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Closed-form pricing formula for foreign equity option with credit risk

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Abstract

Since credit risk in the over-the-counter (OTC) market has undoubtedly become very important issue, credit risk has to be considered when the options in the OTC market are priced. In this paper, we consider the valuation of foreign equity options with credit risk. In order to derive a closed-form pricing formula of this option, we adopt the partial differential equation (PDE) approach and use the Mellin transform method to solve the PDE. Specifically, triple Mellin transforms are used, and the pricing formula is presented as 3-dimensional normal cumulative distribution functions. Finally, we verify that our closed-form formula is accurate by comparing it with the numerical result from the Monte-Carlo simulation.

Keywords: Foreign equity option; Mellin transforms; Credit risk; Structural model

1 Introduction

Since the financial crisis in 2008, credit risk has been definitely the most important issue for researchers and practitioners in the financial market. Black–Scholes model [1], which is used widely for pricing of the financial derivatives, assumes that the counterparty has no credit risk. However, there is credit risk of the conunterparty in the over-the-counter (OTC) markets when trading various derivatives. Since the OTC markets have grown tremendously, it is very important to consider the credit risk when pricing the financial derivatives.

The options which are considered the credit risk of the counterparty have been called vulnerable options. Vulnerable options first were considered by Johnson and Stulz [2]. They assumed that the credit risk depends on the potential liability of the option writer. Klein [3] improved the result of Johnson and Stulz by allowing for the correlation between the asset of the option writer and the underlying asset of the option. Klein and Inglis [4] considered the stochastic interest rate model when the vulnerable option is priced. Additionally, Liao and Huang [5] studied the valuation of vulnerable options with early counterparty risk. The analytic pricing formula of vulnerable American options was provided under the Black–Scholes model by Chang and Hung [6]. Recently, many researchers have studied the pricing of vulnerable options with the improvement of dynamics of underlying assets to overcome the limit of the Black–Scholes model. In fact, two types models have been studied for the improved asset dynamics: the jump–diffusion models and the stochastic volatility models. In [7–10], the jump–diffusion models of underlying assets were considered for valuing of the vulnerable options. In [11–14], the stochastic volatility models, which describe the volatility smile in the real market, were used for the improvement of the vulnerable option pricing.

We study the vulnerable options with multiple assets in this paper. Specifically, we derive a closed-form solution of foreign equity option price with credit risk based on the partial differential equation (PDE) approach. Kwok and Wong [15] first derived the pricing formulas of different foreign equity options under the Black–Scholes model. Since the study of [15], there have been the extended results under various extensions of the Black–Scholes model, such as the Lévy process [16], stochastic volatility [17, 18], regime switching [19] and jump diffusion [20, 21]. However, there have been no studies on foreign equity options with credit risk. We adopt the structural model of Klein [3] for modeling of credit risk and the PDE approach to find the pricing formula of options. Moreover, we use the Mellin transforms to solve the PDE for the price of foreign equity options with credit risk. The Mellin transforms have been widely used for pricing of vulnerable options by many researchers. Yoon and Kim [22] first used the Mellin transforms to obtain vulnerable European option prices. Recently, many studies showed that the Mellin transforms are useful to solve the PDE for various types of financial derivatives with credit risk (Asian option [23], exchange option [24], path-dependent option [25, 26], dynamic fund protection [27] European option with early credit risk [28], lookback option [29]). This paper deals with the valuation of foreign equity option price with credit risk based on the PDE approach and provides a closed-form pricing formula of the options using the Mellin transforms.

The rest of this paper is organized as follows. Section 2 introduces the model used in this paper and indicates the PDE for the foreign equity option with credit risk. Section 3 presents the pricing formula of foreign equity options with credit risk solving the PDE with the Mellin transforms and shows the accuracy of our formula by a comparison between the price by the derived pricing formula and Monte-Carlo simulation price. Section 4 presents concluding remarks. Finally, in Appendices A and B, we provide the detailed components and supplements for our theorem.

2 Model

Let $S^f(t)$ and $S^d(t)$ be the prices of foreign and domestic stocks, respectively. We denote the exchange rate specified in domestic currency per unit of the foreign currency at time t by $Y(t)$, so that the relation between $S^f(t)$ and $S^d(t)$ is formulated as $S^d(t) = Y(t)S^f(t)$. We also assume that r^d and r^f are the domestic and foreign risk-free interest rates, respectively. The value processes for $S^f(t)$ and $S^d(t)$ are given by

$$\mathrm{d}S^d(t) = \left(r^d - q\right)S^d(t)\,\mathrm{d}t + \sigma_S S^d(t)\,\mathrm{d}W_t^1, \tag{1}$$

$$\mathrm{d}S^f(t) = \left(r^f - q - \rho_{13}\sigma_S\sigma_Y\right)S^d(t)\,\mathrm{d}t + \sigma_S S^f(t)\,\mathrm{d}W_t^1, \tag{2}$$

where q is the dividend of the stock, σ_S is the volatility of the foreign stock and W_t^1 is the standard Brownian motion under risk-neutral probability measure $\mathbb{P}^*$ In addition, we assume that value process of the firm V_t is given by

$$\mathrm{d}V(t) = r^d V(t)\,\mathrm{d}t + \sigma_V V(t)\,\mathrm{d}W_t^2, \tag{3}$$

where σ_V is the volatility of the firm value. As mentioned in [15], under the risk-neutral measure $\mathbb{P}^*$, the dynamic of the exchange rate $Y(t)$ is given by

$$\mathrm{d}Y(t) = \left(r^d - r^f\right)Y(t)\,\mathrm{d}t + \sigma_Y Y(t)\,\mathrm{d}W_t^3, \tag{4}$$

where σ_Y is the volatility of the exchange rate, and W_t^2 and W_t^3 are standard Brownian motions under risk-neutral probability measure $\mathbb{P}^*$ satisfying $\mathrm{d}\langle W_t^1, W_t^2\rangle = \rho_{12}\,\mathrm{d}t$, $\mathrm{d}\langle W_t^2, W_t^3\rangle = \rho_{23}\,\mathrm{d}t$, and $\mathrm{d}\langle W_t^1, W_t^3\rangle = \rho_{13}\,\mathrm{d}t$.

Let Π be the portfolio value in the domestic market and P be a no-arbitrage price of a European vulnerable call option. Then the portfolio value is given by

$$\Pi = P - \alpha_1 S^f - \alpha_2 V - \alpha_3 Y,$$

where $\alpha_1, \alpha_2, \alpha_3$ are the number of units of S^f, V, Y, respectively.

Applying Ito's lemma, we obtain the following stochastic differential equation:

$$\mathrm{d}\Pi = \mathrm{d}P - \alpha_1\,\mathrm{d}S^f - \alpha_2\,\mathrm{d}V - \alpha_3\,\mathrm{d}Y - \alpha_1\left(r^d - r^f + q + \rho_{13}\sigma_S\sigma_Y\right)S^f\,\mathrm{d}t - \alpha_3 r^f Y\,\mathrm{d}t. \tag{5}$$

From the equation $\mathrm{d}\Pi = r^d\Pi\,\mathrm{d}t$, the governing PDE for the foreign equity option price in foreign currency with credit risk of a contingent claim $C = C(t,s,v,y) \triangleq E^*[e^{-r_d(T-t)}u(S_T^f, V_T, Y_T) \mid S_t^f = s, V_t = v, Y_t = y]$, where u is the pay off function, is obtained. The PDE is

$$\begin{aligned}
&\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_S^2 S^{f2}\frac{\partial^2 C}{\partial S^{f2}} + \frac{1}{2}\sigma_V^2 V^2\frac{\partial^2 C}{\partial V^2} + \frac{1}{2}\sigma_Y^2 Y^2\frac{\partial^2 C}{\partial Y^2} \\
&\quad + \rho_{12}\sigma_S\sigma_V S^f V\frac{\partial^2 C}{\partial S^f\partial V} + \rho_{13}\sigma_S\sigma_Y S^f Y\frac{\partial^2 C}{\partial S^f\partial Y} + \rho_{23}\sigma_V\sigma_Y VY\frac{\partial^2 C}{\partial V\partial Y} \\
&\quad + \left(r^f - q - \rho_{13}\sigma_f\sigma_Y\right)S^f\frac{\partial C}{\partial S^f} + r^d V\frac{\partial C}{\partial V} + \left(r^d - r^f\right)Y\frac{\partial C}{\partial Y} - r^d C = 0,
\end{aligned} \tag{6}$$

with the terminal condition

$$C\left(T, S^f, V, Y\right) = Y(T)\left(S^f(T) - K\right)^+\left(1_{\{V(T)>D\}} + 1_{\{V(T)<D\}}\frac{(1-\alpha)V(T)}{D}\right),$$

where T is the maturity, K is the strike price, α is the deadweight cost related with the bankruptcy, and D is the value of the liabilities of the option issuer.

