdi, H.: The solution of high-order nonlinear ordinary differential equations by Chebyshev series. *Appl. Math. Comput.* **217**(12), 5658–5666 (2011)
41. Ramadan, M.A., Abd El Salam, M.A.: Spectral collocation method for solving continuous population models for single and interacting species by means of exponential Chebyshev approximation. *Int. J. Biomath.* **11**(08), 1850109 (2018)
42. Yüzbaşı, Ş.: Bessel collocation approach for solving continuous population models for single and interacting species. *Appl. Math. Model.* **36**(8), 3787–3802 (2012)
43. Ozturk, Y., Gulsu, M.: An efficient algorithm for solving nonlinear system of differential equations and applications. *New Trends Math. Sci.* **3**(3), 192 (2015)
44. Pamuk, S.: The decomposition method for continuous population models for single and interacting species. *Appl. Math. Comput.* **163**(1), 79–88 (2005)
45. Pamuk, S., Pamuk, N.: He's homotopy perturbation method for continuous population models for single and interacting species. *Comput. Math. Appl.* **59**(2), 612–621 (2010)
46. Shawagfeh, N.T.: Nonperturbative approximate solution for Lane–Emden equation. *J. Math. Phys.* **34**(9), 4364–4369 (1993)
47. Aslanov, A.: Determination of convergence intervals of the series solutions of Emden–Fowler equations using polytropes and isothermal spheres. *Phys. Lett. A* **372**(20), 3555–3561 (2008)
48. Parand, K., Dehghan, M., Rezaei, A.R., Ghaderi, S.M.: An approximation algorithm for the solution of the nonlinear Lane–Emden type equations arising in astrophysics using Hermite functions collocation method. *Comput. Phys. Commun.* **181**(6), 1096–1108 (2010)
49. Agarwal, P., Berdyshev, A., Karimov, E.: Solvability of a nonlocal problem with integral transmitting condition for mixed type equation with Caputo fractional derivative. *Results Math.* **71**(3–4), 1235–1257 (2017)
50. Agarwal, P., Ntouyas, S.K., Jain, S., Chand, M., Singh, G.: Fractional kinetic equations involving generalized k-Bessel function via Sumudu transform. *Alex. Eng. J.* **57**(3), 1937–1942 (2018)
51. Agarwal, P., Deniz, S., Jain, S., Alderremy, A.A., Aly, S.: A new analysis of a partial differential equation arising in biology and population genetics via semi analytical techniques. *Phys. A, Stat. Mech. Appl.* **542**, 122769 (2020)

Submit your manuscript to a SpringerOpen® journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)
