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Generalized homogeneous q-difference equations for q-polynomials and their applications to generating functions and fractional q-integrals

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Abstract

In this paper, our aim is to build generalized homogeneous q-difference equations for q-polynomials. We also consider their applications to generating functions and fractional q-integrals by using the perspective of q-difference equations. In addition, we also reveal relevant relations of various special cases of our main results involving some known results.

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1 Introduction

The objective of this paper is to give an extension of know results on generalized Verma–Jain polynomials [13] and the Hahn polynomials [1, 6, 12, 30]. Here, we will give and prove generating functions for the q-polynomials $\omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$, $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$, and several q-identities by using the q-difference equations and the fractional q-integrals. In this article, we begin our investigation by reviewing some definitions as in [33] with $0<q<1$. The basic hypergeometric function ${}_{\mathfrak{r}}\Phi_{\mathfrak{s}}$ is defined in [16, 25] (see also for details [24, Chap. 3] and [32, p. 347, Eq. (272)]):

$$ {}_{\mathfrak{r}}\Phi_{\mathfrak{s}}\left[\begin{matrix} a_1,a_2,\ldots,a_{\mathfrak{r}}; \\ b_1,b_2,\ldots,b_{\mathfrak{s}}; \end{matrix}\ q;z\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{\mathfrak{r}};q)_n}{(q,b_1,b_2,\ldots,b_{\mathfrak{s}};q)_n}\Big[(-1)^n q^{\binom{n}{2}}\Big]^{1+\mathfrak{s}-\mathfrak{r}} z^n. \qquad (1.1) $$

For all z if $\mathfrak{r}\le\mathfrak{s}$ and for $|z|<1$ if $\mathfrak{r}=\mathfrak{s}+1$, the basic hypergeometric function converges absolutely. (See [31] for some recent applications of the basic hypergeometric function.) For any real or complex parameter x, the q-shifted factorials of ${}_{\mathfrak{r}}\Phi_{\mathfrak{s}}$ are defined, respectively, by

$$(x;q)_0 = 1, \qquad (x;q)_n = \prod_{k=0}^{n-1}\bigl(1-xq^k\bigr), n \geq 1, \quad x \in \mathbb{C} \tag{1.2}$$

and

$$(x;q)_\infty = \prod_{k=0}^{\infty}\bigl(1-xq^k\bigr). \tag{1.3}$$

For $m \in \{1,2,3,\ldots\}$, the product of several q-shifted factorials are given by

$$(x_1,x_2,\ldots,x_m;q)_n = (x_1;q)_n(x_2;q)_n\ldots(x_m;q)_n,$$
$$(x_1,x_2,\ldots,x_m;q)_\infty = (x_1;q)_\infty(x_2;q)_\infty\ldots(x_m;q)_\infty.$$

Taking $x = aq^{-n}, a \neq q^n$ in (1.2), we have the following relation:

$$\bigl(aq^{-n};q\bigr)_n = \frac{(aq^{-n};q)_\infty}{(a;q)_\infty} = (q/a;q)_n(-a)^n q^{-n-\binom{n}{2}}. \tag{1.4}$$

The q-binomial coefficient is defined as [16]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{-n};q)_k}{(q;q)_k}(-1)^k q^{nk-\binom{k}{2}}, \quad 0 \leq k \leq n. \tag{1.5}$$

Chen *et al.* [15] introduced the homogeneous q-difference operator D_{xy}, Saad and Sukhi [23] introduced the dual homogeneous q-difference operator θ_{xy} as

$$D_{xy}\bigl\{f(x,y)\bigr\} := \frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}, \qquad \theta_{xy}\bigl\{f(x,y)\bigr\} := \frac{f(q^{-1}x,y)-f(x,qy)}{q^{-1}x-y}. \tag{1.6}$$

Al-Salam and Carlitz [2, Eqs. (1.11) and (1.15)] have introduced the following polynomials:

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (a;q)_k x^k \quad \text{and} \quad \psi_n^{(a)}(x|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)}\bigl(aq^{1-k};q\bigr)_k x^k. \tag{1.7}$$

Since then, these polynomials are called "*Al-Salam–Carlitz polynomials*" by many authors. Because of their considerable role in the theories of q-series and q-orthogonal polynomials, many authors investigated an extension of the Al-Salam–Carlitz polynomials (see [7, 12, 28, 35]).

Recently, Cao [7, Eq. (4.7)] has introduced two families of generalized Al-Salam–Carlitz polynomials,

$$\phi_n^{(a,b,c)}(x,y|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a,b;q)_k}{(c;q)_k}x^k y^{n-k}, \tag{1.8}$$

$$\psi_n^{(a,b,c)}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1)^k q^{\binom{k+1}{2}-nk}(a,b;q)_k}{(c;q)_k}x^k y^{n-k}, \tag{1.9}$$

together with the following generating functions [7, Eqs. (4.10) and (4.11)]:

$$\sum_{n=0}^{\infty}\phi_n^{(a,b,c)}(x,y|q)\frac{t^n}{(q;q)_n}=\frac{1}{(xt;q)_\infty}{}_2\Phi_1\left[\begin{matrix}a,b;\\c;\end{matrix}\ q;yt\right]\quad\bigl(\max\bigl\{|yt|,|xt|\bigr\}<1\bigr), \tag{1.10}$$

$$\sum_{n=0}^{\infty}\psi_n^{(a,b,c)}(x,y|q)\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}=(xt;q)_\infty{}_2\Phi_1\left[\begin{matrix}a,b;\\c;\end{matrix}\ q;yt\right]\quad\bigl(|xt|<1\bigr). \tag{1.11}$$

Motivated by the work of Cao [7], the authors [12] introduced a new extension of the Al-Salam–Carlitz polynomials $\phi_n^{\binom{a,b,c}{d,e}}(x,y|q)$, $\psi_n^{\binom{a,b,c}{d,e}}(x,y|q)$,

$$\phi_n^{\binom{a,b,c}{d,e}}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(a,b,c;q)_k}{(d,e;q)_k}x^{n-k}y^k, \tag{1.12}$$

$$\psi_n^{\binom{a,b,c}{d,e}}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1)^k q^{k(k-n)}(a,b,c;q)_k}{(d,e;q)_k}x^{n-k}y^k, \tag{1.13}$$

and obtained the following results.