3 Pricing of foreign equity option in foreign currency with credit risk

In this section, we provide the exact pricing formula of foreign equity options using Mellin transform methods to solve PDE (6). Specifically, we provide the closed-form formula of options and present the implications with the Monte Carlo simulations to show the accuracy of the pricing formulas.

3.1 Pricing of foreign equity option

In this subsection, we derive the closed-form pricing formula of foreign equity options in foreign currency with credit risk. To derive the formula, we rewrite the dynamics introduced in the previous section. The dynamics used for valuing of the options is as follows:

$$dS^d(t) = (r^d - q)S^d(t)\,dt + \sigma_s S^d(t)\,dW_t^1, \tag{7}$$

$$dS^f(t) = (r^f - q - \rho_{13}\sigma_s\sigma_y)S^d(t)\,dt + \sigma_s S^f(t)\,dW_t^1, \tag{8}$$

$$dV(t) = r^d V(t)\,dt + \sigma_v V(t)\,dW_t^2, \tag{9}$$

$$dY(t) = (r^d - r^f)Y(t)\,dt + \sigma_y Y(t)\,dW_t^3. \tag{10}$$

Let us define $C_n(t,s,v,y)$ as $C_n(t,s,v,y) = C(t,s,v,y) \wedge n$ so that the boundedness of C_n is ensured. Due to its construction, the governing PDE for $C_n(t,s,v,y)$ is analogously derived as

$$\begin{cases} \mathcal{L}C_n = 0 & \text{in } [0,T) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times [\bar{y},\infty), \\ C_n = h(s,v,y) & \text{on } \{t = T\} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}, \end{cases} \tag{11}$$

where $\bar{y} \in \mathbb{R}$ and the differential operator $\mathcal{L}$ is given by

$$\begin{aligned} &\frac{\partial}{\partial t} + \frac{1}{2}\sigma_S^2 S^{f2}\frac{\partial^2}{\partial S^{f2}} + \frac{1}{2}\sigma_V^2 V^2\frac{\partial^2}{\partial V^2} + \frac{1}{2}\sigma_Y^2 Y^2\frac{\partial^2}{\partial Y^2} \\ &\quad + \rho_{12}\sigma_S\sigma_V S^f V\frac{\partial^2}{\partial S^f\partial V} + \rho_{13}\sigma_S\sigma_Y S^f Y\frac{\partial^2 C}{\partial S^f\partial Y} + \rho_{23}\sigma_V\sigma_Y VY\frac{\partial^2}{\partial V\partial Y} \\ &\quad + (r^f - q - \rho_{13}\sigma_f\sigma_Y)S^f\frac{\partial}{\partial S^f} + r^d V\frac{\partial}{\partial V} + (r^d - r^f)Y\frac{\partial}{\partial Y} - r^d I = 0. \end{aligned}$$

The terminal condition is expressed as

$$\begin{aligned} C_n(T,S^f,V,Y) &= Y(T)(S^f(T) - K)^+\left(1_{\{V(T)>D\}} + 1_{\{V(T)<D\}}\frac{(1-\alpha)V(T)}{D}\right) \\ &\triangleq h(S(T),V(T),Y(T)), \end{aligned}$$

where α, D, and K are nonnegative constants. We define a sequence of functions $h_n(s,v,y)$ for $n = 1,2,\ldots$ such that $h_n(s,v,y) \to h(s,v,y)$ as $n \to \infty$. That is,

$$h_n(s,v,y) = h_n^1(s)h_n^2(v)h_n^3(y), \tag{12}$$

where

$$h_n^1(s) \triangleq \begin{cases} \forall s \in [K,n), & s - K, \\ \forall s \notin [K,n), & 0, \end{cases}$$

$$h_n^2(v) \triangleq \begin{cases} \forall v \in [D,n), & 1, \\ \forall v \notin [D,n), & (1-\alpha)v/D, \end{cases}$$

$$h_n^3(y) \triangleq \begin{cases} \forall y \in [\bar{y},n), & y, \\ \forall y \notin [\bar{y},n), & 0. \end{cases}$$

Let us define the triple Mellin transform for $C_n(t,s,v,y)$ by $\hat{C}_n(t,s_*,v_*,y_*)$. Then the relation between $C_n(t,s,v,y)$ and $\hat{C}_n(t,s_*,v_*,y_*)$ becomes

$$C_n(t,s,v,y) = \frac{1}{(2\pi i)^3}\int_{c_3-i\infty}^{c_3+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_1-i\infty}^{c_1+i\infty} s^{-s^*}v^{-v^*}y^{-y^*}\hat{C}_n(t,s_*,v_*,y_*)\,\mathrm{d}s_*\,\mathrm{d}v_*\,\mathrm{d}y_* \quad (13)$$

for $c_1,c_2,c_3 \in \mathbb{R}$.

Assigning (13) into (11), we have the differential equation for $\hat{C}_n$:

$$\begin{aligned}&\partial_t\hat{C}_n + \biggl[\frac{1}{2}\sigma_s^2(s_*+1)s_* + \frac{1}{2}\sigma_v^2(v_*+1)v^* + \frac{1}{2}\sigma_y^2(y_*+1)y_* \\ &\quad + \rho_{12}\sigma_s\sigma_v s_* v_* + \rho_{13}\sigma_s\sigma_y s_* y_* + \rho_{23}\sigma_v\sigma_y v_* y_* \\ &\quad - \bigl(r^f - q - \rho_{13}\sigma_s\sigma_y\bigr)s_* - r^d v^* - \bigl(r^d - r^f\bigr)y_* - r^d\biggr]\hat{C}_n = 0. \end{aligned} \quad (14)$$

Let us replace the coefficient of $\hat{C}_n$ in (14) by $\Phi(s_*,v_*,y_*)$. Then we have

$$\begin{aligned}\Phi(s_*,v_*,y_*) &= \frac{\sigma_s^2}{2}s_*^2 + \frac{\sigma_v^2}{2}v_*^2 + \frac{\sigma_y^2}{2}y_*^2 + \rho_{12}\sigma_s\sigma_v s_* v_* + \rho_{13}\sigma_s\sigma_y s_* y_* + \rho_{23}\sigma_v\sigma_y v_* y_* \\ &\quad - \left(r_s - \frac{\sigma_s^2}{2}\right)s_* - \left(r_v - \frac{\sigma_v^2}{2}\right)v_* - \left(r_y - \frac{\sigma_y^2}{2}\right)y_* - r_v. \end{aligned} \quad (15)$$

Here, $r_s \triangleq r^f - q - \rho_{13}\sigma_s\sigma_y$, $r_v \triangleq r^d$, and $r_y \triangleq r^d - r^f$. Since Eq. (14) is the ordinary differential equation (ODE) in a time variable, $\hat{C}_n$ satisfies

$$\hat{C}_n(t) = \hat{h}_n(T)e^{\Phi(s_*,v_*,y_*)(T-t)}.$$

By the inverse triple Mellin transform, $\hat{C}_n$ becomes

$$\begin{aligned}&C_n(t,s,v,y) \\ &\quad = \frac{1}{(2\pi i)^3}\int_{c_3-i\infty}^{c_3+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_1-i\infty}^{c_1+i\infty} \hat{h}_n(T)e^{\Phi(s_*,v_*,y_*)(T-t)}s^{-s_*}v^{-v_*}y^{-y_*}\,\mathrm{d}s_*\,\mathrm{d}v_*\,\mathrm{d}y_*. \end{aligned} \quad (16)$$

To compute Eq. (16), we define $\mathcal{C}(t,s,v,y)$ as

$$\mathcal{C}(t,s,v,y) \triangleq \frac{1}{(2\pi i)^3}\int_{c_3-i\infty}^{c_3+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_1-i\infty}^{c_1+i\infty} e^{\Phi(s^*,v^*,y^*)(T-t)}s^{-s^*}v^{-v^*}y^{-y^*}\,\mathrm{d}s^*\,\mathrm{d}v^*\,\mathrm{d}y^* \quad (17)$$

which is the inverse triple Mellin transform of $\exp[\Phi(s_*,v_*,y_*)(T-t)]$. Now, we introduce the lemmas for computing of the inverse triple Mellin transform.

