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A new aspect of generalized integral operator and an estimation in a generalized function theory

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Abstract

In this paper we investigate certain integral operator involving Jacobi–Dunkl functions in a class of generalized functions. We utilize convolution products, approximating identities, and several axioms to allocate the desired spaces of generalized functions. The existing theory of the Jacobi–Dunkl integral operator (Ben Salem and Ahmed Salem in Ramanujan J. 12(3):359–378, 2006) is extended and applied to a new addressed set of Boehmians. Various embeddings and characteristics of the extended Jacobi–Dunkl operator are discussed. An inversion formula and certain convergence with respect to δ and Δ convergences are also introduced.

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1 Introduction and preliminaries

We start with some background and notations from the Jacobi–Dunkl function theory, supplementing the material in the Introduction. We recapitulate some results related to the harmonic analysis associated with the Jacobi–Dunkl differential-difference operator $\Delta_{\alpha,\beta}$ and the Jacobi–Dunkl kernel function $\Psi_\lambda^{\alpha,\beta}$. We denote by $\mathbb{C},\mathbb{R}$, and $\mathbb{N}$ the sets of complex numbers, real numbers, and positive integers, respectively. For $\alpha,\beta\in\mathbb{R},\alpha\geq\beta\geq-\frac{1}{2}$, and $\alpha\neq-\frac{1}{2}$, we denote by $\Delta_{\alpha,\beta}$ the Jacobi–Dunkl differential-difference operator defined by [1]

$$\Delta_{\alpha,\beta}\psi(\zeta)=\acute{\psi}(\zeta)+\bigl((2\alpha+1)\coth\zeta+(2\beta+1)\tanh\zeta\bigr)\biggl(\frac{\psi(\zeta)-\psi(-\zeta)}{2}\biggr). \tag{1}$$

For $\lambda^2=\mu^2+\rho^2,\lambda\in\mathbb{C},\zeta\in\mathbb{R}$, and $\rho=\alpha+\beta+1$, we denote by $\Psi_\lambda^{\alpha,\beta}$ the Jacobi–Dunkl kernel function [1]

$$\Psi_\lambda^{\alpha,\beta}(\zeta)=\varphi_\mu^{\alpha,\beta}(\zeta)+i\frac{\lambda}{2(\alpha+1)}\sinh\zeta\cosh\zeta\,\varphi_\mu^{\alpha+1,\beta+1}(\zeta). \tag{2}$$

Then the Jacobi–Dunkl kernel function $\Psi_\lambda^{\alpha,\beta}$ is the C^∞-solution of the differential-difference equation

$$\left.\begin{array}{l}\Delta_{\alpha,\beta}\Psi_\lambda^{\alpha,\beta} = i\lambda\Psi_\lambda^{\alpha,\beta}, \quad \lambda\in\mathbb{C},\\ \Psi_\lambda^{\alpha,\beta}(0)=1,\end{array}\right\}$$

where $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function defined by

$$\varphi_\mu^{\alpha,\beta}(\zeta) = F\left(\frac{\rho+i\mu}{2}, \frac{\rho-i\mu}{2}, \alpha+1, -(\sinh\zeta)^2\right), \tag{3}$$

F being a Gauss hypergeometric function. For $\alpha \geq \beta \geq -\frac{1}{2}$ and $\lambda\in\mathbb{C}$, the function $\Psi_\lambda^{\alpha,\beta}$ is an eigenfunction of the differential-difference operator $\Delta_{\alpha,\beta}$ that satisfies the product formula [1, (2.10)]

$$\Psi_\lambda^{\alpha,\beta}(\zeta)\Psi_\lambda^{\alpha,\beta}(y) = \int_{-\infty}^{\infty}\Psi_\lambda^{\alpha,\beta}(u)\,d\mu_{\zeta,y}^{\alpha,\beta}(u),$$

where

$$d\mu_{\zeta,y}^{\alpha,\beta}(u)\begin{cases}K_{\alpha,\beta}(\zeta,y,u)X_{\alpha,\beta}(u)\,du, & \zeta y\neq 0,\\ \delta_x, & y=0,\\ \delta_y, & \zeta=0,\end{cases}$$

and

$$K_{\alpha,\beta}(\zeta,y,u) = M_{\alpha,\beta}\left(\sinh|\zeta|\sinh|y|\sinh|u|\right)^{-2\alpha}1_{I_{\zeta,y}}(u) \times\int_0^\pi \gamma^z(\zeta,y,u)g(\zeta,y,u,z)_\zeta^{\alpha-\beta-1}\sin^{2\beta}z\,dz,$$

$1_{I_{\zeta,y}}$ being the indicator of $I_{\zeta,y} = [-|\zeta|-|y|, -||\zeta|-|y||U[||\zeta|-|y||, |\zeta|+|y|]]$, $\gamma^z(\zeta,y,u) = 1-\sigma_{\zeta,y,u}^z+\sigma_{u,y,\zeta}^z+\sigma_{u,\zeta,y}^z$, and

$$\sigma_{\zeta,y,u}^z = \begin{cases}\frac{-\cosh u\cosh\zeta-\cosh\zeta\cosh y}{\sinh\zeta\sinh y}, & \zeta y\neq 0,\\ 0, & \zeta y=0,\end{cases}$$

$M_{\alpha,\beta}$ being the classical function given by [1, (1.15)]. We denote by D the set of C^∞-functions whose supports over $\mathbb{R}$ are bounded. By $L^1_{\alpha,\beta}(\mathbb{R}, X_{\alpha,\beta}(\zeta)\,d\zeta)$, or $L^1_{\alpha,\beta}$, we denote the measurable space of functions over $\mathbb{R}$ that satisfies the norms [2]

$$\|\psi\|_{\alpha,\beta} = \int_{-\infty}^{\infty}\left|\psi(\zeta)\right|X_{\alpha,\beta}(\zeta)\,d\zeta<\infty, \tag{4}$$

where

$$X_{\alpha,\beta}(\zeta) = 2^{2\rho}\left(\sinh|\zeta|\right)^{2\alpha+1}(\cosh\zeta)^{2\beta+1}, \quad \rho=\alpha+\beta+1. \tag{5}$$

The convolution product of the arbitrary functions ψ_1 and ψ_2 is defined by [1, Def. (3.1)]

$$\psi_1 *_{\alpha,\beta} \psi_2(\zeta) = \int_{-\infty}^{\infty} T^{\zeta}_{\alpha,\beta}(\psi_1)(-y)\psi_2(y)X_{\alpha,\beta}(y)\,dy, \tag{6}$$

where $T^{\zeta}_{\alpha,\beta}, \zeta \in \mathbb{R}$, is the transformation operator defined by [1, Def. (2.8)]

$$T^{\zeta}_{\alpha,\beta}\psi_1(y) = \mu^{\alpha,\beta}_{\zeta(y)}(\psi_1) = \int_{-\infty}^{\infty} \psi_1\,d\mu^{\alpha,\beta}_{\zeta(y)}. \tag{7}$$

However, the product $*_{\alpha,\beta}$ of the suitable functions ψ_1 and ψ_2 satisfies several results as follows [1, p. 375].

