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q-Hardy type inequalities for quantum integrals

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Abstract

The aim of this work is to obtain quantum estimates for q-Hardy type integral inequalities on quantum calculus. For this, we establish new identities including quantum derivatives and quantum numbers. After that, we prove a generalized q-Minkowski integral inequality. Finally, with the help of the obtained equalities and the generalized q-Minkowski integral inequality, we obtain the results we want. The outcomes presented in this paper are q-extensions and q-generalizations of the comparable results in the literature on inequalities. Additionally, by taking the limit $q \to 1^-$, our results give classical results on the Hardy inequality.

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1 Introduction

Hardy's integral inequality, proved by G.H. Hardy in 1920 [4] is

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(t)\,dt, \tag{1.1}$$

where $p > 1$, $x > 0$, f is a nonnegative measurable function on $(0,\infty)$ and $\int_0^\infty f^p(t)\,dt$ is convergent. Also the constant $(\frac{p}{p-1})^p$ is the best possible.

Hardy's type inequalities have been studied by a large number of authors during the 20th century and has motivated some important lines of study which are currently active. Over the last 20 years a large number of papers have appeared in the literature which deal with the simple proofs, various generalizations and discrete analogues of Hardy's inequality and its generalizations; see [5, 8, 11, 12, 15, 17–19].

The inequalities have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Recently, the Hermite–Hadamard type inequality has become the subject of intensive research. For recent results, refinements, counterparts, generalizations and new Hadamard's-type inequalities, see [1, 7, 10, 14, 16, 20].

On the other hand, the study of calculus without limits is known as quantum calculus or q-calculus. The famous mathematician Euler initiated the study q-calculus in the 18th century by introducing the parameter q in Newton's work of infinite series. In the early 20th century, Jackson [6] has started a symmetric study of q-calculus and introduced q-definite integrals. The subject of quantum calculus has numerous applications in various areas of mathematics and physics, such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions, quantum theory, mechanics and in theory of relativity. This subject has received outstanding attention by many researchers and hence it is considered as an in-corporative subject between mathematics and physics. The reader is referred to [2, 3, 9] for some current advances in the theory of quantum calculus and theory of inequalities in quantum calculus.

The purpose of this work is to establish quantum estimates for q-Hardy type integral inequalities on quantum calculus. For this, we establish new identities including quantum derivatives and quantum numbers. After that, we prove a generalized q-Minkowski integral inequality. Finally, with the help of the obtained equalities and the generalized q-Minkowski integral inequality, we obtain the results we want. The outcomes presented in this paper are q-extensions and q-generalizations of the comparable results in the literature on inequalities. In addition, by taking the limit $q \to 1^-$, our results give classical results on the Hardy inequality.

2 Preliminaries and definitions of q-calculus

Throughout this paper, let $a < b$ and $0 < q < 1$ be a constant. The following definitions, notations and theorems for q-derivative and q-integral of a function f on $[a,b]$ are given in [2, 3, 9].

The notation $[z]_q$ is defined by

$$[z]_q = \frac{1-q^n}{1-q} \quad \bigl(z \in \mathbb{C}; q \in \mathbb{C}\backslash\{1\}; q^z \neq 1\bigr). \tag{2.1}$$

A special case of (2.1) when $z \in \mathbb{N}$ is

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} \quad (n \in \mathbb{N}).$$

Also

$$[-n]_q = -\frac{1}{q^n}[n]_q \quad (n \in \mathbb{N}). \tag{2.2}$$

Definition 1 Let $f : [a,b] \to \mathbb{R}$ be a continuous function, then q-derivative of f at $x \in [a,b]$ is characterized by the expression

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0. \tag{2.3}$$

Since $f : [a,b] \to \mathbb{R}$ is a continuous function, thus we have $D_q f(a) = \lim_{x\to a} D_q f(x)$ The function f is said to be q- differentiable on $[a,b]$ if $D_q f(t)$ exists for all $x \in [a,b]$. Also $\lim_{q\to 1^-} D_q f(x) = f'(x)$ is classic derivative.

Theorem 1 *Assume that $f, g : I \subset \mathbb{R} \to \mathbb{R}$ are continuous functions, then we have the properties of the q-derivative*:

(I) $D_q\bigl(af(x) \pm bg(x)\bigr) = aD_q f(x) \pm bD_q g(x)$.

(II) $D_q\big(f(x)g(x)\big) = f(qx)D_q g(x) + g(x)D_q f(x)$.

(III) $D_q\left(\frac{f(x)}{g(x)}\right) = \frac{f(qx)D_q g(x) + g(x)D_q f(x)}{g(x)g(qx)}$.

Definition 2 Suppose $0 < a < b$. The definite q-integral is defined as

$$\int_0^b f(t)\,d_q t = (1-q)b\sum_{n=0}^{\infty} q^n f\big(q^n b\big) \tag{2.4}$$

and

$$\int_a^b f(t)\,d_q t = \int_0^b f(t)\,d_q t - \int_0^a f(t)\,d_q t,$$

where $\sum_{n=0}^{\infty} q^n f(q^n b)$ and $\sum_{n=0}^{\infty} q^n f(q^n a)$ are convergent.

