Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A study on multiterm hybrid multi-order fractional boundary value problem coupled with its stability analysis of Ulam–Hyers type

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Abstract

In this research work, a newly-proposed multiterm hybrid multi-order fractional boundary value problem is studied. The existence results for the supposed hybrid fractional differential equation that involves Riemann–Liouville fractional derivatives and integrals of multi-orders type are derived using Dhage's technique, which deals with a composition of three operators. After that, its stability analysis of Ulam–Hyers type and the relevant generalizations are checked. Some illustrative numerical examples are provided at the end to illustrate and validate our obtained results.

MSC: Primary 34A08; secondary 34A12

Keywords: Hybrid boundary problem; Riemann–Liouville derivative; Dhage's technique; Stability

1 Introduction

Most researchers in the history of mathematics place the origin of fractional calculus in a work by Leibniz where he introduces the notation of the nth derivative of an arbitrary function y; that is, $\frac{d^n y}{dx^n}$ with $n \in \mathbb{N}$. But does it make sense to extend the values of n in that expression to other numeric fields?. The idea of fractional derivative materialized in 1695, when L'Hopital asked what $\frac{d^n y}{dx^n}$ means if $n = \frac{1}{2}$. After such an idea having appeared, many extended definitions of this concept have been constructed under two conceptions: global (classical) and local. In the first conception, the fractional derivative is defined as integral, Fourier or Mellin transformations, which means that its nature is not local and has a memory effect. The second conception of fractional derivative is based on a local definition through certain incremental ratios. The global formulation is associated with the appearance of the fractional calculus itself, going back to the pioneering work of Euler, Laplace, Lacroix, Fourier, Abel, Liouville, etc. until the establishment of the classic definitions of Riemann–Liouville and Caputo. Thus, the classical theory of fractional calculus constitutes a mathematical analysis tool applied to the investigation of arbitrary order integrals and derivatives, which extends the concepts of integer-order differentiation and n-fold integration.

Furthermore, the study of the practical and theoretical elements of fractional differential equations has became a base of academic advanced research [1–4]. Many fractional differential equations, particularly boundary value problems, have gathered the research interests of researchers in applied mathematics, theoretical physics, and engineering due to their nonlocality and their powerful flexibility in modeling complex scientific and physical phenomena that show the memory effect. The dynamics and behavior of certain physical systems can be explained better with respect to fractional derivatives and fractional integrals than for classical integer-order systems. In recent years, the great potential of these integrals and derivatives has been revealed in various fields of natural sciences and technology, such as biology, fluid mechanics, biomathematics, physics, image processing, chemistry, and entropy theory [5–36].

Recently, the fractional formulation of boundary value problems related to hybrid differential equations has received the interests of the most researchers. The 1th order hybrid differential equations involving first and second kind of disturbances have been discussed, using the Riemann–Liouville derivatives in [37, 38]. In [39], the authors turned to the existence property of solutions to hybrid fractional differential equations by terms of both types applying the Caputo derivative. In 2015, similar results for fractional initial value problems involving hybrid integro-differential equations are established [40] by Sitho et al. The existence problems of mild solution for hybrid fractional differential equations involving the Caputo fractional derivative of arbitrary order are investigated in [41] by Mahmudov in 2017. For similar research, refer to [42–44].

In [45], Ben Chikh et al. proved the unique solution's existence and various stability's types for a boundary value problem involving Riemann–Liouville integrals and then in [46], they implemented same results for a newly-formulated four-point Caputo-conformable fractional problem involving boundary conditions of the Riemann–Liouville conformable type (for more background information about conformable derivative, refer to [47, 48]) formulated as

$$\begin{cases} \varepsilon^{*CC}\mathcal{D}_{t_0}^{k^*,\varrho}\rho(t) + {}^{CC}\mathcal{D}_{t_0}^{\theta^*,\varrho}\rho(t) = \hat{\hbar}(t,\rho(t)), \quad (t\in[t_0,K], k^*\in(2,3]), \\ \rho(t_0) = 0, \qquad \tau_1^{*CC}\mathcal{D}_{t_0}^{\beta_1^*,\varrho}\rho(K) + {}^{CC}\mathcal{D}_{t_0}^{\beta_2^*,\varrho}\rho(\eta) = \delta_1, \\ \tau_2^{*RC}\mathcal{I}_{t_0}^{m_1^*,\varrho}\rho(K) + {}^{RC}\mathcal{I}_{t_0}^{m_2^*,\varrho}\rho(\nu) = \delta_2, \end{cases} \tag{1}$$

where $\nu,\eta\in[t_0,K]$, $2<\theta^*<k^*$, $0<\varepsilon^*,\tau_1^*,\tau_2^*\le 1$, $0\le\beta_1^*,\beta_2^*<k^*-\theta^*$, $m_1^*,m_2^*\in\mathbb{R}^+$, $\varrho\in(0,1]$ and $t_0\ge 0$, with the map $\hat{\hbar}:[t_0,K]\times\mathbb{R}\to\mathbb{R}$ is continuous.

Due to the importance and flexibility of hybrid differential equations in modeling of electromagnetic waves, deflection of a curved beam, gravity driven by flows etc., Baleanu et al. [49] designed a hybrid fractional boundary value problem of thermostat control model and discussed required existence specifications of its solutions in the form

$$\begin{cases} {}^{c}\mathcal{D}^{\gamma}\left(\frac{\vartheta(\mathfrak{t})}{\mathfrak{g}(\mathfrak{t},\vartheta(\mathfrak{t}))}\right) + \mathfrak{h}(\mathfrak{t},\vartheta(\mathfrak{t})) = 0, \quad (\mathfrak{t}\in\mathbb{I}:=[0,1]), \\ {}^{c}\mathcal{D}^{1}\left(\frac{\vartheta(\mathfrak{t})}{\mathfrak{g}(\mathfrak{t},\vartheta(\mathfrak{t}))}\right)|_{\mathfrak{t}=0} = 0, \\ k{}^{c}\mathcal{D}^{\gamma-1}\left(\frac{\vartheta(\mathfrak{t})}{\mathfrak{g}(\mathfrak{t},\vartheta(\mathfrak{t}))}\right)|_{\mathfrak{t}=1} + \left(\frac{\vartheta(\mathfrak{t})}{\mathfrak{g}(\mathfrak{t},\vartheta(\mathfrak{t}))}\right)|_{\mathfrak{t}=\eta} = 0, \end{cases} \tag{2}$$

where $\gamma\in(1,2]$, $\gamma-1\in(0,1]$, $\eta\in\mathbb{I}$, ${}^{c}\mathcal{D}^{1}=\frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}$, $k>0$. Inspired by the above previous work, we investigate in this work a generalized hybrid problem: indeed, we prove the existence

of solution for the hybrid fractional differential equation including a finite number of Riemann–Liouville derivatives and Riemann–Liouville integrals of different orders of the following form:

$$\begin{cases} \varepsilon^* \mathcal{D}^k [\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}] + (1-\varepsilon^*)\sum_{i=1}^m \mathcal{D}^{\theta_i}\rho(t) = \sum_{i=1}^n \mathcal{I}^{\sigma_i}\mathbb{K}(t,\rho(t)), \\ [\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=0} = 0, \\ \tau_1^* \mathcal{D}^{\beta_1^*} [\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=K} \\ \quad + (1-\tau_1^*)\mathcal{D}^{\beta_2^*} [\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=K} = \delta_1, \\ \tau_2^* \mathcal{I}^{m_1^*} [\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=K} \\ \quad + (1-\tau_2^*)\mathcal{I}^{m_2^*} [\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=K} = \delta_2, \end{cases} \tag{3}$$

where $k \in (2,3]$, $2 < \theta_i < k$ $(i = 1,\dots,m)$, $0 < \varepsilon^*, \tau_1^*, \tau_2^* < 1$, $0 < \beta_1^*, \beta_2^* < k - \theta_i$, m_1^*, m_2^*, α_i, $\vartheta_i, \sigma_i > 0$, $i = 1,\dots,n$ and $t \in J := [0,K]$. Also, $\mathcal{D}^\tau$ represents the τth Riemann–Liouville fractional derivative, $\mathcal{I}^\eta$ denotes the ηth Riemann–Liouville integral, and

$$\begin{aligned} \mathbb{H}\big(t,\rho(t)\big) &:= \mathcal{H}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\dots,\mathcal{I}^{k_n}\rho(t)\big), \\ \mathbb{A}\big(t,\rho(t)\big) &:= \mathcal{A}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\dots,\mathcal{I}^{k_n}\rho(t)\big), \\ \mathbb{K}\big(t,\rho(t)\big) &:= \hat{\hbar}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\dots,\mathcal{I}^{k_n}\rho(t)\big), \end{aligned}$$

where $k_i > 0$ and the maps $\mathcal{H}$, and $\hat{\hbar} : [0,K] \times \mathbb{R}^{n+1} \to \mathbb{R}$ and $\mathcal{A} : [0,K] \times \mathbb{R}^{n+1} \to \mathbb{R}/\{0\}$ are continuous. In the above suggested structure given by (3), we have several nonlinear functions depending on their components. This type of hybrid fractional boundary value problem can be employed in description and modeling non-homogeneous physical processes. The Dhage technique, based on some nonlinear operators, will be used here regarding the existence property of given fractional boundary value problem (3). In spite of some previous standard work regarding solutions of the fractional differential equation, we here aim to study some qualitative properties of solutions to a novel hybrid fractional boundary value problem which is a more complicated system. Naturally, if one can analyze the behavior of such a hybrid system, then we will be able to simulate other real phenomena based on these hybrid fractional differential equations.

The scheme of the paper is organized in such an order: In the next section, we present some essential fractional calculus definitions and notions that will be applied. Next, the existence results for the multiterm hybrid fractional differential equation are established in Sect. 3. Ulam–Hyers type stability and other generalizations for proposed system are checked in Sect. 4. At the end, some illustrative examples are included to illustrate our obtained results. Conclusive remarks are expressed in Sect. 6.

2 Essential preliminaries

Some essential fractional calculus definitions and notions that will be used later are presented in this section.

Definition 2.1 ([1]) The ςth Riemann–Liouville fractional integral of a given mapping $\psi : (0,\infty) \to \mathbb{R}$ is expressed as

$$\mathcal{I}^{\varsigma}\psi(t) = \int_0^t \frac{(t-r)^{\varsigma-1}}{\Gamma(\varsigma)}\psi(r)\,dr, \quad \varsigma > 0$$

if the R.H.S. exists.

Definition 2.2 ([1]) The ςth Riemann–Liouville fractional derivative of a given function $\psi : (0,\infty) \to \mathbb{R}$ is expressed as

$$\mathcal{D}^{\varsigma}\psi(t) = \frac{1}{\Gamma(n-\varsigma)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{\psi(r)}{(t-r)^{\varsigma-n+1}}\,dr,$$

where $n = [\varsigma] + 1$, if the R.H.S. exists.

From the definition of the Riemann–Liouville fractional derivative, we get the following.

Lemma 2.3 ([1]) *Assume that $\theta^* > \eta^* > 0$. Then*

$$\mathcal{D}^{\theta^*}\left(\mathcal{I}^{\theta^*}\psi\right)(t) = \psi(t), \qquad \mathcal{D}^{\eta^*}\left(\mathcal{I}^{\theta^*}\psi\right)(t) = \mathcal{I}^{\theta^*-\eta^*}\psi(t).$$

Lemma 2.4 ([1]) *Assume that $k^* > 0$ and $\rho(t) \in C(0,1) \cap L(0,1)$. Then the linear fractional differential equation $\mathcal{D}^{k^*}\rho(t) = 0$ possesses a solution uniquely as*

$$\rho(t) = A_1 t^{k^*-1} + A_2 t^{k^*-2} + \cdots + A_n t^{k^*-n},$$

where $A_i \in \mathbb{R}$ and $n - 1 < k^* < n$.

Lemma 2.5 ([1]) *Assume that $k^* > 0$ is the same above. Then, for $\rho(t) \in C(0,1) \cap L(0,1)$, we have*

$$\mathcal{I}^{k^*}\mathcal{D}^{k^*}\rho(t) = \rho(t) + A_1 t^{k^*-1} + A_2 t^{k^*-2} + \cdots + A_n t^{k^*-n},$$

where $A_i \in \mathbb{R}$.

We shall establish our main existence criterion by the aid of the next theorem known as Dhage's technique.

Theorem 2.6 ([50]) *Assume that $S \neq \emptyset$ is a convex closed bounded set in the Banach algebra X, $\psi_1, \psi_2 : X \to X$, and $\psi_3 : S \to X$ are three operators along with*:

(1) *ψ_1 and ψ_2 are Lipschitz via constants $\mathfrak{l}_1^*$ and $\mathfrak{l}_2^*$,*
(2) *ψ_3 has two specifications: continuity and compactness,*
(3) *$\rho = \psi_1\rho\psi_3\nu + \psi_2\rho \Rightarrow \rho \in S$ for all $\nu \in S$,*
(4) *$\mathfrak{l}_1^*\Delta^* + \mathfrak{l}_2^* < 1$, where $\Delta^* = \|\psi_3(S)\|$.*

Then it is found a solution in S for the operator equation $\psi_1\rho\psi_3\rho + \psi_2\rho = \rho$.

3 Results regarding to the existence property

We turn to the investigation of our required existence criteria in the current situation. The notation $\mathcal{C} = C(J, \mathbb{R})$ represents the space of all continuous mappings from $J = [0, K]$ to $\mathbb{R}$ with actions

$$\|\rho\| = \sup_{t\in J} \big|\rho(t)\big|, \qquad \big(\rho \cdot \rho'\big)(t) = \rho(t)\rho'(t) \quad \forall \rho, \rho' \in \mathcal{C}.$$

$(\mathcal{C}, \|\cdot\|, \cdot)$ is a Banach algebra. The next lemma is key.

