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Solvability for generalized nonlinear two dimensional functional integral equations via measure of noncompactness

Soniya Singh[1], Bhupander Singh[2], Kottakkaran Sooppy Nisar[3*], Abd-Allah Hyder[4,5] and M. Zakarya[4,6]

*Correspondence:
n.sooppy@psau.edu.sa
[3]Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this article, we provide the existence result for functional integral equations by using Petryshyn's fixed point theorem connecting the measure of noncompactness in a Banach space. The results enlarge the corresponding results of several authors. We present fascinating examples of equations.

MSC: 45A05; 45H05

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1 Introduction

FIEs play a very significant role in many areas of fixed point theory, and they have many applications in various areas of mathematical physics, engineering, mathematical biology, population dynamics, natural science, and mechanics (see [1, 7, 15, 19, 20, 26, 33]). It has been seen that integral equations have a large number of applications to finding the existence solution of integro-differential equations, differential equations, and fractional differential equations. Recently, many authors have used the MNC technique associated with Darbo's fixed point theorem [3] to examine the existence and uniqueness results of various types of FIEs. The details of this type of work can be found in these articles (see [4–6, 8, 9, 11–14, 17, 18, 24, 25, 30, 32, 34, 35] and the references therein).

In this work, we use Petryshyn's fixed point theorem [29] instead of Darbo's fixed point theorem to establish the existence of solutions for the following FIE:

$$z(s,\zeta) = G(s,\zeta) + F\Big(s,\zeta,f\big(s,\zeta,z(s,\zeta)\big), \int_0^s \int_0^\zeta g\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi, \int_0^c \int_0^d h\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi\Big), \tag{1}$$

where $(s,\zeta) \in I = [0,c] \times [0,d]$. Recently several authors used Petryshyn's fixed point theorem to find the existence of solutions for nonlinear FIEs in Banach spaces as well as Banach algebra (for instance see [10, 21, 22, 31] and the references therein). The following statements explain the main causes why we use equation (1) and what is the perfection of our work. The first is that the conditions in various papers will be analyzed, and the second reason is that this paper unifies the relevant work in this area. The third condition is the bounded condition shows that the "sublinear condition" that has been discussed in several literature works does not have a significant role.

The paper is divided into five sections including the introduction. In Sect. 2, we present some preliminaries and define the concept of MNC. Section 3 states and proves an existence result for equations including condensing operators using Petryshyn's fixed point theorem. In Sect. 4 we give examples that test the utilization of this kind of FIE. Finally, Sect. 5 concludes the paper.

2 Preliminaries

In this work, X is a real Banach space and $B_{\tilde{r}}$ denotes closed ball center at 0 with radius $\tilde{r}$ and $\partial B_r = \{z \in X : \|z\| = \tilde{r}\}$ for the sphere in X around 0 with radius $\tilde{r} > 0$. MNCs are valuable tools in the analysis of existence in the operator equations and theory of fixed point in X.

Definition 2.1 ([23]) Let $Y \in M_X$ and

$$\mu(Y) = \inf\left\{\epsilon > 0 : Y = \bigcup_{i=1}^{n} Y_i \text{ with } \operatorname{diam} Y_i \le \epsilon, i = 1, 2, \dots, n\right\}.$$

Hence, $0 \le \vartheta(Y) < \infty$. $\vartheta(Y)$ is called the Kuratowski MNC.

Definition 2.2 ([16]) The Hausdorff MNC

$$\vartheta(Y) = \inf\{\epsilon > 0 : \text{there exists a finite } \epsilon\text{-net for Y in } X\}, \tag{2}$$

where from a finite ϵ-net for Y in X that means a set $\{z_1, z_2, \dots, z_n\} \subset X$ such that the ball $B_\epsilon(X, z_1), B_\epsilon(X, z_2), \dots, B_\epsilon(X, z_n)$ over Y. These MNCs are mutually equivalent in the sense that

$$\vartheta(Y) \le \hat{\beta}(Y) \le 2\vartheta(Y)$$

for a bounded set $Y \subset X$.

Theorem 2.1 *Let $Y, \hat{Y} \in M_X$ and $\lambda \in \mathbb{R}$. Then*

(i) $\vartheta(Y) = 0$ *if and only if* $Y \in M_X$;
(ii) $Y \subseteq \hat{Y}$ *implies* $\vartheta(Y) \le \vartheta(\hat{Y})$;
(iii) $\vartheta(\operatorname{Conv} Y) = \vartheta(Y)$;
(iv) $\vartheta(Y \cup \hat{Y}) = \max\{\vartheta(Y), \vartheta(\hat{Y})\}$;
(v) $\vartheta(\lambda Y) = |\lambda| \vartheta(Y)$;
(vi) $\vartheta(Y + \hat{Y}) \le \vartheta(Y) + \vartheta(\hat{Y})$.