Definition 3 ([9]) The improper q-integral of $f(t)$ on $[0,\infty)$ is defined by

$$\int_0^{\infty} f(t)\,d_q t = \sum_{n=-\infty}^{\infty}\int_{q^{n+1}}^{q^n} f(t)\,d_q t = (1-q)\sum_{n=-\infty}^{\infty} q^n f\big(q^n\big) \quad (0 < q < 1)$$

and

$$\int_0^{\infty} f(t)\,d_q t = \sum_{n=-\infty}^{\infty}\int_{q^n}^{q^{n+1}} f(t)\,d_q t = \frac{q-1}{q}\sum_{n=-\infty}^{\infty} q^n f\big(q^n\big) \quad (1 < q),$$

where $\sum_{n=-\infty}^{\infty} q^n f(q^n)$ is convergent.

We have the following properties of the q-integral of (2.4):

(I) $D_q \int_a^x f(t)\,d_q t = f(x)$.

(II) $\int_a^x D_q f(t)\,d_q t = f(x) - f(a)$.

(III) $\int_a^x \big[f(t) \pm g(t)\big]\,d_q t = \int_a^x f(t)\,d_q t \pm \int_a^x g(t)\,d_q t$.

(IV) $\int_0^x t^{\alpha}\,d_q t = \frac{x^{\alpha+1}}{[\alpha+1]_q}$, for $\alpha \in \mathbb{R}\backslash\{-1\}$.

(V) The integration by parts rule of the q-integral:

$$\int_c^x f(t)D_q g(t)\,d_q t = f(t)g(t)|_c^x - \int_c^x g(qt)D_q f(t)\,d_q t. \tag{2.5}$$

Theorem 2 (q-Hölder inequality) *Let f, g be q-integrable on $[a,b]$ and $0 < q < 1$ and $\frac{1}{s} + \frac{1}{r} = 1$ with $s > 1$. Then we have*

$$\int_a^b \big|f(t)g(t)\big|\,d_q t \le \left(\int_a^b \big|f(t)\big|^s\,d_q t\right)^{\frac{1}{s}}\left(\int_a^b \big|f(t)g(t)\big|^r\,d_q t\right)^{\frac{1}{r}}.$$

3 Auxiliary results

The following results which will be used. There is no general change of variables property for the q-integral. However, the variable can be changed as follows.

Lemma 1 (q-Change of variables property) *Let $f : I \to \mathbb{R}$ be a function and $0 < q < 1$. Then we have*

$$\int_0^1 f(sb)\,d_q s = \frac{1}{b}\int_0^b f(t)\,d_q t, \tag{3.1}$$

where $b \neq 0$ and $\int_0^b f(t)\,d_q t$ is convergent.

Proof From the definition of the q-integral, we have

$$\begin{aligned}
&\int_0^1 f(sb)\,d_q s \\
&\quad = (1-q)(1-0)\sum_{n=0}^{\infty} q^n f\bigl(\bigl[q^n 1 + \bigl(1-q^n\bigr)0\bigr]b\bigr) \\
&\quad = \frac{1}{b}\int_0^b f(t)\,d_q t
\end{aligned}$$

as desired. □

A general chain rule for q-derivative does not exist. However, a chain rule of $(h(t))^p$ and $(h(t))^{\frac{1}{p}}$ can be calculated as follows.

Lemma 2 *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a function $p \in \mathbb{Z}$ and $0 < q < 1$. Then we have*

$$D_q\bigl(h(t)\bigr)^p = \left(\sum_{i=0}^{p-1}\bigl[h(t)\bigr]^{p-1-i}\bigl[h(qt)\bigr]^i\right)D_q h(t). \tag{3.2}$$

In (3.2) if we choose $q \to 1^-$ we have the classical derivative of $(h(t))^p$,

$$\lim_{q\to 1^-} D_q\bigl(h(t)\bigr)^p = p\bigl(h(t)\bigr)^{p-1}h'(t) = \bigl[\bigl(h(t)\bigr)^p\bigr]'.$$

Proof By the definition of the q-derivative we have

$$\begin{aligned}
&D_q\bigl(h(t)\bigr)^p \\
&\quad = \frac{[h(t)]^p - [h(qt)]^p}{(1-q)t} \\
&\quad = \frac{(h(t) - h(qt))}{(1-q)t}\sum_{i=0}^{p-1}\bigl[h(t)\bigr]^{p-1-i}\bigl[h(qt)\bigr]^i \\
&\quad = \left(\sum_{i=0}^{p-1}\bigl[h(t)\bigr]^{p-1-i}\bigl[h(qt)\bigr]^i\right)D_q h(t)
\end{aligned}$$

as desired. □

Lemma 3 *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a function $p \in \mathbb{Z}$ and $0 < q < 1$. Then we have*

$$D_q\big(h(t)\big)^{\frac{1}{p}} = \frac{D_q h(t)}{\sum_{i=0}^{p-1} (h(t))^{\frac{p-1-i}{p}} (h(qt))^{\frac{i}{p}}}. \tag{3.3}$$

In (3.3) if we choose $q \to 1^-$ we have the classical derivative of $(h(t))^{\frac{1}{p}}$,

$$\lim_{q\to 1^-} D_q\big(h(t)\big)^{\frac{1}{p}} = \frac{h'(t)}{p(h(t))^{\frac{p-1}{p}}} = \big[\big(h(t)\big)^{\frac{1}{p}}\big]'.$$