Proposition 1 *Let $\psi_1, \psi_2, \psi_3 \in L^1_{\alpha,\beta}(\mathbb{R}, X_{\alpha,\beta}(\zeta)\,d\zeta)$. Then the undermentioned relations hold true.*

(i) $\psi_1 *_{\alpha,\beta} \psi_2 = \psi_2 *_{\alpha,\beta} \psi_1$,

(ii) $(\psi_1 *_{\alpha,\beta} \psi_2) *_{\alpha,\beta} \psi_3 = \psi_1 *_{\alpha,\beta} (\psi_2 *_{\alpha,\beta} \psi_3)$.

Consequently, for $p = q = r = 1$, [1, Prop. (3.2)] leads to the following fruitful result.

Proposition 2 *Let $\psi_2, \psi_1 \in L^1_{\alpha,\beta}(\mathbb{R}, X_{\alpha,\beta}(\zeta)\,d\zeta)$. Then the following hold*:

(i) $T^{x}_{\alpha,\beta}$ *is defined a.e. on $\mathbb{R}$. Moreover, it is a member of $L^1_{\alpha,\beta}(\mathbb{R}, X_{\alpha,\beta}(\zeta)\,d\zeta)$ and*

$$\left\| T^{\zeta}_{\alpha,\beta}\psi_1 \right\|_{\alpha,\beta} \leq 4\|\psi_1\|_{\alpha,\beta}.$$

(ii) $\psi_1 *_{\alpha,\beta} \psi_2 \in L^1_{\alpha,\beta}(\mathbb{R}, X_{\alpha,\beta}(\zeta)\,d\zeta)$, *and*

$$\|\psi_1 *_{\alpha,\beta} \psi_2\|_{\alpha,\beta} \leq 4\|\psi_1\|_{\alpha,\beta}\|\psi_2\|_{\alpha,\beta}.$$

The Jacobi–Dunkl operator for a suitable function ψ_1 is defined over $\mathbb{R}$ by [1, Def. 3.3]

$$J^{d}_{\alpha,\beta}(\psi_1)(\lambda) = \int_{-\infty}^{\infty} \psi_1(\zeta)\Psi^{\alpha,\beta}_{-\lambda}(\zeta)X_{\alpha,\beta}(\zeta)\,d\zeta. \tag{8}$$

Moreover, for $\psi_2, \psi_1 \in L^1_{\alpha,\beta}(\mathbb{R}, X_{\alpha,\beta}(\zeta)\,d\zeta)$ and $\lambda \in \mathbb{R}$, Prop. (3.6) of [1, p. 376] reveals

$$J^{d}_{\alpha,\beta}(\psi_1 *_{\alpha,\beta} \psi_2)(\lambda) = J^{d}_{\alpha,\beta}(\psi_1)(\lambda)J^{d}_{\alpha,\beta}(\psi_2)(\lambda). \tag{9}$$

The Plancherel formula for the $J^{d}_{\alpha,\beta}$ transform is defined as

$$\int_{-\infty}^{\infty} \left|\psi_1(\zeta)\right|^2 X_{\alpha,\beta}(\zeta)\,d\zeta = \int_{-\infty}^{\infty} \left|J^{d}_{\alpha,\beta}(\psi_1)(\lambda)\right|^2 d\Pi_{\alpha,\beta}(\lambda),$$

where

$$d\Pi_{\alpha,\beta}(\lambda) = \frac{|\lambda|d\lambda}{8\pi\sqrt{\lambda^2 - p^2}|c_{\alpha,\beta}\sqrt{\lambda^2 - p^2}|} 1_{\mathbb{R}(-p,p)}(\lambda)$$

is the Plancherel measure [1, p. 376]. For more illustrations about this theory, readers are referred to [1–4, 41, 43] and [5–20] and the references cited therein. However, this research is organized in the following format. In Sect. 1, we present some definitions and results associated with the Jacobi–Dunkl function theory. In Sect. 2, we establish the generalized spaces $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ and $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times, \bar{\Delta})$. In Sect. 3, we extend the Jacobi–Dunkl function theory to the generalized spaces of generalized functions.

2 The spaces $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ and $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times, \bar{\Delta})$

The concepts of the Boehmian spaces are obtained by following an algebraic approach that hires convolutions and delta sequences, which are approximating identities. When the structure is allowed to be a function space and the space multiplication is interpreted as a convolution product, the new structure yields a space of Boehmians (see, e.g., [2, 21–35, 44] and [1, 8, 11, 33, 36–40, 42]). Let Δ be the set of all sequences (δ_n) in D such that the following properties hold:

$$\int_{-\infty}^{\infty} \delta_n(\zeta) X_{\alpha,\beta}(\zeta)\, d\zeta = 1, \tag{10}$$

$$\int_{-\infty}^{\infty} \left|\delta_n(\zeta)\right| X_{\alpha,\beta}(\zeta)\, d\zeta < M, \quad M \in \mathbb{R}, \tag{11}$$

$$\operatorname{supp}(\delta_n) \subset (0, a_n), \quad a_n \to 0 \text{ as } n \to \infty. \tag{12}$$

The following result shows that Δ is a set of delta sequences.