Here, we consider the Banach space $C(I, \mathbb{R})$ with the usual norm

$$\|z\| = \max\left\{\left|z(s, \zeta)\right| : (s, \zeta) \in I\right\}.$$

Let $X \in C(I, \mathbb{R})$. Given $\epsilon > 0$, the modulus of continuity of $z \in Y$ is defined as

$$\omega(z, \epsilon) = \sup\{|z(s, \zeta) - z(\hat{s}, \hat{\zeta})| : s, \hat{s} \in [0, c], \zeta, \hat{\zeta} \in [0, d], |s - \hat{s}| \leq \epsilon, |\zeta - \hat{\zeta}| \leq \epsilon\}.$$

Further

$$\omega(Y, \epsilon) = \sup\{\omega(z, \epsilon) : z \in Y\}, \qquad \omega_0(Y) = \lim_{\epsilon \to 0} \omega(Y, \epsilon).$$

Theorem 2.2 ([21]) *The Hausdorff MNC is similar to*

$$\mu(Y) = \lim_{\epsilon \to 0} \sup \omega(z, \epsilon) \tag{3}$$

for all bounded set $Y \subset C(I, \mathbb{R})$.

Theorem 2.3 ([27]) *Let* $H : X \to X$ *be a continuous mapping of* X. H *is called a* k *set contraction if, for all* $D \subset X$ *with* D *bounded,* $H(D)$ *is bounded and* $\hat{\beta}(HD) \leq k\hat{\beta}(D)$, $k \in (0, 1)$. *If* $\hat{\beta}(HD) < \hat{\beta}(D)$ *for all* $\hat{\beta}(D) > 0$, *then* H *is called densifying or condensing map.*

Theorem 2.4 ([29]) *Let* $H : B_{\tilde{r}} \to X$ *be a condensing function which fulfills the boundary condition if* $H(z) = kz$ *for some* $z \in \partial B_r$, *then* $k \leq 1$. *Then* $F(H)$ *in* $B_{\tilde{r}}$ *is nonempty, where* $F(H)$ *is the set of fixed points of* H.

3 Main results

Now, we study the main aim of equation (1). Namely, we assume the following assumptions:

(1) $G \in C(I, \mathbb{R})$, $F \in C(I_1 \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f \in C(I \times \mathbb{R}, \mathbb{R})$, $g, h \in C(I_2 \times \mathbb{R}, \mathbb{R})$, where

$$I = I_c \times I_d, \qquad I_1 = \{(s, \zeta, f) : 0 \leq s \leq c, 0 \leq \zeta \leq d, \xi \in \mathbb{R}\},$$
$$I_2 = \{(s, t, \xi, \eta) \in I^2 : 0 \leq \xi \leq s \leq c, 0 \leq \eta \leq \zeta \leq d\};$$

(2) There exist nonnegative constants $k_1, k_2, k_3, k_4, k_1k_4 < 1$ such that

$$|F(s, \zeta, z, u, x) - F(s, \zeta, \hat{z}, \hat{u}, \hat{x}| \leq k_1|z - \hat{z}| + k_2|u - \hat{u}| + k_3|x - \hat{x}|;$$
$$|f(s, \zeta, z) - f(s, \zeta, \hat{z}| \leq k_4|z - \hat{z}|;$$

(3) There exists $\tilde{r} > 0$ such that the resulting bounded condition is fulfilled

$$\sup\{|G(s, \zeta) : (s, \zeta) \in I| + |F(s, \zeta, z, u, x)| : (s, \zeta) \in I, z \in [-\tilde{r}, \tilde{r}],$$
$$u \in [-cdM_1, cdM_1], x \in [-cdM_2, cdM_2]\} \leq \tilde{r},$$

where

$$M_1 = \sup\{|g(s, \zeta, \xi, \eta, z)| : \text{for all } (s, \zeta, \xi, \eta) \in I_2 \text{ and } z \in [-\tilde{r}, \tilde{r}]\},$$
$$M_2 = \sup\{|h(s, \zeta, \xi, \eta, z)| : \text{for all } (s, \zeta, \xi, \eta) \in I_2 \text{ and } z \in [-\tilde{r}, \tilde{r}]\}.$$

Theorem 3.1 *Under assumptions* (1)–(3) *with* $k_1k_4 < 1$, *equation* (1) *has at least one solution in* X.