Proof We consider

$$y(t) = \big(h(t)\big)^{\frac{1}{p}},$$
$$\big(y(t)\big)^p = \big(h(t)\big),$$

such that

$$D_q\big(y(t)\big)^p = D_q\big(h(t)\big), \tag{3.4}$$

and from (3.2) we know

$$D_q\big(y(t)\big)^p = \left(\sum_{i=0}^{p-1}\big[y(t)\big]^{p-1-i}\big[y(qt)\big]^i\right) D_q y(t) = D_q\big(h(t)\big). \tag{3.5}$$

Thus, we get

$$D_q y(t) = \frac{D_q h(t)}{\sum_{i=0}^{p-1} (h(t))^{\frac{p-1-i}{p}} (h(qt))^{\frac{i}{p}}}$$

as desired. □

Similarly, we have more general result as follows.

Lemma 4 *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a function $\frac{n}{m} \in \mathbb{Q}$ and $0 < q < 1$. Then we have*

$$D_q\big(h(t)\big)^{\frac{n}{m}} = \frac{\sum_{i=0}^{n-1} (h(t))^{n-1-i} (h(qt))^i}{\sum_{i=0}^{m-1} (h(t))^{\frac{n(m-1-i)}{m}} (h(qt))^{\frac{ni}{m}}} D_q h(t). \tag{3.6}$$

In (3.6) if we choose $q \to 1^-$ we have the classical derivative of $(h(t))^{\frac{n}{m}}$,

$$\lim_{q\to 1^-} D_q\big(h(t)\big)^{\frac{n}{m}} = \frac{n}{m}\big(h(t)\big)^{\frac{n}{m}-1} h'(t) = \big[\big(h(t)\big)^{\frac{n}{m}}\big]'.$$

Proof We consider

$$y(t) = \big(h(t)\big)^{\frac{n}{m}},$$
$$\big(y(t)\big)^m = \big(h(t)\big)^n,$$

such that

$$D_q\big(y(t)\big)^m = D_q\big(h(t)\big)^n,$$

and from (3.2) we have

$$\begin{aligned}&\left(\sum_{i=0}^{m-1}\big[y(t)\big]^{m-1-i}\big[y(qt)\big]^i\right)D_q y(t)\\&\quad=\left(\sum_{i=0}^{n-1}\big[h(t)\big]^{n-1-i}\big[h(qt)\big]^i\right)D_q\big(h(t)\big).\end{aligned}$$

Thus, we get

$$\begin{aligned}D_q y(t) &= \frac{\sum_{i=0}^{n-1}[h(t)]^{n-1-i}[h(qt)]^i}{\sum_{i=0}^{m-1}[y(t)]^{m-1-i}[y(qt)]^i}D_q h(t)\\&=\frac{\sum_{i=0}^{n-1}(h(t))^{n-1-i}(h(qt))^i}{\sum_{i=0}^{m-1}(h(t))^{\frac{n(m-1-i)}{m}}(h(qt))^{\frac{ni}{m}}}D_q h(t)\end{aligned}$$

as desired. □

4 Main results

Firstly, we will prove the generalized q-Minkokski type integral inequality which will be used in the next theorem.

Theorem 3 (Generalized q-Minkowski integral inequality) *Let* $\alpha \in (0,1]$, $1 \leq p \leq \infty$, $f : [a,b] \times [c,d] \to \mathbb{R}$ *be a q-integrable function. Then the following inequality holds*:

$$\left(\int_a^b\left|\int_c^d f(x,y)\,d_qy\right|^p d_qx\right)^{\frac{1}{p}} \leq \int_c^d\left(\int_a^b\big|f(x,y)\big|^p\,d_qx\right)^{\frac{1}{p}}d_qy, \tag{4.1}$$

where $q \in (0,1)$.

Proof The case $p = 1$ corresponds to Fubini's theorem. For the case $p = \infty$ we just notice that

$$\left(\int_a^b\left|\left(\int_c^d f(x,y)\,d_qy\right)_q\right|^p d_qx\right)^{\frac{1}{p}} \leq \int_c^d \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\big|f(x,y)\big|\,d_qy.$$

Now assume that $1 < p < \infty$ and we can write

$$\begin{aligned}&\int_a^b\left|\int_c^d f(x,y)\,d_qy\right|^p d_qx\\&\quad=\int_a^b\left|\int_c^d f(x,y)\,d_qy\right|^{p-1}\left|\int_c^d f(x,y)\,d_qy\right|d_qx\\&\quad\leq\int_a^b\left|\int_c^d f(x,t)\,d_qt\right|^{p-1}\left(\int_c^d\big|f(x,y)\big|\,d_qy\right)d_qx\end{aligned}$$

$$= \int_a^b \left(\int_c^d \left| \int_c^d f(x,t)\, d_qt \right|^{p-1} \left| f(x,y) \right| d_qy \right) d_qx$$
$$= \int_c^d \left(\int_a^b \left| \int_c^d f(x,t)\, d_qt \right|^{p-1} \left| f(x,y) \right| d_qx \right) d_qy$$

