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Common fixed point theorem on Proinov type mappings via simulation function

Badr Alqahtani[1], Sara Salem Alzaid[1], Andreea Fulga[2] and Seher Sultan Yeşilkaya[3*]

*Correspondence:
yesilkaya@tdmu.edu.vn;
sultanseher20@gmail.com
[3]Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam
Full list of author information is available at the end of the article

Abstract

In this paper, we aim to discuss the common fixed point of Proinov type mapping via simulation function. The presented results not only generalize, but also unify the corresponding results in this direction. We also consider an example to indicate the validity of the obtained results.

MSC: Primary 47H10; secondary 54H25

Keywords: Proinov type mappings; Simulation function; Common fixed point

1 Introduction and preliminaries

Fixed point theory is one of the dynamic research topics of the last decades due to its vast application potential on several distinct disciplines; see e.g. [1–4]. Very recently, Proinov [5] introduced new classes of auxiliary function to propose a new metric fixed point theorem that covers many existing fixed point theorems, mostly having appeared in the last decades. Proinov [5] also showed that recently declared theorems are in fact equivalent to the special cases of Skof's theorem [6]. Recently, Proinov type contractions have attracted the attention of some authors; see e.g. [7–9] On the other hand, another interesting improvement was reported in 2015: simulation functions were proposed first by Khojasteh *et al.* [10] to unify some well-known fixed point theorems. This approach has been considered and improved by several authors; see e.g. [11–20].

In this paper, we combine the notions of simulation functions and Proinov type contraction to get a more general framework to guarantee the existence of a fixed point. We investigate the common fixed point of new types mapping under this construction in the context of complete metric space.

We shall first recall the notations we shall use: $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ for the reals, nonnegative real numbers and natural numbers, $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\} = [0,\infty)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\Theta = \{\vartheta : (0,\infty) \to \mathbb{R}\}$.

Definition 1 (See [10]) A function $\eta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is called a *simulation function* if the following conditions hold:

(η_1) $\eta(t,s) < s - t$ for all $t,s \in \mathbb{R}_0^+$;

(η_2) if $\{t_m\}, \{s_m\}$ in $(0,\infty)$ are two sequences such that $\lim_{m\to\infty} t_m = \lim_{m\to\infty} s_m > 0$, then

$$\limsup_{m\to\infty} \eta(t_m, s_m) < 0. \tag{1.1}$$

For the set of all functions simulation functions η, we employ the symbol Z.

Theorem 2 ([5]) *Let the metric space (X, d) and the mapping $P : X \to X$ such that*

$$\vartheta\big(\mathsf{d}(P\chi, Py)\big) \le \psi\big(\mathsf{d}(\chi, y)\big) \quad \textit{for all } \chi, y \in X \textit{ with } \mathsf{d}(P\chi, Py) > 0,$$

where $\vartheta, \psi : (0,\infty) \to \mathbb{R}$ are such that the following conditions hold:

(a) $\psi(s) < \vartheta(s)$, *for any* $s > 0$;

(b) $\inf_{s>s_0} \vartheta(s) > -\infty$, *for any* $s_0 > 0$;

(c) *if the sequences* $\{\vartheta(a_m)\}$ *and* $\{\psi(a_m)\}$ *are convergent with the same limit and* $\{\vartheta(a_m)\}$ *is strictly decreasing, then* $a_m \to 0$ *as* $m \to \infty$;

(d) $\limsup_{s\to s_0+} \psi(s) < \liminf_{s\to s_0} \vartheta(s)$ *or* $\limsup_{s\to s_0} \psi(s) < \liminf_{s\to s_0+} \vartheta(s)$ *for any* $s_0 > 0$;

(e) $\limsup_{s\to 0+} \psi(s) < \liminf_{s\to s_0} \vartheta(s)$ *for any* $s_0 > 0$.

Then the mapping P possesses exactly one fixed point.

We mention here the following lemmas which will be useful in the sequel.

Lemma 3 ([21]) *Let $\{\chi_m\}$ be a sequence in a metric space (X, d) such that $\lim_{m\to\infty} \mathsf{d}(\chi_m, \chi_{m+1}) = 0$. If the sequence $\{\chi_{2m}\}$ is not Cauchy then there exist $\mathsf{e}_0 > 0$ and the sequences $\{m_l\}$, $\{p_l\}$ of positive integers such that m_l is the smallest index for which $m_l > p_l > l$, $\mathsf{d}(\chi_{2p_l}, \chi_{2m_l}) \ge \mathsf{e}_0$ and*

$$\begin{aligned}\lim_{l\to\infty} \mathsf{d}(\chi_{2p_l}, \chi_{2m_l+1}) &= \lim_{l\to\infty} \mathsf{d}(\chi_{2p_l-1}, \chi_{2m_l+1}) \\ &= \lim_{l\to\infty} \mathsf{d}(\chi_{2p_l}, \chi_{2m_l}) = \lim_{l\to\infty} \mathsf{d}(\chi_{2p_l-1}, \chi_{2m_l}) = \mathsf{e}_0.\end{aligned} \tag{1.2}$$

Lemma 4 ([5]) *For $\vartheta : (0,\infty) \to \mathbb{R}$ the following conditions are equivalent*:

(1) $\inf_{s>e} \vartheta(s) > -\infty$ *for every* $e > 0$.

(2) $\liminf_{s>e+} \vartheta(s) > -\infty$ *for every* $e > 0$.

(3) $\liminf_{m\to\infty} \vartheta(a_m) = -\infty$ *implies* $\lim_{m\to\infty} a_m = 0$.

2 Main results

In what follows, we shall consider that $P, Q : X \to X$ and $\mathsf{m}, \mathsf{r}_1, \mathsf{r}_2 : X \times X \to \mathbb{R}_0^+$ are defined as

$$\mathsf{m}(\chi, y) = \max\left\{\mathsf{d}(\chi, y), \mathsf{d}(\chi, P\chi), \mathsf{d}(y, Qy), \frac{\mathsf{d}(\chi, Qy) + \mathsf{d}(y, P\chi)}{2}\right\}, \quad \chi, y \in X; \tag{2.1}$$

$$\mathsf{r}_1(\chi, y) = \max\left\{\begin{array}{l}\mathsf{d}(\chi, y), \frac{\mathsf{d}(y,Qy)(1+\mathsf{d}(\chi,P\chi))}{1+\mathsf{d}(\chi,y)}, \\ \frac{\mathsf{d}(\chi,Qy)+\mathsf{d}(y,P\chi)}{2}\end{array}\right\}, \quad \chi, y \in X, \tag{2.2}$$

and

$$\mathsf{r}_2(x,y) = \max\left\{\mathsf{d}(x,y), \frac{\mathsf{d}(x,Px)\mathsf{d}(x,Qy) + \mathsf{d}(y,Qy)\mathsf{d}(y,Px)}{\max\{\mathsf{d}(x,Qy),\mathsf{d}(y,Px)\}}\right\}, \tag{2.3}$$

for any $x, y \in X$ such that $\max\{\mathsf{d}(x,Qy),\mathsf{d}(y,Px)\} \neq 0$.

Theorem 5 *Let* (X,d) *be a complete metric space and two mappings* $P, Q : X \to X$. *Assume that there exists a function* $\eta \in Z$ *such that*

$$\eta\big(\vartheta\big(\mathsf{d}(Px,Qy)\big), \psi\big(\mathsf{m}(x,y)\big)\big) \geq 0, \quad \text{for any } x, y \in X \text{ with } \mathsf{d}(Px,Qy) > 0, \tag{2.4}$$

where $\vartheta, \psi \in \Theta$. *Then the mappings* P, Q *have a unique fixed point provided that the following conditions are satisfied*:

(a_1) $\psi(s) < \vartheta(s)$, *for any* $s > 0$;

(a_2) $\inf_{s>s_0} \vartheta(s) > -\infty$, *for any* $s_0 > 0$;

(a_3) *if* $\{a_m\}$, $\{b_m\}$ *are two convergent sequences with* $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m > 0$ *then the sequences* $\{\vartheta(a_m)\}$, $\{\vartheta(b_m)\}$ *are convergent and* $\lim_{m\to\infty} \vartheta(a_m) = \lim_{m\to\infty} \vartheta(b_m) > 0$.

Proof Let $x_0 \in X$ be an arbitrary, but fixed point and the sequence $\{x_m\}$ defined as follows:

$$x_1 = Px_0, \qquad x_2 = Qx_1, \quad \ldots, \quad x_{2m+1} = Px_{2m}, \qquad x_{2m+2} = Qx_{2m+1},$$

for each $m \in \mathbb{N}_0$. First of all, let us remark that, if there exists $m_0 \in \mathbb{N}$ such that $x_{m_0} = x_{m_0+1}$, then x_{m_0} is a fixed point of P (in the case that m_0 is even) or Q (if m_0 is odd). Moreover, supposing, for example, that x_{m_0} is a fixed point of the mapping P but is not a common fixed point of P and Q (this means $\mathsf{d}(x_{m_0}, Qx_{m_0}) > 0$), we get $\mathsf{d}(Px_{m_0}, Qx_{m_0}) > 0$ and

$$\begin{aligned} 0 &\leq \eta\big(\vartheta\big(\mathsf{d}(Px_{m_0}, Qx_{m_0})\big), \psi\big(\mathsf{m}(x_{m_0}, x_{m_0})\big)\big) \\ &= \eta\big(\vartheta\big(\mathsf{d}(x_{m_0}, Qx_{m_0})\big), \psi\big(\mathsf{m}(x_{m_0}, x_{m_0})\big)\big) \\ &< \psi\big(\mathsf{m}(x_{m_0}, x_{m_0})\big) - \vartheta\big(\mathsf{d}(x_{m_0}, Qx_{m_0})\big); \end{aligned}$$