Lemma 1 *Given* $z_0,z_1 \in \mathbb{C}$ *such that* $\mathrm{Re}(z_0) \geq 0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{z_0(w+z_1)^2}x^{-w}\,\mathrm{d}w = \frac{1}{2\sqrt{\pi z_0}}x^{z_1}\exp\left[-\frac{1}{4z_0}(\ln x)^2\right]. \quad (18)$$

Proof Refer to Yoon and Kim [22]. □

Lemma 2 *The pricing kernel* $\mathcal{C}(t,s,v,y)$ *in Eq.* (17) *is given by*

$$\mathcal{C}(\tau,s,v,y) = \exp\left[-\frac{1}{2}\left(\frac{\ln s}{\sigma_s\sqrt{\tau}}\right)^2 - \frac{1}{2}\left(\frac{\ln v}{\sigma_v\sqrt{R_{12}\tau}}\right)^2 - \frac{1}{2}\left(\frac{\ln y}{\sigma_y\sqrt{R\tau}}\right)^2 + \theta_3(\tau,s,v)\right] \times \frac{s^{\theta_0}}{\sigma_s\sqrt{2\pi\tau}}\frac{v^{\theta_1(\tau,s)}}{\sigma_v\sqrt{2\pi R_{12}\tau}}\frac{y^{\theta_2(\tau,s,v)}}{\sigma_y\sqrt{2\pi R\tau}}, \tag{19}$$

where the parameters used for simplicity are

$$\begin{aligned} &p(x_1,x_2,x_3) \triangleq R_{23}x_1^2 + R_{13}x_2^2 + R_{12}x_3^2 - 2(\hat{\rho}_{12}x_1x_2 + \hat{\rho}_{13}x_1x_3 + \hat{\rho}_{23}x_2x_3), \\ &R_{23} \triangleq 1-\rho_{23}^2, \qquad R_{13} \triangleq 1-\rho_{13}^2, \qquad R_{12} \triangleq 1-\rho_{12}^2, \\ &\hat{\rho}_{12} \triangleq \rho_{12}-\rho_{13}\rho_{23}, \qquad \hat{\rho}_{13} \triangleq \rho_{13}-\rho_{12}\rho_{23}, \qquad \hat{\rho}_{23} \triangleq \rho_{23}-\rho_{12}\rho_{13}, \\ &p \triangleq \frac{(1-\alpha)v}{D}. \end{aligned} \tag{20}$$

Proof From Lemma 1, we define

$$\mathcal{C}_1(\tau,s,v_*,y_*) \triangleq \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} e^{\Phi(s_*,v_*,y_*)\tau}\,\mathrm{d}s_*.$$

Then $\mathcal{C}_1(\tau,s,v_*,y_*)$ becomes

$$\begin{aligned} \mathcal{C}_1(\tau,s,v_*,y_*) &= e^{\phi_1(\tau,v_*,y_*)} \times \left(\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} \exp\left[\frac{1}{2}\sigma_s^2\tau(s_*+f_1)^2\right]s^{-s_*}\,\mathrm{d}s_*\right) \\ &= \exp\left[\phi_1(\tau,v_*,y_*) + \rho_{12}'(\ln s)v_* + \rho_{13}'(\ln s)y_*\right] \\ &\quad \times \frac{s^{\theta_0}}{\sigma_s\sqrt{2\pi\tau}}\exp\left[-\frac{1}{2}\left(\frac{\ln s}{\sigma_s\sqrt{\tau}}\right)^2\right], \end{aligned}$$

where $k_1 \triangleq 2r_s/\sigma_s^2, f_1(v_*,y_*) \triangleq \rho_{12}'v_* + \rho_{13}'y_* + \theta_0$, and

$$\begin{aligned} \phi_1(\tau,v_*,y_*) &\triangleq \frac{\sigma_v^2(1-\rho_{12}^2)}{2}v_*^2\tau \\ &\quad + \left\{\frac{\rho_{12}\sigma_s\sigma_v(k_1-1)}{2} - \left(r_v - \frac{\sigma_v^2}{2}\right) + \sigma_v\sigma_y(\rho_{23}-\rho_{12}\rho_{13})y_*\right\}v_*\tau \\ &\quad + \frac{\sigma_y^2(1-\rho_{13}^2)}{2}y_*^2\tau \\ &\quad + \left\{\frac{\rho_{13}\sigma_s\sigma_y(k_1-1)}{2} - \left(r_y - \frac{\sigma_y^2}{2}\right)\right\}y_*\tau - \left(r_v + \frac{\sigma_s^2(k_1-1)^2}{8}\right)\tau, \\ &\triangleq \frac{1}{2}\sigma_v^2R_{12}v_*^2\tau + (H_1 + \sigma_v\sigma_y\hat{\rho}_{23}y_*)v_*\tau + \frac{1}{2}\sigma_y^2R_{13}y_*^2\tau + H_2y_*\tau + \alpha\tau. \end{aligned}$$

In this way,

$$\begin{aligned} \mathcal{C}_2(\tau,s,v,y_*) &\triangleq \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\mathcal{C}_1(\tau,s,v_*,y_*)\,\mathrm{d}v_* \\ &= \frac{s^{\theta_0}}{\sigma_s\sqrt{2\pi\tau}}\exp\left[-\frac{1}{2}\left(\frac{\ln s}{\sigma_s\sqrt{\tau}}\right)^2\right]e^{\phi_2} \end{aligned}$$

$$
\begin{aligned}
&\times\left(\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\exp\left[\frac{1}{2}\sigma_v^2R_{12}\tau\left(v_*+f_2(\tau,s,y_*)\right)^2\right]\mathrm{d}v_*\right)\\
&=\frac{s^{\theta_0}}{\sigma_s\sqrt{2\pi\tau}}\frac{v^{\theta_1(\tau,s)}}{\sigma_v\sqrt{2\pi R_{12}\tau}}\exp\left[\phi_2(\tau,s,y_*)+\frac{\hat{\rho}_{23}\sigma_y}{\sigma_vR_{12}}(\ln v)y_*\right]\\
&\quad\times\exp\left[-\frac{1}{2}\left(\frac{\ln s}{\sigma_s\sqrt{\tau}}\right)^2-\frac{1}{2}\left(\frac{\ln v}{\sigma_v\sqrt{R_{12}\tau}}\right)^2\right],
\end{aligned}
$$

where

$$
\begin{aligned}
&H_1\triangleq\frac{1}{2}\rho_{12}\sigma_s\sigma_v(k_1-1)-\left(r_v-\frac{1}{2}\sigma_v^2\right),\\
&\theta_1(\tau,s)\triangleq\frac{H_1\tau+\rho_{12}'\ln s}{\sigma_v^2R_{12}\tau},\\
&f_2(\tau,s,y_*)\triangleq\frac{H_1\tau+\rho_{12}'\ln s}{\sigma_v^2R_{12}\tau}+\frac{\hat{\rho}_{23}\sigma_y}{\sigma_vR_{12}}y_*,
\end{aligned}
$$

