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New coincidence point results for generalized graph-preserving multivalued mappings with applications

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Abstract

This research aims to investigate a novel coincidence point (cp) of generalized multivalued contraction (gmc) mapping involved a directed graph in *b*-metric spaces (b-ms). An example and some corollaries are derived to strengthen our main theoretical results. We end the manuscript with two important applications, one of them is interested in finding a solution to the system of nonlinear integral equations (nie) and the other one relies on the existence of a solution to fractional integral equations (fie).

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1 Introduction and preliminaries

Fractional calculus is the official modeling tool to represent the conduct of nanofluids, the dynamics of ions over the membrane and many more. In past years, this branch [1, 2] has attracted great interest. There exist many kinds of proposed fractional operators, for instance, we have the well-known Caputo, Riemann–Liouville, and Grunwald–Letnikov derivatives. Among all the papers dealing with fractional derivatives, fractional differential equations, as an important research field, have attained a great deal of attention from many researchers; see [3–8].

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, physics, engineering and biology. Recently, a large amount of literature developed concerning the application of fractional differential equations in nonlinear dynamics [9, 10].

Metric fixed point theory has enormous applications in distinct sciences, one of them on graphs was shown by Echenique [11]. After that, a lot of papers concerning the concept of graph contractions, techniques of fixed points on the graph theory and generalizations of the Banach contraction principle in the directed graph were discussed in various spaces [11–15].

The idea of a metric was expanded by the problem of the convergence with respect to a measure of the measurable functions. So by this notion, the concept of b-ms was introduced by Czerwik [16] and Bakhtin [17]. Interesting results under the mentioned space are obtained. For expansion, we recommend to refer to [18–20].

Under the mentioned space, this manuscript is interested in studying some of cp for gmc mapping involved a directed graph. After that, we used the theoretical results obtained in studying the solution of nie and fde [21–27].

We consider here all sets and subsets are non-empty and Ω is a directed graph with vertex set $\Im(\Omega)$ which is coincided with a b-ms β and edge set $\Upsilon(\Omega)$ which contains Ξ, where Ξ is the diagonal of $\beta \times \beta$. Also, we suppose that Ω has no parallel edges. We say that a mapping $\zeta : \beta \to \beta$ is an Ω-contraction if

- for all $c, e \in \beta$, $(c, e) \in \Upsilon(\Omega)$ implies $(\zeta c, \zeta e) \in \Upsilon(\Omega)$, i.e., ζ preserves edges of Ω,
- for $c, e \in \beta$, there is $0 < \varrho < 1$, so that $(c, e) \in \Upsilon(\Omega)$ implies $\xi(\zeta(c), \zeta(e)) \leq \varrho\xi(c, e)$.

Definition 1.1 ([16]) Let $\beta \neq \emptyset$ be a set and $s \geq 1$ be a given real number. A function $\xi : \beta \times \beta \to [0, +\infty)$ is a b-metric iff, for all $c, e, f \in \Upsilon$, the following stipulations are realized:

- $\xi(c, e) = 0 \Rightarrow e = c$;
- $\xi(c, e) = \xi(e, c)$;
- $\xi(c, f) \leq s(\xi(c, e) + \xi(e, f))$.

The triplet (β, ξ, s) is called a b-ms with coefficient s.

For more details about topological properties of this space see [16, 17].

Let (β, ξ, s) be a b-ms. Denote by $\chi_{b,cl}(\beta)$ the set of all non-empty bounded closed set in β. Define the function $\urcorner : \chi_{b,cl}(\beta) \times \chi_{b,cl}(\beta) \to \mathbb{R}^+$ as

$$\urcorner(M_1, M_2) = \max\left\{ \sup_{\varphi \in M_1} \xi(\varphi, M_2), \sup_{\varphi \in M_2} \xi(\varphi, M_1) \right\},$$

where $\xi(\varkappa, M_1) = \inf\{\varpi(\varkappa, \varphi) : \varphi \in M_1\}$, for $M_1, M_2 \in \chi_{b,cl}(\beta)$. $\urcorner$ is the famous Hausdorff metric.

Consider

$$\delta(M_1, M_2) = \sup\left\{\xi(\varphi, \varkappa) : \varphi \in M_1, \varkappa \in M_2\right\}$$

and

$$\Delta(M_1, M_2) = \inf\left\{\xi(\varphi, \varkappa) : \varphi \in M_1, \varkappa \in M_2\right\}.$$

The properties below can be deduced from the definition of δ, for $M_1, M_2, M_3 \in \chi_{b,cl}(\beta)$,

- $\delta(M_1, M_2) = \delta(M_2, M_1)$;
- $\delta(M_1, M_2) = 0$ iff $M_1 = M_2 = \{\varphi\}$;
- $\delta(M_1, M_3) \leq \delta(M_1, M_2) + \delta(M_2, M_3)$;
- $\delta(M_1, M_1) = \operatorname{diam} M_1$.

Lemma 1.2 ([16]) *Let (β, ξ, s) be a b-ms with a coefficient $s \geq 1$. For any $M_1, M_2, M_3 \in \chi_{b,cl}(\beta)$ and $c, e \in \beta$, the following assertions are valid*:

- $\xi(c, M_2) \leq \xi(c, e)$ *for* $e \in M_2$;

- $\delta(M_1, M_2) \leq \urcorner(M_1, M_2)$;
- $\xi(c, M_2) \leq \urcorner(M_1, M_2)$;
- $\urcorner(M_1, M_1) = 0$;
- $\urcorner(M_1, M_2) = \urcorner(M_2, M_1)$;
- $\urcorner(M_1, M_3) \leq s[\urcorner(M_1, M_2) + \urcorner(M_2, M_3)]$;
- $\xi(c, M_1) \leq s[\xi(c, e) + \xi(e, M_1)]$.

Lemma 1.3 ([28]) *Let a trio (β, ξ, s) be a b-metric space and $\varpi_1, \varpi_2 \in \chi_{b,cl}(\beta)$. then, for $\hbar_1 \geq 1$, $c \in \varpi_1$ there exists $e(c) \in \varpi_2$ such that $\xi(c, e) \leq \hbar_1 \urcorner(\varpi_1, \varpi_2)$.*

Definition 1.4 ([29]) For a set β, assume that $\Omega = (\Im(\Omega), \Upsilon(\Omega))$ is a graph with $\Im(\Omega) = \beta$, then:

- A multivalued map $\wp : \beta \to \chi_{b,cl}(\beta)$ is called graph preserving if for all $\theta \in \wp(c)$, $\vartheta \in \wp(e)$ we have if $(c, e) \in \Upsilon(\Omega)$, then $(\theta, \vartheta) \in \Upsilon(\Omega)$, i.e., it preserves the edges.
- Two multivalued maps $\wp_1, \wp_2 : \beta \to \chi_{b,cl}(\beta)$ are called mixed graph preserving respect to a_1 and a_2 if $(a_1(c), a_2(e)) \in \Upsilon(\Omega)$, then $(\sigma_1, \sigma_2) \in \Upsilon(\Omega)$ for all $\sigma_1 \in \wp_1(c)$ and $\sigma_2 \in \wp_2(e)$ and if $(a_2(e), a_1(c)) \in \Upsilon(\Omega)$, then $(\mu_1, \mu_2) \in \Upsilon(\Omega)$ for all $\mu_1 \in \wp_2(c)$ and $\mu_2 \in \wp_1(e)$.