since

$$\begin{aligned} \mathsf{m}(x_{m_0}, x_{m_0}) &= \max\left\{\begin{matrix} \mathsf{d}(x_{m_0}, x_{m_0}), \mathsf{d}(x_{m_0}, Px_{m_0}), \mathsf{d}(x_{m_0}, Qx_{m_0}), \\ \frac{\mathsf{d}(x_{m_0}, Qx_{m_0}) + \mathsf{d}(x_{m_0}, Px_{m_0})}{2} \end{matrix}\right\} \\ &= \mathsf{d}(x_{m_0}, Qx_{m_0}) \end{aligned}$$

and taking (a_1) into account we deduce that

$$\begin{aligned} 0 &\leq \psi\big(\mathsf{m}(x_{m_0}, x_{m_0})\big) - \vartheta\big(\mathsf{d}(x_{m_0}, Qx_{m_0})\big) < \vartheta\big(\mathsf{m}(x_{m_0}, x_{m_0})\big) - \vartheta\big(\mathsf{d}(x_{m_0}, Qx_{m_0})\big) \\ &= \vartheta\big(\mathsf{d}(x_{m_0}, Qx_{m_0})\big) - \vartheta\big(\mathsf{d}(x_{m_0}, Qx_{m_0})\big) = 0, \end{aligned}$$

which is a contradiction. Therefore, without loss of generality, we can suppose that $x_m \neq x_{m+1}$ for any $m \in \mathbb{N}_0$. Thus, supposing that $m = 2i$, we have $\mathsf{d}(Px_{2i}, Qx_{2i+1}) > 0$ and from

(2.4) and (η_1), we have

$$0 \leq \eta\big(\vartheta\big(\mathsf{d}(P\varkappa_{2i}, Q\varkappa_{2i+1})\big), \psi\big(\mathsf{m}(\varkappa_{2i}, \varkappa_{2i+1})\big)\big) < \psi\big(\mathsf{m}(\varkappa_{2i}, \varkappa_{2i+1})\big) - \vartheta\big(\mathsf{d}(P\varkappa_{2i}, Q\varkappa_{2i+1})\big) \tag{2.5}$$

and using (a_1) we deduce

$$\begin{aligned}\vartheta\big(\mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i+2})\big) &= \vartheta\big(\mathsf{d}(P\varkappa_{2i}, Q\varkappa_{2i+1})\big) < \psi\big(\mathsf{m}(\varkappa_{2i}, \varkappa_{2i+1})\big) < \vartheta\big(\mathsf{m}(\varkappa_{2i}, \varkappa_{2i+1})\big)\\ &= \vartheta\left(\max\left\{\begin{matrix}\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1}), \mathsf{d}(\varkappa_{2i}, P\varkappa_{2i}), \mathsf{d}(\varkappa_{2i+1}, Q\varkappa_{2i+1}),\\ \frac{\mathsf{d}(\varkappa_{2i}, Q\varkappa_{2i+1}) + \mathsf{d}(\varkappa_{2i+1}, P\varkappa_{2i})}{2}\end{matrix}\right\}\right)\\ &= \vartheta\left(\max\left\{\begin{matrix}\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1}), \mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i+2}),\\ \frac{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+2}) + \mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i+1})}{2}\end{matrix}\right\}\right)\\ &= \vartheta\big(\max\big\{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1}), \mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i+2})\big\}\big).\end{aligned} \tag{2.6}$$

In the case that there exists $i_0 \in \mathbb{N}$ such that $\mathsf{d}(\varkappa_{2i_0}, \varkappa_{2i_0+1}) \leq \mathsf{d}(\varkappa_{2i_0+1}, \varkappa_{2i_0+2})$, the inequality (2.6) leads to $\vartheta(\mathsf{d}(\varkappa_{2i_0+1}, \varkappa_{2i_0+2})) < \vartheta(\mathsf{d}(\varkappa_{2i_0+1}, \varkappa_{2i_0+2}))$, which is a contradiction. Accordingly,

$$\mathsf{m}(\varkappa_{2i}, \varkappa_{2i+1}) = \max\big\{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1}), \mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i+2})\big\} = \mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1}) \tag{2.7}$$

and then, for any even natural number m, the sequence $\{\mathsf{d}(\varkappa_m, \varkappa_{m+1})\}$ is non-increasing and positive. Of course, using the same argument, there follows a similar conclusion when m is an odd natural number. Therefore, we can find $\mathsf{D} \geq 0$ such that $\lim_{m\to\infty} \mathsf{d}(\varkappa_m, \varkappa_{m+1}) = \lim_{m\to\infty} \mathsf{m}(\varkappa_m, \varkappa_{m+1}) = \mathsf{D}$. Assuming that $\mathsf{D} > 0$ by (2.6) we have

$$\vartheta\big(\mathsf{d}(\varkappa_{m+1}, \varkappa_{m+2})\big) < \psi\big(\mathsf{m}(\varkappa_m, \varkappa_{m+1})\big) < \vartheta\big(\mathsf{d}(\varkappa_m, \varkappa_{m+1})\big), \tag{2.8}$$

which shows us that the sequence $\{\vartheta(\mathsf{d}(\varkappa_m, \varkappa_{m+1}))\}$ is decreasing and moreover, taking (a_2) into account, it is bounded below. Thus, letting $m \to \infty$ in (2.8), it follows that the sequences $\{\vartheta(\mathsf{d}(\varkappa_m, \varkappa_{m+1}))\}$ and $\{\psi(\mathsf{m}(\varkappa_m, \varkappa_{m+1}))\}$ are convergent to the same limit. Therefore, by (η_2) we get

$$\limsup_{m\to\infty} \eta\big(\vartheta\big(\mathsf{d}(\varkappa_m, \varkappa_{m+1})\big), \psi\big(\mathsf{m}(\varkappa_m, \varkappa_{m+1})\big)\big) < 0. \tag{2.9}$$

On the other hand, taking (η_1) into account, (2.4) implies

$$\eta\big(\vartheta\big(\mathsf{d}(\varkappa_m, \varkappa_{m+1})\big), \psi\big(\mathsf{m}(\varkappa_m, \varkappa_{m+1})\big)\big) \geq 0$$

and

$$\limsup_{m\to\infty} \eta\big(\vartheta\big(\mathsf{d}(\varkappa_m, \varkappa_{m+1})\big), \psi\big(\mathsf{m}(\varkappa_m, \varkappa_{m+1})\big)\big) \geq 0,$$

which contradicts (2.9). Thus,

$$\mathsf{D} = \lim_{m\to\infty} \mathsf{d}(\varkappa_m, \varkappa_{m+1}) = 0. \tag{2.10}$$

Next, we claim that the sequence $\{x_m\}$ is Cauchy. Reasoning by contradiction, if $\{x_{2m}\}$ is not Cauchy, by Lemma 3, we can find $e_0 > 0$ and the sequences $\{m_l\}$, $\{p_l\}$ of positive integers such that the equalities (1.2) hold, where m_l is smallest index for which $m_l > p_l > l$, for all $l \geq 1$. Replacing in (2.1) x by x_{2m_l} and y by x_{2p_l-1} we have

$$\begin{aligned}\mathsf{m}(x_{2m_l}, x_{2p_l-1}) &= \max\left\{\begin{matrix}\mathsf{d}(x_{2m_l}, x_{2p_l-1}), \mathsf{d}(x_{2m_l}, Px_{2m_l}), \mathsf{d}(x_{2p_l-1}, Qx_{2p_l-1}),\\ \frac{\mathsf{d}(x_{2m_l}, Qx_{2p_l-1})+\mathsf{d}(x_{2p_l-1}, Px_{2m_l})}{2}\end{matrix}\right\}\\ &= \max\left\{\begin{matrix}\mathsf{d}(x_{2m_l}, x_{2p_l-1}), \mathsf{d}(x_{2m_l}, x_{2m_l+1}), \mathsf{d}(x_{2p_l-1}, x_{2p_l}),\\ \frac{\mathsf{d}(x_{2m_l}, x_{2p_l})+\mathsf{d}(x_{2p_l-1}, x_{2m_l+1})}{2}\end{matrix}\right\}\end{aligned}$$

and taking into account (2.1) and (2.10) it follows that

$$\lim_{m\to\infty} \mathsf{m}(x_{2m_l}, x_{2p_l-1}) = e_0. \tag{2.11}$$

So, $\lim_{m\to\infty} \mathsf{d}(x_{2m_l+1}, x_{2p_l}) = e_0 = \lim_{m\to\infty} \mathsf{m}(x_{2m_l}, x_{2p_l-1})$ and by (a_3) we get

$$\lim_{m\to\infty} \vartheta\big(\mathsf{d}(x_{2m_l+1}, x_{2p_l})\big) = \lim_{m\to\infty} \vartheta\big(\mathsf{m}(x_{2m_l}, x_{2p_l-1})\big). \tag{2.12}$$

Since by (2.4) we have

$$0 \leq \eta\big(\vartheta\big(\mathsf{d}(Px_{2m_l}, Qx_{2p_l-1})\big), \psi\big(\mathsf{m}(x_{2m_l}, x_{2p_l-1})\big)\big), \tag{2.13}$$

or, taking (η_1) and (a_1) into account,

$$\vartheta\big(\mathsf{d}(x_{2m_l+1}, x_{2p_l})\big) = \vartheta\big(\mathsf{d}(Px_{2m_l}, Qx_{2p_l-1})\big) < \psi\big(\mathsf{m}(x_{2m_l}, x_{2p_l-1})\big) < \vartheta\big(\mathsf{m}(x_{2m_l}, x_{2p_l-1})\big).$$