Proposition 1 ([12, Theorem 4]) *Let $f(a,b,c,d,e,x,y)$ be a seven-variable analytic function in a neighborhood of $(a,b,c,d,e,x,y)=(0,0,0,0,0,0,0)\in\mathbb{C}^7$.*

(I) $f(a,b,c,d,e,x,y)$ can be expanded in terms of $\phi_n^{\binom{a,b,c}{d,e}}(x,y|q)$ if and only if

$$\begin{aligned}
&x\bigl\{f(a,b,c,d,e,x,y)-f(a,b,c,d,e,x,yq)\\
&\quad-(d+e)q^{-1}\bigl[f(a,b,c,d,e,x,yq)-f\bigl(a,b,c,d,e,x,yq^2\bigr)\bigr]\\
&\quad+deq^{-2}\bigl[f\bigl(a,b,c,d,e,x,yq^2\bigr)-f\bigl(a,b,c,d,e,x,yq^3\bigr)\bigr]\bigr\}\\
&=y\bigl\{\bigl[f(a,b,c,d,e,x,y)-f(a,b,c,d,e,xq,y)\bigr]\\
&\quad-(a+b+c)\bigl[f(a,b,c,d,e,x,yq)-f(a,b,c,d,e,xq,yq)\bigr]\\
&\quad+(ab+ac+bc)\bigl[f\bigl(a,b,c,d,e,x,yq^2\bigr)-f\bigl(a,b,c,d,e,xq,yq^2\bigr)\bigr]\\
&\quad-abc\bigl[f\bigl(a,b,c,d,e,x,yq^3\bigr)-f\bigl(a,b,c,d,e,xq,yq^3\bigr)\bigr]\bigr\}.
\end{aligned} \tag{1.14}$$

(II) $f(a,b,c,d,e,x,y)$ can be expanded in terms of $\psi_n^{\binom{a,b,c}{d,e}}(x,y|q)$ if and only if

$$\begin{aligned}
&x\bigl\{f(a,b,c,d,e,xq,y)-f(a,b,c,d,e,xq,yq)\\
&\quad-(d+e)q^{-1}\bigl[f(a,b,c,d,e,xq,yq)-f\bigl(a,b,c,d,e,xq,yq^2\bigr)\bigr]\\
&\quad+deq^{-2}\bigl[f\bigl(a,b,c,d,e,xq,yq^2\bigr)-f\bigl(a,b,c,d,e,xq,yq^3\bigr)\bigr]\bigr\}\\
&=y\bigl\{\bigl[f(a,b,c,d,e,xq,yq)-f(a,b,c,d,e,x,yq)\bigr]\\
&\quad-(a+b+c)\bigl[f\bigl(a,b,c,d,e,xq,yq^2\bigr)-f\bigl(a,b,c,d,e,x,yq^2\bigr)\bigr]\\
&\quad+(ab+ac+bc)\bigl[f\bigl(a,b,c,d,e,xq,yq^3\bigr)-f\bigl(a,b,c,d,e,x,yq^3\bigr)\bigr]
\end{aligned}$$

$$- abc\big[f\big(a,b,c,d,e,xq,yq^4\big) - f\big(a,b,c,d,e,x,yq^4\big)\big]\big\}. \tag{1.15}$$

Subsequently, Cao *et al.* [13], gave another extension of Al-Salam–Carlitz polynomials called "*generalized Verma–Jain polynomials*",

$$\omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a,b,c;q)_k}{(d,e;q)_k} P_{n-k}(x,y) z^k, \tag{1.16}$$

$$\mu_n^{\binom{a,b,c}{d,e}}(x,y,z|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a,b,c;q)_k}{(d,e;q)_k} P_{n-k}(y,x) z^k, \tag{1.17}$$

where

$$P_n(x,y) = (x-y)(x-qy)\ldots\big(x-q^{n-1}y\big) = (y/x;q)_n x^n \tag{1.18}$$

are the Cauchy polynomials.

Remark 2 Upon setting $(y,z) = (0,y)$, the polynomial (1.16) reduces to (1.12).

Motivated by the recent work of Cao [7], Cao *et al.* [12, 13] and with the aid of the polynomials (1.17), we introduce the q-polynomials $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$.

Definition 3 The q-polynomials $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ are defined by

$$\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k}{2}}(a,b,c;q)_k}{(d,e;q)_k} P_{n-k}(y,x) z^k. \tag{1.19}$$

Remark 4 The q-polynomials (1.19) can be viewed as a general form of the Hahn polynomials.

(1) Taking $r = s = 3$, $\mathbf{a} = (a,b,c)$ and $\mathbf{b} = (d,e,0)$ in [29, Definition 1], the q-polynomials (1.19) is a special case of the generalized q-hypergeometric polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$, i.e.,

$$\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q) = (-1)^n q^{\binom{n}{2}} \Psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q).$$

(2) Upon setting $(y,z) = (0,y)$, the q-polynomials $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ defined in (1.19) reduce to the polynomials $\psi_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ [12],

$$\zeta_n^{\binom{a,b,c}{d,e}}(x,0,y|q) = (-1)^n q^{-\binom{n}{2}} \psi_n^{\binom{a,b,c}{d,e}}(x,y|q).$$

(3) For $b = c = d = e = 0$ and $z = -b$, the q-polynomials $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ reduce to the generalized Hahn polynomials $h_n(x,y,a,b|q)$ [30],

$$\zeta_n^{\binom{a,0,0}{0,0}}(x,y,-b|q) = h_n(x,y,a,b|q).$$

(4) Setting $a = b = c = d = e = 0$ and $z = -z$, the q-polynomials $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ reduce to the trivariate q-polynomials $F_n(x,y,z;q)$ [1],

$$\zeta_n^{\binom{0,0,0}{0,0}}(x,y,-z|q) = (-1)^n q^{\binom{n}{2}} F_n(x,y,z;q).$$

(5) If we let $a = b = c = d = e = 0, y = ax$ and $z = -y$, the q-polynomials $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ reduce to $\psi_n^{(a)}(x,y|q)$ [6],

$$\zeta_n^{\binom{0,0,0}{0,0}}(x,ax,-y|q) = (-1)^n q^{\binom{n}{2}} \psi_n^{(a)}(x,y|q).$$

(6) For $b = c = d = e = 0, x = 0, y = x$ and $z = -y$, the q-polynomials $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ reduce to the polynomials $P_n(x,y,a)$ [3],

$$\zeta_n^{\binom{a,0,0}{0,0}}(0,x,-y|q) = P_n(x,y,a).$$

(7) Also, $a = b = c = d = e = 0, y = ax$ and $z = -1$, the q-polynomials $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ reduce to Hahn polynomials $\psi_n^{(a)}(x|q)$ [2],

$$\zeta_n^{\binom{0,0,0}{0,0}}(x,ax,-1|q) = (-1)^n q^{\binom{n}{2}} \psi_n^{(a)}(x|q).$$

The paper is organized as follows. In Sect. 2, we give and prove our main results to be used in the sequel. In Sect. 3, we obtain generating function for q-polynomials. In Sect. 4, we obtain the Srivastava–Agarwal type generating function for q-hypergeometric polynomials. In Sect. 5, we deduce mixed generating functions for the Rajković–Marinković–Stanković polynomials. In Sect. 6, we derive $U(n+1)$ generalizations of the generating functions for q-hypergeometric polynomials.