Lemma 3 *Let (δ_n) and (θ_n) be in Δ. Then $(\delta_n *_{\alpha,\beta} \theta_n)$ is in Δ.*

Proof Let (δ_n) and (θ_n) be in Δ. Then, to prove this lemma, we have to prove that Eqs. (10)–(12) hold for $(\delta_n *_{\alpha,\beta} \theta_n)$. By [1, Eq. (2.2)] we infer that $\psi_\lambda^{\alpha,\beta}(\zeta) = 1$ for $\lambda = 0$. Therefore, we obtain

$$J^d_{\alpha,\beta}(\psi_1)(0) = \int_{-\infty}^{\infty} \psi_1(\zeta) \Psi_0^{\alpha,\beta}(\zeta) X_{\alpha,\beta}(\zeta)\, d\zeta = \int_{-\infty}^{\infty} \psi_1(\zeta) X_{\alpha,\beta}(\zeta)\, d\zeta.$$

Hence, by the convolution theorem, Eq. (9) and Eq. (10) give

$$\begin{aligned} J^d_{\alpha,\beta}(\delta_n *_{\alpha,\beta} \theta_n)(0) &= J^d_{\alpha,\beta}(\delta_n)(0) J^d_{\alpha,\beta}(\theta_n)(0) \\ &= \int_{-\infty}^{\infty} \delta_n(\zeta) X_{\alpha,\beta}(\zeta)\, d\zeta \int_{-\infty}^{\infty} \theta_n(y) X_{\alpha,\beta}(y)\, dy \\ &= 1. \end{aligned}$$

This proves that Eq. (10) holds for $(\delta_n *_{\alpha,\beta} \theta_n)$. To show that Eq. (11) holds for $(\delta_n *_{\alpha,\beta} \theta_n)$, we use Proposition 2 to obtain

$$\|\delta_n *_{\alpha,\beta} \theta_n\|_{\alpha,\beta} \le 4 \|\delta_n\|_{\alpha,\beta} \|\theta_n\|_{\alpha,\beta} \le 4M^2.$$

Finally, the proof of the fact that $(\delta_n *_{\alpha,\beta} \theta_n)$ satisfies Eq. (12) is straightforward. The proof is therefore completed.

Hence, the necessary axioms for establishing the Boehmians space $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$:

(i) $\psi_1 *_{\alpha,\beta} \psi_2 = \psi_2 *_{\alpha,\beta} \psi_1$, $\psi_1 \in L^1_{\alpha,\beta}$ and $\psi_2 \in D$;
(ii) $\psi_1 *_{\alpha,\beta} (\psi_2 *_{\alpha,\beta} \psi_3) = (\psi_1 *_{\alpha,\beta} \psi_2) *_{\alpha,\beta} \psi_3$, $\psi_1 \in L^1_{\alpha,\beta}$ and $\psi_2, \psi_3 \in D$;
(iii) $\psi_1 *_{\alpha,\beta} \psi_2 \in L^1_{\alpha,\beta}$, $\psi_1 \in L^1_{\alpha,\beta}$, $\psi_2 \in D$,

are justified by Propositions 1 and 2. Hence, we omit the details. □

Theorem 4 *Let $\psi_3, \psi_1, \psi_n \in L^1_{\alpha,\beta}$ and $\psi_2 \in D$. Then the undermentioned relations hold.*
(i) *$\psi_n *_{\alpha,\beta} \psi_2 \to \psi_1 *_{\alpha,\beta} \psi_2$ as $n \to \infty$ as $\psi_n \to \psi_1$.*
(ii) *$(\psi_1 + \psi_3) *_{\alpha,\beta} \psi_2 = \psi_1 *_{\alpha,\beta} \psi_2 + \psi_3 *_{\alpha,\beta} \psi_2$.*

Proof of this lemma can be easily obtained from using Eq. (6). Hence, the details are deleted.

To complete the process of establishing the space $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$, we derive the following relation.

Lemma 5 *Let $(\delta_n) \in \Delta$ and $\psi \in L^1_{\alpha,\beta}$. Then we have $\psi *_{\alpha,\beta} \delta_n \to \psi$ as $n \to \infty$.*

Proof It has already been verified that $\psi_1 *_{\alpha,\beta} \delta_n \in L^1_{\alpha,\beta}$. Therefore, from definitions we get

$$\|\psi *_{\alpha,\beta} \delta_n - \psi\|_{\alpha,\beta} = \int_{-\infty}^{\infty} \bigl|(\psi *_{\alpha,\beta} \delta_n - \psi)(\zeta)\bigr| X_{\alpha,\beta}(\zeta)\, d\zeta.$$

Thus, by employing Eq. (10), we obtain

$$\|\psi *_{\alpha,\beta} \delta_n - \psi\|_{\alpha,\beta} \le \int_{-\infty}^{\infty} \bigl|\delta_n(y)\bigr| \bigl|T^{\zeta}_{\alpha,\beta}\psi(-y) - \psi(\zeta)\bigr| X_{\alpha,\beta}(\zeta)\, d\zeta\, X_{\alpha,\beta}(y)\, dy. \tag{13}$$

Again by Part (i) of Proposition 2, Eq. (13) gives

$$\|\psi *_{\alpha,\beta} \delta_n - \psi\|_{\alpha,\beta} \le \bigl(4\|\psi\|_{\alpha,\beta} - \|\psi\|_{\alpha,\beta}\bigr) \int_{-\infty}^{\infty} \bigl|\delta_n(y) X_{\alpha,\beta}(y)\, dy\bigr|.$$

Since (δ_n) is of compact support, we by Eq. (12) get

$$\|\psi *_{\alpha,\beta} \delta_n - \psi\|_{\alpha,\beta} \le 3\|\psi\|_{\alpha,\beta} A a_n \to 0$$

as $n \to \infty$.