In this manuscript, some coincidence points for generalized multivalued contraction mappings implicating a directed graph in the setting of b-metric spaces are established. Also, to support our theoretical results, we present an example and some consequences. Ultimately, the existence solution of the system of nonlinear integral and fractional integral equations are derived.

2 Main results

We will start this part by defining the concept of the generalized Ω-multivalued mapping.

Definition 2.1 Assume that $\Omega = (\Im(\Omega), \Upsilon(\Omega))$ is a graph with vertex set $\Im(\Omega) = \beta$ and edges set $\Upsilon(\Omega)$ such that $\Xi \subseteq \Upsilon(\Omega)$ on a b-ms (β, ξ, s). For $a_1, a_2 : \beta \to \beta$ and $\wp_1, \wp_2 : \beta \to \chi_{b,cl}(\beta)$; $\wp_1$, $\wp_2$ are called generalized $(a_1, a_2) - \Omega$-multivalued mappings if the following assumptions hold:

- with respect to a_1, a_2, $\wp_1$, $\wp_2$ are mixed graph preserving;
- for $c, e \in \beta$ with $(a_1(c), a_2(e)) \in \Upsilon(\Omega)$, there is $\ell \geq 0$ such that, for

$$\Re\big(a_1(c), a_2(e)\big) = \max\left\{\xi\big(a_1(c), a_2(e)\big), \frac{\Delta(a_1(c), \wp_2 c).\Delta(a_2(e), \wp_1 e)}{1 + \xi(a_1(c), a_2(e))}\right\}$$

and

$$\Re^*\big(b_1(c), b_2(e)\big) = \min\left\{\begin{array}{c}\xi(a_1(c), a_2(e)), \Delta(a_1(c), \wp_2 c), \Delta(a_2(e), \wp_1 e), \\ \Delta(a_1(c), \wp_1 e), \Delta(a_2(e), \wp_2 c)\end{array}\right\},$$

we get

$$\Theta\big(s\urcorner(\wp_1 c, \wp_2 e)\big) \leq \lambda\big(\Theta\big(\Re\big(a_1(c), a_2(e)\big)\big)\big)\Theta\big(\Re\big(a_1(c), a_2(e)\big)\big) + \ell\big(\Re^*\big(a_1(c), a_2(e)\big)\big),$$

and if $(a_2(c), a_1(e)) \in \Upsilon(\Omega)$, then we get

$$\Theta\big(s\daleth(\wp_1 c, \wp_2 e)\big) \le \lambda\big(\Theta\big(\Re\big(a_2(c), a_1(e)\big)\big)\big)\Theta\big(\Re\big(a_2(c), a_1(e)\big)\big) + \ell\big(\Re^*\big(a_2(c), a_1(e)\big)\big),$$

where

- $\Theta : [0,\infty) \to [0,\infty)$ is a non-decreasing and continuous function with $\Theta(\alpha\varkappa) \le \alpha\Theta(\varkappa)$ for all $\alpha > 1$ and $\Theta(0) = 0$,
- $\lambda : [0,\infty) \to [0,1)$ is an increasing function with $\lambda(0) = 0$.

Theorem 2.2 *Suppose that (β, ξ, s) is a complete b-ms with $s \ge 1$ involved with a graph $\Omega = (\Im(\Omega), \Upsilon(\Omega))$, $\wp_1, \wp_2 : \beta \to \chi_{b,cl}(\beta)$ generalized $(a_1, a_2)-\Omega$-multivalued mappings with surjective mappings $a_1, a_2 : \beta \to \beta$. Then there are $\theta, \vartheta \in \beta$ such that $a_1(\theta) \in \wp_1(\theta)$ or $a_2(\vartheta) \in \wp_2(\vartheta)$, provided that*

$(\dagger_i)$ *for some $\theta \in \wp_1(c_\circ)$, there is $c_\circ \in \beta$ such that $(a_1(c_\circ), \theta) \in \Upsilon(\Omega)$,*

$(\dagger_{ii})$ *if $(a_1(c), a_2(e)) \in \Upsilon(\Omega)$, then $(\sigma_1, \sigma_2) \in \Upsilon(\Omega)$ for all $\sigma_1 \in \wp_1(c)$ and $\sigma_2 \in \wp_2(e)$ and if $(a_2(e), a_1(c)) \in \Upsilon(\Omega)$, then $(\mu_1, \mu_2) \in \Upsilon(\Omega)$ for all $\mu_1 \in \wp_2(e)$ and $\mu_2 \in \wp_1(e)$,*

$(\dagger_{iii})$ *for $\{c_n\}_{n\in\mathbb{N}} \in \beta$, if $\lim_{n\to\infty} c_n = c$ and $(c_n, c_{n+1}) \in \Upsilon(\Omega)$, for $n \in \mathbb{N}$, then there exists a subsequence $\{c_{n_j}\}_{n_j\in\mathbb{N}}$ so that $(c_{n_j}, c) \in \Upsilon(\Omega)$ for $n_j \in \mathbb{N}$.*

Proof Since a_2 is surjective, there is $c_1 \in \beta$ so that $a_2(c_1) \in \wp_2 c_\circ$ and $(a_1(c_\circ), a_2(c_1)) \in \Upsilon(\Omega)$. Put $\zeta = \frac{1}{\sqrt{\lambda(\Theta(\xi(a_1(c_\circ), a_2(c_1))))}}$, then $\zeta > 1$, so that

$$0 < \Delta\big(a_2(c_1), \wp_1(c_1)\big) \le \daleth\big(\wp_2(c_\circ), \wp_1(c_1)\big) \le \zeta\daleth\big(\wp_2(c_\circ), \wp_1(c_1)\big)\big).$$

It follows from Lemma 1.3, $(\dagger_{ii})$ and a_1 being surjective that there is $c_2 \in \beta$ with $a_1(c_2) \in \wp_1(c_1)$ and $(a_2(c_1), a_1(c_2)) \in \Upsilon(\Omega)$, thus

$$\begin{aligned}
\sqrt{s}\Theta\big(\xi\big(a_2(c_1), a_1(c_2)\big)\big) &\le \Theta\big(s\xi\big(a_2(c_1), a_1(c_2)\big)\big) \\
&< \Theta\big(s\zeta\daleth\big(\wp_2(c_\circ), \wp_1(c_1)\big)\big)\big) \\
&\le \zeta\Theta\big(s\daleth\big(\wp_2(c_\circ), \wp_1(c_1)\big)\big)\big) \\
&\le \zeta\lambda\big(\Theta\big(\Re\big(a_1(c_\circ), a_2(c_1)\big)\big)\big)\Theta\big(\Re\big(a_1(c_\circ), a_2(c_1)\big)\big) \\
&\quad + \zeta\ell\big(\Re^*\big(a_1(c_\circ), a_2(c_1)\big)\big),
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
\Re\big(a_1(c_\circ), a_2(c_1)\big) &= \max\left\{\xi\big(a_1(c_\circ), a_2(c_1)\big), \frac{\Delta(a_1(c_\circ), \wp_2(c_\circ)).\Delta(a_2(c_1), \wp_1(c_1))}{1 + \xi(a_1(c_\circ), a_2(c_1))}\right\} \\
&\le \max\big\{\xi\big(a_1(c_\circ), a_2(c_1)\big), \xi\big(a_2(c_1), a_1(c_2)\big)\big\}
\end{aligned}$$