2 Proof of main results

In this section, we will give and prove our main results to be used in the sequel.

Theorem 5 *Let $f(a,b,c,d,e,x,y,z)$ be an eight-variable analytic function in a neighborhood of $(a,b,c,d,e,x,y,z) = (0,0,0,0,0,0,0,0) \in \mathbb{C}^8$.*

(I) If $f(a,b,c,d,e,x,y,z)$ can be expanded in terms of $\omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ if and only if

$$\begin{aligned}
&(x - q^{-1}y)\big\{\big[f(a,b,c,d,e,x,y,z) - f(a,b,c,d,e,x,y,qz)\big]\\
&\quad - (d+e)q^{-1}\big[f(a,b,c,d,e,x,y,qz) - f\big(a,b,c,d,e,x,y,q^2z\big)\big]\\
&\quad + deq^{-2}\big[f\big(a,b,c,d,e,x,y,q^2z\big) - f\big(a,b,c,d,e,x,y,q^3z\big)\big]\big\}\\
&= z\big\{\big[f\big(a,b,c,d,e,x,q^{-1}y,z\big) - f(a,b,c,,d,e,qx,y,z)\big]\\
&\quad - (a+b+c)\big[f\big(a,b,c,d,e,x,q^{-1}y,qz\big) - f(a,b,c,d,e,qx,y,qz)\big]\\
&\quad + (ab+ac+bc)\big[f\big(a,b,c,d,e,x,q^{-1}y,q^2z\big) - f\big(a,b,c,d,e,qx,y,q^2z\big)\big]\\
&\quad - abc\big[f\big(a,b,c,d,e,x,q^{-1}y,q^3z\big) - f\big(a,b,c,d,e,qx,y,q^3z\big)\big]\big\}.
\end{aligned} \tag{2.1}$$

(II) If $f(a,b,c,d,e,x,y,z)$ *can be expanded in terms of* $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$ *if and only if*

$$\begin{aligned}
&(q^{-1}x-y)\{[f(a,b,c,d,e,x,y,z)-f(a,b,c,d,e,x,y,qz)]\\
&\quad-(d+e)q^{-1}[f(a,b,c,d,e,x,y,qz)-f(a,b,c,d,e,x,y,q^2z)]\\
&\quad+deq^{-2}[f(a,b,c,d,e,x,y,q^2z)-f(a,b,c,d,e,x,y,q^3z)]\}\\
&=z\{[f(a,b,c,d,e,x,qy,qz)-f(a,b,c,d,e,q^{-1}x,y,qz)]\\
&\quad-(a+b+c)[f(a,b,c,d,e,x,qy,q^2z)\\
&\quad-f(a,b,c,d,e,q^{-1}x,y,q^2z)]\\
&\quad+(ab+ac+bc)[f(a,b,c,d,e,x,qy,q^3z)-f(a,b,c,d,e,q^{-1}x,y,q^3z)]\\
&\quad-abc[f(a,b,c,d,e,x,qy,q^4z)-f(a,b,c,d,e,q^{-1}x,y,q^4z)]\}.
\end{aligned} \tag{2.2}$$

Remark 6 For $(y,z)=(0,y)$, Eq. (2.1) reduces to (1.14).

To prove Theorem 5, we need the following lemmas.

Lemma 7 ([17, Hartogs theorem]) *If a complex-valued function is holomorphic* (*analytic*) *in each variable separately in an open domain* $D\in\mathbb{C}^n$, *then it is holomorphic* (*analytic*) *in* D.

Lemma 8 ([20, Proposition 1]) *If* $f(x_1,x_2,\ldots,x_k)$ *is analytic at the origin* $(0,0,\ldots,0)\in\mathbb{C}^k$, *then* f *can be expanded in an absolutely convergent power series*,

$$f(x_1,x_2,\ldots,x_k)=\sum_{n_1,n_2,\ldots,n_k=0}^{\infty}\alpha_{n_1,n_2,\ldots,n_k}x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k}.$$

Proof of Theorem 5 (I) From Lemmas 7 and 8, we assume that there exists a sequence $\{A_n\}$ such that

$$f(a,b,c,d,e,x,y,z)=\sum_{n=0}^{\infty}A_n(a,b,c,d,e,x,y)z^n. \tag{2.3}$$

First, substituting (2.3) into Eq. (2.1), we have

$$\begin{aligned}
&(x-q^{-1}y)\sum_{n=0}^{\infty}[1-q^n-(d+e)q^{n-1}+(d+e)q^{2n-1}+deq^{2n-2}-deq^{3n-2}]\\
&\quad\times A_n(a,b,c,d,e,x,y)z^n\\
&=\sum_{n=0}^{\infty}[1-(a+b+c)q^n+(ab+bc+ac)q^{2n}-abcq^{3n}]\\
&\quad\times[A_n(a,b,c,d,e,x,q^{-1}y)-A_n(a,b,c,d,e,qx,y)]z^{n+1},
\end{aligned}$$

which is equal to

$$(x-q^{-1}y)\sum_{n=0}^{\infty}(1-q^n)(1-dq^{n-1})(1-eq^{n-1})A_n(a,b,c,d,e,x,y)z^n$$

$$= \sum_{n=0}^{\infty} (1-aq^n)(1-bq^n)(1-cq^n) \\ \times \left[A_n(a,b,c,d,e,x,q^{-1}y) - A_n(a,b,c,d,e,qx,y)\right] z^{n+1}. \tag{2.4}$$

Comparing coefficients of $z^n n \geq 1$, on both sides of Eq. (2.4), we readily find that

$$\begin{aligned} &(x-q^{-1}y)(1-q^n)(1-dq^{n-1})(1-eq^{n-1})A_n(a,b,c,d,e,x,y) \\ &\quad = (1-aq^{n-1})(1-bq^{n-1})(1-cq^{n-1}) \\ &\qquad \times \left[A_{n-1}(a,b,c,d,e,x,q^{-1}y) - A_{n-1}(a,b,c,d,e,qx,y)\right], \end{aligned}$$

which is equivalent to

$$A_n(a,b,c,d,e,x,y) = \frac{(1-aq^{n-1})(1-bq^{n-1})(1-cq^{n-1})}{(1-q^n)(1-dq^{n-1})(1-eq^{n-1})} D_{xy} A_{n-1}(a,b,c,d,e,x,y).$$

By iteration, we obtain

$$A_n(a,b,c,d,e,x,y) = \frac{(a,b,c;q)_n}{(q,d,e;q)_n} D_{xy}^n \left\{A_0(a,b,c,d,e,x,y)\right\}. \tag{2.5}$$