Using (2.12) we get $\lim_{m\to\infty} \vartheta(\mathsf{d}(x_{2m_l+1}, x_{2p_l})) = \lim_{m\to\infty} \psi(\mathsf{m}(x_{2m_l}, x_{2p_l-1})) > 0$. Thus, by (η_2) we have

$$\limsup_{m\to\infty} \eta\big(\vartheta\big(\mathsf{d}(x_{2m_l+1}, x_{2p_l})\big), \psi\big(\mathsf{m}(x_{2m_l}, x_{2p_l-1})\big)\big) < 0, \tag{2.14}$$

which leads to a contradiction, since by (2.13), we have

$$0 \leq \limsup_{m\to\infty} \eta\big(\vartheta\big(\mathsf{d}(x_{2m_l+1}, x_{2p_l})\big), \psi\big(\mathsf{m}(x_{2m_l}, x_{2p_l-1})\big)\big).$$

Thereupon, $\{x_m\}$ is a Cauchy sequence. Moreover, since X is a complete metric space, we can find $x_* \in X$ such that

$$\lim_{m\to\infty} \mathsf{d}(x_m, x_*) = 0 \tag{2.15}$$

and we claim that this is a common fixed point of the mappings Q and P. From the point of view of a previous remark, it is enough to prove that x_* is a fixed point of Q (or P). Indeed, supposing $\mathsf{d}(x_*, Qx_*) > 0$, we see that $\mathsf{d}(Px_{2m}, Qx_*) = \mathsf{d}(x_{2m+1}, Qx_*) \to \mathsf{d}(x_*, Qx_*)$ as $m \to \infty$ and then $\mathsf{d}(Px_{2m}, Qx_*) > 0$ for infinitely many values of $m \in \mathbb{N}$. Hence, from (2.4) we have

$$0 \leq \eta(\vartheta\big(\mathsf{d}(Px_{2m}, Qx_*), \psi\big(\mathsf{m}(x_{2m}, x_*)\big)\big) < \psi\big(\mathsf{m}(x_{2m}, x_*)\big) - \vartheta\big(\mathsf{d}(x_{2m+1}, Qx_*)\big),$$

or

$$\vartheta\left(\mathsf{d}(P\varkappa_{2m}, Q\varkappa_*)\right) < \psi\left(\mathsf{m}(\varkappa_{2m}, \varkappa_*)\right) < \vartheta\left(\mathsf{m}(\varkappa_{2m}, \varkappa_*)\right), \tag{2.16}$$

where

$$\begin{aligned}\mathsf{m}(\varkappa_{2m}, \varkappa_*) &= \max\left\{\mathsf{d}(\varkappa_{2m}, \varkappa_*), \mathsf{d}(\varkappa_{2m}, P\varkappa_{2m}), \mathsf{d}(\varkappa_*, Q\varkappa_*), \frac{\mathsf{d}(\varkappa_{2m}, Q\varkappa_*) + \mathsf{d}(\varkappa_*, P\varkappa_{2m})}{2}\right\}\\ &= \max\left\{\mathsf{d}(\varkappa_{2m}, \varkappa_*), \mathsf{d}(\varkappa_{2m}, \varkappa_{2m+1}), \mathsf{d}(\varkappa_*, Q\varkappa_*), \frac{\mathsf{d}(\varkappa_{2m}, Q\varkappa_*) + \mathsf{d}(\varkappa_*, \varkappa_{2m+1})}{2}\right\}.\end{aligned}$$

Thus,

$$\lim_{m\to\infty} \mathsf{d}(P\varkappa_{2m}, Q\varkappa_*) = \lim_{m\to\infty} \mathsf{m}(\varkappa_{2m}, \varkappa_*) = \mathsf{d}(\varkappa_*, Q\varkappa_*) > 0,$$

and in view of (a_4), $\lim_{m\to\infty} \vartheta(\mathsf{d}(P\varkappa_{2m}, Q\varkappa_*)) = \lim_{m\to\infty} \vartheta(\mathsf{m}(\varkappa_{2m}, \varkappa_*)) > 0$. Therefore, letting $m \to \infty$ in (2.16), we get $\lim_{m\to\infty} \vartheta(\mathsf{d}(P\varkappa_{2m}, Q\varkappa_*)) = \lim_{m\to\infty} \psi(\mathsf{m}(\varkappa_{2m}, \varkappa_*)) > 0$ and using (η_1) and (η_2) we obtain

$$0 \le \limsup_{m\to\infty} \eta\left(\vartheta\left(\mathsf{d}(P\varkappa_{2m}, Q\varkappa_*)\right), \psi\left(\mathsf{m}(\varkappa_{2m}, \varkappa_*)\right)\right) < 0,$$

a contradiction. Thereupon, $\mathsf{d}(\varkappa_*, Q\varkappa_*) = 0$, which means that $\varkappa_*$ is a fixed point of Q and then a common fixed point of P and Q.

Finally, we have to show the uniqueness of this point. If on the contrary, there exists another point $y_* \in X$, different by $\varkappa_*$, such that $Qy_* = Py_*$, since $\mathsf{d}(P\varkappa_*, Qy_*) > 0$, we have

$$0 \le \eta\left(\vartheta\left(\mathsf{d}(P\varkappa_*, Qy_*)\right), \psi\left(\mathsf{m}(\varkappa_*, y_*)\right)\right)$$

which in view of (η_1) becomes

$$\begin{aligned}\vartheta\left(\mathsf{d}(\varkappa_*, y_*)\right) &= \vartheta\left(\mathsf{d}(P\varkappa_*, Qy_*)\right) < \psi\left(\mathsf{m}(P\varkappa_*, Qy_*)\right) < \vartheta\left(\mathsf{m}(\varkappa_*, y_*)\right)\\ &= \vartheta\left(\max\left\{\begin{matrix}\mathsf{d}(\varkappa_*, y_*), \mathsf{d}(\varkappa_*, P\varkappa_*), \mathsf{d}(y_*, Qy_*),\\ \frac{\mathsf{d}(\varkappa_*, Qy_*) + \mathsf{d}(y_*, P\varkappa_*)}{2}\end{matrix}\right\}\right)\\ &= \vartheta\left(\mathsf{d}(\varkappa_*, y_*)\right),\end{aligned}$$

which is obviously a contradiction. □

Corollary 6 *Let* (X, d) *be a complete metric space and a mapping* $P : X \to X$*. Assume that there exists a function* $\eta \in Z$ *such that*

$$\begin{aligned}&\eta\Bigg(\vartheta\left(\mathsf{d}(P\varkappa, Py)\right),\\ &\quad\psi\left(\max\left\{\mathsf{d}(\varkappa, y), \mathsf{d}(\varkappa, P\varkappa), \mathsf{d}(y, Py), \frac{\mathsf{d}(\varkappa, Py) + \mathsf{d}(y, P\varkappa)}{2}\right\}\right)\Bigg) \ge 0,\end{aligned} \tag{2.17}$$

for any $\varkappa, y \in X$ *with* $\mathsf{d}(P\varkappa, Py) > 0$, *where* $\vartheta, \psi \in \Theta$. *Then the mapping* P *has a unique fixed point provided that the following conditions are satisfied*:

(a_1) $\psi(s) < \vartheta(s)$, *for any* $s > 0$;
(a_2) $\inf_{s>s_0} \vartheta(s) > -\infty$, *for any* $s_0 > 0$;
(a_3) *if* $\{a_m\}$, $\{b_m\}$ *are two convergent sequences with* $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m > 0$ *then the sequences* $\{\vartheta(a_m)\}$, $\{\vartheta(b_m)\}$ *are convergent and* $\lim_{m\to\infty} \vartheta(a_m) = \lim_{m\to\infty} \vartheta(b_m) > 0$.

Proof Put $Q = P$ in Theorem 5. □

Theorem 7 *Let* (X, d) *be a complete metric space, two mappings* $P, Q : X \to X$ *and a function* $\eta \in Z$ *such that*

$$\eta\big(\vartheta\big(\mathsf{d}(P\chi, Qy)\big), \psi\big(\mathsf{r}_1(\chi, y)\big)\big) \geq 0, \quad \text{for any } \chi, y \in X \text{ with } \mathsf{d}(P\chi, Qy) > 0, \tag{2.18}$$

where $\vartheta, \psi \in \Theta$. *Suppose that*
(a_1) $\psi(s) < \vartheta(s)$, *for any* $s > 0$;
(a_2) $\inf_{s>s_0} \vartheta(s) > -\infty$, *for any* $s_0 > 0$;
(a_3) *if* $\{a_m\}$, $\{b_m\}$ *are convergent sequences with with* $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m > 0$ *then the sequences* $\{\vartheta(a_m)\}$, $\{\vartheta(b_m)\}$, *are convergent and* $\lim_{m\to\infty} \vartheta(a_m) = \lim_{m\to\infty} \vartheta(b_m)$.
Then the mappings P, Q *have a unique fixed point.*

Proof Let $\chi_0 \in X$ be an arbitrary point and the sequence $\{\chi_m\}$ in X, defined as follows:

$$\chi_1 = P\chi_0, \qquad \chi_2 = Q\chi_1, \quad \ldots, \quad \chi_{2m-1} = P\chi_{2m-2}, \qquad \chi_{2m} = Q\chi_{2m-1} \tag{2.19}$$

for every $m \in \mathbb{N}$. In what follows, we shall suppose that $\chi_m \neq \chi_{m+1}$ for any $m \in \mathbb{N}$ (using the same arguments as in the previous proof).