Proof Define $H : B_{\tilde{r}} \to X$ in the following form:

$$(Hz)(s,\zeta) = G(s,\zeta) + F\biggl(s,\zeta,f\bigl(s,\zeta,z(s,\zeta)\bigr), \int_0^s \int_0^\zeta g\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \int_0^c \int_0^d h\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr).$$

Now, we show that H is continuous on the ball $B_{\tilde{r}}$. Take $\epsilon > 0$ and $z,x \in B_{\tilde{r}}$ such that $\|z - x\| < \epsilon$. We get

$$\begin{aligned}
&\bigl|(Hz)(s,\zeta) - (Hx)(s,\zeta)\bigr| \\
&\quad = \biggl|G(s,\zeta) + F\biggl(s,\zeta,f\bigl(s,\zeta,z(s,\zeta)\bigr), \int_0^s \int_0^\zeta g\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\qquad \int_0^c \int_0^d h\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr) \\
&\qquad - G(s,\zeta) - F\biggl(s,\zeta,f\bigl(s,\zeta,x(s,\zeta)\bigr), \int_0^s \int_0^\zeta g\bigl(s,\zeta,\xi,\eta,x(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\qquad \int_0^c \int_0^d h\bigl(s,\zeta,\xi,\eta,x(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr)\biggr| \\
&\quad \le k_1\bigl|f\bigl(s,\zeta,z(s,\zeta)\bigr) - f\bigl(s,\zeta,x(s,\zeta)\bigr)\bigr| \\
&\qquad + k_2 \int_0^s \int_0^\zeta \bigl|g\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr) - g\bigl(s,\zeta,\xi,\eta,x(\xi,\eta)\bigr)\bigr|\,d\eta\,d\xi \\
&\qquad + k_3 \int_0^c \int_0^d \bigl|h\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr) - h\bigl(s,\zeta,\xi,\eta,x(\xi,\eta)\bigr)\bigr|\,d\eta\,d\xi \\
&\quad \le k_1k_4\bigl|z(s,\zeta) - x(s,\zeta)\bigr| + k_2cd\omega(g,\epsilon) + k_3cd\omega(h,\epsilon) \\
&\quad \le k_1k_4\|z - x\| + k_2cd\omega(g,\epsilon) + k_3cd\omega(h,\epsilon),
\end{aligned}$$

where, for $\epsilon > 0$, we denote

$$\omega(g,\epsilon) = \sup\bigl\{\bigl|g(s,\zeta,\xi,\eta,z) - g(s,\zeta,\xi,\eta,x)\bigr| : (s,\zeta,\xi,\eta) \in I_2, z,x \in [-\tilde{r},\tilde{r}], \|z - x\| \le \epsilon\bigr\},$$

$$\omega(h,\epsilon) = \sup\bigl\{\bigl|h(s,\zeta,\xi,\eta,z) - h(s,\zeta,\xi,\eta,x)\bigr| : (s,\zeta,\xi,\eta) \in I_2, z,x \in [-\tilde{r},\tilde{r}], \|z - x\| \le \epsilon\bigr\}.$$

Now, from the uniform continuity of $g(s,\zeta,\xi,\eta,z)$ and $h(s,\zeta,\xi,\eta,z)$ on $I_2 \times [-\epsilon,\epsilon]$ respectively, then $\omega(g,\epsilon)$ and $\omega(h,\epsilon)$ as $\epsilon \to 0$. Hence, we decide that H is continuous on $B_{\tilde{r}}$.

Next, we prove that H fulfills the densifying condition. Select $\epsilon > 0$ and take $z \in Y$, where Y is a bounded subset of X, $(s_1,\zeta_1),(s_2,\zeta_2) \in I$ with $s_1 \le s_2$, $\zeta_1 \le \zeta_2$ such that $s_1 - s_2 \le \epsilon$, $\zeta_1 - \zeta_2 \le \epsilon$, we obtain

$$\begin{aligned}
&\bigl|(Hz)(s_2,\zeta_2) - (Hz)(s_1,\zeta_1)\bigr| \\
&\quad = \biggl|G(s_2,\zeta_2) + F\biggl(s_2,\zeta_2,f\bigl(s_2,\zeta_2,z(s_2,\zeta_2)\bigr), \int_0^{s_2} \int_0^{\zeta_2} g\bigl(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi,
\end{aligned}$$