This ends the proof of the lemma. □

Therefore the Boehmian space $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ is obtained. The sum of the Boehmians $[\frac{\varphi_n}{\delta_n}]$ and $[\frac{\psi_n}{\varepsilon_n}]$ is given in $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ as

$$\left[\frac{\varphi_n}{\delta_n}\right] + \left[\frac{\psi_n}{\varepsilon_n}\right] = \left[\frac{\varphi_n *_{\alpha,\beta} \delta_n + \psi_n *_{\alpha,\beta} \delta_n}{\delta_n *_{\alpha,\beta} \varepsilon_n}\right],$$

whereas the multiplication of a Boehmian $\frac{\varphi_n}{\delta_n}$ or sometimes $[\frac{\varphi_n}{\delta_n}]$ in $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ by a complex number γ is defined as $\gamma[\frac{\varphi_n}{\delta_n}] = [\frac{\gamma\varphi_n}{\delta_n}]$. On the other hand, the extension of $*_{\alpha,\beta}$ and D^α to $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ is introduced as

$$\left[\frac{\varphi_n}{\delta_n} *_{\alpha,\beta} \frac{\psi_n}{\varepsilon_n}\right] = \left[\frac{\varphi_n *_{\alpha,\beta} \psi_n}{\delta_n *_{\alpha,\beta} \varepsilon_n}\right] \quad \text{and} \quad D^\alpha\left[\frac{\varphi_n}{\delta_n}\right] = \left[\frac{D^\alpha \varphi_n}{\delta_n}\right], \quad \alpha \in \mathbb{R}.$$

Moreover, an extension of $*_{\alpha,\beta}$ to $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta) *_{\alpha,\beta} L^1_{\alpha,\beta}(\mathbb{R}^2)$, where (φ_n/δ_n) is in $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ and ω in $L^1_{\alpha,\beta}$, is given as

$$\left[\frac{\varphi_n}{\delta_n} *_{\alpha,\beta} \omega\right] = \left[\frac{\varphi_n *_{\alpha,\beta} \omega}{\delta_n}\right].$$

If $\beta_n, \beta \in B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$, $n = 1, 2, 3, \dots$, then $\{\beta_n\}$ converges in δ type to β, namely $\delta - \lim_{n\to\infty} \beta_n = \beta(\beta_n \xrightarrow{\delta} \beta)$, provided there can be found a delta sequence $\{\delta_n\}$ such that

(a) $(\beta_n *_{\alpha,\beta} \delta_k)$ and $(\beta *_{\alpha,\beta} \delta_k) \in L^1_{\alpha,\beta}$ for all $n, k \in \mathbb{N}$,

(b) $\lim_{n\to\infty} \beta_n *_{\alpha,\beta} \delta_k = \beta *_{\alpha,\beta} \delta_k$ in $L^1_{\alpha,\beta}$ for every $k \in \mathbb{N}$.

Or, equivalently, $\delta - \lim_{n\to\infty} \beta_n = \beta$ if and only if there are $\varphi_{n,k}$, $\varphi_k \in L^1_{\alpha,\beta}$ and $\{\delta_k\} \in \Delta$ such that

(i) $\beta_n = \varphi_{n,k}/\delta_k$, $\beta = \varphi_k/\delta_k$

(ii) $\lim_{n\to\infty} \varphi_{n,k} = \varphi_k \in L^1_{\alpha,\beta}$ to every $k \in \mathbb{N}$.

If $\beta_n, \beta \in B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ for $n = 1, 2, 3, \dots$, then the sequence $\{\beta_n\}$ converges in Δ type to β, namely Δ-$\lim_{n\to\infty} \beta_n = \beta(\beta_n \xrightarrow{\Delta} \beta)$, provided there can be found a delta sequence $\{\delta_n\}$ such that

(i) $(\beta_n - \beta) *_{\alpha,\beta} \delta_n \in L^1_{\alpha,\beta}$ $(\forall n \in \mathbb{N})$

(ii) $\lim_{n\to\infty}(\beta_n - \beta) *_{\alpha,\beta} \delta_n = 0$ in $L^1_{\alpha,\beta}$.

We turn to the construction of the ultraspace $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times, \bar{\Delta})$ of Boehmians. Let $\bar{D}$, $\bar{L}^1_{\alpha,\beta}$ be the spaces of all Jacobi–Dunkl transforms of the spaces D and $L^1_{\alpha,\beta}$, respectively, and $\bar{\Delta}$ be the set of the Jacobi–Dunkl transforms of the set Δ. Define a product formula $\times_{\alpha,\beta}$ between $\bar{D}$ and $\bar{L}^1_{\alpha,\beta}$ by

$$F \times_{\alpha,\beta} G = \left(J^d_{\alpha,\beta}\psi_1\right)\left(J^d_{\alpha,\beta}\psi_2\right), \quad \text{where } \psi_1 \in D \text{ and } \psi_2 \in L^1_{\alpha,\beta}. \tag{14}$$

With the product $\times_{\alpha,\beta}$, the space $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ can be easily verified as a Boehmian space by virtue of the following result.

Theorem 6 *Let $F_1, F_2 \in \bar{L}^1_{\alpha,\beta}$, $G_1, G \in \bar{D}$. Then the undermentioned relations hold.*

(i) $(F_1 + F_2) \times_{\alpha,\beta} G = F_1 \times_{\alpha,\beta} G + F_2 \times_{\alpha,\beta} G$.

(ii) $F_1 \times_{\alpha,\beta} G = G \times_{\alpha,\beta} F_1$.

(iii) $\lambda(F_1 \times_{\alpha,\beta} G) = (\lambda F_1) \times_{\alpha,\beta} G, \lambda \in \mathbb{C}$.

(iv) *$F_n \times_{\alpha,\beta} G \to F_1 \times_{\alpha,\beta} G$ for every $F_n \in \bar{L}^1_{\alpha,\beta}$.*

(v) *$F_n \to F_1$ as $n \to \infty$ in $\bar{L}^1_{\alpha,\beta}$.*

(vi) $F_1 \times_{\alpha,\beta} (G \times_{\alpha,\beta} G_1) = (F_1 \times_{\alpha,\beta} G) \times_{\alpha,\beta} G_1$.