Lemma 3.1 *Assume that $\bar{\chi}$ is a continuous function on $J = [0, K]$, and $k \in (2, 3]$, $2 < \theta_i < k$, $0 \le \varepsilon^*, \tau_1^*, \tau_2^* \le 1$, $0 < \beta_1^*, \beta_2^* < k - \theta_i$, $m_1^*, m_2^*, \alpha_i, \vartheta_i > 0$, $i = 1, \dots, n$. Then the solution of the hybrid fractional boundary value problem is expressed as*

$$\begin{cases} \varepsilon^* \mathcal{D}^k \big[\frac{\rho(t) - \sum_{i=1}^n \mathcal{I}^{\vartheta_i} \mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} \mathbb{A}(t,\rho(t))}\big] + (1-\varepsilon^*) \sum_{i=1}^m \mathcal{D}^{\theta_i} \rho(t) = \bar{\chi}(t), \\ \big[\frac{\rho(t) - \sum_{i=1}^n \mathcal{I}^{\vartheta_i} \mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} \mathbb{A}(t,\rho(t))}\big]\big|_{t=0} = 0, \\ \tau_1^* \mathcal{D}^{\beta_1^*} \big[\frac{\rho(t) - \sum_{i=1}^n \mathcal{I}^{\vartheta_i} \mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} \mathbb{A}(t,\rho(t))}\big]\big|_{t=K} \\ \quad + (1-\tau_1^*) \mathcal{D}^{\beta_2^*} \big[\frac{\rho(t) - \sum_{i=1}^n \mathcal{I}^{\vartheta_i} \mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} \mathbb{A}(t,\rho(t))}\big]\big|_{t=K} = \delta_1, \\ \tau_2^* \mathcal{I}^{m_1^*} \big[\frac{\rho(t) - \sum_{i=1}^n \mathcal{I}^{\vartheta_i} \mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} \mathbb{A}(t,\rho(t))}\big]\big|_{t=K} \\ \quad + (1-\tau_2^*) \mathcal{I}^{m_2^*} \big[\frac{\rho(t) - \sum_{i=1}^n \mathcal{I}^{\vartheta_i} \mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} \mathbb{A}(t,\rho(t))}\big]\big|_{t=K} = \delta_2, \end{cases} \tag{4}$$

satisfies the following equation:

$$\begin{aligned} \rho(t) = {} & \sum_{i=1}^n \mathcal{I}^{\alpha_i} \mathbb{A}\big(t, \rho(t)\big) \Bigg[\sum_{i=1}^m \frac{\varepsilon^* - 1}{\varepsilon^* \Gamma(k^* - \theta_i)} \int_0^t (t-s)^{k-\theta_i - 1} \rho(s)\, ds \\ & + \frac{1}{\varepsilon^* \Gamma(k)} \int_0^t (t-s)^{k-1} \bar{\chi}(t)\, ds + \frac{t^{k-1}}{B} \times \Bigg[\frac{\tau_1^* \Theta_4 (\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^m \mathcal{I}^{k-\theta_i-\beta_1^*} \rho(K) \\ & - \frac{\Theta_2 \tau_2^* (\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^m \mathcal{I}^{k-\theta_i+m_1^*} \rho(K) + \frac{\Theta_4 (1-\tau_1^*)(\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^m \mathcal{I}^{k-\theta_i-\beta_2^*} \rho(K) \\ & - \frac{\Theta_2 (1-\tau_2^*)(\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^m \mathcal{I}^{k-\theta_i+m_2^*} \rho(K) + \frac{\Theta_4 \tau_1^*}{\varepsilon^*} \mathcal{I}^{k-\beta_1^*} \hat{\chi}(K) - \frac{\Theta_2 \tau_2^*}{\varepsilon^*} \mathcal{I}^{k+m_1^*} \bar{\chi}(K) \\ & + \Theta_2 \delta_2 - \Theta_4 \delta_1 + \frac{\Theta_4 (1-\tau_1^*)}{\varepsilon^*} \mathcal{I}^{k-\beta_2^*} \bar{\chi}(K) - \frac{\Theta_2 (1-\tau_2^*)}{\varepsilon^*} \mathcal{I}^{k+m_2^*} \bar{\chi}(K) \Bigg] \\ & - \frac{t^{k-2}}{B} \Bigg[\frac{\tau_1^* \Theta_3 (\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^m \mathcal{I}^{k-\theta_i-\beta_1^*} \rho(K) - \frac{\Theta_1 \tau_2^* (\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^m \mathcal{I}^{k-\theta_i+m_1^*} \rho(K) \\ & + \frac{\Theta_3 (1-\tau_1^*)(\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^m \mathcal{I}^{k-\theta_i-\beta_2^*} \rho(K) - \frac{\Theta_1 (1-\tau_2^*)(\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^m \mathcal{I}^{k-\theta_i+m_2^*} \rho(K) \\ & + \frac{\Theta_3 \tau_1^*}{\varepsilon^*} \mathcal{I}^{k-\beta_1^*} \bar{\chi}(K) - \frac{\Theta_1 \tau_2^*}{\varepsilon^*} \mathcal{I}^{k+m_1^*} \bar{\chi}(K) + \Theta_1^* \delta_2 - \Theta_3 \delta_1 + \frac{\Theta_3 (1-\tau_1^*)}{\varepsilon^*} \mathcal{I}^{k-\beta_2^*} \bar{\chi}(K) \\ & - \frac{\Theta_1 (1-\tau_2^*)}{\varepsilon^*} \mathcal{I}^{k+m_2^*} \bar{\chi}(K) \Bigg] \Bigg] + \sum_{i=1}^n \mathcal{I}^{\vartheta_i} \mathbb{H}\big(t, \rho(t)\big), \end{aligned} \tag{5}$$

where the nonzero constant Θ_i, $i \in \{1,2,3,4\}$ *is defined by*

$$\begin{aligned}
\Theta_1 &= \frac{\tau_1^*\Gamma(k)}{\Gamma(k-\beta_1^*)}K^{k-\beta_1^*-1} + \frac{(1-\tau_1^*)\Gamma(k)}{\Gamma(k-\beta_2^*)}K^{k-\beta_2^*-1},\\
\Theta_2 &= \frac{\tau_1^*\Gamma(k-1)}{\Gamma(k-\beta_1^*-1)}K^{k-\beta_1^*-2} + \frac{(1-\tau_1^*)\Gamma(k-1)}{\Gamma(k-\beta_2^*-1)}K^{k-\beta_2^*-2},\\
\Theta_3 &= \frac{\tau_2^*\Gamma(k)}{\Gamma(k+m_1^*)}K^{k+m_1^*-1} + \frac{(1-\tau_2^*)\Gamma(k)}{\Gamma(k+m_2^*)}K^{k+m_2^*-1},\\
\Theta_4 &= \frac{\tau_2^*\Gamma(k-1)}{\Gamma(k+m_1^*-1)}K^{k+m_1^*-2} + \frac{(1-\tau_2^*)\Gamma(k-1)}{\Gamma(k+m_2^*-1)}T^{k+m_2^*-2},\\
B &= \Theta_3\Theta_2 - \Theta_1\Theta_4. \qquad (6)
\end{aligned}$$

Proof According to the first equation in (4), we obtain

$$\mathcal{D}^k\left(\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}\right) = \frac{\varepsilon^*-1}{\varepsilon^*}\sum_{i=1}^m \mathcal{D}^{\theta_i}\rho(t) + \frac{1}{\varepsilon^*}\bar{\chi}(t). \qquad (7)$$

Let us now take the Riemann–Liouville fractional integral of order k to (7),

$$\begin{aligned}
\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))} &= \sum_{i=1}^m \frac{\varepsilon^*-1}{\varepsilon^*\Gamma(k-\theta_i)}\int_0^t (t-s)^{k-\theta_i-1}\rho(s)\,ds\\
&\quad + \frac{1}{\varepsilon^*\Gamma(k)}\int_0^t (t-s)^{k-1}\bar{\chi}(s)\,ds + A_1t^{k-1} + A_2t^{k-2} + A_3t^{k-3},
\end{aligned}$$

for $A_1, A_2, A_3 \in \mathbb{R}$. Since $2 < k < 3$, the first-boundary condition in (4) indicates that $A_3 = 0$. Thus, we have

$$\begin{aligned}
\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))} &= \sum_{i=1}^m \frac{\varepsilon^*-1}{\varepsilon^*\Gamma(k-\theta_i)}\int_0^t (t-s)^{k-\theta_i-1}\rho(s)\,ds\\
&\quad + \frac{1}{\varepsilon^*\Gamma(k)}\int_0^t (t-s)^{k-1}\bar{\chi}(s)\,ds + A_1t^{k-1} + A_2t^{k-2}. \qquad (8)
\end{aligned}$$

Let us apply the Riemann–Liouville fractional derivative and integral of order γ, q, respectively on both sides of (8) such that $\gamma \in \{\beta_1^*, \beta_2^*\}$, $q \in \{m_1^*, m_2^*\}$, $0 < \gamma < k-\theta_i$ and $2 < \theta_i < k$. We obtain

$$\begin{aligned}
&\mathcal{D}^\gamma\left(\frac{\rho(t)-\sum_{i=1}^n \mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}\right)\\
&\quad = \sum_{i=1}^m \frac{\varepsilon^*-1}{\varepsilon^*\Gamma(k-\theta_i-\gamma)}\int_0^t (t-s)^{k-\theta_i-\gamma-1}\rho(s)\,ds\\
&\qquad + A_1\frac{\Gamma(k)}{\Gamma(k-\gamma)}t^{k-\alpha-1} + \frac{1}{\varepsilon^*\Gamma(k-\gamma)}\int_0^t (t-s)^{k-\gamma-1}\bar{\chi}(s)\,ds + A_2\frac{\Gamma(k-1)}{\Gamma(k-\gamma-1)}t^{k-\gamma-2}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{I}^{q}\left(\frac{\rho(t)-\sum_{i=1}^{n}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}\right)\\
&\quad=\sum_{i=1}^{m}\frac{\varepsilon^{*}-1}{\varepsilon^{*}\Gamma(k-\theta_i+q)}\int_{0}^{t}(t-s)^{k-\theta_i+q-1}\rho(s)\,ds\\
&\qquad+A_1\frac{\Gamma(k)}{\Gamma(k+q)}t^{k+q-1}+\frac{1}{\varepsilon^{*}\Gamma(k+q)}\int_{0}^{t}(t-s)^{k+q-1}\bar{\chi}(s)\,ds+A_2\frac{\Gamma(k-1)}{\Gamma(k+q-1)}t^{k+q-2}.
\end{aligned}$$

By substituting the values $\gamma=\beta_1^{*}$, $\gamma=\beta_2^{*}$, $q=m_1^{*}$ and $q=m_2^{*}$ into the above one and using the second condition in (4), we have

$$\begin{aligned}
A_1={}&\frac{\tau_1^{*}\Theta_4(\varepsilon^{*}-1)}{\varepsilon^{*}}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^{*}}\rho(K)-\frac{\Theta_2\tau_2^{*}(\varepsilon^{*}-1)}{\varepsilon^{*}}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^{*}}\rho(K)\\
&+\frac{\Theta_4(1-\tau_1^{*})(\varepsilon^{*}-1)}{\varepsilon^{*}}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_2^{*}}\rho(K)-\frac{\Theta_2(1-\tau_2^{*})(\varepsilon^{*}-1)}{\varepsilon^{*}}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2^{*}}\rho(K)\\
&+\frac{\Theta_4\tau_1^{*}}{\varepsilon^{*}}\mathcal{I}^{k-\beta_1^{*}}\bar{\chi}(K)-\frac{\Theta_2\tau_2^{*}}{\varepsilon^{*}}\mathcal{I}^{k+m_1^{*}}\bar{\chi}(K)+\Theta_2\delta_2-\Theta_4\delta_1\\
&+\frac{\Theta_4(1-\tau_1^{*})}{\varepsilon^{*}}\mathcal{I}^{k-\beta_2^{*}}\bar{\chi}(K)-\frac{\Theta_2(1-\tau_2^{*})}{\varepsilon^{*}}\mathcal{I}^{k+m_2^{*}}\bar{\chi}(K)
\end{aligned}$$

and

$$\begin{aligned}
A_2={}&\frac{\tau_1^{*}\Theta_3(\varepsilon^{*}-1)}{\varepsilon^{*}}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^{*}}\rho(K)-\frac{\Theta_1\tau_2^{*}(\varepsilon^{*}-1)}{\varepsilon^{*}}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^{*}}\rho(K)\\
&+\frac{\Theta_3(1-\tau_1^{*})(\varepsilon^{*}-1)}{\varepsilon^{*}}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_2^{*}}\rho(K)-\frac{\Theta_1(1-\tau_2^{*})(\varepsilon^{*}-1)}{\varepsilon^{*}}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2^{*}}\rho(K)\\
&+\frac{\Theta_3\tau_1^{*}}{\varepsilon^{*}}\mathcal{I}^{k-\beta_1^{*}}\bar{\chi}(T)-\frac{\Theta_1\tau_2^{*}}{\varepsilon^{*}}\mathcal{I}^{k-m_1^{*}}\bar{\chi}(K)+\Theta_1\delta_2-\Theta_3\delta_1\\
&+\frac{\Theta_3(1-\tau_1^{*})}{\varepsilon^{*}}\mathcal{I}^{k-\beta_2^{*}}\bar{\chi}(K)-\frac{\Theta_1(1-\tau_2^{*})}{\varepsilon^{*}}\mathcal{I}^{k+m_2^{*}}\bar{\chi}(K).
\end{aligned}$$

Let us now substitute the constants' value of A_1 and A_2 into (8) by which Eq. (5) is derived and our proof is ended. □

Some essential hypotheses are presented as follows.