$$
\begin{aligned}
&\int_0^c \int_0^d h\bigl(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr) \\
&- G(s_1,\zeta_1) - F\biggl(s_1,\zeta_1,f\bigl(s_1,\zeta_1,z(s_1,\zeta_1)\bigr),\int_0^{s_1}\int_0^{\zeta_1} g\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\int_0^c \int_0^d h\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr)\biggr| \\
\le{}& \omega_1(G,\epsilon) + \biggl|F\biggl(s_2,\zeta_2,f\bigl(s_2,\zeta_2,z(s_2,\zeta_2)\bigr),\int_0^{s_2}\int_0^{\zeta_2} g\bigl(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\int_0^c \int_0^d h\bigl(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr) \\
&- F\biggl(s_2,\zeta_2,f\bigl(s_2,\zeta_2,z(s_2,\zeta_2)\bigr),\int_0^{s_2}\int_0^{\zeta_2} g\bigl(s_2,\zeta_2,u,\xi,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\int_0^c \int_0^d h\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr)\biggr| \\
&+ \biggl|F\biggl(s_2,\zeta_2,f\bigl(s_2,\zeta_2,z(s_2,\zeta_2)\bigr),\int_0^{s_2}\int_0^{\zeta_2} g\bigl(s_2,\zeta_2,u,\xi,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\int_0^c \int_0^d h\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr)|) \\
&- F\biggl(s_2,\zeta_2,f\bigl(s_2,\zeta_2,z(s_2,\zeta_2)\bigr),\int_0^{s_1}\int_0^{\zeta_1} g\bigl(s_1,\zeta_1,u,\xi,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\int_0^c \int_0^d h\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr| \\
&+ \biggl|F\biggl(s_2,\zeta_2,f\bigl(s_2,\zeta_2,z(s_2,\zeta_2)\bigr),\int_0^{s_1}\int_0^{\zeta_1} g\bigl(s_1,\zeta_1,u,\xi,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\int_0^c \int_0^d h\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr) \\
&- F\biggl(s_2,\zeta_2,f\bigl(s_1,\zeta_1,z(s_1,\zeta_1)\bigr),\int_0^{s_1}\int_0^{\zeta_1} g\bigl(s_1,\zeta_1,u,\xi,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\int_0^c \int_0^d h\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr)\biggr| \\
&+ \biggl|F\biggl(s_2,\zeta_2,f\bigl(s_1,\zeta_1,z(s_1,\zeta_1)\bigr),\int_0^{s_1}\int_0^{\zeta_1} g\bigl(s_1,\zeta_1,u,\xi,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\int_0^c \int_0^d h\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr) \\
&- F\biggl(s_1,\zeta_1,f\bigl(s_1,\zeta_1,z(s_1,\zeta_1)\bigr),\int_0^{s_1}\int_0^{\zeta_1} g\bigl(s_1,\zeta_1,u,\xi,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \\
&\int_0^c \int_0^d h\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr)\biggr| \\
\le{}& k_3\biggl|\int_0^c \int_0^d h\bigl(s_2,\zeta_2,u,\xi,z(u,\xi)\bigr)\,d\eta\,d\xi - \int_0^c \int_0^d h\bigl(s_1,\zeta_1,u,\xi,z(u,\xi)\bigr)\,d\eta\,d\xi\biggr| \\
&+ k_2\biggl|\int_0^{s_2}\int_0^{\zeta_2} g\bigl(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi - \int_0^{s_1}\int_0^{t_1} g\bigl(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr|
\end{aligned}
$$