and

$$
\begin{aligned}
&\phi_2(\tau,s,y_*)\triangleq\frac{1}{2}\sigma_y^2R\tau y_*^2+\left(-\frac{\sigma_y\hat{\rho}_{23}}{\sigma_vR_{12}}H_1\tau+H_2\tau+\frac{\sigma_y\hat{\rho}_{13}}{\sigma_sR_{12}}\ln s\right)y_*-\frac{(H_1\tau+\rho_{12}'\ln s)^2}{2\sigma_v^2R_{12}\tau}+\alpha\tau,\\
&R\triangleq\frac{|\Sigma|}{|\Sigma'|}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathcal{C}_3(\tau,s,v,y)&\triangleq\frac{1}{2\pi i}\int_{c_3-i\infty}^{c_3+i\infty}\mathcal{C}_2(\tau,s,v,y_*)\,\mathrm{d}y_*\\
&=\frac{s^{\theta_0}}{\sigma_s\sqrt{2\pi\tau}}\frac{v^{\theta_1(\tau,s)}}{\sigma_v\sqrt{2\pi R_{12}\tau}}\exp\left[-\frac{1}{2}\left(\frac{\ln s}{\sigma_s\sqrt{\tau}}\right)^2-\frac{1}{2}\left(\frac{\ln v}{\sigma_v\sqrt{R_{12}\tau}}\right)^2\right]e^{\phi_3}\\
&\quad\times\left(\frac{1}{2\pi i}\int_{c_3-i\infty}^{c_3+i\infty}e^{\frac{1}{2}\sigma_y^2R\tau(y_*+f_3(\tau,s,v))^2}\,\mathrm{d}y_*\right)\\
&=\frac{s^{\theta_0}}{\sigma_s\sqrt{2\pi\tau}}\frac{v^{\theta_1(\tau,s)}}{\sigma_v\sqrt{2\pi R_{12}\tau}}\frac{y^{\theta_2(\tau,s,v)}}{\sigma_y\sqrt{2\pi R\tau}}\\
&\quad\times\exp\left[-\frac{1}{2}\left(\frac{\ln s}{\sigma_s\sqrt{\tau}}\right)^2-\frac{1}{2}\left(\frac{\ln v}{\sigma_v\sqrt{R_{12}\tau}}\right)^2-\frac{1}{2}\left(\frac{\ln y}{\sigma_y\sqrt{R\tau}}\right)^2+\theta_3(\tau,s,v)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
&\phi_3\triangleq-\frac{1}{2\sigma_y^2R\tau}\left\{-\frac{\sigma_y\hat{\rho}_{23}}{\sigma_vR_{12}}H_1\tau+H_2\tau+\frac{\sigma_y\hat{\rho}_{13}}{\sigma_sR_{12}}\ln s+\frac{\sigma_y\hat{\rho}_{23}}{\sigma_vR_{12}}\ln v\right\}^2-\frac{(H_1\tau+\rho_{12}'\ln s)^2}{2\sigma_v^2R_{12}\tau}+\alpha\tau\\
&\quad\triangleq\theta_3(\tau,s,v),\\
&f_3(\tau,s,v)\triangleq\frac{\hat{\rho}_{13}}{\sigma_s\sigma_y|\Sigma|\tau}\ln s+\frac{\hat{\rho}_{23}}{\sigma_v\sigma_y|\Sigma|\tau}\ln v-\frac{\hat{\rho}_{23}H_1}{\sigma_v\sigma_y|\Sigma|}+\frac{H_2}{\sigma_y^2R}\triangleq\theta_2(\tau,s,v).
\end{aligned}
$$

Therefore,

$$\mathcal{C}(\tau,s,v,y) = \exp\left[-\frac{1}{2}\left(\frac{\ln s}{\sigma_s\sqrt{\tau}}\right)^2 - \frac{1}{2}\left(\frac{\ln v}{\sigma_v\sqrt{R_{12}\tau}}\right)^2 - \frac{1}{2}\left(\frac{\ln y}{\sigma_y\sqrt{R\tau}}\right)^2 + \theta_3(\tau,s,v)\right] \times \frac{s^{\theta_0}}{\sigma_s\sqrt{2\pi\tau}} \frac{v^{\theta_1(\tau,s)}}{\sigma_v\sqrt{2\pi R_{12}\tau}} \frac{y^{\theta_2(\tau,s,v)}}{\sigma_y\sqrt{2\pi R\tau}}. \tag{21}$$

□

Lemma 3 *Let $f,g:\mathbb{R}_+^3 \to \mathbb{C}$. If $F(w_1,w_2,w_3)$ and $G(w_1,w_2,w_3)$ are the triple Mellin transforms of $f(x,y,z)$ and $g(x,y,z)$, respectively. Then the triple Mellin convolution of f and g is given by*

$$\begin{aligned} f(x,y,z) * g(x,y,z) &\triangleq \mathcal{M}^{-1}_{w_1w_2w_3}\left[F(w_1,w_2,w_3)G(w_1,w_2,w_3);x,y,z\right] \\ &= \int_0^\infty \int_0^\infty \int_0^\infty f\left(\frac{x}{u_1},\frac{y}{v},\frac{z}{w}\right)g(u,v,w)\frac{\mathrm{d}u_1}{u_1}\frac{\mathrm{d}u_2}{u_2}\frac{\mathrm{d}u_3}{u_3}. \end{aligned} \tag{22}$$

With the above lemmas and parameters, the price of foreign equity options in foreign currency with credit risk at time t is given by the following theorem.

Theorem 1 *The price of foreign equity options with credit risk is given by*

$$\begin{aligned} C(t,s,v,y) = {} & sy\delta_1\mathbf{N}_3\left[\kappa_1^1(t,s),\kappa_2^1(t,v),\kappa_3^1(t,y)\right] \\ & - yK\delta_2\mathbf{N}_3\left[\kappa_1^2(t,s),\kappa_2^2(t,v),\kappa_3^2(t,y)\right] \\ & + ps\delta_3\mathbf{N}_3\left[\kappa_1^3(t,s),\kappa_2^3(t,v),\kappa_3^3(t,y)\right] \\ & - pK\delta_4\mathbf{N}_3\left[\kappa_1^4(t,s),\kappa_2^4(t,v),\kappa_3^4(t,y)\right], \end{aligned} \tag{23}$$

where $\mathbf{N}_3$ is the 3-dimensional standard normal cumulative function (CDF)[1] *defined by*

$$\mathbf{N}_3(a,b,c) = \frac{1}{(2\pi)^{3/2}|\Sigma|^{1/2}}\int_{-\infty}^a \int_{-\infty}^b \int_{-\infty}^c \exp\left[-\frac{p(x_1,x_2,x_3)}{2|\Sigma|}\right]\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3. \tag{24}$$

Here,

$$\begin{aligned} &\kappa_1^2(t,s), \quad \kappa_2^1(t,v), \quad \kappa_3^1(t,y), \quad \kappa_1^2(t,s), \quad \kappa_2^2(t,v), \quad \kappa_3^2(t,y), \\ &\kappa_1^3(t,s), \quad \kappa_2^3(t,v), \quad \kappa_3^3(t,y), \quad \kappa_1^4(t,s), \quad \kappa_2^4(t,v), \quad \kappa_3^4(t,y), \\ &\delta_1, \quad \delta_2, \quad \delta_3, \quad \delta_4 \end{aligned}$$

are given in Appendix A.

[1] For more details, see Appendix B.