Proof We prove (ii) as the proofs of (i), (iii), (iv), and (v) are similar to those given in literature or are straightforward results from simple integration. Let $\psi_1 \in L^1_{\alpha,\beta}$ and $\psi_2 \in D$ be such that $J^d_{\alpha,\beta}\psi_1 = F_1$ and $J^d_{\alpha,\beta}\psi_2 = G$. Then, by Eq. (9), we write

$$F_1 \times_{\alpha,\beta} G = J^d_{\alpha,\beta}\psi_1 \times_{\alpha,\beta} J^d_{\alpha,\beta}\psi_2 = J^d_{\alpha,\beta}(\psi_1 *_{\alpha,\beta} \psi_2).$$

Hence, since $\psi_1 *_{\alpha,\beta} \psi_2 = \psi_2 *_{\alpha,\beta} \psi_1$, we have

$$F_1 \times G = J^d_{\alpha,\beta}(\psi_2 *_{\alpha,\beta} \psi_1) = G \times_{\alpha,\beta} F_1.$$

This ends the proof of the theorem. □

Theorem 7 *Let $(\bar{\delta}_n), (\bar{\theta}_n) \in \bar{\Delta}$. Then $\bar{\delta}_n \times_{\alpha,\beta} \bar{\theta}_n \in \bar{\Delta}$ for all $n \in \mathbb{N}$.*

Proof Let $(\delta_n), (\theta_n)$ be in Δ such that $\bar{\delta}_n = J^d_{\alpha,\beta}\delta_n$ and $\bar{\theta}_n = J^d_{\alpha,\beta}\theta_n$. Then, by Eq. (9), we have

$$\bar{\delta}_n \times_{\alpha,\beta} \bar{\theta}_n = J^d_{\alpha,\beta}(\delta_n *_{\alpha,\beta}, \theta_n).$$

It is perspicuous that $\bar{\delta}_n \times_{\alpha,\beta} \bar{\theta}_n \in \bar{\Delta}$ as $(\delta_n *_{\alpha,\beta}, \theta_n) \in \Delta$ by Lemma 3. This completes the proof of the theorem. □

Similarly, we proceed to establishing the following theorem.

Theorem 8 *Let $(\bar{\delta}_n) \in \bar{\Delta}$ and $F \in \bar{L}^1_{\alpha,\beta}$. Then we have*

$$F \times_{\alpha,\beta} \bar{\delta}_n \to F \quad \text{as } n \to \infty.$$

The space $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ is an ultra-Boehmian space. For addition, multiplication by a scalar, δ-convergence, and Δ-convergence in the space $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$, see [2, 21–32] and $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ for similar definitions, replacing $*_{\alpha,\beta}$ with $\times_{\alpha,\beta}$.

3 The generalized Jacobi–Dunkl transform

In this section, we aim to introduce the generalized definition of the Jacobi–Dunkl integral operator. Let $[\frac{\psi_n}{\delta_n}] \in B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$, then the generalized Jacobi–Dunkl transform of $[\frac{\psi_n}{\delta_n}]$ is a Boehmian in $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ defined by

$$\bar{F}_{\alpha,\beta}\left[\frac{\psi_n}{\delta_n}\right] = \left[\frac{J^d_{\alpha,\beta}\psi_n}{J^d_{\alpha,\beta}\delta_n}\right]. \tag{15}$$

Theorem 9 *The estimated generalized Jacobi–Dunkl operator $\bar{F}_{\alpha,\beta}$ is well defined and linear from the space $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ into the space $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$.*

Proof Let $[\frac{\varphi_n}{\delta_n}] = [\frac{\psi_n}{\varepsilon_n}] \in B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$. Then we have

$$\varphi_n *_{\alpha,\beta} \varepsilon_m = \psi_m *_{\alpha,\beta} \delta_n = \psi_n *_{\alpha,\beta} \delta_m \quad \text{for all } m, n \in \mathbb{N}.$$

Applying $\bar{F}_{\alpha,\beta}$ to both sides in the preceding equation and making use of Eq. (9) reveal that

$$J^d_{\alpha,\beta}\varphi_n \times_{\alpha,\beta} J^d_{\alpha,\beta}\varepsilon_m = J^d_{\alpha,\beta}\psi_n \times_{\alpha,\beta} J^d_{\alpha,\beta}\delta_m \quad \text{for all } m, n \in \mathbb{N}. \tag{16}$$

In view of the concept of quotients of the sequences of $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$, Eq. (16) gives

$$\left[\frac{J^d_{\alpha,\beta}\varphi_n}{J^d_{\alpha,\beta}\delta_n}\right] = \left[\frac{J^d_{\alpha,\beta}\psi_n}{J^d_{\alpha,\beta}\varepsilon_n}\right].$$

To show that the transform $\bar{F}_{\alpha,\beta} : B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta) \to B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ is linear, let $[\frac{\varphi_n}{\delta_n}], [\frac{\psi_n}{\varepsilon_n}] \in B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$. Then, by the idea of the addition of $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$, Eq. (15), Eq. (9), and the idea of the addition of $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$, we can announce that

$$\bar{F}_{\alpha,\beta}\left(\left[\frac{\varphi_n}{\delta_n}\right] + \left[\frac{\psi_n}{\varepsilon_n}\right]\right) = \bar{F}_{\alpha,\beta}\left(\left[\frac{\varphi_n *_{\alpha,\beta} \varepsilon_n + \psi_n *_{\alpha,\beta} \delta_n}{\delta_n *_{\alpha,\beta} \varepsilon_n}\right]\right)$$

$$= \left[\frac{J_{\alpha,\beta}^{d}(\varphi_n *_{\alpha,\beta} \varepsilon_n) + J_{\alpha,\beta}^{d}(\psi_n *_{\alpha,\beta} \delta_n)}{J_{\alpha,\beta}^{d}\delta_n \times_{\alpha,\beta} J_{\alpha,\beta}^{d}\varepsilon_n}\right]$$

$$= \left[\frac{J_{\alpha,\beta}^{d}\varphi_n \times_{\alpha,\beta} J_{\alpha,\beta}^{d}\varepsilon_n + J_{\alpha,\beta}^{d}\psi_n \times_{\alpha,\beta} J_{\alpha,\beta}^{d}\delta_n}{J_{\alpha,\beta}^{d}r_n \times_{\alpha,\beta} J_{\alpha,\beta}^{d}\varepsilon_n}\right]$$

$$= \left[\frac{J_{\alpha,\beta}^{d}\varphi_n}{J_{\alpha,\beta}^{d}\delta_n}\right] + \left[\frac{J_{\alpha,\beta}^{d}\psi_n}{J_{\alpha,\beta}^{d}\varepsilon_n}\right].$$