and

$$\Re^*\big(a_1(c_\circ), a_2(c_1)\big) = \min\left\{\begin{array}{c} \xi(a_1(c_\circ), a_2(c_1)), \Delta(a_1(c_\circ), \wp_2(c_\circ)), \Delta(a_2(c_1), \wp_1(c_1)), \\ \Delta(a_1(c_\circ), \wp_1(c_1)), \Delta(a_2(c_1), \wp_2(c_\circ)) \end{array}\right\}$$

$$\begin{aligned}
&\leq \min\left\{\begin{array}{c}\xi(a_1(c_\circ),a_2(c_1)),\xi(a_1(c_\circ),a_2(c_1)),\\ \xi(a_2(c_1),a_1(c_2)),\xi(a_1(c_\circ),a_1(c_2)),\xi(a_2(c_1),a_2(c_1))\end{array}\right\}\\
&= 0.
\end{aligned}$$

Now, if $\xi(a_1(c_\circ),a_2(c_1)) < \xi(a_2(c_1),a_1(c_2))$, then, by (2.1), one can write

$$\begin{aligned}
\sqrt{s}\Theta\bigl(\xi\bigl(a_2(c_1),a_1(c_2)\bigr)\bigr) &\leq \zeta\lambda\bigl(\Theta\bigl(\Re\bigl(a_1(c_\circ),a_2(c_1)\bigr)\bigr)\bigr)\Theta\bigl(\Re\bigl(a_1(c_\circ),a_2(c_1)\bigr)\bigr)\\
&\quad + \zeta\ell\bigl(\Re^*\bigl(a_1(c_\circ),a_2(c_1)\bigr)\bigr)\\
&= \zeta\lambda\bigl(\Theta\bigl(\Re\bigl(a_1(c_\circ),a_2(c_1)\bigr)\bigr)\bigr)\Theta\bigl(\Re\bigl(a_1(c_\circ),a_2(c_1)\bigr)\bigr)\\
&= \zeta\lambda\bigl(\Theta\bigl(\xi\bigl(a_2(c_1),a_1(c_2)\bigr)\bigr)\bigr)\Theta\bigl(\xi a_2(c_1),a_1(c_2)\bigr).
\end{aligned}$$

If we take $\zeta = \frac{1}{\sqrt{\lambda(\Theta(\xi(a_2(c_1),a_1(c_2))))}}$, by the definition of Θ and since $\lambda < 1$, we can write

$$\begin{aligned}
\sqrt{s}\Theta\bigl(\xi\bigl(a_2(c_1),a_1(c_2)\bigr)\bigr) &\leq \frac{\lambda(\Theta(\xi(a_2(c_1),a_1(c_2))))}{\sqrt{\lambda(\Theta(\xi(a_2(c_1),a_1(c_2))))}}\Theta\bigl(\xi\bigl(a_2(c_1),a_1(c_2)\bigr)\bigr)\\
&= \sqrt{\lambda\bigl(\Theta\bigl(\xi\bigl(a_2(c_1),a_1(c_2)\bigr)\bigr)\bigr)}\Theta\bigl(\xi\bigl(a_2(c_1),a_1(c_2)\bigr)\bigr).\\
&< \Theta\bigl(\xi\bigl(a_2(c_1),a_1(c_2)\bigr)\bigr),
\end{aligned}$$

this leads $s < 1$ or $\Theta(\xi(a_2(c_1),a_1(c_2))) < 0$. In both cases we get a contradiction, hence we have $\max\{\xi(a_1(c_\circ),a_2(c_1)),\xi(a_2(c_1),a_1(c_2))\} = \xi(a_1(c_\circ),a_2(c_1))$. Substituting in (2.1), we have

$$\Theta\bigl(\xi\bigl(a_2(c_1),a_1(c_2)\bigr)\bigr) \leq \frac{1}{\sqrt{s}}\sqrt{\lambda\bigl(\Theta\bigl(\xi\bigl(a_1(c_\circ),a_2(c_1)\bigr)\bigr)\bigr)}\Theta\bigl(\xi\bigl(a_1(c_\circ),a_2(c_1)\bigr)\bigr). \tag{2.2}$$

Since Θ is non-decreasing, we get

$$\xi\bigl(a_2(c_1),a_1(c_2)\bigr) \leq \frac{1}{\sqrt{s}}\sqrt{\lambda\bigl(\Theta\bigl(\xi\bigl(a_1(c_\circ),a_2(c_1)\bigr)\bigr)\bigr)}\xi\bigl(a_1(c_\circ),a_2(c_1)\bigr). \tag{2.3}$$

Once again $a_1(c_2) \in \wp_1(c_1)$ and $a_2(c_1) \in \wp_2(c_\circ)$. Take

$$\zeta_1 = \frac{1}{\sqrt{\lambda(\Theta(\xi(a_1(c_1),a_2(c_2))))}}.$$

By (2.2), we observe that $\zeta_1 > 1$. If $a_1(c_2) \in \wp_2(c_2)$, then c_2 is a cp for a_1 and $\wp_2$. Assume that $a_1(c_2) \notin \wp_2(c_2)$, we have

$$0 < \Theta\bigl(\xi\bigl(a_1(c_2),\wp_2(c_2)\bigr)\bigr) \leq \Theta\bigl(\daleth\bigl(\wp_1(c_1),\wp_2(c_2)\bigr)\bigr) < \zeta_1\Theta\bigl(\daleth\bigl(\wp_1(c_1),\wp_2(c_2)\bigr)\bigr).$$

Thus, there is $a_2(c_3) \in \wp_2(c_2)$ so that $(a_1(c_2),a_2(c_3)) \in \Upsilon(\Omega)$ and

$$\begin{aligned}
\sqrt{s}\Theta\bigl(\xi\bigl(a_1(c_2),a_2(c_3)\bigr)\bigr) &\leq \Theta\bigl(s\xi\bigl(a_1(c_2),a_2(c_3)\bigr)\bigr)\\
&< \zeta_1\Theta\bigl(s\daleth\bigl(\wp_1(c_1),\wp_2(c_2)\bigr)\bigr)
\end{aligned}$$

$$\begin{aligned}&\leq \zeta_1\lambda\big(\Theta\big(\Re\big(a_1(c_1),a_2(c_2)\big)\big)\big)\Theta\big(\Re\big(a_1(c_1),a_2(c_2)\big)\big)\\&\quad+\zeta_1\ell\big(\Re^*\big(a_1(c_1),a_2(c_2)\big)\big).\end{aligned}$$

In the same manner, we have $\Re(a_1(c_1),a_2(c_2))\leq\xi(a_1(c_1),a_2(c_2))$ and $\Re^*(a_1(c_1),a_2(c_2))=0$. By property of (λ) and (2.2), we get

$$\begin{aligned}\sqrt{s}\Theta\big(\xi\big(a_1(c_2),a_2(c_3)\big)\big)&\leq\frac{\lambda(\Theta(\xi(a_1(c_1),a_2(c_2))))\Theta(\xi(a_1(c_1),a_2(c_2)))}{\sqrt{\lambda(\Theta(\xi(a_1(c_1),a_2(c_2))))}}\\&\leq\sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_1),a_2(c_2)\big)\big)\big)}\Theta\big(\xi\big(a_1(c_1),a_2(c_2)\big)\big)\\&\leq\frac{1}{\sqrt{s}}\sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_1),a_2(c_2)\big)\big)\big)}\\&\quad\times\sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big)\big)}\Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big).\end{aligned}\tag{2.4}$$