Taking $f(a,b,c,d,e,x,y,0) = A_0(a,b,c,d,e,x,y) = \sum_{n=0}^{\infty} \beta_n P_n(x,y)$ yields

$$A_k(a,b,c,d,e,x,y) = \frac{(a,b,c;q)_k}{(q,d,e;q)_k} \cdot \sum_{n=0}^{\infty} \beta_n \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(x,y), \tag{2.6}$$

and we have

$$\begin{aligned} f(a,b,c,d,e,x,y,z) &= \sum_{k=0}^{\infty} \frac{(a,b,c;q)_k}{(q,d,e;q)_k} \sum_{n=0}^{\infty} \beta_n \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(x,y) z^k \\ &= \sum_{n=0}^{\infty} \beta_n \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a,b,c;q)_k}{(d,e;q)_k} P_{n-k}(x,y) z^k \\ &= \sum_{n=0}^{\infty} \beta_n \omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q). \end{aligned}$$

Second, if $f(a,b,c,d,e,x,y,z)$ can be expanded in terms of $\omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$, we can verify that it satisfies (2.1).

In almost the same way, we assume that there exists a sequence $\{B_n\}$ such that

$$f(a,b,c,d,e,x,y,z) = \sum_{n=0}^{\infty} B_n(a,b,c,d,e,x,y) z^n. \tag{2.7}$$

Now, substituting Eq. (2.7) into Eq. (2.2), we have

$$(q^{-1}x - y) \sum_{n=0}^{\infty} \left[1 - q^n - (d+e)q^{n-1} + (d+e)q^{2n-1} + deq^{2n-2} - deq^{3n-2}\right]$$

$$\times B_n(a,b,c,d,e,x,y)z^n$$
$$= \sum_{n=0}^{\infty}\big[q^n - (a+b+c)q^{2n} + (ab+bc+ac)q^{3n} - abcq^{4n}\big]$$
$$\times \big[B_n(a,b,c,d,e,x,qy) - B_n\big(a,b,c,d,e,q^{-1}x,y\big)\big]z^{n+1},$$

which is equal to

$$\begin{aligned}&\big(q^{-1}x-y\big)\sum_{n=0}^{\infty}\big(1-q^n\big)\big(1-dq^{n-1}\big)\big(1-eq^{n-1}\big)B_n(a,b,c,d,e,x,y)z^n\\&\quad=\sum_{n=0}^{\infty}q^n\big(1-aq^n\big)\big(1-bq^n\big)\big(1-cq^n\big)\\&\qquad\times\big[B_n(a,b,c,d,e,x,qy)-B_n\big(a,b,c,d,e,q^{-1}x,y\big)\big]z^{n+1}.\end{aligned} \tag{2.8}$$

Comparing coefficients of $z^n n \geq 1$, on both sides of Eq. (2.8), we readily find that

$$\begin{aligned}&\big(q^{-1}x-y\big)\big(1-q^n\big)\big(1-dq^{n-1}\big)\big(1-eq^{n-1}\big)B_n(a,b,c,d,e,x,y)\\&\quad=q^{n-1}\big(1-aq^{n-1}\big)\big(1-bq^{n-1}\big)\big(1-cq^{n-1}\big)\\&\qquad\times\big[B_{n-1}(a,b,c,d,e,x,qy)-B_{n-1}\big(a,b,c,d,e,q^{-1}x,y\big)\big],\end{aligned}$$

which is equivalent to

$$B_n(a,b,c,d,e,x,y) = -q^{n-1}\frac{(1-aq^{n-1})(1-bq^{n-1})(1-cq^{n-1})}{(1-q^n)(1-dq^{n-1})(1-eq^{n-1})}\theta_{xy}B_{n-1}(a,b,c,d,e,x,y).$$

By iteration, we obtain

$$B_n(a,b,c,d,e,x,y) = \frac{(-1)^n q^{\binom{n}{2}}(a,b,c;q)_n}{(q,d,e;q)_n}\theta_{xy}^n\big\{B_0(a,b,c,d,e,x,y)\big\}. \tag{2.9}$$

Upon setting $f(a,b,c,d,e,x,y,0) = B_0(a,b,c,d,e,x,y) = \sum_{n=0}^{\infty}\beta_n P_n(y,x)$,

$$B_k(a,b,c,d,e,x,y) = \frac{q^{\binom{k}{2}}(a,b,c;q)_k}{(q,d,e;q)_k}\cdot\sum_{n=0}^{\infty}\beta_n\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(y,x), \tag{2.10}$$

we have

$$\begin{aligned}f(a,b,c,d,e,x,y,z) &= \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}(a,b,c;q)_k}{(q,d,e;q)_k}\sum_{n=0}^{\infty}\beta_n\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(x,y)z^k\\&=\sum_{n=0}^{\infty}\beta_n\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{q^{\binom{k}{2}}(a,b,c;q)_k}{(d,e;q)_k}P_{n-k}(x,y)z^k\\&=\sum_{n=0}^{\infty}\beta_n\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q).\end{aligned}$$

Finally, if $f(a,b,c,d,e,x,y,z)$ can be written in terms of $\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)$, we can verify that $f(a,b,c,d,e,x,y,z)$ satisfies (2.2). The proof of Theorem 5 is complete. □

3 Generating function for new generalized q-polynomials

In this section, our aim is to give and prove the generating functions for q-polynomials by means of the q-difference equations.

Theorem 9 *It is asserted that*

$$\sum_{n=0}^{\infty}\omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(xt;q)_\infty}{}_3\Phi_2\left[\begin{matrix}a,b,c;\\ d,e;\end{matrix}\ q;zt\right]\quad\big(\max\{|xt|,|zt|\}<1\big)\tag{3.1}$$

and

$$\sum_{n=0}^{\infty}\zeta_n^{\binom{a,b,c}{d,e}}(x,y,z|q)\frac{t^n}{(q;q)_n}=\frac{(xt;q)_\infty}{(yt;q)_\infty}{}_3\Phi_3\left[\begin{matrix}a,b,c;\\ 0,d,e;\end{matrix}\ q;-zt\right]\quad\big(|yt|<1\big).\tag{3.2}$$

Remark 10 Equations (3.1) and (3.2) reduce to Eqs. (1.10) and (1.11), respectively, when $c=e=y=0$ in Theorem 9.