the last step coming from Fubini's theorem. By applying the q-Hölder inequality to the inner integral with respect to x, we have

$$\begin{aligned}
&\int_a^b \left| \int_c^d f(x,y)\, d_qy \right|^p d_qx \\
&\leq \int_c^d \left\{ \left(\int_a^b \left(\left| \int_c^d f(x,t)\, d_qt \right|^{r(p-1)} d_qx \right)^{\frac{1}{r}} \left(\int_a^b \left| f(x,y) \right|^p d_qx \right)^{\frac{1}{p}} \right) \right\} d_qy \\
&= \int_c^d \left\{ \left(\int_a^b \left(\left| \int_c^d f(x,t)\, d_qt \right|^p d_qx \right)^{\frac{1}{r}} \left(\int_a^b \left| f(x,y) \right|^p d_qx \right)^{\frac{1}{p}} \right) \right\} d_qy \\
&= \left(\int_a^b \left| \int_c^d f(x,t)\, d_qt \right|^p d_qx \right)^{\frac{1}{r}} \int_c^d \left(\int_a^b \left| f(x,y) \right|^p d_qx \right)^{\frac{1}{p}} d_qy.
\end{aligned}$$

Finally dividing both sides by $\int_a^b (| \int_c^d f(x,t)\, d_qt|^p\, d_qx)^{\frac{1}{r}}$ we have

$$\left(\int_a^b \left| \int_c^d f(x,y)\, d_qy \right|^p d_qx \right)^{1-\frac{1}{r}} \leq \int_c^d \left(\int_a^b \left| f(x,y) \right|^p d_qx \right)^{\frac{1}{p}} d_qy$$

i.e.

$$\left(\int_a^b \left| \int_c^d f(x,y)\, d_qy \right|^p d_qx \right)^{\frac{1}{p}} \leq \int_c^d \left(\int_a^b \left| f(x,y) \right|^p d_qx \right)^{\frac{1}{p}} d_qy,$$

which gives the required inequality. □

Theorem 4 (q-Hardy inequality) *If f is a nonnegative function on $(0,\infty)$, $p > 1$ and $\int_0^\infty f^p(t)\, d_qt$ is convergent, then the following inequality holds*:

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, d_qt \right)^p d_qx \right)^{\frac{1}{p}} \leq \frac{1}{[\frac{p-1}{p}]_q} \left(\int_0^\infty f^p(t)\, d_qt \right)^{\frac{1}{p}}, \tag{4.2}$$

where $q \in (0,1)$.

Proof From (3.1) by the q-changing variables $t = xs$ it follows that

$$\frac{1}{x} \int_0^x f(t)\, d_qt = \int_0^1 f(sx)\, d_qs.$$

Thus, we write

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, d_qt \right)^p d_qx \right)^{\frac{1}{p}} = \left(\int_0^\infty \left(\int_0^1 f(xs)\, d_qs \right)^p d_qx \right)^{\frac{1}{p}}. \tag{4.3}$$

From the generalized q-Minkowski integral inequality and by using the q-changing variables $xs = t$, we have

$$\begin{aligned} &\left(\int_0^\infty \left(\int_0^1 f(xs)\,d_qs\right)^p d_qx\right)^{\frac{1}{p}} \qquad (4.4)\\ &\leq \int_0^1 \left(\int_0^\infty f^p(xs)\,d_qx\right)^{\frac{1}{p}} d_qs = \int_0^1 \left(\int_0^\infty \frac{1}{s} f^p(t)\,d_qt\right)^{\frac{1}{p}} d_qs \\ &= \left(\int_0^1 s^{-\frac{1}{p}}\,d_qs\right)\left(\int_0^\infty f^p(t)\,d_qt\right)^{\frac{1}{p}} = \frac{1}{[1-\frac{1}{p}]_q}\left(\int_0^\infty f^p(t)\,d_qt\right)^{\frac{1}{p}} \end{aligned}$$

from (4.3) and (4.4)

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,d_qt\right)^p d_qx\right)^{\frac{1}{p}} \leq \frac{1}{[\frac{p-1}{p}]_q}\left(\int_0^\infty f^p(t)\,d_qt\right)^{\frac{1}{p}}$$

and the proof is completed. □

Remark 1 In (4.2) if we choose $q \to 1^-$ we recapture the classical Hardy inequality.

The following theorem generalizes the q-Hardy type integral inequality by introducing power weights x^r.

Theorem 5 *If f is a nonnegative function on $(0,\infty)$, $p \geq 1$, $r < p-1$ and $\int_0^\infty t^r f^p(t)\,d_qt$ is convergent, then the following inequality holds*:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,d_qt\right)^p x^r\,d_qx \leq \frac{1}{[\frac{p-r-1}{p}]_q^p}\int_0^\infty t^r f^p(t)\,d_qt,$$

where $q \in (0,1)$.