Let $o_m = \mathsf{d}(\chi_m, \chi_{m+1}) > 0$, $m \in \mathbb{N}$. First of all, we claim that $o_{m+1} < o_m$, for all $m \in \mathbb{N}$. For this purpose, we shall distinguish two situations:

(1) If $m = 2i$, $i \in \mathbb{N}$ we have

$$\begin{aligned}
\mathsf{r}_1(\chi_{2i}, \chi_{2i+1}) &= \max\left\{\begin{matrix}\mathsf{d}(\chi_{2i}, \chi_{2i+1}), \frac{\mathsf{d}(\chi_{2i+1}, Q\chi_{2i+1})(1+\mathsf{d}(\chi_{2i}, P\chi_{2i}))}{1+\mathsf{d}(\chi_{2i}, \chi_{2i+1})}, \\ \frac{\mathsf{d}(\chi_{2i}, Q\chi_{2i+1})+\mathsf{d}(\chi_{2i+1}, P\chi_{2i})}{2}\end{matrix}\right\} \\
&= \max\left\{\begin{matrix}\mathsf{d}(\chi_{2i}, \chi_{2i+1}), \frac{\mathsf{d}(\chi_{2i+1}, \chi_{2i+2})(1+\mathsf{d}(\chi_{2i}, \chi_{2i+1}))}{1+\mathsf{d}(\chi_{2i}, \chi_{2i+1})}, \\ \frac{\mathsf{d}(\chi_{2i}, \chi_{2i+2})+\mathsf{d}(\chi_{2i+1}, \chi_{2i+1})}{2}\end{matrix}\right\} \\
&= \max\big\{\mathsf{d}(\chi_{2i}, \chi_{2i+1}), \mathsf{d}(\chi_{2i+1}, \chi_{2i+2})\big\}.
\end{aligned}$$

Since $o_m > 0$ for any $m \in \mathbb{N}$, we see that $\mathsf{d}(P\chi_{2i}, Q\chi_{2i+1}) = \mathsf{d}(\chi_{2i+1}, \chi_{2i+2}) > 0$ and by (2.18) we have

$$\begin{aligned}
0 &\leq \eta\big(\vartheta\big(\mathsf{d}(P\chi_{2i}, Q\chi_{2i+1})\big), \psi\big(\mathsf{r}_1(\chi_{2i}, \chi_{2i+1})\big)\big) \\
&= \eta\big(\vartheta\big(\mathsf{d}(\chi_{2i+1}, \chi_{2i+2})\big), \psi\big(\max\big\{\mathsf{d}(\chi_{2i}, \chi_{2i+1}), \mathsf{d}(\chi_{2i+1}, \chi_{2i+2})\big\}\big)\big).
\end{aligned}$$

Moreover, from (η_1) and (a_1) it follows

$$\begin{aligned}
\vartheta\big(\mathsf{d}(\chi_{2i+1}, \chi_{2i+2})\big) &< \psi\big(\max\big\{\mathsf{d}(\chi_{2i}, \chi_{2i+1}), \mathsf{d}(\chi_{2i+1}, \chi_{2i+2})\big\}\big) \\
&< \vartheta\big(\max\big\{\mathsf{d}(\chi_{2i}, \chi_{2i+1}), \mathsf{d}(\chi_{2i+1}, \chi_{2i+2})\big\}\big).
\end{aligned} \tag{2.20}$$

If $\max\{\mathsf{d}(\chi_{2i_0}, \chi_{2i_0+1}), \mathsf{d}(\chi_{2i_0+1}, \chi_{2i_0+2})\} = \mathsf{d}(\chi_{2i_0+1}, \chi_{2i_0+2})$ for some $i_0 \in \mathbb{N}$, the inequality (2.20) leads to a contradiction. Therefore, $o_{2i} = \mathsf{d}(\chi_{2i}, \chi_{2i+1}) > \mathsf{d}(\chi_{2i+1}, \chi_{2i+2}) = o_{2i+1}$ for any $i \in \mathbb{N}$.

(2) If $m = 2i - 1$, $i \in \mathbb{N}$,

$$\begin{aligned}
\mathsf{r}_1(\chi_{2i}, \chi_{2i-1}) &= \max \left\{ \begin{matrix} \mathsf{d}(\chi_{2i}, \chi_{2i-1}), \frac{\mathsf{d}(\chi_{2i-1}, Q\chi_{2i-1})(1+\mathsf{d}(\chi_{2i}, P\chi_{2i}))}{1+\mathsf{d}(\chi_{2i}, \chi_{2i-1})}, \\ \frac{\mathsf{d}(\chi_{2i}, Q\chi_{2i-1})+\mathsf{d}(\chi_{2i-1}, P\chi_{2i})}{2} \end{matrix} \right\} \\
&= \max \left\{ \begin{matrix} \mathsf{d}(\chi_{2i}, \chi_{2i-1}), \frac{\mathsf{d}(\chi_{2i-1}, \chi_{2i-2})(1+\mathsf{d}(\chi_{2i}, \chi_{2i-1}))}{1+\mathsf{d}(\chi_{2i}, \chi_{2i-1})}, \\ \frac{\mathsf{d}(\chi_{2i}, \chi_{2i})+\mathsf{d}(\chi_{2i-1}, \chi_{2i+1})}{2} \end{matrix} \right\} \\
&= \max\bigl\{\mathsf{d}(\chi_{2i}, \chi_{2i-1}), \mathsf{d}(\chi_{2i+1}, \chi_{2i})\bigr\}
\end{aligned}$$

and using the same arguments it follows that $o_{2i-1} = \mathsf{d}(\chi_{2i-1}, \chi_{2i}) > \mathsf{d}(\chi_{2i}, \chi_{2i+1}) = o_{2i}$, for any $i \in \mathbb{N}$. Therefore, we conclude that the sequence $\{o_m\}$ is convergent with the limit $\mathsf{D} \geq 0$ (being decreasing and bounded below by 0). Moreover, from (2.20) together with (η_1) and we get

$$\vartheta(o_{2i+1}) < \psi(o_{2i}) < \vartheta(o_{2i}). \tag{2.21}$$

From our considerations, we conclude that the sequence $\{\vartheta(o_{2i})\}$ is convergent (being decreasing and taking (a_2) into account). Thereupon, by (2.21), the sequence $\{\psi(o_{2i})\}$) is convergent and has the same limit as $\{\vartheta(o_{2i})\}$). If we suppose that $\mathsf{D} > 0$, on the one hand, by (η_1) we have

$$\lim_{i\to\infty} \eta\bigl(\vartheta(o_{2i+1}), \psi(o_{2i})\bigr) \geq 0.$$

On the other hand, taking (η_2) into account we get

$$\lim_{i\to\infty} \eta\bigl(\vartheta(o_{2i+1}), \psi(o_{2i})\bigr) < 0.$$

This is a contradiction. Therefore $\mathsf{D} = 0$, so,

$$\lim_{m\to\infty} \mathsf{d}(\chi_m, \chi_{m+1}) = 0. \tag{2.22}$$

We shall prove that $\{\chi_m\}$ is a Cauchy sequence. Arguing by contradiction, if $\{\chi_{2m}\}$ is not Cauchy, by Lemma 3, we can find two sequences $\{m_l\}$, $\{p_l\}$ of positive integers and e_0 such that m_l is smallest index for which $m_l > p_l > l$ and (1.2) hold. Letting $\chi = \chi_{2m_l}$, respectively, $y = \chi_{2p_l-1}$ in (2.2) we have

$$\begin{aligned}
\mathsf{r}_1(\chi_{2m_l}, \chi_{2p_l-1}) &= \max \left\{ \begin{matrix} \mathsf{d}(\chi_{2m_l}, \chi_{2p_l-1}), \frac{\mathsf{d}(\chi_{2p_l-1}, Q\chi_{2p_l-1})(1+\mathsf{d}(\chi_{2m_l}, P\chi_{2m_l}))}{1+\mathsf{d}(\chi_{2m_l}, \chi_{2p_l-1})}, \\ \frac{\mathsf{d}(\chi_{2m_l}, Q\chi_{2p_l-1})+\mathsf{d}(\chi_{2p_l-1}, P\chi_{2p_l})}{2} \end{matrix} \right\} \\
&= \max \left\{ \begin{matrix} \mathsf{d}(\chi_{2m_l}, \chi_{2p_l-1}), \frac{\mathsf{d}(\chi_{2p_l-1}, \chi_{2p_l})(1+\mathsf{d}(\chi_{2m_l}, \chi_{2m_l+1}))}{1+\mathsf{d}(\chi_{2m_l}, \chi_{2p_l-1})}, \\ \frac{\mathsf{d}(\chi_{2m_l}, \chi_{2p_l})+\mathsf{d}(\chi_{2p_l-1}, \chi_{2m_l+1})}{2} \end{matrix} \right\}
\end{aligned}$$

and then $\lim_{m\to\infty} \mathsf{r}_1(x_{2m_l}, x_{2p_l-1}) = \lim_{m\to\infty} \mathsf{d}(x_{2m_l}, x_{2p_l-1}) = \lim_{m\to\infty} \mathsf{d}(x_{2m_l+1}, x_{2p_l}) = e_0 > 0$. Moreover, by (a_3),

$$\lim_{m\to\infty} \vartheta\big(\mathsf{d}(x_{2m_l+1}, x_{2p_l})\big) = \lim_{m\to\infty} \vartheta\big(\mathsf{r}_1(x_{2m_l+1}, x_{2p_l})\big). \tag{2.23}$$

Plugging this into (2.18), we have

$$0 \le \eta\big(\vartheta\big(\mathsf{d}(Px_{2m_l}, Qx_{p_l-1})\big), \psi\big(\mathsf{r}_1(x_{2m_l}, x_{2p_l-1})\big)\big), \tag{2.24}$$

or, taking (η_1) and (a_1) into account

$$\vartheta\big(\mathsf{d}(x_{2m_l+1}, x_{2p_l})\big) = \vartheta\big(\mathsf{d}(Px_{2m_l}, Qx_{p_l-1})\big) < \psi\big(\mathsf{r}_1(x_{2m_l}, x_{2p_l-1})\big) < \vartheta\big(\mathsf{r}_1(x_{2m_l}, x_{2p_l-1})\big).$$