Proof By the triple Mellin convolution property in Lemma 2, Eq. (16) yields

$$\begin{aligned} C_n(t,s,v,y) &= \int_0^n \int_0^n \int_0^n h_n(u_1,u_2,u_3)\mathcal{C}\left(\tau, \frac{s}{u_1}, \frac{v}{u_2}, \frac{y}{u_3}\right) \frac{\mathrm{d}u_1}{u_1}\frac{\mathrm{d}u_2}{u_2}\frac{\mathrm{d}u_3}{u_3} \\ &= e^{\alpha\tau} \int_{\bar{y}}^n \int_D^n \int_K^n e^{\psi_1(\tau,u_1,u_2)}(u_1-K)u_3\psi_2(u_1,u_2,u_3)\frac{\mathrm{d}u_1}{u_1}\frac{\mathrm{d}u_2}{u_2}\frac{\mathrm{d}u_3}{u_3} \\ &\quad + \frac{(1-\alpha)e^{\alpha\tau}}{D} \int_{\bar{y}}^n \int_0^D \int_0^K e^{\psi_1(\tau,u_1,u_2)}(u_1-K)u_2u_3 \\ &\quad \times \psi_2(u_1,u_2,u_3)\frac{\mathrm{d}u_1}{u_1}\frac{\mathrm{d}u_2}{u_2}\frac{\mathrm{d}u_3}{u_3}, \end{aligned}$$

where $\psi_1(\tau,u_1,u_2) \triangleq \theta_3(\tau,s/u_1,v/u_2)$ and

$$\begin{aligned} \psi_2(\tau,u_1,u_2,u_3) \triangleq{}& \left(\frac{s}{u_1}\right)^{\theta_0}\left(\frac{v}{u_2}\right)^{\theta_1(\tau,s/u_1)}\left(\frac{y}{u_3}\right)^{\theta_2(\tau,s/u_1,v/u_2)} \\ &\times \frac{e^{-\frac{1}{2}a^2(\tau,s/u_1)}}{\sigma_s\sqrt{2\pi\tau}} \cdot \frac{e^{-\frac{1}{2}b^2(\tau,v/u_2)}}{\sigma_v\sqrt{2\pi R_{12}\tau}} \cdot \frac{e^{-\frac{1}{2}c^2(\tau,y/u_3)}}{\sigma_y\sqrt{2\pi R\tau}}. \end{aligned}$$

If $n \to \infty$, then we have

$$\begin{aligned} &C(t,s,v,y) \\ &\quad = \lim_{n\to\infty} C_n(t,s,v,y) \\ &\quad = \underbrace{e^{\alpha\tau}\int_{\bar{y}}^\infty \int_D^\infty \int_K^\infty e^{\psi_1(\tau,u_1,u_2)}(u_1-K)u_3\psi_2(u_1,u_2,u_3)\frac{\mathrm{d}u_1}{u_1}\frac{\mathrm{d}u_2}{u_2}\frac{\mathrm{d}u_3}{u_3}}_{\triangleq C^1(t,s,v,y)} \\ &\qquad + \underbrace{\frac{(1-\alpha)e^{\alpha\tau}}{D}\int_{\bar{y}}^\infty \int_0^D \int_0^K e^{\psi_1(\tau,u_1,u_2)}(u_1-K)u_2u_3\psi_2(u_1,u_2,u_3)\frac{\mathrm{d}u_1}{u_1}\frac{\mathrm{d}u_2}{u_2}\frac{\mathrm{d}u_3}{u_3}}_{\triangleq C^2(t,s,v,y)}. \end{aligned} \tag{25}$$

Now, we use the following change of variables:

$$x_1 = \frac{\ln(s/u_1)}{\sigma_s\sqrt{\tau}}, \qquad x_2 = \frac{\ln(v/u_2)}{\sigma_v\sqrt{\tau}}, \quad \text{and} \quad x_3 = \frac{\ln(y/u_3)}{\sigma_y\sqrt{\tau}}. \tag{26}$$

This transformation replaces (u_1,u_2,u_3) with (x_1,x_2,x_3). To solve the $C^1(t,s,v,y)$, we apply (26) to (25). If we define $C^{10}(t,s,v,y)$ as

$$\begin{aligned} C^{10}(t,s,v,y) &\triangleq \frac{sy}{(2\pi)^{3/2}|\Sigma|}\int_{-\infty}^{\frac{\ln(y/\bar{y})}{\sigma_y\sqrt{\tau}}}\int_{-\infty}^{\frac{\ln(v/D)}{\sigma_v\sqrt{\tau}}}\int_{-\infty}^{\frac{\ln(s/K)}{\sigma_s\sqrt{\tau}}} e^{\Lambda_1(\tau,x_1,x_2,x_3)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3, \\ C^{20}(t,s,v,y) &\triangleq -\frac{Ky}{(2\pi)^{3/2}|\Sigma|}\int_{-\infty}^{\frac{\ln(y/\bar{y})}{\sigma_y\sqrt{\tau}}}\int_{-\infty}^{\frac{\ln(v/D)}{\sigma_v\sqrt{\tau}}}\int_{-\infty}^{\frac{\ln(s/K)}{\sigma_s\sqrt{\tau}}} e^{\Lambda_2(\tau,x_1,x_2,x_3)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3, \end{aligned}$$

then $C_1(t,s,v,y) = C^{10}(t,s,v,y) + C^{11}(t,s,v,y)$ where

$$\begin{aligned}
\Lambda_1(\tau,x_1,x_2,x_3) &\triangleq -\frac{1}{2|\Sigma|}\left(R_{23}x_1^2 + R_{13}x_2^2 + R_{12}x_3^2\right) \\
&\quad + \left(\sigma_s\theta_0\sqrt{\tau} - \sigma_s\sqrt{\tau} - \frac{\hat{\rho}_{12}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \frac{\hat{\rho}_{13}H_2}{\sigma_y|\Sigma|}\sqrt{\tau}\right)x_1 \\
&\quad + \left(\frac{R_{13}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \frac{\hat{\rho}_{23}H_2}{\sigma_y|\Sigma|}\sqrt{\tau}\right)x_2 \\
&\quad + \left(\frac{H_2}{\sigma_y R}\sqrt{\tau} - \frac{\hat{\rho}_{23}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \sigma_y\sqrt{\tau}\right)x_3 \\
&\quad + \frac{\hat{\rho}_{12}}{|\Sigma|}x_1x_2 + \frac{\hat{\rho}_{13}}{|\Sigma|}x_1x_3 + \frac{\hat{\rho}_{23}}{|\Sigma|}x_2x_3 - \frac{H_3^2\tau}{2\sigma_y^2 R} - \frac{H_1^2\tau}{2\sigma_v^2 R_{12}} + \alpha\tau, \\
\Lambda_2(\tau,x_1,x_2,x_3) &\triangleq -\frac{1}{2|\Sigma|}\left(R_{23}x_1^2 + R_{13}x_2^2 + R_{12}x_3^2\right) \\
&\quad + \left(\sigma_s\theta_0\sqrt{\tau} - \frac{\hat{\rho}_{12}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \frac{\hat{\rho}_{13}H_2}{\sigma_y|\Sigma|}\sqrt{\tau}\right)x_1 \\
&\quad + \left(\frac{R_{13}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \frac{\hat{\rho}_{23}H_2}{\sigma_y|\Sigma|}\sqrt{\tau}\right)x_2 \\
&\quad + \left(\frac{H_2}{\sigma_y R}\sqrt{\tau} - \frac{\hat{\rho}_{23}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \sigma_y\sqrt{\tau}\right)x_3 \\
&\quad + \frac{\hat{\rho}_{12}}{|\Sigma|}x_1x_2 + \frac{\hat{\rho}_{13}}{|\Sigma|}x_1x_3 + \frac{\hat{\rho}_{23}}{|\Sigma|}x_2x_3 - \frac{H_3^2\tau}{2\sigma_y^2 R} - \frac{H_1^2\tau}{2\sigma_v^2 R_{12}} + \alpha\tau.
\end{aligned}$$

