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# Hermite–Hadamard-type inequalities for geometrically *r*-convex functions in terms of Stolarsky's mean with applications to means

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**Abstract**

In this paper, we obtain new Hermite–Hadamard-type inequalities for *r*-convex and geometrically convex functions and, additionally, some new Hermite–Hadamard-type inequalities by using the Hölder–İşcan integral inequality and an improved power-mean inequality.

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## 1 Introduction

The convexity of a mapping $\mathfrak{K}:\mathbb{k}\to\mathcal{R}$ is defined as follows. A function $\mathfrak{K}:\mathbb{k}\to\mathcal{R}$, $\emptyset\neq\mathbb{k}\subseteq\mathcal{R}$, is said to be convex on $\mathbb{k}$ if

$$\mathfrak{K}\big(\mathfrak{u}\mathfrak{x}_1+(1-\mathfrak{u})\mathfrak{y}_1\big)\leq\mathfrak{u}\mathfrak{K}(\mathfrak{x}_1)+(1-\mathfrak{u})\mathfrak{K}(\mathfrak{y}_1)$$

for all $\mathfrak{x}_1,\mathfrak{y}_1\in\mathbb{k}$ and $\mathfrak{u}\in[0,1]$.

A number of papers on inequalities were published using convexity, and one of the most interesting inequalities in mathematical analysis is as follows:

$$\mathfrak{K}\left(\frac{\mathbf{j}+\mathbf{i}}{2}\right)\leq\frac{1}{\mathbf{i}-\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\mathfrak{K}(\mathfrak{x}_1)\,d\mathfrak{x}_1\leq\frac{\mathfrak{K}(\mathbf{j})+\mathfrak{K}(\mathbf{i})}{2},\tag{1.1}$$

where $\mathfrak{K}:\mathbb{k}\subseteq\mathcal{R}\to\mathcal{R}$ is a convex mapping, and $\mathbf{j},\mathbf{i}\in\mathbb{k}$ with $\mathbf{j}<\mathbf{i}$. Inequalities (1.1) are known as the Hermite–Hadamard inequalities and hold in the reversed direction if $\mathfrak{K}$ is concave.

Modern mathematicians attempt to concentrate their efforts on obtaining novel generalizations of convex functions, which has resulted in novel proofs and noticeable extensions, propositions, and improvements. For new Hermite–Hadamard-type inequalities and various applications, we refer the interested reader to a number of books and papers [1–5, 8–26], and the references therein.

Pearce et al. [18] introduced the notion of $r$-convex function as follows.

**Definition 1** ([18]) For $r \in \mathcal{R}$, a function $\mathfrak{K} : \Bbbk \subseteq \mathcal{R} \to \mathcal{R}_+ = (0, \infty)$ is said to be $r$-convex if

$$\mathfrak{K}\big(\lambda\mathfrak{x}_1 + (1-\lambda)\mathfrak{y}_1\big) \leq \begin{cases} [\lambda\mathfrak{K}^r(\mathfrak{x}_1) + (1-\lambda)\mathfrak{K}^r(\mathfrak{y}_1)]^{\frac{1}{r}}, & r \neq 0, \\ \mathfrak{K}^{\lambda}(\mathfrak{x}_1)\mathfrak{K}^{1-\lambda}(\mathfrak{y}_1), & r = 0, \end{cases}$$

for all $\mathfrak{x}_1, \mathfrak{y}_1 \in \Bbbk$ and $\lambda \in [0,1]$, where $\lambda\mathfrak{x}_1 + (1-\lambda)\mathfrak{y}_1$ and $\mathfrak{K}^{\lambda}(\mathfrak{x}_1)\mathfrak{K}^{1-\lambda}(\mathfrak{y}_1)$ are, respectively, the weighted arithmetic mean of two positive numbers $\mathfrak{x}_1$ and $\mathfrak{y}_1$ and the weighted geometric mean of $\mathfrak{K}(\mathfrak{x}_1)$ and $\mathfrak{K}(\mathfrak{y}_1)$.

Many authors studied the properties of $r$-convex functions; we refer the interested readers to [5–7, 23, 25]. A number of inequalities of Hermite–Hadamard type related to $r$-convex functions are proved in [23] and [25].

Xi and Qi [25] defined geometrically $r$-convex functions and established some new Hermite–Hadamard-type inequalities for them.

**Definition 2** ([25]) For $r \in \mathcal{R}$, a function $\mathfrak{K} : \Bbbk \subseteq \mathcal{R}_+ \to \mathcal{R}_+$ is said to be geometrically $r$-convex if

$$\mathfrak{K}\big(\mathfrak{x}_1^{\lambda}\mathfrak{y}_1^{1-\lambda}\big) \leq \begin{cases} [\lambda\mathfrak{K}^r(\mathfrak{x}_1) + (1-\lambda)\mathfrak{K}^r(\mathfrak{y}_1)]^{\frac{1}{r}}, & r \neq 0, \\ \mathfrak{K}^{\lambda}(\mathfrak{x}_1)\mathfrak{K}^{1-\lambda}(\mathfrak{y}_1), & r = 0, \end{cases}$$

for all $\mathfrak{x}_1, \mathfrak{y}_1 \in \Bbbk$ and $\lambda \in [0,1]$.

It is clear that a geometrically $r$-convex function becomes geometrically convex for $r = 0$ and GA-convex for $r = 1$.

*Remark* 1 ([25]) If $\mathfrak{K}(\mathfrak{x}_1)$ is a decreasing geometrically $r$-convex function on $\Bbbk \subseteq \mathcal{R}_+$, then $\mathfrak{K}(\mathfrak{x}_1)$ is also $r$-convex on $\Bbbk$. Conversely, if $\mathfrak{K}(\mathfrak{x}_1)$ is an increasing $r$-convex function on $\Bbbk$, then $\mathfrak{K}(\mathfrak{x}_1)$ is geometrically $r$-convex on $\Bbbk$.