Hence, Eq. (15) leads to

$$\bar{F}_{\alpha,\beta}\left(\left[\frac{\varphi_n}{\delta_n}\right] + \left[\frac{\psi_n}{\varepsilon_n}\right]\right) = \bar{F}_{\alpha,\beta}\left[\frac{\varphi_n}{\delta_n}\right] + \bar{F}_{\alpha,\beta}\left[\frac{\psi_n}{\varepsilon_n}\right].$$

Also, we have

$$\alpha\bar{F}_{\alpha,\beta}\left[\frac{\varphi_n}{\delta_n}\right] = \alpha\left[\frac{J_{\alpha,\beta}^{d}\varphi_n}{J_{\alpha,\beta}^{d}\delta_n}\right] = \left[\frac{J_{\alpha,\beta}^{d}(\alpha\varphi_n)}{J_{\alpha,\beta}^{d}\delta_n}\right] \quad \text{for } \alpha \in \mathbb{C}. \tag{17}$$

Hence,

$$\alpha\bar{F}_{\alpha,\beta}\left[\frac{\varphi_n}{\delta_n}\right] = \bar{F}_{\alpha,\beta}\left(\alpha\left[\frac{\varphi_n}{\delta_n}\right]\right). \tag{18}$$

Equations (17) and (18) end the proof of this theorem. □

Theorem 10 *The mapping* $\bar{F}_{\alpha,\beta} : B(L_{\alpha,\beta}^{1}, D, *_{\alpha,\beta}, \Delta) \to B(\bar{L}_{\alpha,\beta}^{1}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ *is an isomorphism.*

Proof Let $[\frac{J_{\alpha,\beta}^{d}\varphi_n}{\delta_n}] = [\frac{J_{\alpha,\beta}^{d}\psi_n}{\varepsilon_n}] \in B(\bar{L}_{\alpha,\beta}^{1}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$. Then, by using Eq. (9), we get

$$J_{\alpha,\beta}^{d}\varphi_n \times_{\alpha,\beta} J_{\alpha,\beta}^{d}\varepsilon_m = J_{\alpha,\beta}^{d}\psi_m \times_{\alpha,\beta} J_{\alpha,\beta}^{d}\delta_n \quad \text{for all } m, n \in \mathbb{N}.$$

Once again, (14) reveals to have

$$J_{\alpha,\beta}^{d}(\varphi_n *_{\alpha,\beta} \varepsilon_m) = J_{\alpha,\beta}^{d}(\psi_m *_{\alpha,\beta} \delta_n) \quad \text{for all } m, n \in \mathbb{N}.$$

We, thus, obtain $\varphi_n *_{\alpha,\beta} \varepsilon_m = \psi_m *_{\alpha,\beta} \delta_n$ for all $m, n \in \mathbb{N}$. Hence, by the concept of quotients of $B(L_{\alpha,\beta}^{1}, D, *_{\alpha,\beta}, \Delta)$, we have

$$\left[\frac{\varphi_n}{\delta_n}\right] = \left[\frac{\psi_n}{\varepsilon_n}\right] \in B\left(L_{\alpha,\beta}^{1}, D, *_{\alpha,\beta}, \Delta\right).$$

This confirms the injectivity of the mapping. The surjectivity part of $\bar{F}_{\alpha,\beta}$ is very clear as, for every $[\frac{J_{\alpha,\beta}^{d}\varphi_n}{J_{\alpha,\beta}^{d}\delta_n}] \in B(\bar{L}_{\alpha,\beta}^{1}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$, there can be found $[\frac{\varphi_n}{\delta_n}] \in B(\bar{L}_{\alpha,\beta}^{1}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ such that

$$\bar{F}_{\alpha,\beta}\left[\frac{\varphi_n}{\delta_n}\right] = \left[\frac{J_{\alpha,\beta}^{d}\varphi_n}{J_{\alpha,\beta}^{d}\delta_n}\right].$$

The proof of the theorem is ended. □

Definition 11 Let $[\frac{J^d_{\alpha,\beta}\varphi_n}{J^d_{\alpha,\beta}\delta_n}] \in B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$. Then we define the transform inversion formula of $\bar{F}_{\alpha,\beta}$ as

$$(\bar{F}_{\alpha,\beta})^{-1}\left[\frac{J^d_{\alpha,\beta}\varphi_n}{J^d_{\alpha,\beta}\delta_n}\right] = \left[\frac{\varphi_n}{\delta_n}\right] \quad \text{for each } \{\delta_n\} \in \Delta. \tag{19}$$

Theorem 12 *Let* $[\frac{J^d_{\alpha,\beta}\varphi_n}{J^d_{\alpha,\beta}\delta_n}]$ *be in* $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ *for some* $[\frac{\varphi_n}{\delta_n}]$ *in* $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$. *Then, for* $\phi \in \bar{D}(\mathbb{R})$ *and* $\psi \in D$, *we have*

$$(\bar{F}_{\alpha,\beta})^{-1}\left(\left[\frac{J^d_{\alpha,\beta}\varphi_n}{J^d_{\alpha,\beta}\delta_n}\right] \times_{\alpha,\beta} \phi\right) = \left[\frac{\varphi_n}{\delta_n}\right] *_{\alpha,\beta} \theta \quad \textit{and}$$

$$\bar{F}_{\alpha,\beta}\left(\left[\frac{\varphi_n}{\delta_n}\right] *_{\alpha,\beta} \psi\right) = \left[\frac{J^d_{\alpha,\beta}\varphi_n}{J^d_{\alpha,\beta}\delta_n}\right] \times_{\alpha,\beta} \psi$$

for some $\theta \in D$.