Since $\sqrt{\Theta(\xi(a_1(c_\circ),a_2(c_1)))}<1$, by (2.2), we have

$$\Theta\big(\xi\big(a_1(c_1),a_2(c_2)\big)\big)\leq\Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big).\tag{2.5}$$

Since (λ) is increasing, by (2.4) and (2.5), we can write

$$\Theta\big(\xi\big(a_1(c_2),a_2(c_3)\big)\big)\leq\frac{1}{(\sqrt{s})^2}\Big(\sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big)\big)}\Big)^2\Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big).\tag{2.6}$$

Again by (2.3), we have

$$\xi\big(a_1(c_2),a_2(c_3)\big)\leq\frac{1}{(\sqrt{s})^2}\Big(\sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big)\big)}\Big)^2\xi\big(a_1(c_\circ),a_2(c_1)\big).$$

Manifestly $a_1(c_2)$ is distinct from $a_2(c_3)$ since $a_1(c_\circ)\neq a_2(c_1)$. Set

$$\zeta_2=\frac{1}{\sqrt{\lambda(\Theta(\xi(a_1(c_2),a_2(c_3))))}}.$$

Then $\zeta_2>1$.

If $a_2(c_3)\in\wp_1(c_3)$ then c_3 is a cp for a_2 and $\wp_1$. Let $a_2(c_3)\neq\wp_1(c_3)$. Hence

$$0<\Theta\big(\xi\big(a_2(c_3),\wp_1(c_3)\big)\big)\leq\Theta\big(\urcorner\big(\wp_2(c_2),\wp_1(c_3)\big)\big)<\zeta_2\Theta\big(\urcorner\big(\wp_2(c_2),\wp_1(c_3)\big)\big).$$

Thus, there is $a_1(c_4)\in\wp_1(c_3)$ so that $(a_2(c_3),a_1(c_4))\in\Upsilon(\Omega)$ and

$$\begin{aligned}\sqrt{s}\Theta\big(\xi\big(a_2(c_3),a_1(c_4)\big)\big)&<\zeta_2\Theta\big(s\urcorner(\wp_2c_2,\wp_1c_3)\big)\\&\leq\zeta_1\lambda\big(\Theta\big(\Re\big(a_1(c_2),a_2(c_3)\big)\big)\big)\Theta\big(\Re\big(a_1(c_2),a_2(c_3)\big)\big)\\&\quad+\zeta_1\ell\big(\Re^*\big(a_1(c_2),a_2(c_3)\big)\big).\end{aligned}\tag{2.7}$$

In the same scenario, we get $\Re(a_1(c_2),a_2(c_3))\leq\xi(a_1(c_2),a_2(c_3))$ and $\Re^*(a_1(c_2),a_2(c_3))=0$. Consequently by (2.7), one can get

$$\sqrt{s}\Theta\big(\xi\big(a_2(c_3),a_1(c_4)\big)\big)\leq\sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_2),a_2(c_3)\big)\big)\big)}\Theta\big(\xi\big(a_1(c_2),a_2(c_3)\big)\big)$$

$$
\begin{aligned}
&\leq \sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_2),a_2(c_3)\big)\big)\big)}\,\Theta\big(\xi\big(a_1(c_2),a_2(c_3)\big)\big) \\
&\leq \frac{1}{(\sqrt{s})^2}\sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_2),a_2(c_3)\big)\big)\big)} \\
&\quad \times \Big(\sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big)\big)}\Big)^2\Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big). \qquad (2.8)
\end{aligned}
$$

Since $(\sqrt{\lambda(\Theta(\xi(a_1(c_\circ),a_2(c_1))))})^2 < 1$, by (2.6), we get

$$
\Theta\big(\xi\big(a_1(c_2),a_2(c_3)\big)\big) \leq \Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big).
$$

Again, (λ) is increasing and by (2.8), we have

$$
\xi\big(a_2(c_3),a_1(c_4)\big) \leq \frac{1}{(\sqrt{s})^3}\Big(\sqrt{\lambda\big(\Theta\big(\xi\big(a_1(c_\circ),a_2(c_1)\big)\big)\big)}\Big)^3\xi\big(a_1(c_\circ),a_2(c_1)\big).
$$

It is obvious that $a_2(c_3) \neq a_1(c_4)$. Consider

$$
\zeta_3 = \frac{1}{\sqrt{\lambda(\Theta(\xi(a_1(c_3),a_2(c_4))))}}.
$$

Certainly $\zeta_3 > 1$. By continuing in the same manner we can construct a sequence $\{a(c_n)\}$ in β as follows:

$$
a_1(c_{2n}) \in \wp_1(c_{2n-1}), \qquad a_2(c_{2n-1}) \in \wp_2(c_{2n-2}),
$$

and $(a_1(c_{2n}),a_2(c_{2n-1})),(a_2(c_{2n-1}),a_1(c_{2n})) \in \Upsilon(\Omega)$. Let the sequence $\{a(c_n)\}$ be defined as

$$
a(c_n) = \begin{cases} a_1(c_n), & n \text{ even}, \\ a_2(c_n), & n \text{ odd}. \end{cases}
$$

Set $\theta = \sqrt{\lambda(\Theta(\xi(a_1(c_\circ),a_2(c_1))))}$, then $\theta \in (0,1)$. Take $\kappa = \frac{\theta}{\sqrt{s}}$, $s > 1$, we have

$$
\xi\big(a_1(c_n),a_2(c_{n+1})\big) \leq \kappa^n\xi\big(a_1(c_\circ),a_2(c_1)\big).
$$

Since $\kappa \in (0,1)$, it follows that

$$
\sum_{n=0}^{\infty}\xi\big(a_1(c_n),a_2(c_{n+1})\big) \leq \xi\big(a_1(c_\circ),a_2(c_1)\big)\sum_{n=0}^{\infty}\kappa^n < \infty.
$$

This implies that $\{a(c_n)\}$ is a Cauchy sequence in β. The completeness of β leads to there being a point $\tau \in \beta$ so that the sequence $\{a(c_n)\}$ converges to it. Suppose that $\theta,\vartheta \in \beta$ with $a_1(\theta) = \tau = a_2(\vartheta)$. Using $(\dagger_{iii})$, there is $\{a(c_{n_j})\}$ so that $(a(c_{n_j}),a_1(\theta)) \in \Upsilon(\Omega)$, for each $n \in \mathbb{N}$. Now, we claim that $a_1(\theta) \in \wp_2(\theta)$ or $a_2(\vartheta) \in \wp_1(\vartheta)$.