(H1) The given functions $\mathcal{A}: J\times\mathbb{R}^{n+1}\longrightarrow\mathbb{R}\setminus\{0\}$ and $\hat{\hbar}$, $\mathcal{H}: J\times\mathbb{R}^{n+1}\longrightarrow\mathbb{R}$ are continuous.

(H2) $\exists\Phi,\Psi: J\to\mathbb{R}^{+}$ with bounds $\|\Phi\|=\sup_{t\in J}|\Phi(t)|$ and $\|\Psi\|=\sup_{t\in J}|\Psi(t)|$, respectively, such that

$$\left|\mathcal{A}\big(t,u_1(t),\ldots,u_{n+1}(t)\big)-\mathcal{A}\big(t,v_1(t),\ldots,v_{n+1}(t)\big)\right|\le\Phi(t)\left(\sum_{i=1}^{n+1}|u_i-v_i|\right)$$

and

$$\left|\mathcal{H}\big(t,u_1(t),\dots,u_{n+1}(t)\big)-\mathcal{H}\big(t,v_1(t),\dots,v_{n+1}(t)\big)\right|\le\Psi(t)\left(\sum_{i=1}^{n+1}|u_i-v_i|\right),$$

for all $(t,u_1,\dots,u_{n+1}),(t,v_1,\dots,v_{n+1})\in J\times\mathbb{R}^{n+1}$.

(H3) $\exists\mathcal{P}\in L^\infty(J,\mathbb{R}^+)$ and continuous nondecreasing functions $\Xi_j:[0,\infty)\longrightarrow(0,\infty)$, $j=0,\dots,n$ with

$$\left|\hat{\hbar}(t,u_0,\dots,u_n)\right|\le\mathcal{P}(t)\left(\sum_{j=0}^{n}\Xi_j\big(|u_j|\big)\right),$$

for all $t\in J$ and $(u_0,\dots,u_n)\in\mathbb{R}^{n+1}$.

(H4) $\exists\mathcal{R}>0$ such that

$$\frac{\mathcal{A}_0Q^*\sum_{i=0}^{n}\frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}+\mathcal{H}_0\sum_{i=0}^{m}\frac{K^{\theta_i}}{\Gamma(\theta_i+1)}}{1-\|\Psi\|\sum_{i=1}^{n}(\sum_{j=0}^{n}\frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)})-\|\Phi\|\sum_{i=1}^{n}(\sum_{j=0}^{n}\frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)})Q^*}\le\mathcal{R}\tag{9}$$

and

$$\|\Psi\|\sum_{i=1}^{n}\left(\sum_{j=0}^{n}\frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)}\right)+\|\Phi\|\sum_{i=1}^{n}\left(\sum_{j=0}^{n}\frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)}\right)Q^*\le1,\tag{10}$$

where $k_0=0$, $\mathcal{A}_0=\sup_{t\in J}|\mathcal{A}(t,0,\dots,0)|$, $\mathcal{H}_0=\sup_{t\in J}|\mathcal{H}(t,0,\dots,0)|$, and

$$\begin{aligned}Q^*&=\|\mathcal{P}\|\sum_{j=0}^{n}\Xi_j\left(\frac{K^{k_j}}{\Gamma(k_j+1)}\mathcal{R}\right)\mathcal{W}+\mathcal{R}\mathcal{V}+\frac{1}{|B|}\big[K^{k-1}\big(|\Theta_2\delta_2|+|\Theta_4\delta_1|\big)\\&\quad+K^{k-2}\big(|\Theta_1\delta_2|+|\Theta_3\delta_1|\big)\big],\end{aligned}\tag{11}$$

$$\begin{aligned}\mathcal{V}&=\frac{(|\varepsilon^*-1|)(\Theta_4+\Theta_3K^{-1})}{|B|}\\&\quad\times\left(\tau_1^*\sum_{i=1}^{m}\frac{K^{2k-\theta_i-\beta_1^*-1}}{\varepsilon^*\Gamma(k-\theta_i-\beta_1^*+1)}+\sum_{i=1}^{m}\frac{(1-\tau_1^*)K^{2k-\theta_i-\beta_2^*-1}}{\varepsilon^*\Gamma(k-\theta_i-\beta_2^*+1)}\right)\\&\quad+\frac{(|\varepsilon^*-1|)(\Theta_2+\Theta_1K^{-1})}{|B|}\\&\quad\times\left(\tau_2^*\sum_{i=1}^{m}\frac{K^{2k-\theta_i+m_1^*-1}}{\varepsilon^*\Gamma(k-\theta_i+m_1^*+1)}+\sum_{i=1}^{m}\frac{(1-\tau_2^*)K^{2k-\theta_i+m_2^*-1}}{\varepsilon^*\Gamma(k-\theta_i+m_2^*+1)}\right)\\&\quad+\sum_{i=1}^{m}\frac{(|\varepsilon^*-1|)K^{k-\theta_i}}{\varepsilon^*\Gamma(k-\theta_i+1)},\end{aligned}\tag{12}$$

and

$$\mathcal{W}=\frac{\Theta_4+\Theta_3K^{-1}}{|B|}\left(\tau_1^*\sum_{i=1}^{n}\frac{K^{2k+\sigma_i-\beta_1^*-1}}{\varepsilon^*\Gamma(k+\sigma_i-\beta_1^*+1)}+\sum_{i=1}^{n}\frac{(1-\tau_1^*)K^{2k+\sigma_i-\beta_2^*-1}}{\varepsilon^*\Gamma(k+\sigma_i-\beta_2^*+1)}\right)$$

$$
\begin{aligned}
&+\frac{\Theta_2+\Theta_1K^{-1}}{|B|}\\
&\times\left(\tau_2^*\sum_{i=1}^{n}\frac{K^{2k+\sigma_i+m_1^*-1}}{\varepsilon^*\Gamma(k+\sigma_i+m_1^*+1)}+\sum_{i=1}^{n}\frac{(1-\tau_2^*)K^{2k+\sigma_i+m_2^*-1}}{\varepsilon^*\Gamma(k+\sigma_i+m_2^*+1)}\right)\\
&+\sum_{i=1}^{n}\frac{K^{k+\sigma_i}}{\varepsilon^*\Gamma(k+\sigma_i+1)}. \qquad (13)
\end{aligned}
$$

Theorem 3.2 *Assume* $(H1)$–$(H4)$ *hold. Then there is found a solution on* J *for the multiterm hybrid fractional boundary value problem* (3).

Proof Construct the set $\mathcal{B}_{\mathcal{R}}=\{u\in\mathcal{C}:\|\rho\|\leq\mathcal{R}\}\subset\mathcal{C}$. Obviously, $\mathcal{B}_{\mathcal{R}}$ is convex, closed and bounded. By assuming

$$
\mathbb{K}\big(t,\rho(t)\big):=\hat{\hbar}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\ldots,\mathcal{I}^{k_n}\rho(t)\big),
$$

and by Lemma 3.1, the solution of the multiterm hybrid fractional boundary value problem (3) corresponds to the equation

$$
\begin{aligned}
\rho(t)=&\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}\mathbb{A}\big(t,\rho(t)\big)\Bigg[\sum_{i=1}^{m}\frac{\varepsilon^*-1}{\varepsilon^*\Gamma(k^*-\theta_i)}\int_0^t(t-s)^{k^*-\theta_i-1}\rho(s)\,ds\\
&+\sum_{i=1}^{n}\frac{1}{\varepsilon^*\Gamma(k+\sigma_i)}\int_0^t(t-s)^{k+\sigma_i-1}\mathbb{K}\big(s,\rho(s)\big)\,ds\\
&+\frac{t^{k-1}}{B}\times\Bigg[\frac{\tau_1^*\Theta_4(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\rho(K)-\frac{\Theta_2\tau_2^*(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\rho(K)\\
&+\frac{\Theta_4(1-\tau_1^*)(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_2^*}\rho(K)-\frac{\Theta_2(1-\tau_2^*)(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2^*}\rho(K)\\
&+\frac{\Theta_4\tau_1^*}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i-\beta_1^*}\mathbb{K}\big(K,\rho(K)\big)-\frac{\Theta_2\tau_2^*}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i+m_1^*}\mathbb{K}\big(K,\rho(K)\big)\\
&+\Theta_2\delta_2-\Theta_4\delta_1\\
&+\frac{\Theta_4(1-\tau_1^*)}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i-\beta_2^*}\mathbb{K}\big(K,\rho(K)\big)-\frac{\Theta_2(1-\tau_2^*)}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i+m_2^*}\mathbb{K}\big(K,\rho(K)\big)\Bigg]\\
&-\frac{t^{k-2}}{B}\Bigg[\frac{\tau_1^*\Theta_3(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\rho(K)-\frac{\Theta_1\tau_2^*(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\rho(K)\\
&+\frac{\Theta_3(1-\tau_1^*)(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_2^*}\rho(K)-\frac{\Theta_1(1-\tau_2^*)(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2^*}\rho(K)\\
&+\frac{\Theta_3\tau_1^*}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i-\beta_1^*}\mathbb{K}\big(K,\rho(K)\big)-\frac{\Theta_1\tau_2^*}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i+m_1^*}\mathbb{K}\big(K,\rho(K)\big)\\
&+\Theta_1^*\delta_2-\Theta_3\delta_1
\end{aligned}
$$

$$+\frac{\Theta_3(1-\tau_1^*)}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i-\beta_2^*}\mathbb{K}\big(K,\rho(K)\big)-\frac{\Theta_1(1-\tau_2^*)}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i+m_2^*}\mathbb{K}\big(K,\rho(K)\big)\Bigg]\Bigg]$$

$$+\sum_{i=1}^{n}\mathcal{I}^{\vartheta_i}\mathbb{H}\big(t,\rho(t)\big). \tag{14}$$

We build the operators $\mathfrak{A}, \mathfrak{C}:\mathcal{C}\to\mathcal{C}$ and $\mathfrak{B}:\mathcal{B}_{\mathcal{R}}\to\mathcal{C}$ by

$$\mathfrak{A}\rho(t)=\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}\mathbb{A}\big(t,\rho(t)\big), \tag{15}$$

$$\begin{aligned}\mathfrak{B}\rho(t)=&\sum_{i=1}^{m}\frac{\varepsilon^*-1}{\varepsilon^*\Gamma(k^*-\theta_i)}\int_0^t(t-s)^{k-\theta_i-1}\rho(s)\,ds\\&+\sum_{i=1}^{n}\frac{1}{\varepsilon^*\Gamma(k+\sigma_i)}\int_0^t(t-s)^{k+\sigma_i-1}\mathbb{K}\big(s,\rho(s)\big)\,ds\\&+\frac{t^{k-1}}{B}\times\Bigg[\frac{\tau_1^*\Theta_4(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\rho(K)-\frac{\Theta_2\tau_2^*(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\rho(K)\\&+\frac{\Lambda(1-\tau_1^*)(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_2^*}\rho(K)-\frac{\Theta_2(1-\tau_2^*)(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2^*}\rho(K)\\&+\frac{\Theta_4\tau_1^*}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i-\beta_1^*}\mathbb{K}\big(K,\rho(K)\big)-\frac{\Theta_2\tau_2^*}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i+m_1^*}\mathbb{K}\big(K,\rho(K)\big)\\&+\Theta_2\delta_2-\Theta_4\delta_1\\&+\frac{\Theta_4(1-\tau_1^*)}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma-\beta_2^*}\mathbb{K}\big(K,\rho(K)\big)-\frac{\Theta_2(1-\tau_2^*)}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i+m_2^*}\mathbb{K}\big(K,\rho(K)\big)\Bigg]\\&-\frac{t^{k-2}}{B}\Bigg[\frac{\tau_1^*\Theta_3(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\rho(K)-\frac{\Theta_1\tau_2^*(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\rho(K)\\&+\frac{\Theta_3(1-\tau_1^*)(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_2^*}\rho(K)-\frac{\Theta_1(1-\tau_2^*)(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2^*}\rho(K)\\&+\frac{\Theta_3\tau_1^*}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i-\beta_1^*}\mathbb{K}\big(K,\rho(K)\big)-\frac{\Theta_1\tau_2^*}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i+m_1^*}\mathbb{K}\big(K,\rho(K)\big)\\&+\Theta_1^*\delta_2-\Theta_3\delta_1\\&+\frac{\Theta_3(1-\tau_1^*)}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i-\beta_2^*}\mathbb{K}\big(K,\rho(K)\big)\\&-\frac{\Theta_1(1-\tau_2^*)}{\varepsilon^*}\sum_{i=1}^{n}\mathcal{I}^{k+\sigma_i+m_2^*}\mathbb{K}\big(K,\rho(K)\big)\Bigg],\end{aligned} \tag{16}$$

and

$$\mathfrak{C}\rho(t)=\sum_{i=1}^{n}\mathcal{I}^{\vartheta_i}\mathbb{H}\big(t,\rho(t)\big), \tag{17}$$

where $\mathbb{A}$, $\mathbb{K}$ and $\mathbb{H}$ are illustrated before. Then the integral equation (14) can be expressed in a form which is denoted by

$$\rho(t) = \mathfrak{A}\rho(t)\mathfrak{B}\rho(t) + \mathfrak{C}\rho(t).$$

We will prove that all of $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ fulfill all items of Theorem 2.6.