Proof By the q-changing variables $t = xs$ we get

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,d_qt\right)^p x^r\,d_qx\right)^{\frac{1}{p}} = \left(\int_0^\infty \left(\int_0^1 f(xs)x^{\frac{r}{p}}\,d_qs\right)^p d_qx\right)^{\frac{1}{p}}.$$

So, from Minkowski q-integral inequality and by the changing variables $xs = u$ the proof is completed as follows:

$$\begin{aligned} &\left(\int_0^\infty \left(\int_0^1 f(xs)x^{\frac{r}{p}}\,d_qs\right)^p d_qx\right)^{\frac{1}{p}} \\ &\leq \int_0^1 \left(\int_0^\infty x^r f^p(xs)\,d_qx\right)^{\frac{1}{p}} d_qs = \int_0^1 \left(\int_0^\infty \frac{u^r}{s^{r+1}} f^p(u)\,d_qu\right)^{\frac{1}{p}} d_qs \\ &= \left(\int_0^1 s^{\frac{-r-1}{p}}\,d_qs\right)\left(\int_0^\infty u^r f^p(u)\,d_qu\right)^{\frac{1}{p}} \\ &= \frac{1}{[\frac{p-r-1}{p}]_q}\left(\int_0^\infty u^r f^p(u)\,d_qu\right)^{\frac{1}{p}}. \end{aligned}$$

□

Remark 2 In Theorem 5 if we put $r = 0$ we obtain the inequality (4.2).

Definition 4 For a given weight r, we define the modified q-Hardy operator as

$$H_{q,r}f(x) = \frac{1}{xr(x)} \int_0^x r(t)f(t)\,d_qt.$$

The following theorem will be proved using the q-Hardy operator.

Theorem 6 *Assume f is a nonnegative function on $(0,\infty)$, r being an absolutely continuous function on $(0,\infty)$, and $p > 1$. Also assume $\int_0^\infty f^p(x)\,d_qx$ is convergent, and*

$$\frac{[p-1]_q}{p} + \frac{x}{p}\frac{D_q r(x)}{r(qx)} \sum_{i=0}^{p-1} \left[\frac{h_{r,a}f(qx)}{h_{r,a}f(x)}\right]^i \geq \frac{1}{\lambda}, \tag{4.5}$$

for almost every $x > 0$ and for some $\lambda > 0$. Then we have the following inequality:

$$\int_0^\infty \left(H_r f(x)\right)^p d_qx \leq \lambda^p \beta^p \int_0^\infty f^p(x)\,d_qx,$$

where

$$H_{q,r}f(x) = \frac{1}{xr(x)} \int_0^x r(t)f(t)\,d_qt.$$

Proof We assume $0 < a < b < \infty$ and

$$h_{q,r,a}f(x) = \frac{1}{r(x)} \int_a^x r(t)f(t)\,d_qt.$$

Then, defining $H_{r,a}f(x) = \frac{1}{x}h_{r,a}f(x)$, and integrating by parts from (2.5) with $w = (h_{r,a}f(x))^p$ and $D_qg(x) = x^{-p}$ noting that $g(x) = \frac{x^{1-p}}{[1-p]_q}$, we get

$$\begin{aligned}
&\int_a^b \left(H_{q,r,a}f(x)\right)^p d_qx \\
&\quad= \int_a^b \left(h_{q,r,a}f(x)\right)^p x^{-p}\,d_qx \\
&\quad= \int_0^b \left(h_{q,r,a}f(x)\right)^p x^{-p}\,d_qx - \int_0^a \left(h_{q,r,a}f(x)\right)^p x^{-p}\,d_qx \\
&\quad= \int_0^b \left(h_{q,r,a}f(x)\right)^p D_q\frac{x^{1-p}}{[1-p]_q}\,d_qx - \int_0^a \left(h_{q,r,a}f(x)\right)^p D_q\frac{x^{1-p}}{[1-p]_q}\,d_qx \\
&\quad= \left(h_{q,r,a}f(x)\right)^p \frac{x^{1-p}}{[1-p]_q}\bigg|_0^b - \int_0^b \frac{(qx)^{1-p}}{[1-p]_q} D_q\left(h_{q,r,a}f(x)\right)^p d_qx \\
&\qquad- \left(h_{q,r,a}f(x)\right)^p \frac{x^{1-p}}{[1-p]_q}\bigg|_0^a + \int_0^a \frac{(qx)^{1-p}}{[1-p]_q} D_q\left(h_{q,r,a}f(x)\right)^p d_qx \\
&\quad= \left(h_{q,r,a}f(b)\right)^p \frac{b^{1-p}}{[1-p]_q}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{q^{1-p}}{[1-p]_q} \int_0^b x^{1-p} D_q h_{q,r,a} f(x) \left(\sum_{i=0}^{p-1} \left[h_{q,r,a} f(x)\right]^{p-1-i} \left[h_{q,r,a} f(qx)\right]^i \right) d_q x \\
&+ \frac{q^{1-p}}{[1-p]_q} \int_0^a x^{1-p} D_q h_{q,r,a} f(x) \left(\sum_{i=0}^{p-1} \left[h_{q,r,a} f(x)\right]^{p-1-i} \left[h_{q,r,a} f(qx)\right]^i \right) d_q x \\
= {}& \left(h_{q,r,a} f(b)\right)^p \frac{b^{1-p}}{[1-p]_q} \\
&- \frac{q^{1-p}}{[1-p]_q} \int_a^b x^{1-p} D_q h_{q,r,a} f(x) \left(\sum_{i=0}^{p-1} \left[h_{q,r,a} f(x)\right]^{p-1-i} \left[h_{q,r,a} f(qx)\right]^i \right) d_q x.
\end{aligned}
$$