Here, Λ_1 has the form of

$$\begin{aligned}
&\delta_1 - \frac{1}{2|\Sigma|}\left\{R_{23}(x_1+a_1)^2 + R_{13}(x_2+a_2)^2 + R_{12}(x_3+a_3)^2\right\} \\
&\quad + \frac{1}{|\Sigma|}\left\{\hat{\rho}_{12}(x_1+a_1)(x_2+a_2) + \hat{\rho}_{13}(x_1+a_1)(x_3+a_3) + \hat{\rho}_{23}(x_2+a_2)(x_3+a_3)\right\},
\end{aligned}$$

where

$$\begin{aligned}
\delta_1 &= \frac{R_{23}a_1^2 + R_{13}a_2^2 + R_{12}a_3^2 - 2(\hat{\rho}_{12}a_1a_2 + \hat{\rho}_{13}a_1a_3 + \hat{\rho}_{23}a_2a_3)}{2|\Sigma|} + d, \\
a_1 &= \frac{r^f - q + \frac{1}{2}\sigma_s^2}{\sigma_s}\sqrt{\tau}, \\
a_2 &= \frac{\rho_{12}\sigma_s\sigma_v + \rho_{23}\sigma_v\sigma_y + (r^d - \frac{1}{2}\sigma_v^2)}{\sigma_v}\sqrt{\tau}, \\
a_3 &= \frac{\rho_{13}\sigma_s\sigma_y + r^d - r^f + \frac{1}{2}\sigma_y^2}{\sigma_y}\sqrt{\tau}, \\
d &= -\frac{H_3^2\tau}{2\sigma_y^2 R} - \frac{H_1^2\tau}{2\sigma_v^2 R_{12}} + \alpha\tau, \quad \text{and} \quad H_3 = H_2 - \frac{\sigma_y\hat{\rho}_{23}}{\sigma_v R_{12}}H_1.
\end{aligned}$$

These constants can be determined by the method of undetermined coefficients. In other words, to determine these constants, we have to solve the following three-variable system of linear equations:

$$\begin{cases} -\frac{R_{23}}{2|\Sigma|}a_1 + \frac{\hat{\rho}_{12}}{|\Sigma|}a_2 + \frac{\hat{\rho}_{13}}{|\Sigma|}a_3 = x_1 & \text{term in } \Lambda_1 \\ \frac{\hat{\rho}_{12}}{|\Sigma|}a_1 - \frac{R_{13}}{2|\Sigma|}a_2 + \frac{\hat{\rho}_{23}}{|\Sigma|}a_3 = x_2 & \text{term in } \Lambda_1 \\ \frac{\hat{\rho}_{13}}{|\Sigma|}a_1 + \frac{\hat{\rho}_{23}}{|\Sigma|}a_2 - \frac{R_{13}}{2|\Sigma|}a_3 = x_3 & \text{term in } \Lambda_1 \\ \delta_1 = \text{constant} & \text{term in } \Lambda_1. \end{cases} \tag{27}$$

Similarly, Λ_2 has the form of

$$\begin{aligned} &\delta_2 - \frac{1}{2|\Sigma|}\left\{R_{23}(x_1+b_1)^2 + R_{13}(x_2+b_2)^2 + R_{12}(x_3+b_3)^2\right\} \\ &\quad + \frac{1}{|\Sigma|}\left\{\hat{\rho}_{12}(x_1+b_1)(x_2+b_2) + \hat{\rho}_{13}(x_1+b_1)(x_3+b_3) + \hat{\rho}_{23}(x_2+b_2)(x_3+b_3)\right\}, \end{aligned}$$

where

$$\begin{aligned} \delta_2 &= \frac{R_{23}b_1^2 + R_{13}b_2^2 + R_{12}b_3^2 - 2(\hat{\rho}_{12}b_1b_2 + \hat{\rho}_{13}b_1b_3 + \hat{\rho}_{23}b_2b_3)}{2|\Sigma|} + d, \\ b_1 &= \frac{r^f - q - \frac{1}{2}\sigma_s^2}{\sigma_s}\sqrt{\tau}, \\ b_2 &= \frac{\rho_{23}\sigma_v\sigma_y + (r^d - \frac{1}{2}\sigma_v^2)}{\sigma_v}\sqrt{\tau}, \\ b_3 &= \frac{\rho_{13}\sigma_s\sigma_y + r^d - r^f + \frac{1}{2}\sigma_y^2}{\sigma_y}\sqrt{\tau}. \end{aligned}$$

Let us define $\xi_1 = x_1 + a_1$, $\xi_2 = x_2 + a_2$, $\xi_3 = x_3 + a_3$, $\xi_4 = x_1 + b_1$, $\xi_5 = x_2 + b_2$, and $\xi_6 = x_3 + b_3$, then $C_1(t,s,v,y)$ becomes

$$\begin{aligned} C_1(t,s,v,y) &= sy\delta_1\mathbf{N}_3\left[\kappa_1^1(t,x), \kappa_2^1(t,v), \kappa_3^1(t,y)\right] \\ &\quad - yK\delta_2\mathbf{N}_3\left[\kappa_1^2(t,x), \kappa_2^2(t,v), \kappa_3^2(t,y)\right]. \end{aligned} \tag{28}$$

In this way, if we define $C_{20}(t,s,v,y)$ and $C_{21}(t,s,v,y)$ as

$$\begin{aligned} C_{20}(t,s,v,y) &= \frac{(1-\alpha)sv}{(2\pi)^{3/2}D|\Sigma|}\int_{\frac{\ln(y/y_0)}{\sigma_y\sqrt{\tau}}}^{-\infty}\int_{\infty}^{\frac{\ln(v/D)}{\sigma_v\sqrt{\tau}}}\int_{\frac{\ln(s/K)}{\sigma_s\sqrt{\tau}}}^{-\infty} e^{\Lambda_3(\tau,x_1,x_2,x_3)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3, \\ C_{21}(t,s,v,y) &= \frac{(1-\alpha)vK}{(2\pi)^{3/2}D|\Sigma|}\int_{\frac{\ln(y/y_0)}{\sigma_y\sqrt{\tau}}}^{-\infty}\int_{\infty}^{\frac{\ln(v/D)}{\sigma_v\sqrt{\tau}}}\int_{\frac{\ln(s/K)}{\sigma_s\sqrt{\tau}}}^{-\infty} e^{\Lambda_4(\tau,x_1,x_2,x_3)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3, \end{aligned} \tag{29}$$

respectively, then $C_2(t,s,v,y) = C_{20}(t,s,v,y) + C_{21}(t,s,v,y)$ where

$$\begin{aligned}
\Lambda_3(\tau,x_1,x_2,x_3) \triangleq & -\frac{1}{2|\Sigma|}\left(R_{23}x_1^2 + R_{13}x_2^2 + R_{12}x_3^2\right) \\
& + \left(\sigma_s\theta_0\sqrt{\tau} - \sigma_s\sqrt{\tau} - \frac{\hat{\rho}_{12}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \frac{\hat{\rho}_{13}H_2}{\sigma_y|\Sigma|}\sqrt{\tau}\right)x_1 \\
& + \left(\frac{R_{13}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \frac{\hat{\rho}_{23}H_2}{\sigma_y|\Sigma|}\sqrt{\tau} - \sigma_v\sqrt{\tau}\right)x_2 \\
& + \left(\frac{H_2}{\sigma_y R}\sqrt{\tau} - \frac{\hat{\rho}_{23}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \sigma_y\sqrt{\tau}\right)x_3 \\
& + \frac{\hat{\rho}_{12}}{|\Sigma|}x_1x_2 + \frac{\hat{\rho}_{13}}{|\Sigma|}x_1x_3 + \frac{\hat{\rho}_{23}}{|\Sigma|}x_2x_3 - \frac{H_3^2\tau}{2\sigma_y^2 R} - \frac{H_1^2\tau}{2\sigma_v^2 R_{12}} + \alpha\tau, \\
\Lambda_4(\tau,x_1,x_2,x_3) \triangleq & -\frac{1}{2|\Sigma|}\left(R_{23}x_1^2 + R_{13}x_2^2 + R_{12}x_3^2\right) \\
& + \left(\sigma_s\theta_0\sqrt{\tau} - \frac{\hat{\rho}_{12}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \frac{\hat{\rho}_{13}H_2}{\sigma_y|\Sigma|}\sqrt{\tau}\right)x_1 \\
& + \left(\frac{R_{13}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \frac{\hat{\rho}_{23}H_2}{\sigma_y|\Sigma|}\sqrt{\tau} - \sigma_v\sqrt{\tau}\right)x_2 \\
& + \left(\frac{H_2}{\sigma_y R}\sqrt{\tau} - \frac{\hat{\rho}_{23}H_1}{\sigma_v|\Sigma|}\sqrt{\tau} - \sigma_y\sqrt{\tau}\right)x_3 \\
& + \frac{\hat{\rho}_{12}}{|\Sigma|}x_1x_2 + \frac{\hat{\rho}_{13}}{|\Sigma|}x_1x_3 + \frac{\hat{\rho}_{23}}{|\Sigma|}x_2x_3 - \frac{H_3^2\tau}{2\sigma_y^2 R} - \frac{H_1^2\tau}{2\sigma_v^2 R_{12}} + \alpha\tau.
\end{aligned}$$