Thus, by (2.23), we get

$$\lim_{l\to\infty} \psi\big(\mathsf{r}_1(x_{2m_l+1}, x_{2p_l})\big) = \lim_{l\to\infty} \vartheta\big(\mathsf{d}(x_{2m_l+1}, x_{2p_l})\big) > 0,$$

which implies, by (η_2),

$$\limsup_{l\to\infty} \eta\big(\vartheta\big(\mathsf{d}(x_{2m_l+1}, x_{2p_l})\big), \psi\big(\mathsf{r}_1(x_{2m_l+1}, x_{2p_l})\big)\big) < 0.$$

On the other hand, letting $l \to \infty$ in (2.24), we have

$$\limsup_{l\to\infty} \eta\big(\vartheta\big(\mathsf{d}(x_{2m_l+1}, x_{2p_l})\big), \psi\big(\mathsf{r}_1(x_{2m_l+1}, x_{2p_l})\big)\big) \ge 0,$$

which contradicts the previous inequality. Therefore, the sequence $\{x_m\}$ is Cauchy, and by the completeness of the space X it is a convergent sequence. Let $x_* \in X$ such that $\lim_{m\to\infty} x_m = x_*$. We claim that x_* is a common fixed point of P and Q. First of all, we prove that x_* is a fixed point of Q. If for infinitely many values of m, $\mathsf{d}(Px_{2m}, Qx_*) = \mathsf{d}(x_{2m+1}, Qx_*) = 0$, then

$$\mathsf{d}(x_*, Qx_*) \le \mathsf{d}(x_*, Px_{2m}) + \mathsf{d}(Px_{2m}, Qx_*) = \mathsf{d}(x_*, x_{2m+1}) \to 0, \quad \text{as } m \to \infty$$

and $\mathsf{d}(x_*, Qx_*) = 0$, so that $Qx_* = x_*$.

If $\mathsf{d}(Px_{2m}, Qx_*) > 0$ for any $m \in \mathbb{N}$, by (2.18) we have

$$0 \le \eta\big(\vartheta\big(\mathsf{d}(Px_{2m}, Qx_*)\big), \psi\big(\mathsf{r}_1(x_{2m}, x_*)\big)\big) < \psi\big(\mathsf{r}_1(x_{2m}, x_*)\big) - \vartheta\big(\mathsf{d}(Px_{2m}, Qx_*)\big) \tag{2.25}$$

or, equivalently

$$\vartheta\big(\mathsf{d}(Px_{2m}, Qx_*)\big) < \psi\big(\mathsf{r}_1(x_{2m}, x_*)\big) < \vartheta\big(\mathsf{r}_1(x_{2m}, x_*)\big).$$

Since $\lim_{m\to\infty} \mathsf{d}(P\chi_{2m}, Q\chi_*) = \mathsf{d}(\chi_*, Q\chi_*)$ and

$$\begin{aligned}\lim_{m\to\infty} \mathsf{r}_1(\chi_{2m},\chi_*) &= \lim_{m\to\infty} \max\left\{\begin{array}{c}\mathsf{d}(\chi_{2m},\chi_*), \frac{\mathsf{d}(\chi_*,Q\chi_*)(1+\mathsf{d}(\chi_{2m},P\chi_{2m}))}{1+\mathsf{d}(\chi_{2m},\chi_*)},\\ \frac{\mathsf{d}(\chi_{2m},Q\chi_*)+\mathsf{d}(\chi_*,P\chi_{2m})}{2}\end{array}\right\}\\ &= \lim_{m\to\infty} \max\left\{\begin{array}{c}\mathsf{d}(\chi_{2m},\chi_*), \frac{\mathsf{d}(\chi_*,Q\chi_*)(1+\mathsf{d}(\chi_{2m},Q\chi_{2m}))}{1+\mathsf{d}(\chi_{2m},\chi_*)},\\ \frac{\mathsf{d}(\chi_{2m},Q\chi_*)+\mathsf{d}(\chi_*,\chi_{2m+1})}{2}\end{array}\right\}\\ &= \mathsf{d}(\chi_*, Q\chi_*),\end{aligned}$$

we see that $\lim_{m\to\infty} \mathsf{d}(P\chi_{2m}, Q\chi_*) = \lim_{m\to\infty} \mathsf{r}_1(\chi_{2m},\chi_*)$. Therefore, by (a_3) it follows that $\lim_{m\to\infty} \vartheta(\mathsf{d}(P\chi_{2m}, Q\chi_*) = \lim_{m\to\infty} \psi(\mathsf{r}_1(\chi_{2m},\chi_*))$ and taking (η_2) into account,

$$\limsup_{m\to\infty} \eta\big(\vartheta\big(\mathsf{d}(P\chi_{2m}, Q\chi_*)\big), \psi\big(\mathsf{r}_1(\chi_{2m},\chi_*)\big)\big) < 0. \tag{2.26}$$

But, letting $m \to \infty$ in (2.25),

$$\limsup_{m\to\infty} \eta\big(\vartheta\big(\mathsf{d}(P\chi_{2m}, Q\chi_*)\big), \psi\big(\mathsf{r}_1(\chi_{2m},\chi_*)\big)\big) \geq 0.$$

This is a contradiction; consequently, $\mathsf{d}(\chi_*, Q\chi_*) = 0$ and χ_* is a fixed point of Q and we assume, by "reductio ad absurdum", that χ_* is not a fixed point of P. Then $\mathsf{d}(Q\chi_*, P\chi_*) > 0$ and (2.18) gives us

$$0 \leq \eta\big(\vartheta\big(\mathsf{d}(Q\chi_*, P\chi_*)\big), \psi\big(\mathsf{r}_1(\chi_*,\chi_*)\big)\big) < \psi\big(\mathsf{r}_1(\chi_*,\chi_*)\big) - \vartheta\big(\mathsf{d}(P\chi_*,\chi_*)\big),$$

which is equivalent with

$$\begin{aligned}\vartheta\big(\mathsf{d}(\chi_*, P\chi_*)\big) < \psi\big(\mathsf{r}_1(\chi_*,\chi_*)\big) &= \psi\left(\max\left\{\begin{array}{c}\mathsf{d}(\chi_*,\chi_*), \frac{(1+\mathsf{d}(Q\chi_*,\chi_*))\mathsf{d}(\chi_*,P\chi_*)}{1+\mathsf{d}(\chi_*,\chi_*)},\\ \frac{\mathsf{d}(\chi_*,Q\chi_*)+\mathsf{d}(\chi_*,P\chi_*)}{2}\end{array}\right\}\right)\\ &= \psi\big(\mathsf{d}(\chi_*, P\chi_*)\big) < \vartheta\big(\mathsf{d}(\chi_*, P\chi_*)\big),\end{aligned} \tag{2.27}$$

which is a contradiction. Therefore, by (a_4) it follows that $\mathsf{d}(P\chi_*, \chi_*) = 0$ and then χ_* is a common fixed point of P and Q.

As a last step in our proof, we shall prove the uniqueness of the common fixed point. Indeed, if there exists another point, for example y_* such that $Qy_* = y_* = Py_*$ and $y_* \neq \chi_*$, then, since $\mathsf{d}(P\chi_*, Qy_*) = \mathsf{d}(\chi_*, y_*) > 0$, from (2.18) we have

$$\begin{aligned}0 &\leq \eta(\vartheta\big(\mathsf{d}(P\chi_*, Qy_*)\big), \psi\big(\mathsf{r}_1(\chi_*, y_*)\big) < \psi\big(\mathsf{r}_1(\chi_*, y_*)\big) - \vartheta\big(\mathsf{d}(\chi_*, y_*)\big)\\ &< \vartheta\big(\mathsf{r}_1(\chi_*, y_*)\big) - \vartheta\big(\mathsf{d}(\chi_*, y_*)\big) = \vartheta\big(\mathsf{d}(\chi_*, y_*)\big) - \vartheta\big(\mathsf{d}(\chi_*, y_*)\big),\end{aligned}$$

which is a contradiction. Thereupon, $\chi_* = y_*$, so the fixed point of the mappings Q and P is unique. □

Example 8 Let the set $X = \{\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \mathsf{a}_4\}$ and $\mathsf{d} : X \times X \to [0, +\infty)$ be defined as follows:

$$\mathsf{d}(\mathsf{a}_1,\mathsf{a}_2) = \mathsf{d}(\mathsf{a}_2,\mathsf{a}_1) = 2, \qquad \mathsf{d}(\mathsf{a}_1,\mathsf{a}_3) = \mathsf{d}(\mathsf{a}_3,\mathsf{a}_1) = 3, \qquad \mathsf{d}(\mathsf{a}_1,\mathsf{a}_4) = \mathsf{d}(\mathsf{a}_4,\mathsf{a}_1) = 5;$$

$$\mathsf{d}(\mathsf{a}_2,\mathsf{a}_3)=\mathsf{d}(\mathsf{a}_3,\mathsf{a}_2)=5,\qquad \mathsf{d}(\mathsf{a}_3,\mathsf{a}_4)=\mathsf{d}(\mathsf{a}_4,\mathsf{a}_3)=8,\qquad \mathsf{d}(\mathsf{a}_2,\mathsf{a}_4)=\mathsf{d}(\mathsf{a}_4,\mathsf{a}_2)=3;$$
$$\mathsf{d}(\mathsf{a}_1,\mathsf{a}_1)=\mathsf{d}(\mathsf{a}_2,\mathsf{a}_2)=\mathsf{d}(\mathsf{a}_3,\mathsf{a}_3)=\mathsf{d}(\mathsf{a}_4,\mathsf{a}_4)=0.$$