STEP I: We first prove that $\mathfrak{A}$ and $\mathfrak{C}$ are Lipschitz on $\mathcal{C}$. Assume that $\rho, \nu \in \mathcal{C}$. Then, from $(H2)$, for $t \in J$, we obtain

$$\begin{aligned}
&\left|\mathfrak{A}\rho(t) - \mathfrak{A}\nu(t)\right| \\
&= \left|\sum_{i=1}^{n} \mathcal{I}^{\alpha_i}\mathbb{A}\big(t,\rho(t)\big) - \sum_{i=1}^{n} \mathcal{I}^{\alpha_i}\mathbb{A}\big(t,\nu(t)\big)\right| \\
&= \left|\sum_{i=1}^{n} \mathcal{I}^{\alpha_i}\mathcal{A}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\ldots,\mathcal{I}^{k_n}\rho(t)\big) - \sum_{i=1}^{n} \mathcal{I}^{\alpha_i}\mathcal{A}\big(t,\nu(t),\mathcal{I}^{k_1}\nu(t),\ldots,\mathcal{I}^{k_n}\nu(t)\big)\right| \\
&\leq \sum_{i=1}^{n} \mathcal{I}^{\alpha_i}\left|\mathcal{A}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\ldots,\mathcal{I}^{k_n}\rho(t)\big) - \mathcal{A}\big(t,\nu(t),\mathcal{I}^{k_1}\nu(t),\ldots,\mathcal{I}^{k_n}\nu(t)\big)\right| \\
&\leq \sum_{i=1}^{n} \mathcal{I}^{\alpha_i}\Phi(t)\left(\left|\rho(t) - \nu(t)\right| + \left|\mathcal{I}^{k_1}\rho(t) - \mathcal{I}^{k_1}\nu(t)\right| + \cdots + \left|\mathcal{I}^{k_n}\rho(t) - \mathcal{I}^{k_n}\nu(t)\right|\right) \\
&\leq \sum_{i=1}^{n} \mathcal{I}^{\alpha_i}\Phi(t)\left(1 + \frac{t^{k_1}}{\Gamma(1+k_1)} + \cdots + \frac{t^{k_n}}{\Gamma(1+k_n)}\right)\left|\rho(t) - \nu(t)\right| \\
&\leq \sum_{i=1}^{n} \mathcal{I}^{\alpha_i}\Phi(t)\left(\sum_{j=0}^{n} \frac{t^{k_j}}{\Gamma(1+k_j)}\right)\left|\rho(t) - \nu(t)\right| \\
&\leq \|\Phi\| \sum_{i=1}^{n}\left(\sum_{j=0}^{n} \frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i + k_j + 1)}\right)\|\rho - \nu\|,
\end{aligned}$$

$\forall t \in J$ with $k_0 = 0$. So,

$$\|\mathfrak{A}\rho - \mathfrak{A}\nu\| \leq \|\Phi\| \sum_{i=1}^{n}\left(\sum_{j=0}^{n} \frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i + k_j + 1)}\right)\|\rho - \nu\|,$$

for all $u, \nu \in \mathcal{C}$. This ensures that $\mathfrak{A}$ is Lipschitz on $\mathcal{C}$ with constant

$$\|\Phi\| \sum_{i=1}^{n}\left(\sum_{j=0}^{n} \frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i + k_j + 1)}\right) > 0.$$

Now, for $\mathfrak{C} : \mathcal{C} \longrightarrow \mathcal{C}$, $u, \nu \in \mathcal{C}$, we obtain

$$\|\mathfrak{C}\rho - \mathfrak{C}\nu\| \leq \|\Psi\| \sum_{i=1}^{n}\left(\sum_{j=0}^{n} \frac{T^{\vartheta_i+k_j}}{\Gamma(\vartheta_i + k_j + 1)}\right)\|\rho - \nu\|.$$

Hence, $\mathfrak{C}:\mathcal{C}\to\mathcal{C}$ involves the same property on $\mathcal{C}$ with constant

$$\|\Psi\|\sum_{i=1}^{n}\left(\sum_{j=0}^{n}\frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)}\right)>0.$$

STEP II: In this step, we prove the complete continuity of $\mathfrak{B}$ formulated on $\mathcal{B}_{\mathcal{R}}$. First of all, assume that $\{\rho_n\}$ is a sequence in $\mathcal{B}_{\mathcal{R}}$ which converges to a point $\rho\in\mathcal{B}_{\mathcal{R}}$. From

$$\lim_{n\to\infty}\mathbb{K}\big(t,\rho_n(t)\big)=\mathbb{K}\big(t,\rho(t)\big),$$

and by the Lebesque dominated convergence theorem, we immediately get

$$\lim_{n\to\infty}\mathfrak{B}\rho_n(t)=\mathfrak{B}\rho(t),$$

$\forall t\in J$. This proves the continuity of $\mathfrak{B}$ on $\mathcal{B}_{\mathcal{R}}$.

To check the uniform boundedness of $\mathfrak{B}(\mathcal{B}_{\mathcal{R}})$ in $\mathcal{B}_{\mathcal{R}}$, for any $\rho\in\mathcal{B}_{\mathcal{R}}$, we get

$$\begin{aligned}
\big|\mathfrak{B}\rho(t)\big|\leq{}&\|\mathcal{P}\|\sum_{i=0}^{n}\Xi_i\left(\frac{K^{k_i}}{\Gamma(k_i+1)}\|u\|\right)\Bigg[\sum_{i=1}^{n}\frac{K^{k+\sigma_i}}{\varepsilon^*\Gamma(k+\sigma_i+1)}\\
&+\frac{\Theta_4+\Theta_3K^{-1}}{|B|}\left(\tau_1^*\sum_{i=1}^{n}\frac{K^{2k+\sigma_i-\beta_1^*-1}}{\varepsilon^*\Gamma(k+\sigma_i-\beta_1^*+1)}+\sum_{i=1}^{n}\frac{(1-\tau_1^*)K^{2k+\sigma_i-\beta_2^*-1}}{\varepsilon^*\Gamma(k+\sigma_i-\beta_2^*+1)}\right)\\
&+\frac{\Theta_2+\Theta_1K^{-1}}{|B|}\left(\tau_2^*\sum_{i=1}^{n}\frac{K^{2k+\sigma_i+m_1^*-1}}{\varepsilon^*\Gamma(k+\sigma_i+m_1^*+1)}+\sum_{i=1}^{n}\frac{(1-\tau_2^*)K^{2k+\sigma_i+m_2^*-1}}{\varepsilon^*\Gamma(k+\sigma_i+m_2^*+1)}\right)\Bigg]\\
&+\|\rho\|\Bigg[\frac{(|\varepsilon^*-1|)(\Theta_4+\Theta_3T^{-1})}{|B|}\\
&\times\left(\tau_1^*\sum_{i=1}^{m}\frac{K^{2k-\theta_i-\beta_1^*-1}}{\varepsilon^*\Gamma(k-\theta_i-\beta_1^*+1)}+\sum_{i=1}^{m}\frac{(1-\tau_1^*)K^{2k-\theta_i-\beta_2^*-1}}{\varepsilon^*\Gamma(k-\theta_i-\beta_2^*+1)}\right)\\
&+\frac{(|\varepsilon^*-1|)(\Theta_2+\Theta_1K^{-1})}{|B|}\\
&\times\left(\tau_2^*\sum_{i=1}^{m}\frac{K^{2k-\theta_i+m_1^*-1}}{\varepsilon^*\Gamma(k-\theta_i+m_1^*+1)}+\sum_{i=1}^{m}\frac{(1-\tau_2^*)K^{2k-\theta_i+m_2^*-1}}{\varepsilon^*\Gamma(k-\theta_i+m_2^*+1)}\right)\\
&+\sum_{i=1}^{m}\frac{(|\varepsilon^*-1|)K^{k-\theta_i}}{\varepsilon^*\Gamma(k-\theta_i+1)}\Bigg]\\
&+\frac{1}{|B|}\big[K^{k-1}\big(|\Theta_2\delta_2|+|\Theta_4\delta_1|\big)+K^{k-2}\big(|\Theta_1\delta_2|+|\Theta_3\delta_1|\big)\big]\\
\leq{}&\|\mathcal{P}\|\sum_{i=0}^{n}\Xi_i\left(\frac{K^{k_i}}{\Gamma(k_i+1)}\mathcal{R}\right)\mathcal{W}+\mathcal{R}\mathcal{V}+\frac{1}{|B|}\big[K^{k-1}\big(|\Theta_2\delta_2|+|\Theta_4\delta_1|\big)\\
&+K^{k-2}\big(|\Theta_1\delta_2|+|\Theta_3\delta_1|\big)\big].
\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathfrak{B}\rho\| &\leq \|\mathcal{P}\| \sum_{i=0}^{n} \Xi_i \left(\frac{K^{k_i}}{\Gamma(k_i+1)} \mathcal{R} \right) \mathcal{W} + \mathcal{R}\mathcal{V} + \frac{1}{|B|} \big[K^{k-1} \big(|\Theta_2 \delta_2| + |\Theta_4 \delta_1| \big) \\ &\quad + K^{k-2} \big(|\Theta_1 \delta_2| + |\Theta_3 \delta_1| \big) \big] \\ &= Q^* \end{aligned} \tag{18}$$

for all $\rho \in \mathcal{B}_{\mathcal{R}}$ with Q^* illustrated in (11). This yields the required result in this part for $\mathfrak{B}$ on $\mathcal{B}_{\mathcal{R}}$.

Let us now prove that $\mathfrak{B}(\mathcal{B}_{\mathcal{R}})$ is equi-continuous in $\mathcal{C}$. Assume that $t_1 < t_2 \in J$. Then for any $\rho \in \mathcal{B}_{\mathcal{R}}$ we have