$$
\begin{aligned}
&+ k_1\big|f\big(s_2,\zeta_2,z(s_2,\zeta_2)\big) - f\big(s_2,\zeta_2,z(s_1,\eta_1)\big)\big| + k_1\big|f\big(s_2,\zeta_2,z(s_1,\zeta_1)\big) \\
&- s\big(s_1,\zeta_1,z(s_1,\zeta_1)\big)\big| + \omega_1(G,\epsilon) + \omega_1(F,\epsilon) \\
\leq{}& k_3\int_0^c\int_0^d \big|h\big(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\big) - h\big(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\big)\big|\,d\eta\,d\xi \\
&+ k_2\int_0^{s_1}\int_0^{\zeta_1} \big|g\big(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\big) - g\big(s_1,\zeta_1,\xi,\eta,z(\xi,\eta)\big)\big|\,d\eta\,d\xi \\
&+ k_2\int_{s_1}^{s_2}\int_0^{\zeta_1} \big|g\big(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\big)\big|\,d\eta\,d\xi + \omega_1(G,\epsilon) + \omega_1(F,\epsilon) \\
&+ k_2\int_0^{s_1}\int_{\zeta_1}^{\zeta_2} \big|g\big(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\big)\big|\,d\eta\,d\xi + k_1k_4\big|z(s_2,\zeta_2) - z(s_1,\zeta_1)\big| \\
&+ k_2\int_{s_1}^{s_2}\int_{\zeta_1}^{\zeta_2} \big|g\big(s_2,\zeta_2,\xi,\eta,z(\xi,\eta)\big)\big|\,d\eta\,d\xi + k_1\omega_1(f,\epsilon),
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_1(f,\epsilon) &= \sup\big\{\big|f(s,\zeta,z) - f(\hat{s},\hat{\zeta},z)\big| : |s-\hat{s}| \leq \epsilon, |\zeta - \hat{\zeta}| \leq \epsilon, z \in [-\tilde{r},\tilde{r}]\big\}, \\
\omega_1(g,\epsilon) &= \sup\big\{\big|g(s,\zeta,\xi,\eta,z) - g(\hat{s},\hat{\zeta},\xi,\eta,z)\big| : |s-\hat{s}| \leq \epsilon, |\zeta - \hat{\zeta}| \leq \epsilon, \\
&\qquad (s,\zeta,\xi,\eta) \in I_2, z \in [-\tilde{r},\tilde{r}]\big\}, \\
\omega_1(h,\epsilon) &= \sup\big\{\big|h(s,\zeta,\xi,\eta,z) - h(\hat{s},\hat{\zeta},\xi,\eta,z)\big| : |s-\hat{s}| \leq \epsilon, |\zeta - \hat{\zeta}| \leq \epsilon, \\
&\qquad (s,\zeta,\xi,\eta) \in I_2, z \in [-\tilde{r},\tilde{r}]\big\}, \\
\omega_1(F,\epsilon) &= \sup\big\{\big|F(s,\zeta,z,u,x) - s(\hat{s},\hat{\zeta},z,u,x)\big| : |s-\hat{s}| \leq \epsilon, |\zeta - \hat{\zeta}| \leq \epsilon, z_1 \in [-\tilde{r},\tilde{r}], \\
&\qquad u \in [-cdM_1, cdM_1], x \in [-cdM_2, cdM_2]\big\}.
\end{aligned}
$$

Then, using the above relation, we get

$$
\begin{aligned}
&\big|(Hz)(s_2,\zeta_2) - (Hz)(s_1,\zeta_1)\big| \\
&\quad\leq k_1k_4\big|z(s_2,\nu_2) - z(s_1,\zeta_1)\big| + k_1\omega_1(f,\epsilon) + \omega_1(F,\epsilon) \\
&\qquad + k_3c\,d\omega_1(h,\epsilon) + k_2c\,d\omega(g,\epsilon) + \epsilon k_2\,dM_1 + \epsilon k_2cM_1 + \epsilon^2k_2M_1.
\end{aligned}
$$

Applying limit as $\delta \to 0$,

$$
\omega(Hz,\epsilon) \leq k_1k_4\omega(z,\epsilon).
$$

This gives the following relation:

$$
\vartheta(HY) \leq k_1k_4\vartheta(Y),
$$

hence H is a condensing map. Now, let $z \in \partial B_{\tilde{r}}$, and if $Hz = kz$, then $\|Hz\| = k\|z\| = k\tilde{r}$, and by (3), we obtain

$$
\big|Hz(s,\zeta)\big| = G(s,\zeta) + F\bigg(s,\zeta,f\big(s,\zeta,z(s,\zeta)\big), \int_0^s\int_0^\zeta g\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi,
$$

$$\int_0^c \int_0^d h\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi\bigg)$$
$$\le r$$

for all $(s,\zeta) \in I$. Hence $\|Hz\| \le \tilde{r}$ i.e. $k \le 1$. □

Corollary 3.2 *Let*

(1) $G \in C(I,\mathbb{R})$, $F \in C(I_1 \times \mathbb{R} \times \mathbb{R},\mathbb{R})$, $g,h \in C(I_2 \times \mathbb{R},\mathbb{R})$, *where*

$$I = I_c \times I_d, \qquad I_1 = \big\{(s,\zeta,z) : 0 \le s \le c, 0 \le \zeta \le d, s \in \mathbb{R}\big\},$$
$$I_2 = \big\{(s,\zeta,\xi,\eta) \in I^2 : 0 \le \xi \le s \le c, 0 \le \eta \le \zeta \le d\big\};$$