Assume that $\Omega^* = \{n_j : n_j \text{ is even}\}$ and $\Omega = \{n_j : n_j \text{ is odd}\}$. It is clear that $\Omega \cup \Omega^*$ has infinitely many elements (we call it infinite, for simplicity). So at least Ω or Ω^* must be infinite. Let us consider Ω^* to be infinite, for each $a_2(c_{n_j+1})$, $n_j \in \Omega$, one can write, by Lemma 1.2,

$$
\xi\big(a_2(\vartheta),\wp_1(\vartheta)\big) \leq s\xi\big(a_2(\vartheta),a_2(c_{n_j+1})\big) + s\xi\big(a_2(c_{n_j+1}),\wp_1(\vartheta)\big)
$$

$$\begin{aligned}
&\leq s\xi\big(a_2(\vartheta),a_2(c_{n_j+1})\big)+s\urcorner\big(\wp_2(c_{n_j}),\wp_1(\vartheta)\big)\\
&\leq s\xi\big(a_2(\vartheta),a_2(c_{n_j+1})\big)\\
&\quad+\lambda\big(\Theta\big(\Re\big(a_1(c_{n_j}),a_2(\vartheta)\big)\big)\big)\Theta\big(\Re\big(a_1(c_{n_j}),a_2(\vartheta)\big)\big)\\
&\quad+\ell\big(\Re^*\big(a_1(c_{n_j}),a_2(\vartheta)\big)\big),
\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}
&\Re\big(a_1(c_{n_j}),a_2(\vartheta)\big)\\
&\quad=\max\left\{\xi\big(a_1(c_{n_j}),a_2(\vartheta)\big),\frac{\Delta(a_1(c_{n_j}),\wp_2(c_{n_j})).\Delta(a_2(\vartheta),\wp_1(\vartheta))}{1+\xi(a_1(c_{n_j}),a_2(\vartheta))}\right\}\\
&\quad\leq\max\left\{\xi\big(a_1(c_{n_j}),a_2(\vartheta)\big),\frac{\xi(a_1(c_{n_j}),\wp_2(c_{n_j})).\xi(a_2(\vartheta),\wp_1(\vartheta))}{1+\xi(a_1(c_{n_j}),a_2(\vartheta))}\right\}
\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}
&\Re^*\big(a_1(c_{n_j}),a_2(\vartheta)\big)\\
&\quad=\min\left\{\begin{array}{c}\xi(a_1(c_{n_j}),a_2(\vartheta)),\Delta(a_1(c_{n_j}),\wp_2(c_{n_j})),\Delta(a_2(\vartheta),\wp_1(\vartheta)),\\ \Delta(a_1(c_{n_j}),\wp_1(\vartheta)),\Delta(a_2(\vartheta),\wp_2(c_{n_j}))\end{array}\right\}.
\end{aligned}\tag{2.11}$$

Letting $n\to\infty$ in (2.10) and (2.11), we get

$$\Re\big(a_1(c_{n_j}),a_2(\vartheta)\big)=\Re^*\big(a_1(c_{n_j}),a_2(\vartheta)\big)=0.\tag{2.12}$$

Letting $n\to\infty$ in (2.9), $\Theta(0)=0$, and applying (2.12), we have

$$\xi\big(a_2(\vartheta),\wp_1(\vartheta)\big)\leq 0.$$

Because of the distance is non-negative and $\wp_1(\vartheta)$ is closed, this concludes the proof. In a similar way, we can show that $a_1(\theta)\in\wp_2(\theta)$ when Ω is infinite. This completes the proof. Note, if Ω and Ω^* are infinite, then we have the same result. □

The following example supports Definition 2.1 and Theorem 2.2.

Example 2.3 Let $\beta=[0,1]$. Define $\xi:\beta\times\beta\to\mathbb{R}^+$ by

$$\xi(c,e)=|c-e|^2,\quad\forall c,e\in\beta.$$

It is obvious that a trio (β,ξ,s) is a b-ms with $s=2$. Assume that $\Omega=(\Im(\Omega),\Upsilon(\Omega))$ is a directed graph with $\Im(\Omega)=\beta$ and

$$\Upsilon(\Omega)=\left\{(c,c),\left(0,\frac{1}{2}\right),\left(\frac{1}{2},0\right),\left(0,\frac{1}{8}\right),\left(\frac{1}{8},0\right),\left(\frac{1}{2},\frac{1}{8}\right),\left(\frac{1}{8},\frac{1}{2}\right):c\in\beta\right\}.$$

Define $\wp_1,\wp_2:\beta\to\chi_{b,cl}(\beta)$ by

$$\wp_1(c)=\begin{cases}\{\frac{1}{8}\} & \text{if } c\in\{0,\frac{1}{2},\frac{1}{8},1\},\\ \{0,\frac{1}{8}\} & \text{if } c\in(0,1)-\{\frac{1}{8},\frac{1}{2}\},\end{cases}$$

$$\wp_2(c) = \begin{cases} \{\frac{1}{4}\} & \text{if } c = 1, \\ \{0, \frac{1}{4}\} & \text{if } c \in (0,1) - \{\frac{1}{16}, \frac{1}{8}\}, \\ \{\frac{1}{8}\} & \text{if } c \in \{0, \frac{1}{8}, \frac{1}{2}\}, \end{cases}$$

and $a_1, a_2 : \beta \to \beta$ by $a_1(c) = c^3$ and $a_2(c) = c$. Let $\Theta(\omega) = \omega$ and $\lambda(\omega) = \frac{\omega}{2}$, for $\omega \in [0, \infty)$. Because of for every $c, e \in \{0, \frac{1}{2}, \frac{1}{8}\}$, we have $\urcorner(\wp_1(c), \wp_2(e)) = 0$ and $(a_1(c), a_2(e)), (a_1(e), a_2(c)) \in \Upsilon(\Omega)$. Thus $\wp_1$, $\wp_2$ are generalized Ω-multivalued respect to a_1 and a_2. Now, we discuss the case when $c = e = 1$, then $(a_1(c), a_2(e)) = (1,1) \in \Upsilon(\Omega)$ and $\urcorner(\wp_1(c), \wp_2(e)) = \urcorner(\wp_1 1, \wp_2 1) = \frac{1}{64}$. Consider

$$\begin{aligned} \Re\big(a_1(1), a_2(1)\big) &= \max\left\{\xi\big(a_1(1), a_2(1)\big), \frac{\Delta(a_1(1), \wp_2 1).\Delta(a_2(1), \wp_1 1)}{1 + \xi(a_1(1), a_2(1))}\right\} \\ &= \max\left\{0, \frac{\Delta(1, \frac{1}{4}).\Delta(1, \frac{1}{8})}{1}\right\} = \frac{441}{1024} \end{aligned}$$

and

$$\Re^*\big(b_1(1), b_2(1)\big) = \min\left\{\begin{matrix} \xi(a_1(1), a_2(1)), \Delta(a_1(1), \wp_2 1), \Delta(a_2(1), \wp_1 1), \\ \Delta(a_1(1), \wp_1 1), \Delta(a_2(1), \wp_2 1) \end{matrix}\right\} = 0.$$

Then we can write

$$\begin{aligned} \Theta\big(s\urcorner(\wp_1 c, \wp_2 e)\big) &= 2\urcorner(\wp_1 c, \wp_2 e) = 0.031125 \\ &\leq 0.092736 = \frac{1}{2}\left(\frac{441}{1024}\right) \times \frac{441}{1024} \\ &= \lambda\big(\Re\big(a_1(c), a_2(e)\big)\big)\Theta\big(\Re\big(a_1(c), a_2(e)\big)\big) \\ &= \lambda\big(\Re\big(a_1(c), a_2(e)\big)\big)\Theta\big(\Re\big(a_1(c), a_2(e)\big)\big) \\ &\quad + \ell\big(\Re^*\big(a_1(c), a_2(e)\big)\big). \end{aligned}$$

Also, $\wp_1$, $\wp_2$ are generalized Ω-multivalued with respect to a_1 and a_2. Also, the same results can be obtained if $(a_1(c), a_2(e)) \in \Upsilon(\Omega)$, for $c, e \in \beta$. Note that the stipulations $(\dagger_i)$–$(\dagger_{iii})$ of Theorem 2.2 hold. So, there are $\theta, \vartheta \in \beta$ such that $a_1(\theta) \in \wp_1(\theta)$ or $a_2(\vartheta) \in \wp_2(\vartheta)$. Here $\theta = \frac{1}{2}$ or $\vartheta = \frac{1}{8}$.