$$\begin{aligned}&\big|\mathfrak{B}\rho(t_2) - \mathfrak{B}\rho(t_1)\big| \\ &\leq \sum_{i=1}^{m} \frac{|\varepsilon^* - 1|}{\varepsilon^* \Gamma(k-\theta_i)} \left(\left| \int_0^{t_2} (t_2 - s)^{k-\theta_i - 1} \rho(s)\, ds - \int_0^{t_1} (t_1 - s)^{k-\theta_i - 1} \rho(s)\, ds \right| \right) \\ &\quad + \sum_{i=1}^{n} \frac{1}{\varepsilon^* \Gamma(k+\sigma_i)} \\ &\quad \times \left(\left| \int_0^{t_2} (t_2 - s)^{k+\sigma_i - 1} \mathbb{K}\big(s, \rho(s)\big)\, ds - \int_0^{t_1} (t_1 - s)^{k+\sigma_i - 1} \mathbb{K}\big(s, \rho(s)\big)\, ds \right| \right) \\ &\quad + \frac{t_2^{k-1} - t_1^{k-1}}{|B|} \times \left[\left| \frac{\tau_1^* \Theta_4 (\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-\beta_1^*} \rho(K) \right. \right. \\ &\quad - \frac{\Theta_2 \tau_2^* (\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1^*} \rho(K) \\ &\quad + \frac{\Theta_4 (1-\tau_1^*)(\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-\beta_2^*} \rho(K) - \frac{\Theta_2 (1-\tau_2^*)(\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_2^*} \rho(K) \\ &\quad + \frac{\Theta_4 \tau_1^*}{\varepsilon^*} \sum_{i=1}^{n} \mathcal{I}^{k+\sigma_i-\beta_1^*} \mathbb{K}\big(K, \rho(K)\big) - \sum_{i=1}^{n} \frac{\Theta_2 \tau_2^*}{\varepsilon^*} \mathcal{I}^{k+\sigma_i+m_1^*} \mathbb{K}\big(K, \rho(K)\big) + \Theta_2 \delta_2 - \Theta_4 \delta_1 \\ &\quad \left. + \frac{\Theta_4 (1-\tau_1^*)}{\varepsilon^*} \sum_{i=1}^{n} \mathcal{I}^{k+\sigma_i-\beta_2^*} \mathbb{K}\big(K, \rho(K)\big) - \frac{\Theta_2 (1-\tau_2^*)}{\varepsilon^*} \sum_{i=1}^{n} \mathcal{I}^{k+\sigma_i+m_2^*} \mathbb{K}\big(K, \rho(K)\big) \right| \Bigg] \\ &\quad + \frac{t_2^{k-2} - t_1^{k-2}}{|B|} \left| \left[\frac{\tau_1^* \Theta_3 (\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-\beta_1^*} \rho(K) - \frac{\Theta_1 \tau_2^* (\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1^*} \rho(K) \right. \right. \\ &\quad + \frac{\Theta_3 (1-\tau_1^*)(\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-\beta_2^*} \rho(K) - \frac{\Theta_1 (1-\tau_2^*)(\varepsilon^* - 1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_2^*} \rho(K) \\ &\quad + \frac{\Theta_3 \tau_1^*}{\varepsilon^*} \sum_{i=1}^{n} \mathcal{I}^{k+\sigma_i-\beta_1^*} \mathbb{K}\big(K, \rho(K)\big) - \frac{\Theta_1 \tau_2^*}{\varepsilon^*} \sum_{i=1}^{n} \mathcal{I}^{k+\sigma_i-m_1^*} \mathbb{K}\big(K, \rho(K)\big) + \Theta_1 \delta_2 - \Theta_3 \delta_1 \\ &\quad \left. \left. + \frac{\Theta_3 (1-\tau_1^*)}{\varepsilon^*} \sum_{i=1}^{n} \mathcal{I}^{k+\sigma_i-\beta_2^*} \mathbb{K}\big(K, \rho(K)\big) - \frac{\Theta_1 (1-\tau_2^*)}{\varepsilon^*} \sum_{i=1}^{n} \mathcal{I}^{k+\sigma_i+m_2^*} \mathbb{K}\big(K, \rho(K)\big) \right] \right|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{m} \frac{|\varepsilon^* - 1|}{\varepsilon^* \Gamma(k-\theta_i+1)} \left(\left(t_2^{k-\theta_i} - t_1^{k-\theta_i} \right) + 2(t_2 - t_1)^{k-\theta_i} \right) \\
&\quad + \sum_{i=1}^{n} \frac{1}{\varepsilon^* \Gamma(k+\sigma_i+1)} \left(\left(t_2^{k+\sigma_i} - t_1^{k+\sigma_i} \right) + 2(t_2 - t_1)^{k+\sigma_i} \right) + \frac{t_2^{k-1} - t_1^{k-1}}{|B|} \\
&\quad \times \left\{ \mathcal{R} \left[\sum_{i=1}^{m} \frac{K^{k-\theta_i-\beta_1^*}(|\varepsilon^* - 1|)}{\varepsilon^* \Gamma(k-\theta_i-\beta_1^*+1)} + \sum_{i=1}^{m} \frac{K^{k-\theta_i+m_1^*}(|\varepsilon^* - 1|)\Theta_2 \tau_2^*}{\varepsilon^* \Gamma(k-\theta_i+m_1^*+1)} \right.\right. \\
&\quad + \sum_{i=1}^{m} \frac{\Theta_4(1-\tau_1^*)(\varepsilon^* - 1)K^{k-\theta_i-\beta_2^*}}{\varepsilon^* \Gamma(k-\theta_i-\beta_2^*+1)} + \left.\left. \sum_{i=1}^{m} \frac{\Theta_2(1-\tau_2^*)(|\varepsilon^* - 1|)K^{k-\theta_i+m_2^*}}{\varepsilon^* \Gamma(k-\theta_i+m_2^*+1)} \right] \right. \\
&\quad + \|\mathcal{P}\| \sum_{i=0}^{n} \Xi_i \left(\frac{K^{k_i}}{\Gamma(k_i+1)} \mathcal{R} \right) \left[\Theta_4 \tau_1^* \sum_{i=1}^{n} \frac{K^{k\sigma_i-\beta_1^*}}{\varepsilon^* \Gamma(k+\sigma-\beta_1^*+1)} \right. \\
&\quad + \Theta_2 \tau_2^* \sum_{i=1}^{n} \frac{K^{k+\sigma_i+m_1^*}}{\varepsilon^* \Gamma(k+\sigma+m_1^*+1)} \\
&\quad + \left. \sum_{i=1}^{n} \frac{\Theta_4(1-\tau_1^*)K^{k+\sigma_i-\beta_2^*}}{\varepsilon^* \Gamma(k+\sigma_i-\beta_2^*+1)} + \sum_{i=1}^{n} \frac{\Theta_2(1-\tau_2^*)K^{k+\sigma_i+m_2^*}}{\varepsilon^* \Gamma(k+\sigma_i+m_2^*+1)} \right] + |\Theta_2 \delta_2| + |\Theta_4 \delta_1| \Bigg\} \\
&\quad + \frac{t_2^{k-2} - t_1^{k-2}}{|B|} \left\{ \mathcal{R} \left[\tau_1^* \Theta_3 \sum_{i=1}^{m} \frac{K^{k-\theta_i-\beta_1^*}(|\varepsilon^* - 1|)}{\varepsilon^* \Gamma(k-\theta_i-\beta_1^*+1)} + \Theta_1 \tau_2^* \sum_{i=1}^{m} \frac{K^{k-\theta_i+m_1^*}(|\varepsilon^* - 1|)}{\varepsilon^* \Gamma(k-\theta_i+m_1^*+1)} \right.\right. \\
&\quad + \sum_{i=1}^{m} \frac{\Theta_3(1-\tau_1^*)(|\varepsilon^* - 1|)K^{k-\theta_i-\beta_2^*}}{\varepsilon^* \Gamma(k-\theta_i-\beta_2^*+1)} + \left.\left. \sum_{i=1}^{m} \frac{\Theta_1(1-\tau_2^*)(|\varepsilon^* - 1|)K^{k-\theta_i+m_2^*}}{\varepsilon^* \Gamma(k-\theta_i+m_2^*+1)} \right] \right. \\
&\quad + \|\mathcal{P}\| \sum_{i=0}^{n} \Xi_i \left(\frac{K^{k_i}}{\Gamma(k_i+1)} \mathcal{R} \right) \left[\Theta_3 \tau_1^* \sum_{i=0}^{n} \frac{K^{k+\sigma_i-\beta_1^*}}{\varepsilon^* \Gamma(k+\sigma_i-\beta_1^*+1)} \right. \\
&\quad + \Theta_1 \tau_2^* \sum_{i=0}^{n} \frac{K^{k+\sigma_i+m_1^*}}{\varepsilon^* \Gamma(k+\sigma_i+m_1^*+1)} \\
&\quad + \left. \sum_{i=0}^{n} \frac{\Theta_3(1-\tau_1^*)K^{k+\sigma_i-\beta_2^*}}{\varepsilon^* \Gamma(k+\sigma_i-\beta_2^*+1)} + \sum_{i=0}^{n} \frac{\Theta_1(1-\tau_2^*)K^{k+\sigma_i+m_2^*}}{\varepsilon^* \Gamma(k+\sigma_i+m_2^*+1)} \right] + |\Theta_1 \delta_2| + |\Theta_3 \delta_1| \Bigg\}.
\end{aligned}
$$

Hence, we get

$$
\left| \mathfrak{B}\rho(t_2) - \mathfrak{B}\rho(t_1) \right| \to 0 \quad \text{as } t_2 \to t_1.
$$

This implies that

$$
\left\| \mathfrak{B}\rho(t_2) - \mathfrak{B}\rho(t_1) \right\| \to 0 \quad \text{as } t_2 \to t_1.
$$

Thus, from the Arzela–Ascoli theorem, we arrive at the complete continuity of $\mathfrak{B}$ on $\mathcal{B}_{\mathcal{R}}$.

STEP III: The $(H3)$ of Theorem 2.6 is fulfilled.

Assume that $\rho \in \mathcal{C}$ and $\nu \in \mathcal{B}_{\mathcal{R}}$ are arbitrary elements via $\rho = \mathfrak{A}\rho\mathfrak{B}\nu + \mathfrak{C}\rho$. Then, by (11) and (18), we get

$$
\left| \rho(t) \right| \leq \left| \mathfrak{A}\rho(t) \right| \left| \mathfrak{B}\nu(t) \right| + \left| \mathfrak{C}\rho(t) \right|
$$

$$
\begin{aligned}
&\leq \left| \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathcal{A}\big(t, \rho(t), \mathcal{I}^{k_1}\rho(t), \ldots, \mathcal{I}^{k_n}\rho(t)\big) - \mathcal{I}^{\alpha_i}\mathcal{A}(t,0,0,\ldots,0) \right. \\
&\quad \left. + \mathcal{I}^{\alpha_i}\mathcal{A}(t,0,0,\ldots,0) \right| Q^* \\
&\quad + \left| \sum_{i=1}^{n} \mathcal{I}^{\vartheta_i} \mathcal{H}\big(t, \rho(t), \mathcal{I}^{k_1}\rho(t), \ldots, \mathcal{I}^{k_n}\rho(t)\big) - \mathcal{I}^{\vartheta_i}\mathcal{H}(t,0,0,\ldots,0) \right. \\
&\quad \left. + \mathcal{I}^{\vartheta_i}\mathcal{H}(t,0,0,\ldots,0) \right| \\
&\leq \left(\sum_{i=1}^{n} \frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)} \mathcal{A}_0 + \|\Phi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)} \right) \|\rho\| \right) Q^* \\
&\quad + \sum_{i=1}^{m} \frac{K^{\theta_i}}{\Gamma(\theta_i+1)} \mathcal{H}_0 + \|\Psi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)} \right) \|\rho\|,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\big\|\rho(t)\big\| &\leq \left(\sum_{i=1}^{n} \frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)} \mathcal{A}_0 + \|\Phi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)} \right) \|\rho\| \right) Q^* \\
&\quad + \sum_{i=1}^{m} \frac{K^{\theta_i}}{\Gamma(\theta_i+1)} \mathcal{H}_0 + \|\Psi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)} \right) \|\rho\|.
\end{aligned}
$$

In consequence,

$$
\frac{\mathcal{A}_0 Q^* \sum_{i=0}^{n} \frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)} + \mathcal{H}_0 \sum_{i=0}^{m} \frac{K^{\theta_i}}{\Gamma(\theta_i+1)}}{1 - \|\Psi\| \sum_{i=1}^{n} (\sum_{j=0}^{n} \frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)}) - \|\Phi\| \sum_{i=1}^{n} (\sum_{j=0}^{n} \frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)}) Q^*} \leq \mathcal{R}.
$$

We obtain

$$
\big\|\rho(t)\big\| \leq \mathcal{R}.
$$

STEP IV: We prove that $l_1^* \Delta^* + l_2^* < 1$, in which the item (4) of Theorem 2.6 occurs. Since

$$
\Delta^* = \big\|\mathfrak{B}(\mathcal{B}_{\mathcal{R}})\big\| = \sup_{\rho \in \mathcal{B}_{\mathcal{R}}} \Big(\sup_{t \in J} \big|\mathfrak{B}\rho(t)\big| \Big) \leq Q^*,
$$

by the above calculations, we obtain

$$
\begin{aligned}
&\|\Phi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)} \right) \Delta^* + \|\Psi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)} \right) \\
&\quad \leq \|\Phi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)} \right) Q^* + \|\Psi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)} \right) \leq 1,
\end{aligned}
$$

with

$$\mathfrak{l}_1^* = \|\Phi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\alpha_i + k_j}}{\Gamma(\alpha_i + k_j + 1)} \right)$$

and

$$\mathfrak{l}_2^* = \|\Psi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{K^{\vartheta_i + k_j}}{\Gamma(\vartheta_i + k_j + 1)} \right).$$

Therefore, all items of Theorem 2.6 are fulfilled, and so it is found a solution for $\rho = \mathfrak{A}\rho\mathfrak{B}\rho + \mathfrak{C}\rho$ and also for the multiterm hybrid fractional boundary value problem (3) on J. This ends our argument. □

4 Results regarding to stability

In this section, as a special case of the multiterm hybrid fractional boundary value problem (3), we study Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias stability by assuming $\mathbb{A}(t, \rho(t)) = \mathcal{A}(t)$, $\tau_1^* = 1$ and $\tau_2^* = 1$ given by

$$\begin{cases} \varepsilon^* \mathcal{D}^k \left[\frac{\rho(t) - \sum_{i=1}^{n} \mathcal{I}^{\vartheta_i} \mathbb{H}(t, \rho(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathbb{A}(t, \rho(t))} \right] + (1 - \varepsilon^*) \sum_{i=1}^{m} \mathcal{D}^{\theta_i} \rho(t) = \sum_{i=1}^{n} \mathcal{I}^{\sigma_i} \mathbb{K}(t, \rho(t)), \\ \left[\frac{\rho(t) - \sum_{i=1}^{n} \mathcal{I}^{\vartheta_i} \mathbb{H}(t, \rho(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathbb{A}(t, \rho(t))} \right]\Big|_{t=0} = 0, \\ \mathcal{D}^{\beta_1^*} \left[\frac{\rho(t) - \sum_{i=1}^{n} \mathcal{I}^{\vartheta_i} \mathbb{H}(t, \rho(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathbb{A}(t, \rho(t))} \right]\Big|_{t=K} = \delta_1, \\ \mathcal{I}^{m_1^*} \left[\frac{\rho(t) - \sum_{i=1}^{n} \mathcal{I}^{\vartheta_i} \mathbb{H}(t, \rho(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathbb{A}(t, \rho(t))} \right]\Big|_{t=K} = \delta_2. \end{cases} \tag{19}$$

First, we pay attention to some definitions on different versions of the stability [51].

Definition 4.1 ([51]) The multiterm hybrid fractional boundary value problem (19) is Ulam–Hyers stable whenever some $c \in \mathbb{R}^+$ exists so that $\forall \varepsilon > 0$ and $\forall \nu^* \in \mathcal{C}$ as a solution function satisfying the inequality

$$\left| \varepsilon^* \mathcal{D}^k \left[\frac{\nu^*(t) - \sum_{i=1}^{n} \mathcal{I}^{\vartheta_i} \mathbb{H}(t, \nu^*(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathbb{A}(t, \nu^*(t))} \right] + \left(1 - \varepsilon^*\right) \sum_{i=1}^{m} \mathcal{D}^{\theta_i} \nu^*(t) - \sum_{i=1}^{n} \mathcal{I}^{\sigma_i} \mathbb{K}\left(t, \nu^*(t)\right) \right| \leq \varepsilon, \tag{20}$$

there exists another solution function $\rho \in \mathcal{C}$ for the multiterm hybrid fractional boundary value problem (19) with

$$\left|\nu^*(t) - \rho(t)\right| \leq c\varepsilon, \quad \left(t \in [0, K]\right).$$

Definition 4.2 ([51]) The multiterm hybrid fractional boundary value problem (19) is named generalized Ulam–Hyers stable if $\varphi_{\mathcal{I}^{\sigma_i} \mathbb{K}} \in \mathcal{C}_{\mathbb{R}^+}(\mathbb{R}^+)$ exists with $\varphi_{\sum_{i=1}^{n} \mathcal{I}^{\sigma_i} \mathbb{K}}(0) = 0$ so that, for any solution function $\nu^* \in \mathcal{C}$ of inequality (20), another function $\rho \in \mathcal{C}$ exists

satisfying the multiterm hybrid fractional boundary value problem (19) for which

$$\left|v^*(t)-\rho(t)\right| \le \varphi_{\mathcal{I}^{\sigma_i}\mathbb{K}}(\varepsilon), \quad \left(t\in[0,K]\right),$$

is valid.