Λ_3 has the form of

$$\begin{aligned}
& \delta_3 - \frac{1}{2|\Sigma|}\left\{R_{23}(x_1+c_1)^2 + R_{13}(x_2+c_2)^2 + R_{12}(x_3+c_3)^2\right\} \\
& \quad + \frac{1}{|\Sigma|}\left\{\hat{\rho}_{12}(x_1+c_1)(x_2+c_2) + \hat{\rho}_{13}(x_1+c_1)(x_3+c_3) + \hat{\rho}_{23}(x_2+c_2)(x_3+c_3)\right\}.
\end{aligned}$$

By the method of undetermined coefficients we have the system of linear equations,

$$\begin{aligned}
\delta_3 &= \frac{R_{23}c_1^2 + R_{13}c_2^2 + R_{12}c_3^2 - 2(\hat{\rho}_{12}c_1c_2 + \hat{\rho}_{13}c_1c_3 + \hat{\rho}_{23}c_2c_3)}{2|\Sigma|} + d, \\
c_1 &= \frac{\rho_{12}\sigma_s\sigma_v + (r^f - q + \frac{1}{2}\sigma_s^2)}{\sigma_s}\sqrt{\tau}, \\
c_2 &= \frac{\rho_{12}\sigma_s\sigma_v + \rho_{23}\sigma_v\sigma_y + (r^d + \frac{1}{2}\sigma_v^2)}{\sigma_v}\sqrt{\tau}, \\
c_3 &= \frac{\rho_{13}\sigma_s\sigma_y + \rho_{23}\sigma_v\sigma_y + (r^d - r^f + \frac{1}{2}\sigma_y^2)}{\sigma_y}\sqrt{\tau}.
\end{aligned}$$

And, Λ_4 has the form of

$$\delta_4 - \frac{1}{2|\Sigma|}\left\{R_{23}(x_1+d_1)^2 + R_{13}(x_2+d_2)^2 + R_{12}(x_3+d_3)^2\right\}$$
$$+ \frac{1}{|\Sigma|}\left\{\hat{\rho}_{12}(x_1+d_1)(x_2+d_2) + \hat{\rho}_{13}(x_1+d_1)(x_3+d_3) + \hat{\rho}_{23}(x_2+d_2)(x_3+d_3)\right\},$$

where

$$\delta_4 = \frac{R_{23}d_1^2 + R_{13}d_2^2 + R_{12}d_3^2 - 2(\hat{\rho}_{12}d_1d_2 + \hat{\rho}_{13}d_1d_3 + \hat{\rho}_{23}d_2d_3)}{2|\Sigma|} + d,$$
$$d_1 = \frac{\rho_{12}\sigma_s\sigma_v + (r^f - q - \frac{1}{2}\sigma_s^2)}{\sigma_s}\sqrt{\tau},$$
$$d_2 = \frac{\rho_{23}\sigma_v\sigma_y + (r^d + \frac{1}{2}\sigma_v^2)}{\sigma_v}\sqrt{\tau},$$
$$d_3 = \frac{\rho_{23}\sigma_v\sigma_y + (r^d - r^f + \frac{1}{2}\sigma_y^2)}{\sigma_y}\sqrt{\tau}.$$

Therefore, by $\xi_7 = x_1 + c_1$, $\xi_7 = x_2 + c_2$, $\xi_9 = x_3 + c_3$, $\xi_{10} = x_1 + d_1$, $\xi_{11} = x_2 + d_2$, and $\xi_{12} = x_3 + d_3$, $C_2(t,s,v,y)$ becomes

$$C_2(t,s,v,y) = \frac{(1-\alpha)sv}{D}\delta_3\mathbf{N}_3\left[\kappa_1^3(t,x),\kappa_2^3(t,v),\kappa_3^3(t,y)\right]$$
$$- \frac{(1-\alpha)Kv}{D}\delta_4\mathbf{N}_3\left[\kappa_1^4(t,x),\kappa_2^4(t,v),\kappa_3^4(t,y)\right]. \tag{30}$$

Finally, the price of the foreign equity options with credit risk is obtained by combining (28) and (30). □

3.2 Implications

In this subsection, we investigate the accuracy of the closed-form solution of the foreign equity options with credit risk obtained in Theorem 1 using a Monte-Carlo simulation. For the numerical experiments, the model parameters chosen are $S^d(0) = S^f(0) = s = 40$, $K = 40$, $V(0) = v = 100$, $Y(0) = y = 0.44$, $D = 85$, $\alpha = 0.25$, $T - t = 1$, $\bar{y} = 10^{-4}$, $r^d = r^f = 0.05$, $q = 0.011$, $\sigma_S = \sigma_V = \sigma_Y = 0.2$ and $\rho_{12} = \rho_{13} = \rho_{23} = 0.25$. These parameters are based on the work of Klein [3] and Dai et al. [30].

In Table 1, we present the Monte-Carlo value (C_{MC}) according to the number of simulations, the closed-form pricing formula (C) in Theorem 1, and the price difference between them. We also provide Fig. 1 to verify visually the accuracy of our formula. As shown in

Table 1 Comparison between Monte-Carlo simulation result and the closed-form formula. Note that C_{MC} implies the Monte-Carlo results based on the stochastic dynamics presented in (7)–(10)

Number of simulations	C_{MC}	C	$\lvert C_{\text{MC}} - C\rvert$
5000	1.6934	1.6721	0.0213
10,000	1.6764	1.6721	0.0043
15,000	1.6725	1.6721	0.0004
20,000	1.6718	1.6721	0.0003
25,000	1.6725	1.6721	0.0004
30,000	1.6720	1.6721	0.0001

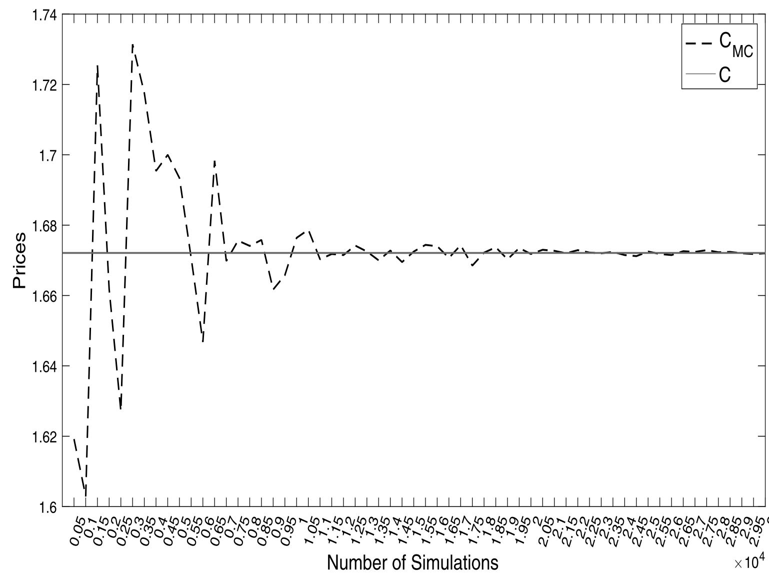

Figure 1 Price convergence of Monte-Carlo simulation for foreign equity option with credit risk

Table 1 and Fig. 1, one can observe that the number of the simulations increases, the price difference $|C_{\mathrm{MC}} - C|$ goes to zero. It implies that the numerical value from the Monte-Carlo simulation, which is regarded as the best approximation of a real-world solution, gets closer to our closed solution. In other words, we conclude that our closed-form pricing formula for the foreign equity options with credit risk is accurately derived.