Theorem 2.2 reduces to the following corollary, if we set $a_1 = a_2 = a$, $\ell = 0$, and $\Re(a_1(c), a_2(e)) = \xi(a(c), a(e))$.

Corollary 2.4 *Let (β, ξ, s) be a complete b-ms with $s \geq 1$ involving a directed graph Ω, $a : \beta \to \beta$ being a surjective map and $\wp_1, \wp_2 : \beta \to \chi_{b,cl}(\beta)$ being a-graph preserving with*

$$\Theta\big(s\urcorner\big(\wp_1(c), \wp_2(e)\big)\big) \leq \lambda\big(\Theta\big(\xi\big(a(c), a(e)\big)\big)\big)\Theta\big(\xi\big(a(c), a(e)\big)\big)$$

for each $c, e \in \beta$ with $(a(c), a(e)) \in \Upsilon(\Omega)$. Assume that

($\ddagger_1$) *for some $\theta \in \wp_1 c_\circ$, there is $c_\circ \in \beta$ such that $(a(c_\circ), \theta) \in \Upsilon(\Omega)$,*

($\ddagger_2$) *for $\{c_n\}_{n\in\mathbb{N}} \in \beta$, if $\lim_{n\to\infty} c_n = c$ and $(c_n, c_{n+1}) \in \Upsilon(\Omega)$, for $n \in \mathbb{N}$, then there exists a subsequence $\{c_{n_j}\}_{n_j\in\mathbb{N}}$ so that $(c_{n_j}, c) \in \Upsilon(\Omega)$ for $n_j \in \mathbb{N}$.*

Then there are $\theta, \vartheta \in \beta$ such that $a(\theta) \in \wp_1(\theta)$ or $a(\vartheta) \in \wp_2(\vartheta)$.

We believe that studying the uniqueness in the current situation is very complicated, so we will reduce Corollary 2.4 to one multivalued mapping and define a partially order set as follows.

Definition 2.5 Suppose that (β,ξ) is a b-metric space with $s>1$ and a partially ordered set $\preceq$. Assume that, for all $\beta_1,\beta_2 \in \beta$, $\beta_1 \preceq \beta_2$ if $m_1 \preceq m_2$ for any $m_1 \in \beta_1$ and $m_2 \in \beta_2$. Let $a:\beta \to \beta$ be a surjective map, and $\wp_1:\beta \to \chi_{b,cl}(\beta)$. $\wp_1$ is called a-increasing if $a(c) \preceq a(e)$ implies $\wp_1(c) \preceq \wp_1(e)$ for any $c,e \in \beta$.

Theorem 2.6 *Let (β,ξ,s) be a complete b-ms with $s \geq 1$ and $\preceq$ be partially order set, $a:\beta \to \beta$ be a surjective map, and $\wp_1:\beta \to \chi_{b,cl}(\beta)$ be a multivalued mapping. Then there is $\theta \in \beta$ such that $a(\theta) \in \wp_1(\theta)$ if the following hypotheses are fulfilled*:

- (h_1) *$\wp_1$ is a-increasing*;
- (h_2) *there are $c_\circ \in \beta$ and $\theta \in \wp_1(c_\circ)$ such that $(a(c_\circ),\theta) \in \Upsilon(\Omega)$*;
- (h_3) *for some $c \in \beta$ and for each sequence c_i such that $a(c_i)$ converges to $a(c)$ and $a(c_i) \preceq a(c_{i+1})$, $i \in \mathbb{N}$, then $a(c_i) \preceq a(c)$*;
- (h_4) *for $c,e \in \beta$ with $a(c) \preceq a(e)$, we get*

$$\Theta\bigl(s\daleth\bigl(\wp_1(c),\wp_1(e)\bigr)\bigr) \leq \lambda\bigl(\Theta\bigl(\xi\bigl(a(c),a(e)\bigr)\bigr)\bigr)\Theta\bigl(\xi\bigl(a(c),a(e)\bigr)\bigr),$$

where Θ and λ are defined in Definition 2.1.

In addition, if $a:\beta \to \beta$ is injective then θ is unique.

Proof Suppose that $\Omega = (\Im(\Omega),\Upsilon(\Omega))$ is a graph with vertex set $\Im(\Omega) = \beta$, and $\Upsilon(\Omega) = \{(c,e): c \preceq e\}$. For $(a(c),a(e)) \in \Upsilon(\Omega)$, we have $a(c) \preceq a(e)$ and by (h_1), we get $\wp_1(c) \preceq \wp_1 e$. For all $\theta \in \wp_1(c)$, $\vartheta \in \wp_1(e)$, we can find $\theta \preceq \vartheta$, thus $(\theta,\vartheta) \in \Upsilon(\Omega)$. This illustrates that $\wp_1$ is a-graph preserving. Hypothesis (h_1) leads to there being $c_\circ \in \beta$ and $\theta \in \wp_1(c_\circ)$ such that $a(c_\circ) \preceq \theta$. Thus $(a(c),\theta) \in \Upsilon(\Omega)$, this fulfills the property $(\ddagger_1)$ of Corollary 2.4. Also the property $(\ddagger_2)$ follows immediately by hypothesis (h_3). Take $\wp_1 = \wp_2$, then $\wp_1$, $\wp_2$ are a-graph-preserving maps and verify

$$\Theta\bigl(s\daleth\bigl(\wp_1(c),\wp_1(e)\bigr)\bigr) \leq \lambda\bigl(\Theta\bigl(\xi\bigl(a(c),a(e)\bigr)\bigr)\bigr)\Theta\bigl(\xi\bigl(a(c),a(e)\bigr)\bigr),$$

for $c,e \in \beta$ with $(a(c),a(e)) \in \Upsilon(\Omega)$. Thus all properties of Corollary 2.4 are fulfilled, so there is $\theta \in \beta$ so that $a(\theta) \in \wp_1(\theta)$. Ultimately, let $\theta,\vartheta \in \beta$ so that $a(\theta) \in \wp_1(\theta)$ or $a(\vartheta) \in \wp_1(\vartheta)$, by contradiction, suppose that $\wp_1(\theta) \neq \wp_1(\vartheta)$ and $\wp_1(\theta) < \wp_1(\vartheta)$ without loss of generality. Since $a(\theta) \in \wp_1(\theta)$ or $a(\vartheta) \in \wp_1(\vartheta)$, it yields $\Delta(a(\theta),\wp_1\theta) = \Delta(a(\vartheta),\wp_1(\vartheta))$ and

$$\begin{aligned}
\Theta\bigl(\xi\bigl(a(\theta),a(\vartheta)\bigr)\bigr) &\leq \Theta(s\xi\bigl(a(\theta),a(\vartheta)\bigr) \\
&\leq \Theta\bigl(s\daleth\bigl(\wp_1(\theta),\wp_1(\vartheta)\bigr)\bigr) \\
&\leq \lambda\bigl(\Theta\bigl(\xi\bigl(a(\theta),a(\vartheta)\bigr)\bigr)\bigr)\Theta\bigl(\xi\bigl(a(\theta),a(\vartheta)\bigr)\bigr) \\
&< \Theta\bigl(\xi\bigl(a(\theta),a(\vartheta)\bigr)\bigr),
\end{aligned}$$

a contradiction, thus $a(\theta) = a(\vartheta)$. Since a is injective, $\theta = \vartheta$. This finishes the required proof. □

The following result is very important in the next section.