Proof of Theorem 9 We denote the right-hand side of Eq. (3.1) by $f(a,b,c,d,e,x,y,z)$. One can verify that $f(a,b,c,d,e,x,y,z)$ satisfies (2.1). So, there exists a sequence $\{\beta_n\}$, such that

$$f(a,b,c,d,e,x,y,z)=\sum_{n=0}^{\infty}\beta_n\omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q).\tag{3.3}$$

Upon setting $z=0$ in Eq. (3.3) and then using the obvious fact $\omega_n^{\binom{a,b,c}{d,e}}(x,y,0|q)=P_n(x,y)$, we have

$$f(a,b,c,d,e,x,y,0)=\sum_{n=0}^{\infty}\beta_nP_n(x,y)=\frac{(yt;q)_\infty}{(xt;q)_\infty}=\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}.$$

So, the function $f(a,b,c,d,e,x,y,z)$ is equivalent to

$$f(a,b,c,d,e,x,y,z)=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q),$$

which equals the right-hand side of Eq. (3.1). Similarly, we prove Eq. (3.2).

The proof of Theorem 9 is complete. □

4 Srivastava–Agarwal type generating function for q-hypergeometric polynomials

We recall that the following Srivastava–Agarwal type generating functions for the Al-Salam–Carlitz polynomials. See also [10, 19] for some recent work on generating functions.

Proposition 11 ([27, Eq. (3.20)] and [5, Eq. (5.4)]) *We have*

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(z|q)\frac{(\lambda;q)_n t^n}{(q;q)_n}=\frac{(\lambda t;q)_\infty}{(t;q)_\infty}{}_2\Phi_1\left[\begin{matrix}\lambda,\alpha;\\ \lambda t;\end{matrix}\; q;zt\right]\quad\bigl(\max\bigl\{|t|,|xt|\bigr\}<1\bigr) \tag{4.1}$$

and

$$\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x|q)(1/\lambda;q)_n\frac{(\lambda tq)^n}{(q;q)_n}=\frac{(xtq;q)_\infty}{(\lambda xtq;q)_\infty}{}_2\Phi_1\left[\begin{matrix}1/\lambda,1/(\alpha x);\\ 1/(\lambda xt);\end{matrix}\; q;\alpha q\right]$$
$$\bigl(\max\bigl\{|\lambda xtq|,|\alpha q|\bigr\}<1\bigr). \tag{4.2}$$

In this section, we state and prove the Srivastava–Agarwal type bilinear generating functions for q-hypergeometric polynomials by the method of homogeneous q-difference equations.

Theorem 12 *It is asserted that*

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)\omega_n^{\binom{a,b,c}{d,e}}(u,v,z|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(vt,\alpha x;q)_\infty}{(ut,x;q)_\infty}\sum_{k=0}^{\infty}\frac{(ut,\alpha;q)_k q^k}{(q/x,vt,q;q)_k}{}_3\Phi_2\left[\begin{matrix}a,b,c;\\ d,e;\end{matrix}\; q;ztq^k\right]$$
$$\bigl(\max\bigl\{|ut|,|zt|,|x|\bigr\}<1\bigr) \tag{4.3}$$

and

$$\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x|q)\zeta_n^{\binom{a,b,c}{d,e}}(u,v,z|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(q/x,uxtq;q)_\infty}{(\alpha q,vxtq;q)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(1/(\alpha x),1/(uxt);q)_n}{(q/x,1/(vxt),q;q)_n}\left(\frac{\alpha uq}{v}\right)^n{}_3\Phi_3\left[\begin{matrix}a,b,c;\\ 0,d,e;\end{matrix}\; q;-zxtq^{1-n}\right]$$
$$\bigl(\max\bigl\{|\alpha q|,|vxt|\bigr\}<1\bigr). \tag{4.4}$$

Remark 13 For $c=e=0, b=d$ and $y=0, x=1$ in Theorem 12, Eq. (4.3) reduces to (4.1)

To prove Theorem 12, the following proposition is necessary.

Proposition 14 ([4, Theorem 5.2] and [16, Eq. (III.4)]) *We have*

$${}_2\Phi_1\left[\begin{matrix}a,b;\\ c;\end{matrix}\; q;z\right]=\frac{(abz/c;q)_\infty}{(az/c;q)_\infty}{}_3\Phi_2\left[\begin{matrix}b,c/a,0;\\ qc/(az),c;\end{matrix}\; q;q\right] \tag{4.5}$$

and

$${}_2\Phi_1\left[\begin{matrix}a,b;\\ c;\end{matrix}\; q;z\right]=\frac{(bz;q)_\infty}{(z;q)_\infty}{}_2\Phi_2\left[\begin{matrix}b,c/a;\\ bz,c;\end{matrix}\; q;az\right]. \tag{4.6}$$

Proof of Theorem 12 If we use $f(a,b,c,d,e,x,y,z)$ to denote the right-hand side of (4.3), we calculate that $f(a,b,c,d,e,x,y,z)$ satisfies (2.1). Thus, there exists a sequence $\{a_n\}$ independent of x, y and z such that

$$f(a,b,c,d,e,x,y,z)=\sum_{n=0}^{\infty}a_n\omega_n^{\binom{a,b,c}{d,e}}(u,v,z|q). \tag{4.7}$$

Letting $z=0$ in Eq. (4.7) and utilizing the obvious fact $\omega_n^{\binom{a,b,c}{d,e}}(u,v,0|q)=P_n(u,v)$, we have

$$\begin{aligned}
f(a,b,c,d,e,u,v,0)&=\sum_{n=0}^{\infty}a_nP_n(u,v)=\frac{(vt,\alpha x;q)_\infty}{(ut,x;q)_\infty}\sum_{k=0}^{\infty}\frac{(ut,\alpha;q)_kq^k}{(q/x,vt,q;q)_k}\\
&=\frac{(vt,\alpha x;q)_\infty}{(ut,x;q)_\infty}{}_3\Phi_2\left[\begin{matrix}ut,\alpha,0;\\ q/x,vt;\end{matrix}\ q;q\right]\quad\text{by (4.5)}\\
&=\frac{(vt;q)_\infty}{(ut;q)_\infty}{}_2\Phi_1\left[\begin{matrix}v/u,\alpha;\\ vt;\end{matrix}\ q;xut\right]\\
&=\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x)P_n(u,v)\frac{t^n}{(q;q)_n}.
\end{aligned}$$

Hence

$$f(a,b,c,d,e,u,v,z)=\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x)\omega_n^{\binom{a,b,c}{d,e}}(u,v,z|q)\frac{t^n}{(q;q)_n},$$

which is equal to the left-hand side of (4.3).