4 Concluding remarks

Foreign equity options belong to the popular exotic options in the over-the-counter markets, and credit risk is an indeed important issue in the OTC market. In this sense, we study the pricing of the foreign equity options with credit risk. To the best of our knowledge, we are first to consider the credit risk when the foreign equity option is priced. Among several foreign equity options, we deal with the foreign equity option in a foreign currency. In this study, we use the PDE approach to obtain a closed-form pricing formula of the foreign equity option with credit risk based on the structural model of Klein [3]. In particular, to solve the PDE problems, the properties of triple Mellin transform are used as an important tool, and they enable us to provide the explicit closed-form pricing formula of the option price with 3-dimensional normal cumulative distribution functions. Finally, we show that our formula is accurate by comparing it to the numerical price by the Monte-Carlo simulation.

Appendix A: Black–Scholes components

The Black–Scholes components presented in Theorem 1 are as follows:

$$\kappa_1^1(t,s) \triangleq \frac{1}{\sigma_s\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right) + \left(r^f - q + \frac{1}{2}\sigma_s^2\right)(T-t)\right],$$

$$\kappa_2^1(t,v) \triangleq \frac{1}{\sigma_v\sqrt{T-t}}\left[\ln\left(\frac{v}{D^*}\right) + \left(\rho_{12}\sigma_s\sigma_v + \rho_{23}\sigma_v\sigma_y + r^d - \frac{1}{2}\sigma_v^2\right)(T-t)\right],$$

$$\kappa_3^1(t,y) \triangleq \frac{1}{\sigma_y\sqrt{T-t}}\left[\ln\left(\frac{y}{\bar{y}}\right) + \left(\rho_{13}\sigma_s\sigma_y + r^d - r^f + \frac{1}{2}\sigma_y^2\right)(T-t)\right],$$

$$\kappa_1^2(t,s) \triangleq \frac{1}{\sigma_s\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right) + \left(r^f - q - \frac{1}{2}\sigma_s^2\right)(T-t)\right],$$

$$\kappa_2^2(t,\nu) \triangleq \frac{1}{\sigma_\nu\sqrt{T-t}}\left[\ln\left(\frac{\nu}{D^*}\right) + \left(\rho_{23}\sigma_\nu\sigma_y + r^d - \frac{1}{2}\sigma_\nu^2\right)(T-t)\right],$$

$$\kappa_3^2(t,y) \triangleq \frac{1}{\sigma_y\sqrt{T-t}}\left[\ln\left(\frac{y}{\bar{y}}\right) + \left(\rho_{13}\sigma_s\sigma_y + r^d - r^f + \frac{1}{2}\sigma_y^2\right)(T-t)\right],$$

$$\kappa_1^3(t,s) \triangleq \frac{1}{\sigma_s\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right) + \left(\rho_{12}\sigma_s\sigma_\nu + r^f - q + \frac{1}{2}\sigma_s^2\right)(T-t)\right],$$

$$\kappa_2^3(t,\nu) \triangleq -\frac{1}{\sigma_\nu\sqrt{T-t}}\left[\ln\left(\frac{\nu}{D^*}\right) + \left(\rho_{12}\sigma_s\sigma_\nu + \rho_{23}\sigma_\nu\sigma_y + r^d + \frac{1}{2}\sigma_\nu^2\right)(T-t)\right],$$

$$\kappa_3^3(t,y) \triangleq \frac{1}{\sigma_y\sqrt{T-t}}\left[\ln\left(\frac{y}{\bar{y}}\right) + \left(\rho_{13}\sigma_s\sigma_y + \rho_{23}\sigma_\nu\sigma_y + r^d - r^f + \frac{1}{2}\sigma_y^2\right)(T-t)\right],$$

$$\kappa_1^4(t,s) \triangleq \frac{1}{\sigma_s\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right) + \left(\rho_{12}\sigma_s\sigma_\nu + r^f - q - \frac{1}{2}\sigma_s^2\right)(T-t)\right],$$

$$\kappa_2^4(t,\nu) \triangleq -\frac{1}{\sigma_\nu\sqrt{T-t}}\left[\ln\left(\frac{\nu}{D^*}\right) + \left(\rho_{23}\sigma_\nu\sigma_y + r^d + \frac{1}{2}\sigma_\nu^2\right)(T-t)\right],$$

$$\kappa_3^4(t,y) \triangleq \frac{1}{\sigma_y\sqrt{T-t}}\left[\ln\left(\frac{y}{\bar{y}}\right) + \left(\rho_{23}\sigma_\nu\sigma_y + r^d - r^f + \frac{1}{2}\sigma_y^2\right)(T-t)\right],$$

$$\delta_1 \triangleq \exp\left[\frac{R_{23}a_1^2 + R_{13}a_2^2 + R_{12}a_3^2 - 2(\hat{\rho}_{12}a_1a_2 + \hat{\rho}_{13}a_1a_3 + \hat{\rho}_{23}a_2a_3)}{2|\Sigma|} + d\right],$$

$$\delta_2 \triangleq \exp\left[\frac{R_{23}b_1^2 + R_{13}b_2^2 + R_{12}b_3^2 - 2(\hat{\rho}_{12}b_1b_2 + \hat{\rho}_{13}b_1b_3 + \hat{\rho}_{23}b_2b_3)}{2|\Sigma|} + d\right],$$

$$\delta_3 \triangleq \exp\left[\frac{R_{23}c_1^2 + R_{13}c_2^2 + R_{12}c_3^2 - 2(\hat{\rho}_{12}c_1c_2 + \hat{\rho}_{13}c_1c_3 + \hat{\rho}_{23}c_2c_3)}{2|\Sigma|} + d\right],$$

$$\delta_4 \triangleq \exp\left[\frac{R_{23}d_1^2 + R_{13}d_2^2 + R_{12}d_3^2 - 2(\hat{\rho}_{12}d_1d_2 + \hat{\rho}_{13}d_1d_3 + \hat{\rho}_{23}d_2d_3)}{2|\Sigma|} + d\right].$$

Here, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, and d are given in the proof of Theorem 1.

Appendix B: *n*-Dimensional normal cumulative distribution

For $n = 1, 2, \ldots$, a random variable $X = [X_1 \cdots X_n]^T$ is said to have a n-dimensional normal distribution with expectation $\mathbb{E}[X] = \mu \in \mathbb{R}^n$ and covariance matrix Σ in the space of symmetric positive definite $n \times n$ matrices if its probability density function is given by

$$f(x) = \frac{1}{(2\pi)^{n/2}(\det\Sigma)^{1/2}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right). \tag{31}$$

If $n = 3$, the standard normal distribution is

$$f(x_1,x_2,x_3) = \frac{1}{(2\pi)^{3/2}(\det\Sigma^*)^{1/2}}\exp\left(\frac{p(x_1,x_2,x_3)}{\det\Sigma^*}\right), \tag{32}$$

where

$$p(x_1,x_2,x_3) \triangleq -R_{23}x_1^2 - R_{13}x_2^2 - R_{12}x_3^2 + 2(\hat{\rho}_{12}x_1x_2 + \hat{\rho}_{13}x_1x_3 + \hat{\rho}_{23}x_2x_3),$$

$$\det \Sigma^* \triangleq 1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23},$$

and R_{23}, R_{13}, R_{12}, $\hat{\rho}_{12}$, $\hat{\rho}_{13}$, $\hat{\rho}_{23}$ are presented in Theorem 1.

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Authors' contributions
DK proved theorems in the paper. J-HY proposed the methods to solve the problems. GK designed and wrote the paper. All authors read and approved the final manuscript.

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