Let $Q,P:X\to X$ be two mappings where

$$P\mathsf{a}_1=P\mathsf{a}_2=P\mathsf{a}_4=\mathsf{a}_1,\qquad P\mathsf{a}_3=\mathsf{a}_2;$$
$$Q\mathsf{a}_1=Q\mathsf{a}_2=Q\mathsf{a}_3=\mathsf{a}_1,\qquad Q\mathsf{a}_4=\mathsf{a}_2,$$

and we choose the functions $\eta\in Z$ and $\vartheta,\psi\in\Theta$, with

$$\eta(t,s)=0.88s-t,\qquad \vartheta(s)=s,\qquad \psi(s)=0.91s.$$

Of course, we can easily see that, with these choices, the assumptions (a_1)–(a_3) of Theorem 7 are obviously satisfied. Thus, we shall check that (2.18) holds for any $\chi,y\in X$, such that $\mathsf{d}(P\chi,Qy)>0$. We discuss then the following situations:

- $\chi=\mathsf{a}_1$, $y=\mathsf{a}_4$,

$$\begin{aligned}\mathsf{d}(P\mathsf{a}_1,Q\mathsf{a}_4)&=\mathsf{d}(\mathsf{a}_1,\mathsf{a}_2)=2,\\ \mathsf{r}_1(\mathsf{a}_1,\mathsf{a}_4)&=\max\left\{\mathsf{d}(\mathsf{a}_1,\mathsf{a}_4),\frac{(1+\mathsf{d}(\mathsf{a}_1,P\mathsf{a}_1))\mathsf{d}(\mathsf{a}_4,Q\mathsf{a}_4)}{1+\mathsf{d}(\mathsf{a}_1,\mathsf{a}_4)},\frac{\mathsf{d}(\mathsf{a}_1,Q\mathsf{a}_4)+\mathsf{d}(\mathsf{a}_4,P\mathsf{a}_1)}{2}\right\}\\ &=\max\left\{\mathsf{d}(\mathsf{a}_1,\mathsf{a}_4),\frac{(1+\mathsf{d}(\mathsf{a}_1,\mathsf{a}_1))\mathsf{d}(\mathsf{a}_4,\mathsf{a}_2)}{1+\mathsf{d}(\mathsf{a}_1,\mathsf{a}_4)},\frac{\mathsf{d}(\mathsf{a}_1,\mathsf{a}_2)+\mathsf{d}(\mathsf{a}_4,\mathsf{a}_1)}{2}\right\}\\ &=\max\left\{5,\frac{5}{6},\frac{7}{2}\right\}=5\end{aligned}$$

  and

$$\eta\big(\vartheta\big(\mathsf{d}(P\mathsf{a}_1,Q\mathsf{a}_4)\big),\psi\big(\mathsf{r}_1(\mathsf{a}_1,\mathsf{a}_4)\big)\big)=0.88\cdot 0.91\cdot 5-2=2.004>0.$$

- $\chi=\mathsf{a}_2$, $y=\mathsf{a}_4$,

$$\begin{aligned}\mathsf{d}(P\mathsf{a}_2,Q\mathsf{a}_4)&=\mathsf{d}(\mathsf{a}_1,\mathsf{a}_2)=2,\\ \mathsf{r}_1(\mathsf{a}_2,\mathsf{a}_4)&=\max\left\{\mathsf{d}(\mathsf{a}_2,\mathsf{a}_4),\frac{(1+\mathsf{d}(\mathsf{a}_2,P\mathsf{a}_2))\mathsf{d}(\mathsf{a}_4,Q\mathsf{a}_4)}{1+\mathsf{d}(\mathsf{a}_2,\mathsf{a}_4)},\frac{\mathsf{d}(\mathsf{a}_2,Q\mathsf{a}_4)+\mathsf{d}(\mathsf{a}_4,P\mathsf{a}_2)}{2}\right\}\\ &=\max\left\{\mathsf{d}(\mathsf{a}_2,\mathsf{a}_4),\frac{(1+\mathsf{d}(\mathsf{a}_2,\mathsf{a}_1))\mathsf{d}(\mathsf{a}_4,\mathsf{a}_2)}{1+\mathsf{d}(\mathsf{a}_2,\mathsf{a}_4)},\frac{\mathsf{d}(\mathsf{a}_1,\mathsf{a}_2)+\mathsf{d}(\mathsf{a}_4,\mathsf{a}_1)}{2}\right\}\\ &=\max\left\{3,\frac{9}{4},\frac{7}{2}\right\}=\frac{7}{2}\end{aligned}$$

  and

$$\eta\big(\vartheta\big(\mathsf{d}(P\mathsf{a}_2,Q\mathsf{a}_4)\big),\psi\big(\mathsf{r}_1(\mathsf{a}_2,\mathsf{a}_4)\big)\big)=0.88\cdot 0,91\cdot 3.5-2=0.8>0.$$

- $\chi=\mathsf{a}_3$, $y=\mathsf{a}_1$,

$$\mathsf{d}(P\mathsf{a}_3,Q\mathsf{a}_1)=\mathsf{d}(\mathsf{a}_2,\mathsf{a}_1)=2,$$

$$\begin{aligned} r_1(a_3,a_1) &= \max\left\{d(a_3,a_1), \frac{(1+d(a_3,Pa_3))d(a_1,Qa_1)}{1+d(a_3,a_1)}, \frac{d(a_3,Qa_1)+d(a_1,Pa_3)}{2}\right\} \\ &= \max\left\{d(a_3,a_1), \frac{(1+d(a_3,a_2))d(a_1,a_1)}{1+d(a_2,a_4)}, \frac{d(a_3,a_1)+d(a_1,a_1)}{2}\right\} \\ &= \max\left\{3,0,\frac{3}{2}\right\} = 3 \end{aligned}$$

and

$$\eta\big(\vartheta\left(d(Pa_3,Qa_1)\right), \psi\left(r_1(a_3,a_1)\right)\big) = 0.88 \cdot 0,91 \cdot 3 - 2 = 0.4024 > 0.$$

- $\chi = a_3$, $y = a_2$,

$$\begin{aligned} d(Pa_3,Qa_2) &= d(a_2,a_1) = 2, \\ r_1(a_3,a_2) &= \max\left\{d(a_3,a_2), \frac{(1+d(a_3,Pa_3))d(a_2,Qa_2)}{1+d(a_3,a_2)}, \frac{d(a_3,Qa_2)+d(a_2,Pa_3)}{2}\right\} \\ &= \max\left\{d(a_3,a_2), \frac{(1+d(a_3,a_2))d(a_1,a_1)}{1+d(a_2,a_4)}, \frac{d(a_3,a_1)+d(a_1,a_1)}{2}\right\} \\ &= \max\left\{5,0,\frac{3}{2}\right\} = 5 \end{aligned}$$

and

$$\eta\big(\vartheta\left(d(Pa_3,Qa_2)\right), \psi\left(r_1(a_3,a_2)\right)\big) = 0.88 \cdot 0,91 \cdot 5 - 2 = 2.004 > 0.$$

- $\chi = a_3$, $y = a_3$,

$$\begin{aligned} d(Pa_3,Qa_3) &= d(a_2,a_1) = 2, \\ r_1(a_3,a_3) &= \max\left\{d(a_3,a_3), \frac{(1+d(a_3,Pa_3))d(a_3,Qa_3)}{1+d(a_3,a_3)}, \frac{d(a_3,Qa_3)+d(a_3,Pa_3)}{2}\right\} \\ &= \max\left\{d(a_3,a_3), \frac{(1+d(a_3,a_2))d(a_3,a_1)}{1+d(a_3,a_3)}, \frac{d(a_3,a_1)+d(a_3,a_1)}{2}\right\} \\ &= \max\left\{0,18,\frac{6}{2}\right\} = 18 \end{aligned}$$

and

$$\eta\big(\vartheta\left(d(Pa_3,Qa_3)\right), \psi\left(r_1(a_3,a_3)\right)\big) = 0.88 \cdot 0,91 \cdot 18 - 2 = 12.41 > 0.$$

- $\chi = a_4$, $y = a_4$,

$$d(Pa_4,Qa_4) = d(a_1,a_2) = 2,$$

$$\begin{aligned}
\mathsf{r}_1(\mathsf{a}_4,\mathsf{a}_4) &= \max\left\{\mathsf{d}(\mathsf{a}_4,\mathsf{a}_4), \frac{(1+\mathsf{d}(\mathsf{a}_4,P\mathsf{a}_4))\mathsf{d}(\mathsf{a}_4,Q\mathsf{a}_4)}{1+\mathsf{d}(\mathsf{a}_4,\mathsf{a}_4)}, \frac{\mathsf{d}(\mathsf{a}_4,Q\mathsf{a}_4)+\mathsf{d}(\mathsf{a}_4,P\mathsf{a}_4)}{2}\right\} \\
&= \max\left\{\mathsf{d}(\mathsf{a}_4,\mathsf{a}_4), \frac{(1+\mathsf{d}(\mathsf{a}_4,\mathsf{a}_1))\mathsf{d}(\mathsf{a}_4,\mathsf{a}_2)}{1+\mathsf{d}(\mathsf{a}_4,\mathsf{a}_4)}, \frac{\mathsf{d}(\mathsf{a}_4,\mathsf{a}_2)+\mathsf{d}(\mathsf{a}_4,\mathsf{a}_1)}{2}\right\} \\
&= \max\{0,18,4\} = 18
\end{aligned}$$

and

$$\eta\big(\vartheta\big(\mathsf{d}(P\mathsf{a}_4,Q\mathsf{a}_4)\big),\psi\big(\mathsf{r}_1(\mathsf{a}_4,\mathsf{a}_4)\big)\big) = 0.88\cdot 0.91\cdot 18 - 2 = 12.41 > 0.$$

Therefore, all the assumptions of Theorem 7 are satisfied; a_1 is the unique common fixed point of the mappings P and Q.