Proof Let $[\frac{J^d_{\alpha,\beta}\varphi_n}{J^d_{\alpha,\beta}\delta_n}] \in B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ and $\phi \in \bar{D}(\mathbb{R})$ be such that $\phi = J^d_{\alpha,\beta}\theta$ for some $\theta \in D$. Then, by Eq. (9), we write

$$\begin{aligned}(\bar{F}_{\alpha,\beta})^{-1}\left(\left[\frac{J^d_{\alpha,\beta}\varphi_n}{J^d_{\alpha,\beta}\delta_n}\right] \times_{\alpha,\beta} \phi\right) &= (\bar{F}_{\alpha,\beta})^{-1}\left(\left[\frac{J^d_{\alpha,\beta}\varphi_n \times_{\alpha,\beta} \phi}{J^d_{\alpha,\beta}\delta_n}\right]\right) \\ &= \left[\frac{(J^d_{\alpha,\beta})^{-1}(J^d_{\alpha,\beta}\varphi_n \times_{\alpha,\beta} J^d_{\alpha,\beta}\theta)}{(J^d_{\alpha,\beta})^{-1}(J^d_{\alpha,\beta}\delta_n)}\right] \\ &= (J^d_{\alpha,\beta})^{-1}\left[\frac{J^d_{\alpha,\beta}(\varphi_n *_{\alpha,\beta} \theta)}{J^d_{\alpha,\beta}\delta_n}\right] \\ &= \left[\frac{\varphi_n}{\delta_n}\right] *_{\alpha,\beta} \theta.\end{aligned}$$

To prove the second identity of this theorem, we make use of Eq. (9) to obtain

$$\bar{F}_{\alpha,\beta}\left(\left[\frac{\varphi_n}{\delta_n}\right] *_{\alpha,\beta} \psi\right) = \bar{F}_{\alpha,\beta}\left(\left[\frac{\varphi_n *_{\alpha,\beta} \psi}{\delta_n}\right]\right) = \left[\frac{J^d_{\alpha,\beta}\varphi_n}{J^d_{\alpha,\beta}\delta_n}\right] \times_{\alpha,\beta} \psi. \tag{20}$$

This ends the proof of the theorem. □

Theorem 13 *The mappings* $\bar{F}_{\alpha,\beta} : B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta) \to B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ *and* $(\bar{F}_{\alpha,\beta})^{-1} : B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta}) \to B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ *are continuous with respect to* δ *and* Δ-*convergence.*

Proof We show that $\bar{F}_{\alpha,\beta}$ and $(\bar{F}_{\alpha,\beta})^{-1}$ are continuous with respect to the convergence of δ type. For this aim, we assume $\beta_n \xrightarrow{\delta} \beta$ in $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ as $n \to \infty$ and verify that $\bar{F}_{\alpha,\beta}\beta_n \to \bar{F}_{\alpha,\beta}\beta$ as $n \to \infty$. Let $\psi_{n,k}$ and ψ_k be in $L^1_{\alpha,\beta}$ such that

$$\beta_n = \left[\frac{\psi_{n,k}}{\phi_k}\right] \quad \text{and} \quad \beta = \left[\frac{\psi_k}{\phi_k}\right]$$

and $\psi_{n,k} \to \psi_k$ as $n \to \infty$ for all $k \in \mathbb{N}$. Then $J^d_{\alpha,\beta}\psi_{n,k} \to J^d_{\alpha,\beta}\psi_k$ as $n \to \infty$ in the space $\bar{L}^1_{\alpha,\beta}$. Therefore,

$$\left[\frac{J^d_{\alpha,\beta}\psi_{n,k}}{J^d_{\alpha,\beta}\phi_k}\right] \to \left[\frac{J^d_{\alpha,\beta}\psi_k}{J^d_{\alpha,\beta}\phi_k}\right] \tag{21}$$

as $n \to \infty$ in $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$.

To prove the second part, let $g_n \xrightarrow{\delta} g$ in $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ as $n \to \infty$. Then, let $g_n = [\frac{J^d_{\alpha,\beta}\psi_{n,k}}{J^d_{\alpha,\beta}\phi_k}]$ and $g = [\frac{J^d_{\alpha,\beta}\psi_k}{J^d_{\alpha,\beta}\phi_k}]$ and $J^d_{\alpha,\beta}\psi_{n,k} \to J^d_{\alpha,\beta}\psi_k$ as $n \to \infty$. Therefore, $\psi_{n,k} \to \psi_k$ in $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ as $n \to \infty$. Hence, $[\frac{\psi_{n,k}}{\phi_k}] \to [\frac{\psi_k}{\phi_k}]$ as $n \to \infty$. Using Eq. (15) reveals

$$(\bar{F}_{\alpha,\beta})^{-1}\left[\frac{J^d_{\alpha,\beta}\psi_{n,k}}{J^d_{\alpha,\beta}\phi_k}\right] \to (\bar{F}_{\alpha,\beta})^{-1}\left[\frac{J^d_{\alpha,\beta}\psi_k}{J^d_{\alpha,\beta}\phi_k}\right] \quad \text{as } n \to \infty.$$

To establish continuity with respect to the convergence of Δ type, we assume $\beta_n \xrightarrow{\Delta} \beta$ in $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$ as $n \to \infty$. Then there exist $\psi_n \in L^1_{\alpha,\beta}$ and $(\phi_n) \in \Delta$ such that $(\beta_n - \beta) *_{\alpha,\beta} \phi_n = [\frac{\psi_n *_{\alpha,\beta} \phi_k}{\phi_k}]$ and $\psi_n \to 0$ as $n \to \infty$. Employing (15) gives

$$\bar{F}_{\alpha,\beta}\big((\beta_n - \beta) *_{\alpha,\beta} \phi_n\big) = \left[\frac{J^d_{\alpha,\beta}(\psi_n *_{\alpha,\beta} \phi_k)}{J^d_{\alpha,\beta}\phi_k}\right]. \tag{22}$$

Hence, we derive

$$\bar{F}_{\alpha,\beta}\big((\beta_n - \beta) *_{\alpha,\beta} \phi_n\big) = \left[\frac{J^d_{\alpha,\beta}\psi_n \times_{\alpha,\beta} J^d_{\alpha,\beta}\phi_k}{J^d_{\alpha,\beta}\phi_k}\right] \simeq J^d_{\alpha,\beta}f_n \to 0$$

as $n \to \infty$ in $\bar{L}^1_{\alpha,\beta}$. Therefore, from Eq. (22) we get

$$\bar{F}_{\alpha,\beta}\big((\beta_n - \beta) *_{\alpha,\beta} \phi_n\big) = (\bar{F}_{\alpha,\beta}\beta_n - \bar{F}_{\alpha,\beta}\beta) \times_{\alpha,\beta} \phi_n \to 0 \quad \text{as } n \to \infty.$$

Hence, $\bar{F}_{\alpha,\beta}\beta_n \xrightarrow{\Delta} \bar{F}_{\alpha,\beta}\beta$ as $n \to \infty$.