(2) *There exist nonnegative constants* $k_1,k_2,k_3,k_4 \in (0,1)$ *such that*

$$\big|F(s,\zeta,z,u,x) - F(s,\zeta,\hat{z},\hat{u},\hat{x}\big| \le k_1|z-\hat{z}| + k_2|u-\hat{u}| + k_3|x-\hat{u}|;$$

(3) *There exists* $\tilde{r} > 0$ *such that resulting bounded fulfills*

$$\sup\big\{\big|G(s,\zeta) : (s,\zeta) \in I\big| + \big|F(s,\zeta,z_1,z_2,z_3)\big| : (s,\zeta) \in I, z_1 \in [-\tilde{r},\tilde{r}],$$
$$z_2 \in [-cdM_1, cdM_1], z_3 \in [-cdM_2, cdM_2]\big\} \le r,$$

here

$$M_1 = \sup\big\{\big|g(s,\zeta,\xi,\eta,z)\big| : \textit{for all } (s,\zeta,\xi,\eta) \in I_2 \textit{ and } z \in [-\tilde{r},\tilde{r}]\big\},$$
$$M_2 = \sup\big\{\big|h(s,\zeta,\xi,\eta,z)\big| : \textit{for all } (s,\zeta,\xi,\eta) \in I_2 \textit{ and } z \in [-\tilde{r},\tilde{r}]\big\}.$$

Then

$$z(s,\zeta) = G(s,\zeta) + F\bigg(s,\zeta,z(s,\zeta), \int_0^s \int_0^\zeta g\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi,$$
$$\int_0^c \int_0^d h\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi\bigg), \tag{4}$$

has at least one solution in X.

Proof The proof is linked to the beginning Theorem 3.1 and the details that follow. □

Corollary 3.3 *Let*

(S_1) $F \in C(I \times \mathbb{R} \times \mathbb{R},\mathbb{R})$, $f \in C(I_1,\mathbb{R})$, $g \in C(I_2 \times \mathbb{R},\mathbb{R})$, $h \in C(I_2 \times \mathbb{R},\mathbb{R})$;

(S_2) *There exist nonnegative constants* μ *and* ν *such that*

$$\big|f(s,\zeta,0)\big| \le \mu; \qquad \big|F(s,\zeta,0,0)\big| \le \nu;$$

(S_3) *There exist nonnegative constants* $k_1,k_2,k_3 \in (0,1)$ *such that*

$$\big|f(s,\zeta,z) - f(s,\zeta,\hat{z}\big| \le k_1|z-\hat{z}|$$
$$\big|F(s,\zeta,z,u) - F(s,\zeta,\hat{z},\hat{u}\big| \le k_2|z-\hat{z}| + k_3|u-\hat{u}|;$$

(S_4) *There exist nonnegative constants c_1, c_2, d_1, and d_2 such that*

$$\left|g(s,\zeta,\xi,\eta,z)\right| \leq c_1 + c_2|z|, \qquad \left|h(s,\zeta,\xi,\eta,z)\right| \leq d_1 + d_2|z|;$$

(S_5) $k_1 + k_2cdc_2 + k_3cdd_2 < 1$.

Then the equation

$$z(s,\zeta) = f\bigl(s,\zeta,z(s,\zeta)\bigr) + F\biggl(s,\zeta,\int_0^s\int_0^\zeta g\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi, \int_0^c\int_0^d h\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr) \tag{5}$$

has at least one solution in X.

Proof Let $\tilde{r} = \frac{N_2}{1-N_1}$, where $N_1 = k_1 + k_2cdc_2 + k_3cdd_2$, $N_2 = \mu + k_2cdc_1 + k_3cdd_1 + \nu$, and

$$G(s,\zeta) = 0, \qquad F(s,\zeta,z,u,x) = z + F(s,\zeta,u,x),$$

where

$$z = f\bigl(s,\zeta,z(s,\zeta)\bigr), \qquad u = \int_0^s\int_0^\zeta g\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi,$$
$$x = \int_0^c\int_0^d h\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi.$$