Corollary 9 *Let (X,d) be a complete metric space, a mapping $P: X\to X$ and a function $\eta\in Z$ such that*

$$\eta\left(\vartheta\big(\mathsf{d}(P\chi,Py)\big),\; \psi\left(\max\left\{\mathsf{d}(\chi,y), \frac{\mathsf{d}(y,Py)(1+\mathsf{d}(\chi,P\chi))}{1+\mathsf{d}(\chi,y)}, \frac{\mathsf{d}(\chi,Py)+\mathsf{d}(y,P\chi)}{2}\right\}\right)\right) \geq 0, \tag{2.28}$$

for any $\chi,y\in X$ with $\mathsf{d}(P\chi,Py)>0$, where $\vartheta,\psi\in\Theta$. Suppose that

(a_1) $\psi(s)<\vartheta(s)$, *for any* $s>0$;

(a_2) $\inf_{s>s_0}\vartheta(s)>-\infty$, *for any* $s_0>0$;

(a_3) *if* $\{a_m\}$, $\{b_m\}$ *are convergent sequences with with* $\lim_{m\to\infty}a_m=\lim_{m\to\infty}b_m>0$ *then the sequences* $\{\vartheta(a_m)\}$, $\{\vartheta(b_m)\}$, *are convergent and* $\lim_{m\to\infty}\vartheta(a_m)=\lim_{m\to\infty}\vartheta(b_m)$.

Then the mapping P possesses a unique fixed point.

Proof Put $Q=P$ in Theorem 7. □

Theorem 10 *Let (X,d) be a complete metric space, two mappings $P,Q: X\to X$, the functions $\vartheta,\psi\in\Theta$ and a function such that*

$$\eta\big(\vartheta\big(\mathsf{d}(P\chi,Qy)\big),\psi\big(\mathsf{r}_2(\chi,y)\big)\big)\geq 0, \quad \text{for any } \chi,y\in X \text{ with } \mathsf{d}(P\chi,Qy)>0, \tag{2.29}$$

when $\max\{\mathsf{d}(\chi,Qy),\mathsf{d}(y,P\chi)\}\neq 0$ *and* $\mathsf{d}(P\chi,Qy)=0$ *when* $\max\{\mathsf{d}(\chi,Qy),\mathsf{d}(y,P\chi)\}=0$. *Suppose that*

(a_1) $\psi(s)<\vartheta(s)$, *for any* $s>0$;

(a_2) $\inf_{s>s_0}\vartheta(s)>-\infty$, *for any* $s_0>0$;

(a_3) *if* $\{a_m\}$, $\{b_m\}$ *are convergent sequences with with* $\lim_{m\to\infty}a_m=\lim_{m\to\infty}b_m>0$ *then the sequences* $\{\vartheta(a_m)\}$, $\{\vartheta(b_m)\}$, *are convergent and* $\lim_{m\to\infty}\vartheta(a_m)=\lim_{m\to\infty}\vartheta(b_m)$;

(a_4) *if* $\{\vartheta(a_m)\}$ *is a strictly decreasing sequence and* $\{\vartheta(a_m)\}$, $\{\psi(a_m)\}$ *are convergent with the same limit then* $\lim_{m\to\infty}a_m=0$;

(a_5) $\limsup_{s\to 0+}\psi(s)<\liminf_{s\to e_0}\vartheta(s)$, *for any* $e_0>0$.

Then the mappings P,Q have a unique fixed point.

Proof First of al, by (2.29) and taking (η_1) into account, we have

$$0 \leq \eta\big(\vartheta\big(\mathsf{d}(P\varkappa, Qy)\big), \psi\big(\mathsf{r}_2(\varkappa, y)\big)\big) < \psi\big(\mathsf{r}_2(\varkappa, y)\big) - \vartheta\big(\mathsf{d}(P\varkappa, Qy)\big),$$

which can be rewritten as

$$\vartheta\big(\mathsf{d}(P\varkappa, Qy)\big) < \psi\big(\mathsf{r}_2(\varkappa, y)\big). \tag{2.30}$$

Let $\{\varkappa_m\}$ be the sequence defined by (2.19). Since $\varkappa_m \neq \varkappa_{m+1}$ for any $m \in \mathbb{N}_0$ (we have already shown this in the proof of Theorem 5), letting $m = 2i$, we have

$$\begin{aligned}
&\mathsf{r}_2(\varkappa_{2i}, \varkappa_{2i+1}) \\
&\quad = \max\left\{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1}), \frac{\mathsf{d}(\varkappa_{2i}, P\varkappa_{2i})\mathsf{d}(\varkappa_{2i}, Q\varkappa_{2i+1}) + \mathsf{d}(\varkappa_{2i+1}, Q\varkappa_{2i+1})\mathsf{d}(\varkappa_{2i+1}, P\varkappa_{2i})}{\max\{\mathsf{d}(\varkappa_{2i}, Q\varkappa_{2i+1}), \mathsf{d}(\varkappa_{2i+1}, P\varkappa_{2i})\}}\right\} \\
&\quad = \max\left\{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1}), \frac{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1})\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+2}) + \mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i+2})\mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i+1})}{\max\{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+2}), \mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i+1})\}}\right\} \\
&\quad = \mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1}),
\end{aligned} \tag{2.31}$$

and letting $m = 2i - 1$,

$$\begin{aligned}
&\mathsf{r}_2(\varkappa_{2i}, \varkappa_{2i-1}) \\
&\quad = \max\left\{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i-1}), \frac{\mathsf{d}(\varkappa_{2i}, P\varkappa_{2i})\mathsf{d}(\varkappa_{2i}, Q\varkappa_{2i-1}) + \mathsf{d}(\varkappa_{2i-1}, Q\varkappa_{2i-1})\mathsf{d}(\varkappa_{2i+1}, P\varkappa_{2i})}{\max\{\mathsf{d}(\varkappa_{2i}, Q\varkappa_{2i-1}), \mathsf{d}(\varkappa_{2i-1}, P\varkappa_{2i})\}}\right\} \\
&\quad = \max\left\{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i-1}), \frac{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1})\mathsf{d}(\varkappa_{2i}, \varkappa_{2i}) + \mathsf{d}(\varkappa_{2i-1}, \varkappa_{2i})\mathsf{d}(\varkappa_{2i-1}, \varkappa_{2i+1})}{\max\{\mathsf{d}(\varkappa_{2i}, \varkappa_{2i}), \mathsf{d}(\varkappa_{2i-1}, \varkappa_{2i+1})\}}\right\} \\
&\quad = \mathsf{d}(\varkappa_{2i}, \varkappa_{2i-1}).
\end{aligned} \tag{2.32}$$

Thus, from (2.30) and keeping (a_1) in mind we get

$$\vartheta(\mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i+2}) = \vartheta(\mathsf{d}(P\varkappa_{2i}, Q\varkappa_{2i+1}) < \psi\big(\mathsf{r}_2(\varkappa_{2i}, \varkappa_{2i+1})\big) = \psi\big(\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1})\big) < \vartheta(\mathsf{d}(\varkappa_{2i}, \varkappa_{2i+1}),$$

and similarly

$$\vartheta(\mathsf{d}(\varkappa_{2i+1}, \varkappa_{2i}) = \vartheta(\mathsf{d}(P\varkappa_{2i}, Q\varkappa_{2i-1}) < \psi\big(\mathsf{r}_2(\varkappa_{2i}, \varkappa_{2i-1})\big) = \psi\big(\mathsf{d}(\varkappa_{2i}, \varkappa_{2i-1})\big) < \vartheta(\mathsf{d}(\varkappa_{2i}, \varkappa_{2i-1}).$$

Denoting $\mathsf{d}(\varkappa_{m+1}, \varkappa_m)$ by o_m, we get

$$\begin{aligned}
&\vartheta(o_{2i}) \leq \psi(o_{2i-1}) < \vartheta(o_{2i-1}), \\
&\vartheta(o_{2i+1}) \leq \psi(o_{2i}) < \vartheta(o_{2i})
\end{aligned}$$

and we conclude that

$$\vartheta(o_{m+1}) < \psi(o_m) < \vartheta(o_m) \tag{2.33}$$

for any $m \in \mathbb{N}$. Consequently, the sequence $\{\vartheta(o_m)\}$ is convergent, being strictly decreasing and bounded below(from (a_2) and Lemma 4). Thus, letting $m \to \infty$ in (2.33) we see that $\{\psi(o_m)\}$ is convergent with the same limit as $\{\vartheta(o_m)\}$. Thereupon, by (a_4),

$$\lim_{m\to\infty} o_m = \lim_{m\to\infty} \mathsf{d}(\varkappa_m, \varkappa_{m+1}) = 0.$$