Corollary 2.7 *Let (β, ξ, s) be a complete b-ms with $s \geq 1$ and graph $\Omega, \wp_1, \wp_2 : \beta \to \beta$ be a generalized mapping so that*

$$\Theta\big(s\xi\big(\wp_1(c), \wp_2(e)\big)\big) \leq \lambda\big(\Theta\big(\xi(c,e)\big)\big)\Theta\big(\xi(c,e)\big), \tag{2.13}$$

where Θ and λ are defined in Theorem 2.2. Assume that, for $c, e \in \beta$, the following assumptions are fulfilled:

(a_1) *there are $c_\circ \in \beta$ and $\theta \in \wp_1(c_\circ)$ such that $(c_\circ, \theta) \in \Upsilon(\Omega)$;*

(a_2) *if $(c,e) \in \Upsilon(\Omega)$, then $(\sigma_1, \sigma_2) \in \Upsilon(\Omega)$ for all $\sigma_1 \in \wp_1(c)$ and $\sigma_2 \in \wp_2(e)$ and if $(e,c) \in \Upsilon(\Omega)$, then $(\mu_1, \mu_2) \in \Upsilon(\Omega)$ for all $\mu_1 \in \wp_2(c)$ and $\mu_2 \in \wp_1(e)$;*

(a_3) *for $\{c_n\}_{n\in\mathbb{N}} \in \beta$, if $\lim_{n\to\infty} c_n = c$ and $(c_n, c_{n+1}) \in \Upsilon(\Omega)$, for $n \in \mathbb{N}$, then there exists a subsequence $\{c_{n_j}\}_{n_j\in\mathbb{N}}$ so that $(c_{n_j}, c) \in \Upsilon(\Omega)$ for $n_j \in \mathbb{N}$.*

Then there are $\theta, \vartheta \in \beta$ so that $\theta \in \wp_1(\theta)$ or $\vartheta \in \wp_2(\vartheta)$.

3 Supportive applications

This section is divided into two parts, the first part discusses the existence and uniqueness of solutions to Fredholm integral equations of the second kind are discussed and in the second one, the existence of solution to the system of (fie) is showed.

First part: Consider the following system:

$$\begin{cases} c(\omega) = \Xi(\omega) + \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi(\omega,\nu,c(\nu))\,d\nu, \\ e(\omega) = \Xi(\omega) + \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi(\omega,\nu,e(\nu))\,d\nu, \end{cases} \qquad \omega \in [r_1, r_2]. \tag{3.1}$$

Let $\nabla = C[r_1, r_2]$ be the set of all continuous function defined on $[r_1, r_2]$. For $c, e \in \nabla$ and $q > 1$, define $\xi : \nabla \times \nabla \to \mathbb{R}^+$ by

$$\xi(c,e) = \Big(\sup_{\omega\in[r_1,r_2]} \big|c(\omega) - e(\omega)\big|\Big)^q.$$

Then (ξ, β, s) is a complete b-ms on ∇ with $s = 2^{q-1}$.

Let Ω be a graph defined by $\Im(\Omega) = \nabla$ and

$$\Upsilon(\Omega) = \big\{(c,e) \in \nabla \times \nabla : c(\omega) \leq e(\omega) \text{ or } e(\omega) \leq c(\omega), \text{for } \omega \in [r_1, r_2]\big\}.$$

To proceed we consider the following theorem.

Theorem 3.1 *Consider Problem (3.1) with the following hypotheses:*

($\heartsuit_1$) $\Pi : [r_1, r_2] \times [r_1, r_2] \times \mathbb{R} \to \mathbb{R}$, $\Lambda(\omega, \nu) : [r_1, r_2] \times [r_1, r_2] \to \mathbb{R}$ *are continuous functions so that*

$$\sup_{\omega\in[r_1,r_2]} \int_{r_1}^{r_2} \Lambda(\omega,\nu)\,d\nu \leq \frac{1}{\sqrt[q]{2}}, \quad q > 1,$$

($\heartsuit_2$) *for all $q > 1$, consider*

$$\big|\Pi\big(\omega,\nu,c(\nu)\big) - \Pi\big(\omega,\nu,e(\nu)\big)\big| \leq \frac{|c(\nu) - e(\nu)|^2}{\sqrt[q]{(1 + |c(\nu) - e(\nu)|^q)}}, \quad q > 1,$$

($\heartsuit_3$) *there is a function* $c_\circ \in \nabla$ *such that*

$$c_\circ(\omega) \leq \Xi(\omega) + \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi\big(\omega,\nu,\wp_2 c_\circ(\nu)\big)\,d\nu, \quad \omega \in [r_1,r_2],$$

($\heartsuit_4$) $\wp_2 c(\nu) \leq \wp_1 e(\nu)$ *and* $\wp_1 c(\nu) \leq \wp_2 e(\nu)$, *provided that* $c(\nu) \leq e(\nu)$, $\wp_1$ *and* $\wp_2$ *are defined in the proof below,*

($\heartsuit_5$) *for* $\{c_n\}_{n\in\mathbb{N}} \in \beta$, *if* $\lim_{n\to\infty} c_n = c$ *and* $c_n(\omega) \leq c_{n+1}(\omega) \in \Upsilon(\Omega)$, *then there is a subsequence* $\{c_{n_j}\}_{n_j\in\mathbb{N}}$ *so that* $c_{n_j}(\omega) \leq c(\omega)$ *for* $n_j \in \mathbb{N}$, $\omega \in [r_1,r_2]$.