(2) is conducted by (S_2). Now, we show that (S_3) is also fulfilled, we have

$$\begin{aligned}
\left|z(s,\zeta)\right| &= \left|f\bigl(s,\zeta,z(s,\zeta)\bigr) + F\biggl(s,\zeta,\int_0^s\int_0^\zeta g\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi,\right.\\
&\qquad \left.\int_0^c\int_0^d h\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\biggr)\right|,\\
&\leq \left|f\bigl(s,\zeta,z(r,\zeta)\bigr) - f(s,\zeta,0)\right| + \left|f(s,\zeta,0)\right|\\
&\quad + k_2\left|\int_0^s\int_0^\zeta g\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\right|\\
&\quad + k_3\left|\int_0^c\int_0^d h\bigl(s,\zeta,\xi,\eta,z(\xi,\eta)\bigr)\,d\eta\,d\xi\right| + \left|F(s,\zeta,0,0)\right|,\\
&\leq k_1\|z\| + \mu + k_2cd\bigl(c_1 + c_2\|z\|\bigr) + k_3cd\bigl(d_1 + d_2\|z\|\bigr) + \nu,\\
&\leq (k_1 + k_2cdc_2 + k_3cdd_2)\|z\| + \mu + k_2cdc_1 + k_3cdd_1 + \nu
\end{aligned}$$

for all $(s,\zeta) \in I$; consequently,

$$\sup\left|F(s,\zeta,z,u,x)\right| \leq N_1r + N_2 = N_1\frac{N_2}{1-N_1} + N_2 = \tilde{r}. \qquad \square$$

Corollary 3.4 ([9]) *Let*

(E_1) $F \in C(I_1 \times \mathbb{R},\mathbb{R})$, $g \in C(I_2 \times \mathbb{R},\mathbb{R})$;

(E_2) *There exist nonnegative constants* m_1 *and* m_2 *such that* $|A(s,\zeta)| \leq m_1$; $|F(s,\zeta,0,0)| \leq m_2$;

(E_3) *There exist nonnegative constants* $k_1, k_2 \in (0,1)$ *such that*

$$|F(s,\zeta,z,u) - F(s,\zeta,\hat{z},\hat{u}| \leq k_1|z-\hat{z}| + k_2|u-\hat{u}|;$$

(E_4) *There exist nonnegative constants* h_1 *and* h_2 *such that* $|g(s,\zeta,\xi,\eta,z)| \leq h_1 + h_2|z|$;

(E_5) $k_1 + k_2cdh_2 < 1$.

Then the equation

$$z(s,\zeta) = A(s,\zeta) + F\left(s,\zeta,z(s,\zeta), \int_0^r \int_0^\zeta g\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta d\xi\right) \tag{6}$$

has at least one solution in X.

Proof Let $\tilde{r} = \frac{F_2}{1-F_1}$, where $F_1 = k_1 + k_2cdh_2$, $F_2 = k_2cdh_1 + m_2 + m_1$, and

$$F(s,\zeta,z,u,x) = F(s,\zeta,z,u),$$

where

$$z = z(s,\zeta), \qquad u = \int_0^s \int_0^t g\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi.$$

(T_2) is handled by (E_2). Now, we show that (E_3) is also fulfilled. We have

$$\begin{aligned}
\big|z(s,\zeta)\big| &= \left|A(s,\zeta) + F\left(s,\zeta,z(s,\zeta), \int_0^s \int_0^\zeta g\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi\right)\right|, \\
&\leq \left|F\left(s,\zeta,z(s,\zeta), \int_0^s \int_0^\zeta g\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi\right) - F(s,\zeta,0,0)\right| \\
&\quad + \big|F(s,\zeta,0,0)\big| + \big|A(s,\zeta)\big|, \\
&\leq k_1\big|z(s,\zeta)\big| + k_2\left|\int_0^s \int_0^\zeta g\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi\right| \\
&\quad + \big|F(s,\zeta,0,0)\big| + \big|A(s,\zeta)\big|, \\
&\leq k_1\|z\| + k_2cd\big(h_1 + h_2|z|\big) + m_2 + m_1, \\
&\leq (k_1 + k_2cdh_2)\|z\| + k_2cdh_1 + m_2 + m_1
\end{aligned}$$

for all $(s,\zeta) \in I$; consequently,

$$\sup\big|F(s,\zeta,z,u,x)\big| \leq F_1\tilde{r} + F_2 = F_1\frac{F_2}{1-F_1} + F_2 = \tilde{r}.$$

□

4 Applications

Example 4.1

$$z(s,\zeta) = g(s,\zeta) + \int_0^s \int_0^\zeta P(s,\zeta,\xi,\eta)Q\big(\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi$$

for $\nu = g(s,\zeta)$ and $h(s,\zeta,\xi,\eta,z(\xi,\eta)) = P(s,\zeta,\xi,\eta)Q(\xi,\eta,z(\xi,\eta))$, which may be regarded as a two dimensional generalization of the famous Hammerstein type FIE (see [28])

$$z(s,\zeta) = g(s,\zeta) + \int_0^1 \int_0^1 h\big(s,\zeta,\xi,\eta,z(\xi,\eta)\big)\,d\eta\,d\xi,$$

which is the famous two dimensional Fredholm FIE examined (e.g. [2]).