Definition 4.3 ([51]) The multiterm hybrid fractional boundary value problem (19) is Ulam–Hyers–Rassias stable which is dependent on $\varphi:[0,K]\to\mathbb{R}^+$ whenever $\exists c_\varphi\in\mathbb{R}^+$ so that $\forall\varepsilon>0$ and $\forall v^*\in\mathcal{C}$ as a solution of the inequality

$$\left|\varepsilon^*\mathcal{D}^k\left[\frac{v^*(t)-\sum_{i=1}^n\mathcal{I}^{\vartheta_i}\mathbb{H}(t,v^*(t))}{\sum_{i=1}^n\mathcal{I}^{\alpha_i}\mathbb{A}(t,v^*(t))}\right] + \left(1-\varepsilon^*\right)\sum_{i=1}^m\mathcal{D}^{\theta_i}v^*(t)-\sum_{i=1}^n\mathcal{I}^{\sigma_i}\mathbb{K}\left(t,v^*(t)\right)\right| \le \varepsilon\varphi(t), \tag{21}$$

there exists another solution function $\rho(t)\in\mathcal{C}$ of the multiterm hybrid fractional boundary value problem (19) satisfying

$$\left|v^*(t)-\rho(t)\right| \le c_\varphi\varepsilon\varphi(t), \quad \left(t\in[0,K]\right).$$

Definition 4.4 ([51]) The multiterm hybrid fractional boundary value problem (19) is said to be generalized Ulam–Hyers–Rassias stable depending on $\varphi:[0,K]\to\mathbb{R}^+$ if $\exists c_\varphi\in\mathbb{R}^+$ so that $\forall\varepsilon>0$ and $\forall v^*\in\mathcal{C}$ as a solution of the inequality

$$\left|\varepsilon^*\mathcal{D}^k\left[\frac{v^*(t)-\sum_{i=1}^n\mathcal{I}^{\vartheta_i}\mathbb{H}(t,v^*(t))}{\sum_{i=1}^n\mathcal{I}^{\alpha_i}\mathbb{A}(t,v^*(t))}\right] + \left(1-\varepsilon^*\right)\sum_{i=1}^m\mathcal{D}^{\theta_i}v^*(t)-\sum_{i=1}^n\mathcal{I}^{\sigma_i}\mathbb{K}\left(t,v^*(t)\right)\right| \le \varphi(t), \tag{22}$$

another solution $\rho(t)\in\mathcal{C}$ exists for the multiterm hybrid fractional boundary value problem (19) satisfying

$$\left|v^*(t)-\rho(t)\right| \le c_\varphi\varphi(t) \quad \left(t\in[0,K]\right).$$

Remark 4.5 ([51]) $v^*(t)\in\mathcal{C}$ is named as a solution for (20) iff some function $g\in\mathcal{C}$ exists which is dependent on v^* and

(*i*) $|g(t)|<\varepsilon$,

(*ii*) $\varepsilon^*\mathcal{D}^k[\frac{v^*(t)-\sum_{i=1}^n\mathcal{I}^{\vartheta_i}\mathbb{H}(t,v^*(t))}{\sum_{i=1}^n\mathcal{I}^{\alpha_i}\mathbb{A}(t,v^*(t))}]+(1-\varepsilon^*)\sum_{i=1}^m\mathcal{D}^{\theta_i}v^*(t)=\sum_{i=1}^n\mathcal{I}^{\sigma_i}\mathbb{K}(t,v^*(t))+g(t)$,

for $t\in[0,K]$.

Theorem 4.6 *Let* $\hat{\hbar}:[0,K]\times\mathbb{R}^{n+1}\to\mathbb{R}^+$ *be continuous and* $\exists L^*\in\mathbb{R}^+$ *so that*

$$\left|\hat{\hbar}\left(t,u_1(t),\ldots,u_{n+1}(t)\right)-\hat{\hbar}\left(t,v_1(t),\ldots,v_{n+1}(t)\right)\right| \le L^*\left(\sum_{i=1}^{n+1}|u_i-v_i|\right). \tag{23}$$

If the second condition of $(H2)$ *holds, then the multiterm hybrid fractional boundary value problem* (19) *is Ulam–Hyers stable on* $[0,K]$ *and accordingly is generalized Ulam–Hyers stable if*

$$\|\mathcal{A}\| \sum_{i=1}^{n} \frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)} \left(L^* \sum_{i=0}^{n} \frac{K^{k_i}}{\Gamma(k_i+1)} \mathcal{W} + \mathcal{V} \right) + \|\Psi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{T^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)} \right) < 1,$$

where $\|\mathcal{A}\| = \sup_{t\in[0,K]} |\mathcal{A}(t)|$.

Proof For $\varepsilon > 0$, and every solution $v^*(t) \in \mathcal{C}$ of the inequality

$$\left| \varepsilon^* \mathcal{D}^k \left[\frac{v^*(t) - \sum_{i=1}^{n} \mathcal{I}^{\vartheta_i} \mathbb{H}(t, v^*(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathbb{A}(t, v^*(t))} \right] + \left(1-\varepsilon^*\right) \sum_{i=1}^{m} \mathcal{D}^{\theta_i} v^*(t) - \sum_{i=1}^{n} \mathcal{I}^{\sigma_i} \mathbb{K}\left(t, v^*(t)\right) \right| \le \varepsilon,$$

there is found a function g with

$$\varepsilon^* \mathcal{D}^k \left[\frac{\rho(t) - \sum_{i=1}^{n} \mathcal{I}^{\vartheta_i} \mathbb{H}(t, \rho(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathbb{A}(t, \rho(t))} \right] + \left(1-\varepsilon^*\right) \sum_{i=1}^{m} \mathcal{D}^{\theta_i} \rho(t) = \sum_{i=1}^{n} \mathcal{I}^{\sigma_i} \mathbb{K}\left(t, \rho(t)\right) + g(t),$$

in which $|g(t)| \le \varepsilon$. So

$$\begin{aligned}
v^*(t) = {} & \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathbb{A}\left(t, v^*(t)\right) \left[\sum_{i=1}^{m} \frac{\varepsilon^*-1}{\varepsilon^* \Gamma(k^*-\theta_i)} \int_0^t (t-s)^{k-\theta_i-1} v^*(s)\, ds \right. \\
& + \frac{1}{\varepsilon^* \Gamma(k)} \int_0^t (t-s)^{k-1} \mathbb{K}\left(t, v^*(t)\right) ds + \frac{1}{\varepsilon^* \Gamma(k)} \int_0^t (t-s)^{k-1} g(t)\, ds \\
& + \frac{t^{k-1}}{B} \times \left[\frac{\Theta_4(\varepsilon^*-1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-\beta_1^*} v^*(T) - \frac{\Theta_2(\varepsilon^*-1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1^*} v^*(K) \right. \\
& + \frac{\Theta_4}{\varepsilon^*} \mathcal{I}^{k-\beta_1^*} \mathbb{K}\left(K, v^*(K)\right) + \frac{\Theta_4}{\varepsilon^*} \mathcal{I}^{k-\beta_1^*} g(K) \\
& \left. - \frac{\Theta_2}{\varepsilon^*} \mathcal{I}^{k+m_1^*} \mathbb{K}\left(K, v^*(K)\right) - \frac{\Theta_2}{\varepsilon^*} \mathcal{I}^{k+m_1^*} g(K) + \Theta_2\delta_2 - \Theta_4\delta_1 \right] \\
& - \frac{t^{k-2}}{B} \left[\frac{\Theta_3(\varepsilon^*-1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-\beta_1^*} v^*(K) - \frac{\Theta_1(\varepsilon^*-1)}{\varepsilon^*} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1^*} v^*(K) \right. \\
& + \frac{\Theta_3}{\varepsilon^*} \mathcal{I}^{k-\beta_1^*} \mathbb{K}\left(K, v^*(K)\right) + \frac{\Theta_3}{\varepsilon^*} \mathcal{I}^{k-\beta_1^*} g(K) - \frac{\Theta_1}{\varepsilon^*} \mathcal{I}^{k+m_1^*} \mathbb{K}\left(K, v^*(K)\right) \\
& \left. \left. - \frac{\Theta_1}{\varepsilon^*} \mathcal{I}^{k+m_1^*} g(K) + \Theta_1^*\delta_2 - \Theta_3\delta_1 \right] \right] + \sum_{i=1}^{n} \mathcal{I}^{\vartheta_i} \mathbb{H}\left(t, v^*(t)\right). \qquad (24)
\end{aligned}$$

Moreover, consider $\rho(t) \in \mathcal{C}$ as the unique solution of the multiterm hybrid fractional boundary value problem (19). Then $\rho(t)$ is illustrated as

$$\begin{aligned}
\rho(t) = {} & \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathbb{A}\left(t, \rho(t)\right) \left[\sum_{i=1}^{m} \frac{\varepsilon^*-1}{\varepsilon^* \Gamma(k^*-\theta_i)} \int_0^t (t-s)^{k-\theta_i-1} \rho(s)\, ds \right. \\
& \left. + \frac{1}{\varepsilon^* \Gamma(k)} \int_0^t (t-s)^{k-1} \mathbb{K}\left(t, \rho(t)\right) ds + \frac{1}{\varepsilon^* \Gamma(k)} \int_0^t (t-s)^{k-1} g(t)\, ds \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^{k-1}}{B}\times\left[\frac{\Theta_4(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\rho(K)-\frac{\Theta_2(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\rho(K)\right.\\
&\left.+\frac{\Theta_4}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}\mathbb{K}\big(K,\rho(K)\big)-\frac{\Theta_2}{\varepsilon^*}\mathcal{I}^{k+m_1^*}\mathbb{K}\big(K,\rho(K)\big)+\Theta_2\delta_2-\Theta_4\delta_1\right]\\
&-\frac{t^{k-2}}{B}\left[\frac{\Theta_3(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\rho(K)-\frac{\Theta_1(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\rho(K)\right.\\
&\left.\left.+\frac{\Theta_3}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}\mathbb{K}\big(K,\rho(K)\big)-\frac{\Theta_1}{\varepsilon^*}\mathcal{I}^{k+m_1^*}\mathbb{K}\big(K,\rho(K)\big)+\Theta_1^*\delta_2-\Theta_3\delta_1\right]\right]\\
&+\sum_{i=1}^{n}\mathcal{I}^{\vartheta_i}\mathbb{H}\big(t,\rho(t)\big). \qquad (25)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\big|\nu^*(t)-\rho(t)\big|\\
&\quad\le\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}\left[\varepsilon\mathcal{W}+L^*\sum_{i=0}^{n}\frac{K^{k_i}}{\Gamma(k_i+1)}\mathcal{W}\big\|\nu^*-\rho\big\|+\mathcal{V}\big\|\nu^*-\rho\big\|\right]\\
&\qquad+\|\Psi\|\sum_{i=1}^{n}\left(\sum_{j=0}^{n}\frac{T^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)}\right)\big\|\nu^*-\rho\big\|. \qquad (26)
\end{aligned}
$$

We get

$$
\begin{aligned}
&\big\|\nu^*(t)-\rho(t)\big\|\\
&\quad\le\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}\left[\varepsilon\mathcal{W}+L^*\sum_{i=0}^{n}\frac{K^{k_i}}{\Gamma(k_i+1)}\mathcal{W}\big\|\nu^*-\rho\big\|+\mathcal{V}\big\|\nu^*-\rho\big\|\right]\\
&\qquad+\|\Psi\|\sum_{i=1}^{n}\left(\sum_{j=0}^{n}\frac{T^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)}\right)\big\|\nu^*-\rho\big\|, \qquad (27)
\end{aligned}
$$

where $\mathcal{W}$ and $\mathcal{V}$ are defined in (12) and (13) with $\tau_1^*=\tau_2^*=1$. In consequence,

$$
\begin{aligned}
&\big\|\nu^*(t)-\rho(t)\big\|\\
&\quad\le\frac{\mathcal{W}\varepsilon\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}}{1-\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}(L^*\sum_{i=0}^{n}\frac{K^{k_i}}{\Gamma(k_i+1)}\mathcal{W}+\mathcal{V})-\|\Psi\|\sum_{i=1}^{n}(\sum_{j=0}^{n}\frac{T^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)})}.
\end{aligned}
$$

If we put

$$
c=\frac{\mathcal{W}\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}}{1-\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}(L^*\sum_{i=0}^{n}\frac{K^{k_i}}{\Gamma(k_i+1)}\mathcal{W}+\mathcal{V})-\|\Psi\|\sum_{i=1}^{n}(\sum_{j=0}^{n}\frac{T^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)})},
$$

then the Ulam–Hyers stability criterion is fulfilled. More generally, for

$$\varphi(\varepsilon)=\frac{\mathcal{W}\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}\varepsilon}{1-\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}(L^*\sum_{i=0}^{n}\frac{K^{k_i}}{\Gamma(k_i+1)}\mathcal{W}+\mathcal{V})-\|\Psi\|\sum_{i=1}^{n}(\sum_{j=0}^{n}\frac{T^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)})},$$

with $\varphi(0)=0$, the generalized Ulam–Hyers stability criterion is also fulfilled. □

Remark 4.7 ([51]) $v^*(t)\in\mathcal{C}$ is named as a solution for (4.3) iff $\exists g\in\mathcal{C}$ depending on v^* so that

(*i*) $|g(t)|<\varepsilon\varphi(t)$,

(*ii*) $\varepsilon^*\mathcal{D}^k[\frac{v^*(t)-\sum_{i=1}^{n}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,v^*(t))}{\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}\mathbb{A}(t,v^*(t))}]+(1-\varepsilon^*)\sum_{i=1}^{m}\mathcal{D}^{\theta_i}v^*(t)=\sum_{i=1}^{n}\mathcal{I}^{\sigma_i}\mathbb{K}(t,v^*(t))+g(t)$,

for $t\in[0,K]$.