We notice that from (2.2)

$$
\left(h_{q,r,a} f(b)\right)^p \frac{b^{1-p}}{[1-p]_q} = -q^{p-1} \left(h_{q,r,a} f(b)\right)^p \frac{b^{1-p}}{[p-1]_q}
$$

is negative since $p-1 \in \mathbb{N}$, $p-1>0$ and $h_{q,r,a} f(b) > 0$ with $b>0$. Also, from the definition of $h_{q,r,a} f(x)$ we have

$$
\begin{aligned}
&D_q h_{q,r,a} f(x) \\
&= D_q \left(\frac{1}{r(x)} \int_a^x r(t) f(t)\, d_q t \right) \\
&= D_q \left(\frac{1}{r(x)} \int_0^x r(t) f(t)\, d_q t \right) - D_q \left(\frac{1}{r(x)} \int_0^a r(t) f(t)\, d_q t \right) \\
&= \frac{1}{r(qx)} D_q \left(\int_0^x r(t) f(t)\, d_q t \right) + \left(\int_0^x r(t) f(t)\, d_q t \right) D_q \frac{1}{r(x)} - \left(\int_0^a r(t) f(t)\, d_q t \right) D_q \frac{1}{r(x)} \\
&= \frac{1}{r(qx)} D_q \left(\int_0^x r(t) f(t)\, d_q t \right) + \left(\int_a^x r(t) f(t)\, d_q t \right) D_q \frac{1}{r(x)} \\
&= \frac{r(x)}{r(qx)} f(x) + \left(\int_a^x r(t) f(t)\, d_q t \right) D_q \frac{1}{r(x)} \\
&= \frac{r(x)}{r(qx)} f(x) - h_{q,r,a} f(x) \frac{D_q r(x)}{r(qx)}.
\end{aligned}
$$

Hence, by $[1-p]_q = -\frac{1}{q^{(p-1)}} [(p-1)]_q$

$$
\begin{aligned}
&[p-1]_q \int_a^b \left(H_{q,r,a} f(x)\right)^p d_q x \\
&\leq \int_a^b x^{1-p} \left(\frac{r(x)}{r(qx)} f(x) - h_{r,a} f(x) \frac{D_q r(x)}{r(qx)} \right) \left(\sum_{i=0}^{p-1} \left[h_{q,r,a} f(x)\right]^{p-1-i} \left[h_{q,r,a} f(qx)\right]^i \right) d_q x \\
&= \int_a^b x^{1-p} \frac{r(x)}{r(qx)} f(x) \left[h_{q,r,a} f(x)\right]^{p-1} \left(\sum_{i=0}^{p-1} \left[\frac{h_{q,r,a} f(qx)}{h_{q,r,a} f(x)} \right]^i \right) d_q x \\
&\quad - \int_a^b x^{1-p} \left[h_{q,r,a} f(x)\right]^p \frac{D_q r(x)}{r(qx)} \left(\sum_{i=0}^{p-1} \left[\frac{h_{q,r,a} f(qx)}{h_{q,r,a} f(x)} \right]^i \right) d_q x,
\end{aligned}
$$

or equivalently

$$\int_a^b \left[[p-1]_q + x\frac{D_q r(x)}{r(qx)} \left(\sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}f(qx)}{h_{q,r,a}f(x)} \right]^i \right) \right] \left(H_{q,r,a}f(x)\right)^p \, d_q x$$
$$\leq \int_a^b \left(\frac{r(x)}{r(qx)} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}f(qx)}{h_{q,r,a}f(x)} \right]^i \right) f(x) \left(H_{q,r,a}f(x)\right)^{p-1} \, d_q x.$$

Now, using (4.5) and the q-Hölder inequality, we have

$$\frac{p}{\lambda} \int_a^b \left(H_{q,r,a}f(x)\right)^p \, d_q x$$
$$\leq \left(\int_a^b \left(\frac{r(x)}{r(qx)} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}f(qx)}{h_{q,r,a}f(x)} \right]^i \right)^p f^p(x) \, d_q x \right)^{\frac{1}{p}} \left(\int_a^b \left[H_{q,r,a}f(x)\right]^{(p-1)p'} \, d_q x \right)^{\frac{1}{p'}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, that is,

$$\int_a^b \left(H_{q,r,a}f(x)\right)^p \, d_q x \leq \frac{\lambda^p}{p^p} \int_0^\infty \left(\frac{r(x)}{r(qx)} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}f(qx)}{h_{q,r,a}f(x)} \right]^i \right)^p f^p(x) \, d_q x.$$

If we take $c > a$, then

$$\int_c^b \left(H_{q,r,a}f(x)\right)^p \, d_q x \leq \int_a^b \left(H_{q,r,a}f(x)\right)^p \, d_q x$$
$$\leq \frac{\lambda^p}{p^p} \int_0^\infty \left(\frac{r(x)}{r(qx)} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}f(qx)}{h_{q,r,a}f(x)} \right]^i \right)^p f^p(x) \, d_q x.$$