Similarly, if we use $f(a,b,c,d,e,x,y,z)$ to denote the right-hand side of (4.4), we test that $g(a,b,c,d,e,x,y,z)$ satisfies (2.2). Thus, there exists a sequence $\{b_n\}$ independent of x, y and z such that

$$g(a,b,c,d,e,u,v,z)=\sum_{n=0}^{\infty}b_n\zeta_n^{\binom{a,b,c}{d,e}}(u,v,z|q). \tag{4.8}$$

Setting $z=0$ in Eq. (4.8), using the obvious fact $\zeta_n^{\binom{a,b,c}{d,e}}(u,v,0|q)=P_n(v,u)$, we have

$$\begin{aligned}
&g(a,b,c,d,e,u,v,0)\\
&\quad=\sum_{n=0}^{\infty}b_nP_n(v,u)=\frac{(q/x,uxtq;q)_\infty}{(\alpha q,vxtq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}(1/(\alpha x),1/(uxt);q)_n}{(q/x,1/(vxt),q;q)_n}\left(\frac{\alpha uq}{v}\right)^n\\
&\quad=\frac{(q/x,uxtq;q)_\infty}{(\alpha q,vxtq;q)_\infty}{}_2\Phi_2\left[\begin{matrix}1/(\alpha x),1/(uxt);\\ q/x,1/(vxt);\end{matrix}\ q;\frac{\alpha uq}{v}\right]\quad\text{by (4.6)}\\
&\quad=\frac{(uxtq;q)_\infty}{(vxtq;q)_\infty}{}_2\Phi_2\left[\begin{matrix}u/v,1/(\alpha x);\\ 1/(vxt);\end{matrix}\ q;\alpha q\right]\\
&\quad=\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x|q)P_n(v,u)\frac{(qt)^n}{(q;q)_n}.
\end{aligned}$$

Hence

$$g(a,b,c,d,e,u,v,z) = \sum_{n=0}^{\infty} \psi_n^{(\alpha)}(x)\zeta_n^{\binom{a,b,c}{d,e}}(u,v,z|q)\frac{(qt)^n}{(q;q)_n},$$

which is equal to the left-hand side of (2.2). This completes the proof of Theorem 12. □

Theorem 15 *For $s \in \mathbb{N}$, we have*

$$\sum_{n=0}^{\infty} \omega_{n+s}^{\binom{a,b,c}{d,e}}(x,y,z|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{t^s(xt;q)_\infty}\sum_{k=0}^{s}\frac{(q^{-s},xt;q)_k q^k}{(q;yt;q)_k}{}_3\Phi_2\left[\begin{matrix} a,b,c; \\ d,e; \end{matrix} q;ztq^k\right]$$
$$\bigl(\max\bigl\{|xt|,|zt|\bigr\} < 1\bigr) \tag{4.9}$$

and

$$\sum_{n=0}^{\infty} \zeta_{n+s}^{\binom{a,b,c}{d,e}}(x,y,z|q)\frac{t^n}{(q;q)_n} = \frac{(xt;q)_\infty}{t^s(yt;q)_\infty}\sum_{k=0}^{s}\frac{(q^{-s},yt;q)_k q^k}{(q;xt;q)_k}{}_3\Phi_3\left[\begin{matrix} a,b,c; \\ d,e,0; \end{matrix} q;-ztq^k\right]$$
$$\bigl(|yt| < 1\bigr). \tag{4.10}$$

Corollary 16 ([35, Eq. (2.1)]) *For $s \in \mathbb{N}$ and $\max\{|z|,|xz|,|b|\} < 1$, we have*

$$\sum_{n=0}^{\infty} \Omega_{n+s}(x;a,b|q)\frac{z^n}{(q;q)_n} = \frac{(b,axz,bzq^s;q)_\infty}{(z,xz,bq^s;q)_\infty}{}_3\Phi_2\left[\begin{matrix} q^{-s},a,x; \\ axz,q^{1-s}/b; \end{matrix} q;\frac{qx}{b}\right]. \tag{4.11}$$

Remark 17 For $c = e = 0, b = d$ and $x = 1$ in Theorem 15, Eq. (4.9) reduces to (4.11).

To prove Theorem 15, we need the following lemma.

Lemma 18 ([16, Eq. (II.6)]) *The q-Chu–Vandermonde formula is given by*

$${}_2\Phi_1\left[\begin{matrix} q^{-n},a; \\ c; \end{matrix} q;q\right] = \frac{(c/a;q)_n}{(c;q)_n}a^n \quad \bigl(n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}\bigr). \tag{4.12}$$

Proof of Theorem 15 If we denote the right-hand side of Eq. (4.9) equivalently by

$$f(a,b,c,d,e,x,y,z) = t^{-s}\sum_{k=0}^{s}\frac{(q^{-s};q)_k q^k}{(q;q)_k}\frac{(ytq^k;q)_\infty}{(xtq^k;q)_\infty}{}_3\Phi_2\left[\begin{matrix} a,b,c; \\ d,e; \end{matrix} q;ztq^k\right],$$

we test that $f(a,b,c,d,e,x,y,z)$ satisfies Eq. (2.1). Thus, there exists a sequence $\{\alpha_n\}$ independent of x, y and z such that

$$f(a,b,c,d,e,x,y,z) = \sum_{n=0}^{\infty}\alpha_n\omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q).$$

We set $z = 0$ in the above equation, using the notable fact $\omega_n^{\binom{a,b,c}{d,e}}(x,y,0|q) = P_n(x,y)$, we have

$$f(a,b,c,d,e,x,y,0) = \sum_{n=0}^{\infty} \beta_n P_n(x,y) = \frac{(yt;q)_\infty}{t^s(xt;q)_\infty} {}_2\Phi_1 \left[\begin{matrix} q^{-s}, xt; \\ yt; \end{matrix} q;q \right] \quad \text{by (4.12)}$$
$$= \frac{(ytq^s;q)_\infty P_s(x,y)}{(xt;q)_\infty} = \sum_{n=0}^{\infty} P_{n+s}(x,y) \frac{t^n}{(q;q)_n} = \sum_{n=s}^{\infty} P_n(x,y) \frac{t^{n-s}}{(q;q)_{n-s}}.$$

We immediately conclude that

$$f(a,b,c,d,e,x,y,z) = \sum_{n=0}^{\infty} \omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q) \frac{t^{n-s}}{(q;q)_{n-s}} = \sum_{n=0}^{\infty} \omega_{n+s}^{\binom{a,b,c}{d,e}}(x,y,z|q) \frac{t^n}{(q;q)_n},$$

which is equal to the left-hand side of (4.9).

Similarly, we get (4.10). This completes the proof of Theorem 15. □

5 Some new mixed generating functions for the Rajković–Marinković–Stanković polynomials

In this section, we give and prove the mixed generating functions for the Rajković–Marinković–Stanković polynomials.