Note that $v^*(t)\in\mathcal{C}$ can be represented by

$$\begin{aligned}v^*(t)=&\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}\mathbb{A}\big(t,v^*(t)\big)\Bigg[\sum_{i=1}^{m}\frac{\varepsilon^*-1}{\varepsilon^*\Gamma(k^*-\theta_i)}\int_0^t(t-s)^{k-\theta_i-1}v^*(s)\,ds\\&+\frac{1}{\varepsilon^*\Gamma(k)}\int_0^t(t-s)^{k-1}\mathbb{K}\big(t,v^*(t)\big)\,ds+\frac{1}{\varepsilon^*\Gamma(k)}\int_0^t(t-s)^{k-1}g(t)\,ds\\&+\frac{t^{k-1}}{B}\times\Bigg[\frac{\Theta_4(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}v^*(T)-\frac{\Theta_2(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}v^*(K)\\&+\frac{\Theta_4}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}\mathbb{K}\big(K,v^*(K)\big)+\frac{\Theta_4}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}g(K)\\&-\frac{\Theta_2}{\varepsilon^*}\mathcal{I}^{k+m_1^*}\mathbb{K}\big(K,v^*(K)\big)-\frac{\Theta_2}{\varepsilon^*}\mathcal{I}^{k+m_1^*}g(K)+\Theta_2\delta_2-\Theta_4\delta_1\Bigg]\\&-\frac{t^{k-2}}{B}\Bigg[\frac{\Theta_3(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}v^*(K)-\frac{\Theta_1(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}v^*(K)\\&+\frac{\Theta_3}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}\mathbb{K}\big(K,v^*(K)\big)+\frac{\Theta_3}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}g(K)-\frac{\Theta_1}{\varepsilon^*}\mathcal{I}^{k+m_1^*}\mathbb{K}\big(K,v^*(K)\big)\\&-\frac{\Theta_1}{\varepsilon^*}\mathcal{I}^{k+m_1^*}g(K)+\Theta_1^*\delta_2-\Theta_3\delta_1\Bigg]\Bigg]+\sum_{i=1}^{n}\mathcal{I}^{\vartheta_i}\mathbb{H}\big(t,v^*(t)\big).\end{aligned}\tag{28}$$

Now, we discuss the Ulam–Hyers–Rassias stability of solution to the problem (19).

Theorem 4.8 *Let* $\hat{\hbar}:[0,K]\times\mathbb{R}^{n+1}\to\mathbb{R}$ *be continuous and the second condition of* (*H*2) *and also* (*H*3) *hold. If* $\exists\mathcal{Q}>0$ *so that*

$$\mathcal{Q}>\frac{\sum_{i=1}^{m}\frac{K^{\theta_i}}{\Gamma(\theta_i+1)}\mathcal{H}_0+\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}(\mathfrak{M}+\|\mathcal{P}\|\sum_{i=0}^{n}\Xi_i(\frac{K^{k_i}}{\Gamma(k_i+1)}\mathcal{Q})\mathcal{W})}{1-\|\Psi\|\sum_{i=1}^{n}(\sum_{j=0}^{n}\frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)})-\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}\mathcal{V}},\tag{29}$$

with

$$\mathfrak{M}=\frac{1}{|B|}\big[K^{k-1}\big(|\Theta_2\delta_2|+|\Theta_4\delta_1|\big)+K^{k-2}\big(|\Theta_1\delta_2|+|\Theta_3\delta_1|\big)\big],$$

and there exists a function g satisfying Remark 4.7 with $2\mathcal{Q} \le |g(t)| \le \varepsilon\varphi(t)$ *for any* $t \in [0,K]$, *and* $(H5)$ $\exists$ *an increasing map* $\varphi \in C([0,K],\mathbb{R}^+)$ *and* $\exists\lambda_\varphi > 0$ *such that for all* $t \in [0,K]$

$$\begin{aligned}&\frac{2}{\varepsilon^*}\Bigg[\mathcal{I}^k\varphi(t) + \frac{t^{k-1}}{B}\big(\Theta_4\mathcal{I}^{k-\beta_1^*}\varphi(t)\\&\quad+\Theta_2\mathcal{I}^{k+m_1^*}\varphi(t)\big) + \frac{t^{k-2}}{B}\big(\Theta_3\mathcal{I}^{k-\beta_1^*}\varphi(t)+\Theta_1\mathcal{I}^{k+m_1^*}\varphi(t)\big)\Bigg] < \lambda_\varphi\varphi(t),\end{aligned}$$

then the multiterm hybrid fractional boundary value problem (19) *is Ulam–Hyers–Rassias stable and accordingly is generalized Ulam–Hyers–Rassias stable*.

Proof Suppose that $\nu^* \in \mathcal{C}$ is a solution of (4.3) and also let $\rho \in \mathcal{C}$ be a solution for the multiterm hybrid fractional boundary value problem (19). Thus, for small

$$\begin{aligned}&\big|\nu^*(t)-\rho(t)\big|\\&\quad=\Bigg|\nu^*(t)-\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}\mathbb{A}\big(t,\nu^*(t)\big)\Bigg[\sum_{i=1}^{m}\frac{\varepsilon^*-1}{\varepsilon^*\Gamma(k^*-\theta_i)}\int_0^t(t-s)^{k-\theta_i-1}\nu^*(s)\,ds\\&\qquad+\frac{1}{\varepsilon^*\Gamma(k)}\int_0^t(t-s)^{k-1}\mathbb{K}\big(t,\nu^*(t)\big)\,ds+\frac{1}{\varepsilon^*\Gamma(k)}\int_0^t(t-s)^{k-1}g(t)\,ds\\&\qquad+\frac{t^{k-1}}{B}\times\Bigg[\frac{\Theta_4(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\nu^*(T)-\frac{\Theta_2(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\nu^*(K)\\&\qquad+\frac{\Theta_4}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}\mathbb{K}\big(K,\nu^*(K)\big)+\frac{\Theta_4}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}g(K)\\&\qquad-\frac{\Theta_2}{\varepsilon^*}\mathcal{I}^{k+m_1^*}\mathbb{K}\big(K,\nu^*(K)\big)-\frac{\Theta_2}{\varepsilon^*}\mathcal{I}^{k+m_1^*}g(K)+\Theta_2\delta_2-\Theta_4\delta_1\Bigg]\\&\qquad-\frac{t^{k-2}}{B}\Bigg[\frac{\Theta_3(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\nu^*(K)-\frac{\Theta_1(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\nu^*(K)\\&\qquad+\frac{\Theta_3}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}\mathbb{K}\big(K,\nu^*(K)\big)+\frac{\Theta_3}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}g(K)-\frac{\Theta_1}{\varepsilon^*}\mathcal{I}^{k+m_1^*}\mathbb{K}\big(K,\nu^*(K)\big)\\&\qquad-\frac{\Theta_1}{\varepsilon^*}\mathcal{I}^{k+m_1^*}g(K)+\Theta_1^*\delta_2-\Theta_3\delta_1\Bigg]\Bigg]+\sum_{i=1}^{n}\mathcal{I}^{\vartheta_i}\mathbb{H}\big(t,\nu^*(t)\big)\Bigg|\\&\qquad+\Bigg|\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}\mathbb{A}\big(t,\nu^*(t)\big)\Bigg[\sum_{i=1}^{m}\frac{\varepsilon^*-1}{\varepsilon^*\Gamma(k^*-\theta_i)}\int_0^t(t-s)^{k-\theta_i-1}\nu^*(s)\,ds\\&\qquad+\frac{1}{\varepsilon^*\Gamma(k)}\int_0^t(t-s)^{k-1}\mathbb{K}\big(t,\nu^*(t)\big)\,ds+\frac{1}{\varepsilon^*\Gamma(k)}\int_0^t(t-s)^{k-1}g(t)\,ds\\&\qquad+\frac{t^{k-1}}{B}\times\Bigg[\frac{\Theta_4(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\nu^*(T)-\frac{\Theta_2(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\nu^*(K)\\&\qquad+\frac{\Theta_4}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}\mathbb{K}\big(K,\nu^*(K)\big)+\frac{\Theta_4}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}g(K)\\&\qquad-\frac{\Theta_2}{\varepsilon^*}\mathcal{I}^{k+m_1^*}\mathbb{K}\big(K,\nu^*(K)\big)-\frac{\Theta_2}{\varepsilon^*}\mathcal{I}^{k+m_1^*}g(K)+\Theta_2\delta_2-\Theta_4\delta_1\Bigg]\end{aligned}$$

$$
\begin{aligned}
&- \frac{t^{k-2}}{B}\left[\frac{\Theta_3(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-\beta_1^*}\nu^*(K) - \frac{\Theta_1(\varepsilon^*-1)}{\varepsilon^*}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1^*}\nu^*(K)\right.\\
&+ \frac{\Theta_3}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}\mathbb{K}\big(K,\nu^*(K)\big) + \frac{\Theta_3}{\varepsilon^*}\mathcal{I}^{k-\beta_1^*}g(K) - \frac{\Theta_1}{\varepsilon^*}\mathcal{I}^{k+m_1^*}\mathbb{K}\big(K,\nu^*(K)\big)\\
&\left.- \frac{\Theta_1}{\varepsilon^*}\mathcal{I}^{k+m_1^*}g(K) + \Theta_1^*\delta_2 - \Theta_3\delta_1\right]\Bigg] + \sum_{i=1}^{n}\mathcal{I}^{\vartheta_i}\mathbb{H}\big(t,\nu^*(t)\big)\Bigg| + \big|\rho(t)\big|\\
&\le \|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}\left\{\mathcal{W}\varepsilon\varphi(t) + \|\mathcal{P}\|\sum_{i=0}^{n}\Xi_i\left(\frac{K^{k_i}}{\Gamma(k_i+1)}\big\|\nu^*\big\|\right)\mathcal{W} + \big\|\nu^*\big\|\mathcal{V}\right.\\
&+ \frac{1}{|B|}\big[K^{k-1}\big(|\Theta_2\delta_2| + |\Theta_4\delta_1|\big) + K^{k-2}\big(|\Theta_1\delta_2| + |\Theta_3\delta_1|\big)\big]\\
&+ \|\mathcal{P}\|\sum_{i=0}^{n}\Xi_i\left(\frac{K^{k_i}}{\Gamma(k_i+1)}\|\rho\|\right)\mathcal{W} + \|\rho\|\mathcal{V}\\
&\left.+ \frac{1}{|B|}\big[K^{k-1}\big(|\Theta_2\delta_2| + |\Theta_4\delta_1|\big) + K^{k-2}\big(|\Theta_1\delta_2| + |\Theta_3\delta_1|\big)\big]\right\}\\
&+ \sum_{i=1}^{m}\frac{K^{\theta_i}}{\Gamma(\theta_i+1)}\mathcal{H}_0 + \|\Psi\|\sum_{i=1}^{n}\left(\sum_{j=0}^{n}\frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)}\right)\big\|\nu^*\big\|\\
&+ \sum_{i=1}^{m}\frac{K^{\theta_i}}{\Gamma(\theta_i+1)}\mathcal{H}_0 + \|\Psi\|\sum_{i=1}^{n}\left(\sum_{j=0}^{n}\frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)}\right)\|\rho\|\\
&\le \|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}\mathcal{W}\varepsilon\varphi(t) + 2\mathcal{Q} + \varepsilon\lambda_\varphi\varphi(t)\\
&\le \|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}\mathcal{W}\varepsilon\varphi(t) + \varepsilon\varphi(t) + \varepsilon\lambda_\varphi\varphi(t),
\end{aligned}
$$

it yields

$$
\big|\nu^*(t) - \rho(t)\big| \le \varepsilon\left(\|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}\mathcal{W} + 1 + \lambda_\varphi\right)\varphi(t).
$$

For the sake of simplicity, we take

$$
c_\varphi = \|\mathcal{A}\|\sum_{i=1}^{n}\frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)}\mathcal{W} + 1 + \lambda_\varphi.
$$

Then

$$
\big|\nu^*(t) - \rho(t)\big| \le \varepsilon c_\varphi\varphi(t).
$$

Thus, the multiterm hybrid fractional boundary value problem (19) is Ulam–Hyers–Rassias stable. In addition, if we set $\varepsilon = 1$, then the multiterm hybrid fractional boundary value problem (19) is generalized Ulam–Hyers–Rassias stable. □

5 Numerical examples

Some illustrative numerical examples will be given in this section to apply and validate our theoretical results.