Finally, let $g_n \xrightarrow{\Delta} g$ in $B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$ as $n \to \infty$. Then we find $J^d_{\alpha,\beta}\psi_k \in \bar{L}^1_{\alpha,\beta}$ such that $(g_n - g) \times_{\alpha,\beta} \phi_k = [\frac{J^d_{\alpha,\beta}\psi_k \times_{\alpha,\beta}\phi_k}{\phi_k}]$ and $J^d_{\alpha,\beta}\psi_k \to 0$ as $n \to \infty$ for some $\phi_k = J^d_{\alpha,\beta}\theta_k, \theta_k \in \Delta$. Now, using Definition 11, we obtain

$$(\bar{F}_{\alpha,\beta})^{-1}\big((g_n - g) \times_{\alpha,\beta} \phi_k\big) = \left[\big(J^d_{\alpha,\beta}\big)^{-1}\frac{(J^d_{\alpha,\beta}\psi_k \times_{\alpha,\beta} J^d_{\alpha,\beta}\theta_k)}{J^d_{\alpha,\beta}\theta_k}\right]. \tag{23}$$

That is,

$$(\bar{F}_{\alpha,\beta})^{-1}\big((g_n - g) \times_{\alpha,\beta} \phi_k\big) = \left[\frac{\psi_k *_{\alpha,\beta} \theta_k}{\theta_k}\right] = \psi_k \to 0 \quad \text{as } k \to \infty \text{ in } L^1_{\alpha,\beta}.$$

Thus, Eq. (23) gives

$$(\bar{F}_{\alpha,\beta})^{-1}\big((g_n - g) \times_{\alpha,\beta} \phi_k\big) = \big((\bar{F}_{\alpha,\beta})^{-1}g_n - (\bar{F}_{\alpha,\beta})^{-1}g\big) *_{\alpha,\beta} \theta_k \to 0 \quad \text{as } n \to \infty.$$

Due to the above equation, we infer that $(\bar{F}_{\alpha,\beta})^{-1}g_n \xrightarrow{\Delta} (\bar{F}_{\alpha,\beta})^{-1}g$ for large values of n in $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$.

This ends the proof of the theorem. □

Theorem 14 *The extended $\bar{F}_{\alpha,\beta}$ transform and the classical $J^d_{\alpha,\beta}$ transform are consistent.*

Proof For every $\psi \in L^1_{\alpha,\beta}$, assume that β is its representative in $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$. This indeed reveals that $\beta = [\frac{\psi *_{\alpha,\beta} \delta_n}{\delta_n}]$, where $(\delta_n) \in \Delta, n \in \mathbb{N}$. It is obvious that the delta sequence (δ_n) is independent from the representative for all $n \in \mathbb{N}$. Consequently,

$$\bar{F}_{\alpha,\beta}(\beta) = \bar{F}_{\alpha,\beta}\left(\left[\frac{\psi *_{\alpha,\beta} \delta_n}{\delta_n}\right]\right) = \left[\frac{J^d_{\alpha,\beta}(\psi *_{\alpha,\beta} \delta_n)}{J^d_{\alpha,\beta}\delta_n}\right] = \left[\frac{J^d_{\alpha,\beta}\psi \times_{\alpha,\beta} J^d_{\alpha,\beta}\delta_n}{J^d_{\alpha,\beta}\delta_n}\right],$$

which is the representative of $J^d_{\alpha,\beta}\psi$ in $\bar{L}^1_{\alpha,\beta}$.

Hence, the proof of the theorem is ended. □

Theorem 15 *Let $[\frac{\psi_n}{\delta_n}] \in B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$. Then the condition for $[\frac{\psi_n}{\delta_n}]$, which is necessary and sufficient, to be in the range of $\bar{F}_{\alpha,\beta}$ is that ψ_n is in the range of $J^d_{\alpha,\beta}$ for every $n \in \mathbb{N}$.*

Proof If $[\frac{\psi_n}{\delta_n}]$ is in the range of $\bar{F}_{\alpha,\beta}$, then indeed ψ_n is in the range of $J^d_{\alpha,\beta}$ for all $n \in \mathbb{N}$. For the converse, if ψ_n is in the range of $J^d_{\alpha,\beta}$ for all $n \in \mathbb{N}$, then we can find $f_n \in L^1_{\alpha,\beta}$ so that $J^d_{\alpha,\beta}f_n = \psi_n$ for all $n \in \mathbb{N}$. Since $[\frac{\psi_n}{\delta_n}] \in B(\bar{L}^1_{\alpha,\beta}, \bar{D}, \times_{\alpha,\beta}, \bar{\Delta})$,

$$\psi_n \times_{\alpha,\beta} \delta_m = \psi_m \times_{\alpha,\beta} \delta_n \quad \text{for all } m, n \in \mathbb{N}. \tag{24}$$

Therefore, for some $f_n \in L^1_{\alpha,\beta}$ and $\varphi_n \in \Delta$, we find

$$J^d_{\alpha,\beta}(f_n *_{\alpha,\beta} \varphi_n) = J^d_{\alpha,\beta}(f_m *_{\alpha,\beta} \varphi_n) \quad \text{for all } m, n \in \mathbb{N}.$$

The fact that $J^d_{\alpha,\beta}$ is injective, implies that $f_n *_{\alpha,\beta} \varphi_m = f_m *_{\alpha,\beta} \varphi_n, m, n \in \mathbb{N}$.

Thus, $\frac{f_n}{\varphi_n}$ is a quotient of the sequences in $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$. Hence,

$$\bar{F}_{\alpha,\beta}\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\psi_n}{\psi_n}\right] \quad \text{for some } \left[\frac{f_n}{\varphi_n}\right] \in B\bigl(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta\bigr).$$

Hence the theorem is proved. □

4 Conclusion

The classical theory of the Jacobi–Dunkl integral operator of [1] is extended to a class of Boehmians. Every element of the classical space $L^1_{\alpha,\beta}$ is identified as a member of the Boehmian space $B(L^1_{\alpha,\beta}, D, *_{\alpha,\beta}, \Delta)$. Various embeddings and characteristics of the extended integral operator including an inversion formula are given in a generalized sense. Convergence with respect to δ and Δ is also discussed.

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