Invoking the dominated convergence theorem, taking $a \to \infty$, we get

$$\int_c^b \left(H_{q,r}f(x)\right)^p \, d_q x \leq \frac{\lambda^p}{p^p} \int_0^\infty \left(\frac{r(x)}{r(qx)} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}f(qx)}{h_{q,r,a}f(x)} \right]^i \right)^p f^p(x) \, d_q x$$

for all $c, b > 0$. Finally, letting $b \to \infty$ and $c \to 0$,

$$\int_0^\infty \left(H_{q,r}f(x)\right)^p \, d_q x \leq \frac{\lambda^p}{p^p} \int_0^\infty \left(\frac{r(x)}{r(qx)} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}f(qx)}{h_{q,r,a}f(x)} \right]^i \right)^p f^p(x) \, d_q x. \qquad \square$$

In Theorem 6 if we take the limit $q \to 1^-$ we obtain the following theorem, proved by N. Levinson in 1964 (cf. [13, Theorem 4]).

Remark 3 Let f be a nonnegative function on $(0, \infty)$, r being absolutely continuous function on $(0, \infty)$ and $p > 1$. Also assume $\int_0^\infty (f(x))^p \, dx$ is convergent, and

$$\frac{p-1}{p} + x\frac{r'}{r} \geq \frac{1}{\lambda},$$

for almost every $x > 0$ and for some $\lambda > 0$. Then we have the following inequality:

$$\int_0^\infty \big(H_r f(x)\big)^p \, dx \le \lambda^p \int_0^\infty f^p(x)\, dx,$$

where

$$H_r f(x) = \frac{1}{xr(x)} \int_0^x r(t) f(t)\, dt.$$

Theorem 7 *Assume f is a nonnegative function on $(0,\infty)$, u is absolutely continuous function on $(0,\infty)$ and $p > 1$. Also assume $\int_a^b (f(x))^p \, d_q x$ is convergent, and*

$$\frac{[p-1]_q}{p} - \frac{x}{p}\frac{D_q u(x)}{u(x)} \sum_{i=0}^{p-1} \left(\frac{u(qx)}{u(x)}\right)^{\frac{i}{p}} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}g(qx)}{h_{q,r,a}g(x)}\right]^i \ge \frac{1}{\lambda}, \tag{4.6}$$

for almost every $x > 0$ and for some $\lambda > 0$. Then we have the following inequality:

$$\int_0^\infty \big(H_q f(x)\big)^p u(x)\, d_q x \le \frac{\lambda^p}{p^p} \int_0^\infty \left(\sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}g(qx)}{h_{q,r,a}g(x)}\right]^i\right)^p f^p(x) u(qx)\, d_q x, \tag{4.7}$$

where

$$H_q f(x) = \frac{1}{x} \int_0^x f(t)\, d_q t.$$

Proof If we consider $r(x) = (\frac{1}{u(x)})^{\frac{1}{p}}$, then

$$f(x) = r(x) g(x) = \left(\frac{1}{u(x)}\right)^{\frac{1}{p}} g(x)$$

and we apply Theorem 6 to g, we assume $0 < a < b < \infty$ and

$$h_{q,r,a}g(x) = \frac{1}{r(x)} \int_a^x r(t) g(t)\, d_q t = \big(u(x)\big)^{\frac{1}{p}} \int_a^x f(t)\, d_q t.$$

Then, defining $H_{q,r,a}g(x) = \frac{1}{x} h_{q,r,a}g(x)$, and integrating by parts from (2.5) with $w = (h_{q,r,a}g(x))^p$ and $D_q v(x) = x^{-p}$ noting that $v(x) = \frac{x^{1-p}}{[1-p]_q}$ we get

$$\begin{aligned}
&\int_a^b \big(H_{q,r,a}g(x)\big)^p \, d_q x \\
&\quad = \big(h_{q,r,a}g(b)\big)^p \frac{b^{1-p}}{[1-p]_q} \\
&\qquad - \frac{q^{1-p}}{[1-p]_q} \int_0^b x^{1-p} D_q h_{q,r,a}g(x) \left(\sum_{i=0}^{p-1} \big[h_{q,r,a}g(x)\big]^{p-1-i} \big[h_{q,r,a}g(qx)\big]^i\right) d_q x \\
&\qquad + \frac{q^{1-p}}{[1-p]_q} \int_0^a x^{1-p} D_q h_{q,r,a}g(x) \left(\sum_{i=0}^{p-1} \big[h_{q,r,a}g(x)\big]^{p-1-i} \big[h_{q,r,a}g(qx)\big]^i\right) d_q x
\end{aligned}$$

$$
\begin{aligned}
&= \left(h_{q,r,a}g(b)\right)^p \frac{b^{1-p}}{[1-p]_q} \\
&\quad - \frac{q^{1-p}}{[1-p]_q} \int_a^b x^{1-p} D_q h_{q,r,a}g(x) \left(\sum_{i=0}^{p-1} \left[h_{q,r,a}g(x)\right]^{p-1-i} \left[h_{q,r,a}g(qx)\right]^i \right) d_q x.
\end{aligned}
$$