Example 5.1 Consider the multiterm hybrid fractional boundary value problem in the format

$$\begin{cases} 0.7\mathcal{D}^{2.8}[\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^{2}\mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}] + 0.3\sum_{i=1}^{2}\mathcal{D}^{\theta_i}\rho(t) = \sum_{i=1}^{2}\mathcal{I}^{\sigma_i}\mathbb{K}(t,\rho(t)), \\ [\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^{2}\mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=0} = 0, \\ 0.01\mathcal{D}^{\beta_1^*}[\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^{2}\mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=1.3} \\ \quad + 0.99\mathcal{D}^{\beta_2^*}[\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^{2}\mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=1.3} = 0.58, \\ 0.06\mathcal{I}^{0.25}[\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^{2}\mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=1.3} \\ \quad + 0.94\mathcal{I}^{9.25}[\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\sum_{i=1}^{2}\mathcal{I}^{\alpha_i}\mathbb{A}(t,\rho(t))}]|_{t=1.3} = 0.85, \end{cases} \tag{30}$$

where $\mathbb{H},\mathbb{K}: J\times\mathbb{R}\to\mathbb{R}$ and $\mathbb{A}: J\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ are formulated by

$$\begin{aligned} &\mathbb{H}\big(t,\rho(t)\big) = \mathcal{H}\big(t,\rho(t),\mathcal{I}^{2.24}\rho(t),\mathcal{I}^{3.21}\rho(t)\big), \\ &\mathbb{A}\big(t,\rho(t)\big) = \mathcal{A}\big(t,\rho(t),\mathcal{I}^{2.24}\rho(t),\mathcal{I}^{3.21}\rho(t)\big), \\ &\mathbb{K}\big(t,\rho(t)\big) = \hat{h}\big(t,\rho(t),\mathcal{I}^{2.24}\rho(t),\mathcal{I}^{3.21}\rho(t)\big), \end{aligned}$$

and we set $m=n=2$, $k=2.8$, $\theta_1=2.11$, $\theta_2=2.14$, $\varepsilon^*=0.7$, $\tau_1^*=0.01$, $\tau_2^*=0.06$, $\delta_1=0.58$, $\delta_2=0.85$, $m_1^*=0.25$, $m_2^*=9.25$, $\alpha_1=5.23$, $\alpha_2=0.12$, $\vartheta_1=0.25$, $\vartheta_2=0.56$, $k_1=2.24$, $k_2=3.21$, $K=1.3$ and define

$$\begin{aligned} &\hat{h}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\mathcal{I}^{k_2}\rho(t)\big) \\ &\quad = \exp(-2t)\sin\big(\rho(t)\big) + \frac{\exp(-2t)}{1+t^2}\frac{|\mathcal{I}^{k_1}\rho(t)|}{1+|\mathcal{I}^{k_1}\rho(t)|} + \frac{\exp(-2t)\mathcal{I}^{k_2}\rho(t)}{\sqrt{(8+t)}}, \\ &\mathcal{A}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\mathcal{I}^{k_2}\rho(t)\big) \\ &\quad = \frac{1}{4+t^2}\bigg(\frac{\rho(t)+\mathcal{I}^{k_1}\rho(t)+\mathcal{I}^{k_2}\rho(t)}{\rho(t)+1+\mathcal{I}^{k_1}\rho(t)+\mathcal{I}^{k_2}\rho(t)} + \frac{\exp(-t)}{10}\bigg), \\ &\mathcal{H}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\mathcal{I}^{k_2}\rho(t)\big) \\ &\quad = \frac{\exp(-t^2)}{(3+t)^2}\sin\big(\mathcal{I}^{k_1}\rho(t)+\mathcal{I}^{k_2}\rho(t)\big) + \frac{\exp(-t^2)\rho(t)}{(6+2t)^2} + \frac{1}{100}. \end{aligned}$$

We see that

$$\begin{aligned} &\big|\hat{h}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\mathcal{I}^{k_2}\rho(t)\big)\big| \le \exp(-2t)\big[\big|\rho(t)\big| + \big|\mathcal{I}^{k_1}\rho(t)\big| + \big|\mathcal{I}^{k_2}\rho(t)\big|\big], \\ &\big|\mathcal{A}\big(t,\rho_1(t),\rho_2(t),\rho_3(t)\big) - \mathcal{A}\big(t,\nu_1(t),\nu_2(t),\nu_3(t)\big)\big| \le \frac{1}{4+t^2}\bigg[\sum_{j=1}^{3}\big|\rho_j(t)-\nu_j(t)\big|\bigg], \\ &\big|\mathcal{H}\big(t,\rho_1(t),\rho_2(t),\rho_3(t)\big) - \mathcal{H}\big(t,\nu_1(t),\nu_2(t),\nu_3(t)\big)\big| \le \frac{\exp(-t^2)}{(3+t)^2}\bigg[\sum_{j=1}^{3}\big|\rho_j(t)-\nu_j(t)\big|\bigg], \end{aligned}$$

where $\Xi_1(|\rho(t)|) = |\rho(t)|$, $\Xi_2(|\mathcal{I}^{k_1}\rho(t)|) = |\mathcal{I}^{k_1}\rho(t)|$, $\Xi_3(|\mathcal{I}^{k_2}\rho(t)|) = |\mathcal{I}^{k_2}\rho(t)|$ and $\mathcal{P}(t) = \exp(-2t)$. Hence, we obtain

$$\Phi(t) = \frac{1}{4+t^2}, \qquad \Psi(t) = \frac{\exp(-t^2)}{(3+t)^2}.$$

Then $\|\Phi\| = \frac{1}{4}$, $\|\Psi\| = \frac{1}{9}$, $\|\mathcal{P}\| = 1$ and

$$\mathcal{A}_0 = \sup_{t\in J}\big|\mathcal{A}(t,0,\dots,0)\big| = \frac{1}{40}, \qquad \mathcal{A}_0 = \sup_{t\in J}\big|\mathcal{A}(t,0,\dots,0)\big| = \frac{1}{100}.$$

From MATLAB software and by (9) and (10), we have $1.9502 < \mathcal{R} < 37.3794$. As all items of Theorem 3.2 are fulfilled, the multiterm hybrid fractional boundary value problem (30) admits a solution on $[0, 1.3]$.

For the special case (19), we provide the following examples.

Example 5.2 Consider the multiterm hybrid fractional boundary value problem in the format

$$\begin{cases} 0.7\mathcal{D}^{2.8}[\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\mathbb{A}(t,\rho(t))}] + 0.3\sum_{i=1}^{2}\mathcal{D}^{\theta_i}\rho(t) = \sum_{i=1}^{2}\mathcal{I}^{\sigma_i}\mathbb{K}(t,\rho(t)), \\ [\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\mathbb{A}(t,\rho(t))}]|_{t=0} = 0, \\ \mathcal{D}^{\beta_1^*}[\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\mathbb{A}(t,\rho(t))}]|_{t=1.3} = 0.58, \\ \mathcal{I}^{0.25}[\frac{\rho(t)-\sum_{i=1}^{2}\mathcal{I}^{\vartheta_i}\mathbb{H}(t,\rho(t))}{\mathbb{A}(t,\rho(t))}]|_{t=1.3} = 0.85, \end{cases} \tag{31}$$

where $\mathbb{H}, \mathbb{K} : J \times \mathbb{R} \to \mathbb{R}$ are formulated by

$$\begin{aligned} &\mathbb{H}\big(t,\rho(t)\big) = \mathcal{H}\big(t,\rho(t),\mathcal{I}^{2.24}\rho(t),\mathcal{I}^{3.21}\rho(t)\big), \\ &\mathbb{A}\big(t,\rho(t)\big) = \mathcal{A}(t) = \frac{\exp(-(t-6)^2)}{100}\sin(t+10), \\ &\mathbb{K}\big(t,\rho(t)\big) = \hat{h}\big(t,\rho(t),\mathcal{I}^{2.24}\rho(t),\mathcal{I}^{3.21}\rho(t)\big), \end{aligned}$$

and we set $m = n = 2$, $k = 2.8$, $\theta_1 = 2.11$, $\theta_2 = 0.01$, $\varepsilon^* = 0.7$, $\delta_1 = 0.01$, $\delta_2 = 0.99$, $m_1^* = 0.06$, $\beta_1 = 0.01$, $\alpha_1 = 5.25$, $\alpha_2 = 8.56$, $\vartheta_1 = 0.5$, $\vartheta_2 = 11.12$, $k_1 = 2.24$, $k_2 = 0.21$, $K = 1.34$ and define

$$\begin{aligned} &\hat{h}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\mathcal{I}^{k_2}\rho(t)\big) = \frac{1}{\exp(t)+9}\left(1+\frac{|\rho(t)+\sum_{i=1}^{2}\mathcal{I}^{k_i}\rho(t)|}{|\rho(t)+\sum_{i=1}^{2}\mathcal{I}^{k_i}\rho(t)|+1}\right), \\ &\mathcal{H}\big(t,\rho(t),\mathcal{I}^{k_1}\rho(t),\mathcal{I}^{k_2}\rho(t)\big) = \frac{\exp(-t^2)}{(3+t)^2}\sin\big(\mathcal{I}^{k_1}\rho(t)+\mathcal{I}^{k_2}\rho(t)\big) + \frac{\exp(-t^2)\rho(t)}{(6+2t)^2} + \frac{10}{11}. \end{aligned}$$

We see that

$$\begin{aligned} &\big|\hat{h}\big(t,\rho_1(t),\rho_2(t),\rho_3(t)\big) - \hat{h}\big(t,\nu_1(t),\nu_2(t),\nu_3(t)\big)\big| \le \frac{1}{10}\left[\sum_{j=1}^{3}\big|\rho_j(t)-\nu_j(t)\big|\right], \\ &\big|\mathcal{H}\big(t,\rho_1(t),\rho_2(t),\rho_3(t)\big) - \mathcal{H}\big(t,\nu_1(t),\nu_2(t),\nu_3(t)\big)\big| \le \frac{\exp(-t^2)}{(3+t)^2}\left[\sum_{j=1}^{3}\big|\rho_j(t)-\nu_j(t)\big|\right], \end{aligned}$$

and we obtain

$$L^* = \frac{1}{10}, \quad \text{and} \quad \Psi(t) = \frac{\exp(-t^2)}{(3+t)^2}.$$

Then $\|\Psi\| = \frac{1}{9}$ and $\|\mathcal{A}\| = \frac{\exp(-(K-4)^4)}{100}$ and

$$\|\mathcal{A}\| \sum_{i=1}^{n} \frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)} \left(L^* \sum_{i=0}^{n} \frac{K^{k_i}}{\Gamma(k_i+1)} \mathcal{W} + \mathcal{V} \right) + \|\Psi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{T^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)} \right) \simeq 0.3520 < 1.$$

The conditions of Theorem 4.6 imply that the aforementioned problem (31) is Ulam–Hyers stable and also accordingly is generalized Ulam–Hyers stable.

Example 5.3 We again take the same above example by changing the function $\hat{\hbar}$ as the form

$$\hat{\hbar}\big(t, \rho(t), \mathcal{I}^{k_1}\rho(t), \mathcal{I}^{k_2}\rho(t)\big) = \frac{1}{t^2+5} \left(\frac{(|\rho(t)| + |\mathcal{I}^{k_1}\rho(t)| + |\mathcal{I}^{k_2}\rho(t)|)|\rho(t)|}{|\rho(t)|+1} + 3 \right). \tag{32}$$

Then we have

$$\big|\hat{\hbar}\big(t, \rho(t), \mathcal{I}^{k_1}\rho(t), \mathcal{I}^{k_2}\rho(t)\big)\big| \le \frac{1}{t^2+5} \big(|\rho(t)| + |\mathcal{I}^{k_1}\rho(t)| + |\mathcal{I}^{k_2}\rho(t)| + 3 \big).$$

Put $\mathcal{P}(t) = \frac{1}{t^2+5}$ and $\Xi_1(|\rho|) = |\rho| + 1$, $\Xi_2(|\mathcal{I}^{k_1}\rho|) = |\mathcal{I}^{k_1}\rho| + 1$ and $\Xi_2(|\mathcal{I}^{k_2}\rho|) = |\mathcal{I}^{k_2}\rho| + 1$. Select $\mathcal{Q} > 2.5876$ so that

$$\mathcal{Q} > \frac{\sum_{i=1}^{m} \frac{K^{\theta_i}}{\Gamma(\theta_i+1)} \mathcal{H}_0 + \|\mathcal{A}\| \sum_{i=1}^{n} \frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)} (\mathfrak{M} + \|\mathcal{P}\| \sum_{i=0}^{n} \Xi_i(\frac{K^{k_i}}{\Gamma(k_i+1)} \mathcal{Q}) \mathcal{W})}{1 - \|\Psi\| \sum_{i=1}^{n} (\sum_{j=0}^{n} \frac{K^{\vartheta_i+k_j}}{\Gamma(\vartheta_i+k_j+1)}) - \|\mathcal{A}\| \sum_{i=1}^{n} \frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)} \mathcal{V}}.$$

By defining $g(t) = 2\exp(\frac{t+2}{3})^2$ and $\mathcal{Q} = 3$, we reach an inequality $2\mathcal{Q} \le g(t)$ for any $t \in [0, 1.34]$. Now, we set $\varphi(t) = \exp(\frac{t+2}{3})^2$ and we obtain $c_\varphi = \|\mathcal{A}\| \sum_{i=1}^{n} \frac{K^{\alpha_i}}{\Gamma(\alpha_i+1)} \mathcal{W} + 1 + \lambda_\varphi > 0$. Hence, Theorem 4.8 implies that the multiterm hybrid fractional boundary value problem (31) with $\hat{\hbar}$ defined in (32) is Ulam–Hyers–Rassias stable and also accordingly is generalized Ulam–Hyers–Rassias stable on $[0, 1.34]$ for $\varepsilon = 1$.

6 Conclusion

The existence results for the proposed multiterm hybrid fractional boundary value problem that involves the Riemann–Liouville operators of finitely many orders have been successfully investigated. With the help of three operators having specific properties, we implemented the defined method in Dhage's technique for ensuring the existence of solutions. The stability criteria in different versions are checked for a special case. Some relevant numerical examples are provided to validate our obtained theoretical results. The supposed hybrid fractional boundary value problem (3) is thoroughly abstract and general but involves some special formats by assuming some specific parameters. One can extend

it to the differential inclusion by terms of multi-valued version of Dhage's technique in future work. In the next work, one can use generalized fractional operators with singular or non-singular kernels to model real hybrid systems such as the thermostat equation, the pantograph equation, and the Langevin equation, and to analyze their qualitative behaviors theoretically and numerically.

Acknowledgements
The fourth and fifth authors were supported by Azarbaijan Shahid Madani University. The authors express their gratitude to the unknown referees for their helpful suggestions, which improved the final version of this paper.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Publisher's Note

Received: 1 May 2021 Accepted: 8 July 2021 Published online: 21 July 2021

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