Let a and b be two real numbers, the Thomae–Jackson q-integral is defined as [16, 18, 34]

$$\int_a^b f(x)\,\mathrm{d}_q x = (1-q) \sum_{n=0}^{\infty} \bigl[bf(bq^n) - af(aq^n) \bigr] q^n. \tag{5.1}$$

Assume that $\alpha \in \mathbb{R}^+$ and $0 < a < x < 1$, the generalized Riemann–Liouville fractional q-integral operator is defined by [22] (see [11])

$$\bigl(I_{q,a}^{\alpha} f\bigr)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x;q)_{\alpha-1} f(t)\,\mathrm{d}_q t. \tag{5.2}$$

Due to the q-integral (5.1), we rewrite fractional q-integral (5.2) equivalently as follows (see [9, 11, 14]):

$$\bigl(I_{q,a}^{\alpha} f\bigr)(x) = \frac{x^{\alpha-1}(1-q)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} \bigl[x\bigl(q^{n+1};q\bigr)_{\alpha-1} f\bigl(xq^n\bigr) - a\bigl(aq^{n+1}/x;q\bigr)_{\alpha-1} f\bigl(aq^n\bigr) \bigr] q^n. \tag{5.3}$$

Recall that the Rajković–Marinković–Stanković polynomials are defined [22] (see [8, 11]) by

$$\mathcal{P}_n(\alpha,a,x|q) = I_{q,a}^{\alpha}\bigl\{x^n\bigr\} = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q! a^{n-k}}{\Gamma_q(\alpha+k+1)} x^{\alpha+k} (a/x;q)_{\alpha+k}, \tag{5.4}$$

where $\alpha \in \mathbb{R}^*$ and $0 < a < x < 1$.

We have the following lemmas.

Lemma 19 ([8, Lemma 10]) *For* $\alpha \in \mathbb{R}^+, 0 < a < x < 1$, *we have*

$$\sum_{n=0}^{\infty} \mathcal{P}_n(\alpha, a, x|q) \frac{w^n}{(q;q)_n} = \frac{(1-q)^\alpha}{(aw;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k} w^k}{(q;q)_{\alpha+k}}. \tag{5.5}$$

Lemma 20 ([8, Theorem 3]) *For* $\alpha \in \mathbb{R}^+, 0 < a < x < 1$ *and if* $\max\{|at|, |az|\} < 1$, *we have*

$$I_{q,a}^{\alpha}\left\{\frac{(bxz, tx;q)_\infty}{(xs, xz;q)_\infty}\right\} = \frac{(1-q)^\alpha (abz, at;q)_\infty}{(as, az;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k (q;q)_{\alpha+k}} {}_3\Phi_2\left[\begin{matrix} q^{-k}, as, az; \\ at, abz; \end{matrix}\ q;q\right]. \tag{5.6}$$

Remark 21 Upon taking $z = 0$ in (5.6) and by the means of the q-Chu–Vandermonde formula (4.12), we obtain

$$I_{q,a}^{\alpha}\left\{\frac{(xt;q)_\infty}{(xs;q)_\infty}\right\} = \frac{(1-q)^\alpha (at;q)_\infty}{(as;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{(q;q)_{\alpha+k}} \frac{(t/s;q)_k s^k}{(at;q)_k}, \quad |as| < 1. \tag{5.7}$$

Using Lemma 20 and the theory of q-difference equations, we are able to deduce the following new mixed generating functions for the Rajković–Marinković–Stanković polynomials.

Theorem 22 *For* $\alpha \in \mathbb{R}^+, 0 < a < x < 1$, *and* $\max\{|aws|, |awt|\} < 1$, *we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathcal{P}_n(\alpha, a, x|q) \omega_n^{\binom{a_1, b_1, c_1}{d_1, e_1}}(s,t,r|q) \frac{w^n}{(q;q)_n} \\ &= \frac{(1-q)^\alpha (awt;q)_\infty}{(aws;q)_\infty} \sum_{n=0}^{\infty} \frac{(a_1, b_1, c_1, awt;q)_n (r/s)^n}{(q, d_1, e_1, t/s;q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, aws;q)_k q^k}{(q, awt;q)_k} \\ &\quad \times \sum_{m=0}^{\infty} \frac{x^{\alpha+m}(a/x;q)_{\alpha+m}}{a^m (q;q)_{\alpha+m}} {}_3\Phi_2\left[\begin{matrix} q^{-m}, awsq^k, awtq^n; \\ awt, awtq^k; \end{matrix}\ q;q\right] \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathcal{P}_n(\alpha, a, x|q) \zeta_n^{\binom{a_1, b_1, c_1}{d_1, e_1}}(s,t,r|q) \frac{w^n}{(q;q)_n} \\ &= \frac{(1-q)^\alpha (aws;q)_\infty}{(awt;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(a_1, b_1, c_1, aws;q)_n (r/t)^n}{(q, d_1, e_1, s/t;q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, awt;q)_k q^k}{(q, aws;q)_k} \\ &\quad \times \sum_{m=0}^{\infty} \frac{x^{\alpha+m}(a/x;q)_{\alpha+m}}{a^m (q;q)_{\alpha+m}} {}_3\Phi_2\left[\begin{matrix} q^{-m}, awtq^k, awsq^n; \\ aws, awsq^k; \end{matrix}\ q;q\right]. \end{aligned} \tag{5.9}$$

Proof of Theorem 22 The LHS of Eq. (5.8) is equal to

$$I_{q,a}^{\alpha}\left\{\sum_{n=0}^{\infty} \omega_n^{\binom{a_1, b_1, c_1}{d_1, e_1}}(s,t,r|q) \frac{(xw)^n}{(q;q)_n}\right\}$$

$$
\begin{aligned}
&= I_{q,a}^{\alpha}\left\{\frac{(xwt;q)_\infty}{xws;q_\infty}{}_3\Phi_2\left[\begin{matrix} a_1,b_1,c_1; \\ d_1,e_1; \end{matrix} q;rxw\right]\right\} \\
&= I_{q,a}^{\alpha}\left\{\frac{(xwt;q)_\infty}{xws;q_\infty}\sum_{n=0}^{\infty}\frac{(a_1,b_1,c_1;q)_n}{(q,d_1,e_1;q)_n}(rw)^n x^n\right\} \\
&= I_{q,a}^{\alpha}\left\{\frac{(xwt;q)_\infty}{xws;q_\infty}\sum_{n=0}^{\infty}\frac{(a_1,b_1,c_1;q)_n(rw)^n}{(q,d_1,e_1;q)_n}\frac{(xwt;q)_n}{(t/s;q)_n(ws)^n}\sum_{k=0}^{n}\frac{(q^{-n},xws;q)_k}{(q,xwt;q)_k}q^k\right\} \\
&= I_{q,a}^{\alpha}\left\{\sum_{n=0}^{\infty}\frac{(a_1,b_1,c_1;q)_n}{(q,d_1,e_1;q)_n}(r/s)^n\frac{(xwt;q)_n}{(t/s;q)_n}\sum_{k=0}^{n}\frac{(xwtq^k;q)_\infty(q^{-n};q)_k}{(xwsq^k;q)_\infty(q;q)_k}q^k\right\} \\
&= \sum_{n=0}^{\infty}\frac{(a_1,b_1,c_1;q)_n}{(q,d_1,e_1;q)_n}\frac{(r/s)^n}{(t/s;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q;q)_k}I_{q,a}^{\alpha}\left\{\frac{(xwt,xwtq^k;q)_\infty}{(xwtq^n,xwsq^k;q)_\infty}\right\} \\
&= \frac{(1-q)^{\alpha}(awt;q)_\infty}{(aws;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,b_1,c_1,awt;q)_n(r/s)^n}{(q,d_1,e_1,t/s;q)_n}\sum_{k=0}^{n}\frac{(q^{-n},aws;q)_k q^k}{(q,awt;q)_k} \\
&\quad\times\sum_{m=0}^{\infty}\frac{x^{\alpha+m}(a/x;q)_{\alpha+m}}{a^m(q;q)_{\alpha+m}}{}_3\Phi_2\left[\begin{matrix} q^{-m},awsq^k,awtq^n; \\ awt,awtq^k; \end{matrix} q;q\right],
\end{aligned}
$$