We claim that the sequence $\{\varkappa_m\}$ is Cauchy, reasoning by contradiction. Indeed, if we suppose that $\{\varkappa_{2m}\}$ is not Cauchy, by Lemma 3, we can find $e_0 > 0$ and the sequences $\{m_l\}$, $\{p_l\}$ of positive integers such that the equalities (1.2) hold, where m_l is the smallest index for which $m_l > p_l > l$, for all $l \geq 1$. We have

$$\begin{aligned} \mathsf{r}_2(\varkappa_{2m_l}, \varkappa_{2p_l-1}) &= \max\left\{ \begin{array}{c} \mathsf{d}(\varkappa_{2m_l}, \varkappa_{2p_l-1}), \\ \frac{\mathsf{d}(\varkappa_{2m_l}, P\varkappa_{2m_l})\mathsf{d}(\varkappa_{2m_l}, Q\varkappa_{2p_l-1}) + \mathsf{d}(\varkappa_{2p_l-1}, Q\varkappa_{2p_l-1})\mathsf{d}(\varkappa_{2p_l-1}, P\varkappa_{2m_l})}{\max\{\mathsf{d}(\varkappa_{2m_l}, Q\varkappa_{2p_l-1}), \mathsf{d}(\varkappa_{2p_l-1}, P\varkappa_{2m_l})\}} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \mathsf{d}(\varkappa_{2m_l}, \varkappa_{2p_l-1}), \\ \frac{\mathsf{d}(\varkappa_{2m_l}, \varkappa_{2m_l+1})\mathsf{d}(\varkappa_{2m_l}, \varkappa_{2p_l}) + \mathsf{d}(\varkappa_{2p_l-1}, \varkappa_{2p_l})\mathsf{d}(\varkappa_{2p_l-1}, \varkappa_{2m_l+1})}{\max\{\mathsf{d}(\varkappa_{2m_l}, \varkappa_{2p_l}), \mathsf{d}(\varkappa_{2p_l-1}, \varkappa_{2m_l+1})\}} \end{array} \right\}, \end{aligned}$$

and setting $u_l = \mathsf{d}(\varkappa_{2m_l+1}, \varkappa_{2p_l})$ and $v_l = \mathsf{r}_2(\varkappa_{2m_l}, \varkappa_{2p_l-1})$ it follows that $\lim_{l\to\infty} u_l = \lim_{l\to\infty} v_l = e_0$. Moreover, by (a_3), $\lim_{l\to\infty} \vartheta(u_l) = \lim_{l\to\infty} \vartheta(v_l)$. Consequently, by (2.33), $\lim_{l\to\infty} \vartheta(u_l) = \lim_{l\to\infty} \psi(v_l)$ and using (η_2),

$$\limsup_{l\to\infty} \eta\big(\vartheta(u_l), \psi(v_l)\big) < 0, \tag{2.34}$$

which is a contradiction, since by (2.29)

$$0 \leq \limsup_{l\to\infty} \eta(\vartheta\big(\mathsf{d}(P\varkappa_{2m_l}, Q\varkappa_{2p_l-1}), \psi\big(\mathsf{r}_2(\varkappa_{2m_l}, \varkappa_{2p_l-1})\big)\big) = \limsup_{l\to\infty} \eta\big(\vartheta(u_l), \psi(v_l)\big).$$

In this way we proved that $\{\varkappa_m\}$ is a Cauchy sequence on a complete metric space, so there exists $\varkappa_* \in X$ such that $\lim_{m\to\infty} \varkappa_m = \varkappa_*$.

We shall show that $\varkappa_*$ is a common fixed point of P and Q. First of all, we remark that, if $\mathsf{d}(P\varkappa_*, Q\varkappa_{2m-1}) = 0$ for infinitely many values of m, then

$$\begin{aligned} \mathsf{d}(P\varkappa_*, \varkappa_*) &\leq \mathsf{d}(P\varkappa_*, \varkappa_{2m}) + \mathsf{d}(\varkappa_{2m}, \varkappa_*) \\ &= \mathsf{d}(P\varkappa_*, Q\varkappa_{2m-1}) + \mathsf{d}(\varkappa_{2m}, \varkappa_*) \to \mathsf{d}(P\varkappa_*, \varkappa_*) \quad \text{as } m \to \infty. \end{aligned}$$

That means that $P\varkappa_* = \varkappa_*$.

Therefore, we can suppose that $\mathsf{d}(P\varkappa_*, Q\varkappa_{2m-1}) > 0$ for infinitely many values of m, by (2.29) we have

$$0 \leq \eta\big(\vartheta\big(\mathsf{d}(P\varkappa_*, Q\varkappa_{2m-1})\big), \psi\big(\mathsf{r}_2(\varkappa_*, \varkappa_{2m-1})\big)\big) < \psi\big(\mathsf{r}_2(\varkappa_*, \varkappa_{2m-1})\big) - \vartheta\big(\mathsf{d}(P\varkappa_*, Q\varkappa_{2m-1})\big),$$

or equivalently, by (a_1)

$$\vartheta\big(\mathsf{d}(P\varkappa_*, \varkappa_{2m})\big) = \vartheta\big(\mathsf{d}(P\varkappa_*, Q\varkappa_{2m-1})\big) < \psi\big(\mathsf{r}_2(\varkappa_*, \varkappa_{2m-1})\big) < \vartheta\big(\mathsf{r}_2(\varkappa_*, \varkappa_{2m-1})\big), \tag{2.35}$$

where

$$\begin{aligned}
&\mathsf{r}_2(\chi_*, \chi_{2m-1}) \\
&\quad = \max\left\{\mathsf{d}(\chi_*, \chi_{2m-1}), \frac{\mathsf{d}(\chi_*, P\chi_*)\mathsf{d}(\chi_*, Q\chi_{2m-1}) + \mathsf{d}(\chi_{2m-1}, Q\chi_{2m-1})\mathsf{d}(\chi_{2m-1}, P\chi_*)}{\max\{\mathsf{d}(\chi_*, Q\chi_{2m-1}), \mathsf{d}(\chi_{2m-1}, P\chi_*)\}}\right\} \\
&\quad = \max\left\{\mathsf{d}(\chi_*, \chi_{2m-1}), \frac{\mathsf{d}(\chi_*, P\chi_*)\mathsf{d}(\chi_*, \chi_{2m}) + \mathsf{d}(\chi_{2m-1}, \chi_{2m})\mathsf{d}(\chi_{2m-1}, P\chi_*)}{\max\{\mathsf{d}(\chi_*, \chi_{2m}), \mathsf{d}(\chi_{2m-1}, P\chi_*)\}}\right\} \to 0,
\end{aligned}$$

as $m \to \infty$. Let $\lim_{m\to\infty} \mathsf{d}(P\chi_*, \chi_{2m}) = \mathsf{d}(P\chi_*, \chi_*) = e$. If we suppose that $e > 0$, from (2.35) and (a_5) we have

$$\begin{aligned}
\liminf_{s\to e} \vartheta(s) &\le \liminf_{m\to\infty} \vartheta\bigl(\mathsf{d}(P\chi_*, \chi_{2m})\bigr) \le \liminf_{m\to\infty} \psi\bigl(\mathsf{r}_2(\chi_*, \chi_{2m-1})\bigr) \\
&\le \limsup_{s\to 0} \psi(s) < \liminf_{s\to e} \vartheta(s),
\end{aligned}$$

which is a contradiction. Therefore, $\mathsf{d}(P\chi_*, \chi_*) = 0$.

Similarly, choosing $\chi = \chi_{2m}$ and $y = \chi_*$ in (2.29), we can show that $\mathsf{d}(\chi_*, Q\chi_*) = 0$ and we conclude that $P\chi_* = \chi_* = Q\chi_*$.

To prove the uniqueness of the common fixed point, we will assume that, on the contrary, there exists another point $y_* \in X$ such that $Py_* = y_* = Qy_*$ and $\chi_* \neq y_*$. Since $\mathsf{d}(\chi_*, y_*) = \mathsf{d}(P\chi_*, Qy_*) > 0$, we have

$$\begin{aligned}
0 &\le \eta\bigl(\vartheta\bigl(\mathsf{d}(P\chi_*, Qy_*)\bigr), \psi\bigl(\mathsf{r}_2(\chi_*, y_*)\bigr)\bigr) < \psi\bigl(\mathsf{r}_2(\chi_*, y_*)\bigr) - \vartheta\bigl(\mathsf{d}(P\chi_*, Qy_*)\bigr) \\
&< \vartheta\bigl(\mathsf{r}_2(\chi_*, y_*)\bigr) - \vartheta\bigl(\mathsf{d}(\chi_*, y_*)\bigr) \\
&= \vartheta\bigl(\mathsf{d}(\chi_*, y_*)\bigr) - \vartheta\bigl(\mathsf{d}(\chi_*, y_*)\bigr),
\end{aligned}$$

which is a contradiction. Hence $\chi_* = y_*$. □

Corollary 11 *Let (X, d) be a complete metric space, a mapping $P : X \to X$, the functions $\vartheta, \psi \in \Theta$ and a function $\eta \in Z$ such that*

$$\eta(\vartheta\bigl(\mathsf{d}(P\chi, Py)\bigr), \psi\left(\max\left\{\mathsf{d}(\chi.y), \frac{\mathsf{d}(\chi, P\chi)\mathsf{d}(\chi, Py) + \mathsf{d}(y, Py)\mathsf{d}(y, P\chi)}{\max\{\mathsf{d}(\chi, Py), \mathsf{d}(y, P\chi)\}}\right\}\right) \ge 0, \qquad (2.36)$$

for any $\chi, y \in X$ with $\mathsf{d}(P\chi, Py) > 0$, when $\max\{\mathsf{d}(\chi, Py), \mathsf{d}(y, P\chi)\} \neq 0$ and $\mathsf{d}(P\chi, Py) = 0$ when $\max\{\mathsf{d}(\chi, Py), \mathsf{d}(y, P\chi)\} = 0$. Suppose that

(a_1) *$\psi(s) < \vartheta(s)$, for any $s > 0$;*

(a_2) *$\inf_{s>s_0} \vartheta(s) > -\infty$, for any $s_0 > 0$;*

(a_3) *if $\{a_m\}$, $\{b_m\}$ are convergent sequences with with $\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m > 0$ then the sequences $\{\vartheta(a_m)\}$, $\{\vartheta(b_m)\}$, are convergent and $\lim_{m\to\infty} \vartheta(a_m) = \lim_{m\to\infty} \vartheta(b_m)$;*

(a_4) *if $\{\vartheta(a_m)\}$ is a strictly decreasing sequence and $\{\vartheta(a_m)\}$, $\{\psi(a_m)\}$ are convergent with the same limit then $\lim_{m\to\infty} a_m = 0$;*

(a_5) *$\limsup_{s\to 0+} \psi(s) < \liminf_{s\to e_0} \vartheta(s)$, for any $e_0 > 0$.*

Then the mapping P has exactly one fixed point.

Proof Put $Q = P$ in Theorem 10. □

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Author details
[1]Department of Mathematics, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia. [2]Department of Mathematics and Computer Sciences, Universitatea Transilvania, Brasov, Romania. [3]Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam.

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