Example 4.2 Consider the following two dimensional-FIE:

$$\begin{aligned} z(s,\zeta) &= \frac{s^2}{2(1+s^2\zeta^2)}e^{-s^2\zeta} + \frac{1}{2}\left(\frac{1+s\zeta^2}{3+4s^2\zeta^2}\right)\cos z(s,\zeta) + \frac{1}{2}\int_0^s \int_0^\zeta \xi\eta^2 \cos z(\xi,\eta)\,d\eta\,d\xi \\ &\quad + \frac{1}{2}\int_0^1\int_0^1 \arctan\left(\frac{|z(\xi,\eta)|}{1+|z(\xi,\eta)|}\right)d\eta\,d\xi \end{aligned} \tag{7}$$

for $(s,\zeta) \in I = [0,1] \times [0,1]$. Here, we put

$$\begin{aligned} F(s,\zeta,z,u,\eta) &= \frac{1}{2}z_+\frac{1}{2}u + \frac{1}{2}\eta, \\ f(s,\zeta,z) &= \frac{1+s\zeta^2}{3+4s^2\zeta^2}\cos z(s,\zeta), \\ g(s,\zeta,\xi,\eta,z) &= \xi\eta^2\cos z(\xi,\eta), \\ h(s,\zeta,\xi,\eta,z) &= \arctan\left(\frac{|z(\xi,\eta)|}{1+|z(\xi,\eta)|}\right). \end{aligned}$$

It can clearly be noticed that F, f, g, h are continuous functions on the respective domain and

$$\begin{aligned} &\big|F(s,\zeta,z,u,x) - F(s,\zeta,\hat{z},\hat{u},\hat{x})\big| \le \frac{1}{2}|z-\hat{z}| + \frac{1}{2}|u-\hat{u}| + \frac{1}{2}|x-\hat{x}|, \\ &|f(s,\zeta,z) - f(s,\zeta,\hat{z}| \le \frac{1}{3}|z-\hat{z}|. \end{aligned}$$

Here, $k_1 = k_2 = k_3 = k_4 = \frac{1}{2}$. It is seen that these functions satisfy (1) and (2). Now, we check that (3) also holds. Take $r = 3$, then we get $M_1 = M_2 \le 1$ and

$$\begin{aligned} &\sup\big\{\big|G(s,\zeta) + F(s,\zeta,z,u,\eta)\big| : s,\zeta \in [0,1], z \in [-3,3], u,\eta \in [-1,1]\big\} \\ &\quad \le \sup\left|\left(\frac{s^2}{2(1+s^2\zeta^2)}e^{-s^2\zeta} + \frac{1+s\zeta^2}{2(3+2s^2\zeta^2)}\cos z(s,\zeta) + \frac{1}{2}\int_0^s\int_0^\zeta \xi\eta^2\cos z(\xi,\eta)\,d\eta\,d\xi \right.\right. \\ &\quad\quad \left.\left. + \frac{1}{2}\int_0^1\int_0^1 \arctan\left(\frac{|z(\xi,\eta)|}{1+|z(\xi,\eta)|}\right)d\eta\,d\xi\right)\right| \\ &\quad \le 3. \end{aligned}$$

All assumptions (1)–(3) are satisfied. Hence, by Theorem 3.1, equation (7) has at least one solution in $C(I)$.

5 Conclusion

By unifying and enlarging the earlier results of [9, 11, 18, 35] and using Petryshyn's fixed point Theorem 3.1, in the third section, we obtained a new method to prove the existence of solutions for some functional integral equations. The merit of Theorem 3.1 among the others (Darbo's and Schauder's fixed point theorems) lies in that in applying the theorem, one does not need to confirm that the involved operator maps a closed convex subset onto itself. For future work, the interested researchers can obtain the existence of solution of equation (1) in different Banach function spaces e.g. Sobolev space, Hölder space, etc.

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Author details
[1]Department of Applied Sciences and Engineering, IIT Roorkee, Roorkee 247667, India. [2]Department of Mathematics, Meerut College Meerut, Meerut 250001, India. [3]Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawaser, Saudi Arabia. [4]Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia. [5]Department of Engineering Mathematics and Physics, Faculty of Engineering, Al-Azhar University, Cairo, Egypt. [6]Department of Mathematics, Faculty of Science, Al-Azhar University, 71524, Assiut, Egypt.

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