which equals the RHS of Eq. (5.8) after using (5.6). Similarly, we get (5.9). This completes the proof of Theorem 22. □

Remark 23 For $(t,r)=(0,0)$ in Theorem 22, we get (5.5).

6 The $U(n+1)$ generalizations of generating functions for q-hypergeometric polynomials

Lemma 24 ([21, Theorem 5.42]) *Let b,z and $x_1,\dots,x_n$ be indeterminate, and let $n\ge 1$. Suppose that none of the denominators in the following identity vanishes, $0<|q|<1$ and $|z|<|x_1,\dots,x_n||x_m|^{-n}|q|^{(n-1)/2}$, for $m=1,2,\dots,n$. Then we have*

$$
\begin{aligned}
&\sum_{\substack{y_k\ge 0\\ k=1,2,\dots,n}}\left\{\prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1}\prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\cdots+y_n)}(-1)^{(n-1)(y_1+\cdots+y_n)}\right. \\
&\quad\left.\times q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)[\binom{y_1}{2}+\cdots+\binom{y_n}{2}]-e_2(y_1,\dots,y_n)}(b;q)_{y_1+\cdots+y_n}z^{y_1+\cdots+y_n}\right\} \\
&= \frac{(bz;q)_\infty}{(z;q)_\infty},
\end{aligned}
\tag{6.1}
$$

where $e_2(y_1,\dots,y_n)$ is the second elementary symmetric function of $\{y_1,\dots,y_n\}$.

In this part, using the method of homogeneous q-difference equations, we derive the following $U(n+1)$ type generating functions for q-hypergeometric polynomials.

Theorem 25 *Let $b,z,x_1,\dots x_n,n\ge 1$ be indeterminate. Suppose that none of the denominators in the following identity vanishes, and that $0<|q|<1$, and $|z|<|x_1,\dots,x_n|\times$*

$|x_m|^{-n}|q|^{(n-1)/2}$, *for* $m = 1, 2, \ldots, n$. *Then we have the following*:

$$\sum_{\substack{y_k\geq 0\\ k=1,2,\ldots,n}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s};q \right)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i-(y_1+\cdots+y_n)} (-1)^{(n-1)(y_1+\cdots+y_n)} \right.$$
$$\left. \times q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)[\binom{y_1}{2}+\cdots+\binom{y_n}{2}]-e_2(y_1,\ldots,y_n)} \omega_{s+y_1+\cdots+y_n}^{\binom{a,b,c}{d,e}}(x,y,z|q) t^{y_1+\cdots+y_n} \right\}$$
$$= \frac{(yt;q)_\infty}{t^s(xt;q)_\infty} \sum_{k=0}^{s} \frac{(q^{-s},xt;q)_k q^k}{(q,yt;q)_k} {}_3\Phi_2 \left[\begin{matrix} a,b,c; \\ d,e; \end{matrix} \, q; ztq^k \right], \tag{6.2}$$

where $a = q^{-M}$ *and* $|xt| < 1$.

Remark 26 Setting $n = 1$ in Theorem 25, the assertion (6.2) reduces to (4.9).

Proof of Theorem 25 Upon taking $(b,z) = (yq^s/x, xt)$ in Eq. (6.1), we obtain

$$\sum_{\substack{y_k\geq 0\\ k=1,2,\ldots,n}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s};q \right)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i-(y_1+\cdots+y_n)} (-1)^{(n-1)(y_1+\cdots+y_n)} \right.$$
$$\left. \times q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)[\binom{y_1}{2}+\cdots+\binom{y_n}{2}]-e_2(y_1,\ldots,y_n)} P_{s+y_1+\cdots+y_n}(x, yq^s) t^{y_1+\cdots+y_n} \right\}$$
$$= \frac{(ytq^s;q)_\infty}{(xt;q)_\infty}. \tag{6.3}$$

If we use $f(a,b,c,d,e,x,y,z)$ to denote the left-hand side of Eq. (6.2), we can verify that $f(a,b,c,d,e,x,y,z)$ satisfies Eq. (2.1). There exists a sequence $\{\beta_n\}$ such that

$$f(a,b,c,d,e,x,y,z) = \sum_{n=0}^{\infty} \beta_n \omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q). \tag{6.4}$$

Setting $z = 0$ in Eq. (6.4) and then, using the obvious fact $\omega_n^{\binom{a,b,c}{d,e}}(x,y,0|q) = P_n(x,y)$, we have

$$f(a,b,c,d,e,x,y,0)$$
$$= \sum_{n=0}^{\infty} \beta_n P_n(x,y)$$
$$= \sum_{\substack{y_k\geq 0\\ k=1,2,\ldots,n}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s};q \right)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i-(y_1+\cdots+y_n)} (-1)^{(n-1)(y_1+\cdots+y_n)} \right.$$
$$\left. \times q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)[\binom{y_1}{2}+\cdots+\binom{y_n}{2}]-e_2(y_1,\ldots,y_n)} P_{s+y_1+\cdots+y_n}(x, yq^s) t^{y_1+\cdots+y_n} \right\}$$
$$= \frac{P_s(x,y)(ytq^s;q)_\infty}{(xt;q)_\infty}$$
$$= \sum_{n=0}^{\infty} P_{s+n}(x,y) \frac{t^n}{(q;q)_n}.$$

Hence

$$f(a,b,c,d,e,x,y,z)=\sum_{n=0}^{\infty}\omega_n^{\binom{a,b,c}{d,e}}(x,y,z|q)\frac{t^{n-s}}{(q;q)_{n-s}},$$

which is equal to the right-hand side of (6.2) by (4.9). The proof of Theorem 25 is complete. □

We remark in passing that, in a recently-published survey-cum-expository review article, the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [26, p. 340]).

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