Then there are $\theta, \vartheta \in \nabla$ *so that*

$$\theta(\omega) = \Xi(\omega) + \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi\big(\omega,\nu,\theta(\nu)\big)\,d\nu$$

or

$$\vartheta(\omega) = \Xi(\omega) + \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi\big(\omega,\nu,\vartheta(\nu)\big)\,d\nu.$$

Proof Define $\wp_1, \wp_2 : C[r_1,r_2] \to C[r_1,r_2]$ by

$$\begin{cases} \wp_1 c(\omega) = \Xi(\omega) + \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi(\omega,\nu,c(\nu))\,d\nu, \\ \wp_2 e(\omega) = \Xi(\omega) + \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi(\omega,\nu,e(\nu))\,d\nu, \end{cases}$$

for each $\omega \in [r_1,r_2]$ and $c, e \in \nabla$. By hypotheses ($\heartsuit_1$) and ($\heartsuit_2$), we can write

$$\begin{aligned}
&2^{q-1}\xi\big(\wp_1 c(\omega), \wp_2 e(\omega)\big) \\
&\quad = 2^{q-1}\Big(\sup_{\omega\in[r_1,r_2]} \big|\wp_1 c(\omega) - \wp_2 c(\omega)\big|\Big)^q \\
&\quad = 2^{q-1}\bigg(\sup_{\omega\in[r_1,r_2]} \bigg|\int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi\big(\omega,\nu,c(\nu)\big)\,d\nu - \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi\big(\omega,\nu,e(\nu)\big)\,d\nu\bigg|\bigg)^q \\
&\quad = 2^{q-1}\bigg(\sup_{\omega\in[r_1,r_2]} \int_{r_1}^{r_2} \Lambda(\omega,\nu)\big|\Pi\big(\omega,\nu,c(\nu)\big) - \Pi\big(\omega,\nu,e(\nu)\big)\big|\,d\nu\bigg)^q \\
&\quad \leq 2^{q-1}\bigg(\sup_{\omega,\nu\in[r_1,r_2]} \int_{r_1}^{r_2} \Lambda(\omega,\nu)\,d\nu\bigg)^q \times \sup_{\omega,\nu\in[r_1,r_2]} \bigg(\frac{|c(\nu)-e(\nu)|^2}{\sqrt[q]{(1+|c(\nu)-e(\nu)|^q)}}\bigg)^q \\
&\quad \leq 2^{q-1}\bigg(\frac{1}{\sqrt[q]{2}}\bigg)^q \bigg(\sup_{\nu\in[r_1,r_2]} \frac{|c(\nu)-e(\nu)|^{2q}}{1+|c(\nu)-e(\nu)|^q}\bigg) \\
&\quad = \frac{1}{2}\bigg(\sup_{\nu\in[r_1,r_2]} \frac{|c(\nu)-e(\nu)|^q}{1+|c(\nu)-e(\nu)|^q}\bigg)\Big(\sup_{\nu\in[r_1,r_2]} \big|c(\nu)-e(\nu)\big|^q\Big) \\
&\quad \leq \lambda\Big(\sup_{\nu\in[r_1,r_2]} \big|c(\nu)-e(\nu)\big|^q\Big)\Big(\sup_{\nu\in[r_1,r_2]} \big|c(\nu)-e(\nu)\big|^q\Big) \\
&\quad = \lambda\big(\Theta\big(\xi(c,e)\big)\big)\Theta\big(\xi(c,e)\big).
\end{aligned}$$

Thus, Condition (2.13) of Corollary 2.7 holds by taking $\lambda(\omega) = \frac{1}{2}\frac{\omega}{1+\omega}$ and $\Theta(\omega) = \omega$. It follows that by hypotheses ($\heartsuit_3$), ($\heartsuit_4$) and the definition of the graph Ω that the stipulations

(a_1) and (a_2) of Corollary 2.7 are fulfilled. Also hypothesis $(\heartsuit_5)$ immediately fulfills stipulation (a_3) of Corollary 2.7. So, we get

$$\theta(\omega) = \Xi(\omega) + \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi\big(\omega,\nu,\theta(\nu)\big)\,d\nu,$$

or

$$\vartheta(\omega) = \Xi(\omega) + \int_{r_1}^{r_2} \Lambda(\omega,\nu)\Pi\big(\omega,\nu,\vartheta(\nu)\big)\,d\nu.$$

Second part: Consider the problem below:

$$\begin{cases} {}^{c}\mathrm{D}^{\iota}\theta(\omega) + \phi(\vartheta(\omega)) = 0, & \iota \in (1,2], \omega \in [1,0], \\ {}^{c}\mathrm{D}^{\iota}\vartheta(\omega) + \psi(\theta(\omega)) = 0, & \iota \in (1,2], \omega \in [1,0], \\ \theta(0) = \vartheta(0) = \alpha, & \theta(1) = \vartheta(1) = \alpha^*, \end{cases} \tag{3.2}$$

where α and α^* are constants, $\phi,\psi : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^+$ and ${}^{c}\mathrm{D}^{\iota}$ is the famous Caputo derivative of order ι. □

The equivalent form of the Problem (3.2) is given by

$$\begin{cases} \theta(\omega) = \Xi(\omega) + \int_0^1 \Upsilon(\omega,\kappa)\phi(\vartheta(\kappa))\,d\kappa, & \omega \in [0,1], \\ \vartheta(\tau) = \Xi(\omega) + \int_0^1 \Upsilon(\omega,\kappa)\psi(\theta(\kappa))\,d\kappa, & \omega \in [0,1]. \end{cases} \tag{3.3}$$

Here the Green's function $\Upsilon(\omega,\kappa)$ on $[0,1]\times[0,1]$ is continuous and is defined as

$$\Upsilon(\omega,\kappa) = \begin{cases} \frac{(\omega-\kappa)^{\vartheta-1}-\omega(1-\kappa)^{\vartheta-1}}{\Gamma(\iota)}, & 0 \le \kappa \le \omega \le 1, \\ \frac{-\omega(1-\kappa)^{\vartheta-1}}{\Gamma(\iota)}, & 0 \le \omega \le \kappa \le 1. \end{cases}$$

Now, put $\Pi(\omega,\nu,\theta(\nu)) = \phi(\vartheta(\kappa))$ and $\Pi(\omega,\nu,\vartheta(\nu)) = \psi(\theta(\kappa))$, then the System (3.3) turns into

$$\begin{cases} \theta(\omega) = \Xi(\omega) + \int_0^1 \Upsilon(\omega,\kappa)\Pi(\omega,\nu,\theta(\nu))\,d\nu, & \omega \in [0,1], \\ \vartheta(\tau) = \Xi(\omega) + \int_0^1 \Upsilon(\omega,\kappa)\Pi(\omega,\nu,\vartheta(\nu))\,d\mu, & \omega \in [0,1]. \end{cases} \tag{3.4}$$

System (3.4) is equivalent to the System (3.1) when $r_1 = 0$ and $r_2 = 1$. So by using Theorem 3.1, there are $\theta,\vartheta \in \nabla$ so that

$$\theta(\omega) = \Xi(\omega) + \int_0^1 \Lambda(\omega,\nu)\Pi\big(\omega,\nu,\theta(\nu)\big)\,d\nu$$

or

$$\vartheta(\omega) = \Xi(\omega) + \int_0^1 \Lambda(\omega,\nu)\Pi\big(\omega,\nu,\vartheta(\nu)\big)\,d\nu,$$

i.e., Problem (3.2) has a solution, and by this we have finished this section.

4 Conclusion

The fixed point technique is considered an essential branch and a good tool in the field of nonlinear analysis, and due to its multiple applications in various fields of the mathematical and engineering sciences, it has received great interest from researchers. To be clear, we found in this work the solution to a system of nonlinear equations and fractional differential equations in the framework of the Caputo fractional derivative via new coincidence points of generalized multivalued contraction mappings involving a directed graph in b-metric spaces. The main benefit of our paper is to provide a solution to such types of equations as ones that serve numerical analysis scholars in finding a numeric solution, in addition to the new contributions made by the manuscript in the field of fixed point theory that serve many researchers in this field.

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