We notice that from (2.2)

$$
\left(h_{q,r,a}g(b)\right)^p \frac{b^{1-p}}{[1-p]_q} = -q^{p-1} \left(h_{q,r,a}g(b)\right)^p \frac{b^{1-p}}{[p-1]_q}
$$

is negative since $p-1 \in \mathbb{N}$, $p-1>0$ and $h_{q,r,a}g(b)>0$ with $b>0$. Also, from the definition of $h_{q,r,a}g(x)$ we have

$$
\begin{aligned}
&D_q h_{q,r,a}g(x) \\
&= D_q \left(\left(u(x)\right)^{\frac{1}{p}} \int_a^x f(t)\, d_q t \right) \\
&= D_q \left(\left(u(x)\right)^{\frac{1}{p}} \int_0^x f(t)\, d_q t \right) - D_q \left(\left(u(x)\right)^{\frac{1}{p}} \int_0^a f(t)\, d_q t \right) \\
&= \left(u(qx)\right)^{\frac{1}{p}} D_q \left(\int_0^x f(t)\, d_q t \right) + \left(\int_0^x f(t)\, d_q t \right) D_q \left(u(x)\right)^{\frac{1}{p}} - \left(\int_0^a f(t)\, d_q t \right) D_q \left(u(x)\right)^{\frac{1}{p}} \\
&= \left(u(qx)\right)^{\frac{1}{p}} D_q \left(\int_0^x f(t)\, d_q t \right) + \left(\int_a^x f(t)\, d_q t \right) D_q \left(u(x)\right)^{\frac{1}{p}} \\
&= \left(u(qx)\right)^{\frac{1}{p}} f(x) + \frac{h_{q,r,a}g(x)}{\left(u(x)\right)^{\frac{1}{p}}} \frac{D_q u(x)}{\sum_{i=0}^{p-1} \left(u(x)\right)^{\frac{p-1-i}{p}} \left(u(qx)\right)^{\frac{i}{p}}} \\
&= \left(u(qx)\right)^{\frac{1}{p}} f(x) + \frac{h_{q,r,a}g(x)}{u(x)} D_q u(x) \sum_{i=0}^{p-1} \left(\frac{u(qx)}{u(x)} \right)^{\frac{i}{p}}.
\end{aligned}
$$

Hence, by $[1-p]_q = -\frac{1}{q^{(p-1)}} [(p-1)]_q$

$$
\begin{aligned}
&[p-1]_q \int_a^b \left(H_{q,r,a}g(x)\right)^p d_q x \\
&\leq \int_a^b x^{1-p} \left(u(qx)\right)^{\frac{1}{p}} f(x) \left[h_{q,r,a}g(x)\right]^{p-1} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}g(qx)}{h_{q,r,a}g(x)} \right]^i d_q x \\
&\quad + \int_a^b x^{1-p} \frac{(h_{q,r,a}g(x))^p}{u(x)} D_q u(x) \sum_{i=0}^{p-1} \left(\frac{u(qx)}{u(x)} \right)^{\frac{i}{p}} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}g(qx)}{h_{q,r,a}g(x)} \right]^i d_q x
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&\int_a^b \left[[p-1]_q - x \frac{D_q u(x)}{u(x)} \sum_{i=0}^{p-1} \left(\frac{u(qx)}{u(x)} \right)^{\frac{i}{p}} \sum_{i=0}^{p-1} \left(\frac{h_{q,r,a}g(qx)}{h_{q,r,a}g(x)} \right)^i \right] \left(H_{q,r,a}g(x)\right)^p d_q x \\
&\leq \int_a^b \left(u(qx)\right)^{\frac{1}{p}} \sum_{i=0}^{p-1} \left[\frac{h_{q,r,a}g(qx)}{h_{q,r,a}g(x)} \right]^i f(x) \left[H_{q,r,a}g(x)\right]^{p-1} d_q x.
\end{aligned}
$$

Finally, by using (4.6) and the q-Hölder inequality, we have

$$\int_0^\infty \left(H_{q,r}f(x)\right)^p d_qx \le \frac{\lambda^p}{p^p}\int_0^\infty \left(\sum_{i=0}^{p-1}\left[\frac{h_{q,r,a}g(qx)}{h_{q,r,a}g(x)}\right]^i\right)^p f^p(x)u(qx)\,d_qx$$

and

$$\int_0^\infty \left(H_qf(x)\right)^p u(x)\,d_qx \le \frac{\lambda^p}{p^p}\int_0^\infty \left(\sum_{i=0}^{p-1}\left[\frac{h_{q,r,a}g(qx)}{h_{q,r,a}g(x)}\right]^i\right)^p f^p(x)u(qx)\,d_qx,$$

and this completes the proof. □

In Theorem 7 if we take the limit $q \to 1^-$ we obtain the following result, proved by N. Levinson in 1964 [13] on continuous analysis.

Remark 4 Assume that f is a nonnegative function on $(0,\infty)$, u is absolutely continuous function on $(0,\infty)$, and $p > 1$. Also assume $\int_a^b (f(x))^p\,dx$ is convergent, and

$$\frac{p-1}{p} - px\frac{u'}{u} \ge \frac{1}{\lambda},$$

for almost every $x > 0$ and for some $\lambda > 0$. Then we have the following inequality:

$$\int_0^\infty \left(Hf(x)\right)^p u(x)\,dx \le \lambda^p \int_0^\infty f^p(x)u(x)\,dx,$$

where

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt.$$

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