*Remark* 2 ([10, Theorem 16, p. 26.]) If the right-hand side in Definition 2 is denoted by $\mathfrak{M}_r(\mathfrak{K}(\mathfrak{x}_1), \mathfrak{K}(\mathfrak{y}_1))$, then

$$\mathfrak{M}_{r_1}\big(\mathfrak{K}(\mathfrak{x}_1), \mathfrak{K}(\mathfrak{y}_1)\big) \leq \mathfrak{M}_{r_2}\big(\mathfrak{K}(\mathfrak{x}_1), \mathfrak{K}(\mathfrak{y}_1)\big)$$

for $r_1, r_2 \in \mathcal{R}$ with $r_1 < r_2$. Consequently, if $r_1, r_2 \in \mathcal{R}$ with $r_1 < r_2$ and $\mathfrak{K}(\mathfrak{x}_1)$ is a geometrically $r_1$-convex function on $\Bbbk \subseteq \mathcal{R}_+$, then $\mathfrak{K}(\mathfrak{x}_1)$ is geometrically $r_2$-convex on $\Bbbk$.

*Remark* 3 ([25]) Let $\mathfrak{K} : \Bbbk \subseteq \mathcal{R}_+ \to \mathcal{R}_+$ be a geometrically $r$-convex function for $r \in \mathcal{R}$, let $g(u) = \mathfrak{K}(e^u)$, and let $u \in \ln\Bbbk = \{\ln u : u \in \Bbbk\}$. Then $g$ is $r$-convex if and only if $\mathfrak{K}$ is geometrically $r$-convex.

The purpose of this paper is to provide new geometrically $r$-convex inequalities of the Hermite–Hadamard type by new methods.

## 2 Main Results

**Proposition 1** *For $r \in \mathcal{R}$, let $\mathfrak{K} : [\mathbf{j}, \mathbf{i}] \subseteq \mathcal{R}_+ \to \mathcal{R}_+$ be a geometrically r-convex function, and let $\mathfrak{K} \in L([\mathbf{j}, \mathbf{i}])$. Then*

$$\frac{1}{2n} \int_0^n \mathfrak{K}\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\, d\mathfrak{u} \leq \begin{cases} E(\mathfrak{K}^r(\mathbf{j}), \mathfrak{K}^r(\mathbf{i}); r, r+1) + \frac{r(\mathfrak{K}^{r+1}(\mathbf{j}) - [A(\mathfrak{K}^r(\mathbf{j}), \mathfrak{K}^r(\mathbf{i}))]^{1+\frac{1}{r}})}{(r+1)(\mathfrak{K}^r(\mathbf{i}) - \mathfrak{K}^r(\mathbf{j}))}, & r \neq 0, \\ \sqrt{\mathfrak{K}(\mathbf{i})} E(\mathfrak{K}(\mathbf{j}), \mathfrak{K}(\mathbf{i}); 0, 1), & r = 0, \end{cases} \tag{2.1}$$

*where $E(u, v; r, s)$ is Stolarsky's mean defined by*

$$E(u, v; r, s) = \left[\frac{r(v^s - u^s)}{s(v^r - u^r)}\right]^{\frac{1}{s-r}}, \quad rs(r-s)(u-v) \neq 0,$$

$$E(u, v; 0, s) = \left[\frac{v^s - u^s}{s(\ln v - \ln u)}\right]^{\frac{1}{s}}, \quad s(u-v) \neq 0,$$

$$E(u, v; r, r) = \frac{1}{e^{\frac{1}{r}}} \left(\frac{u^{u^r}}{v^{v^r}}\right)^{\frac{1}{u^r - v^r}}, \quad r(u-v) \neq 0,$$

$$E(u, v; 0, 0) = \sqrt{uv}, \quad u \neq v,$$

$$E(u, u; r, s) = u, \quad u = v,$$

*$L(u, v)$ is the logarithmic mean defined by*

$$E(u, v; 0, 1) = L(u, v),$$

*and $A(\mathfrak{K}^r(\mathbf{j}), \mathfrak{K}^r(\mathbf{i}))$ is the arithmetic mean of $\mathfrak{K}^r(\mathbf{j})$ and $\mathfrak{K}^r(\mathbf{i})$ for $(u, v) \in \mathcal{R}_+^2$, $(r, s) \in \mathcal{R}^2$.*

*Proof* By the geometric $r$-convexity of $\mathfrak{K}$ we have

**Case I:** For $r = 0$,

$$\begin{aligned} \frac{1}{2n} \int_0^n \mathfrak{K}\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\, d\mathfrak{u} &\leq \frac{1}{2n} \int_0^n \big[\mathfrak{K}(\mathbf{j})\big]^{\frac{n-\mathfrak{u}}{2n}} \big[\mathfrak{K}(\mathbf{i})\big]^{\frac{n+\mathfrak{u}}{2n}}\, d\mathfrak{u} \\ &= \frac{\sqrt{\mathfrak{K}(\mathbf{i})}(\mathfrak{K}(\mathbf{i}) - \mathfrak{K}(\mathbf{j}))}{\ln \mathfrak{K}(\mathbf{i}) - \ln \mathfrak{K}(\mathbf{j})} \\ &= \sqrt{\mathfrak{K}(\mathbf{i})} E\big(\mathfrak{K}(\mathbf{j}), \mathfrak{K}(\mathbf{i}); 0, 1\big). \end{aligned} \tag{2.2}$$

**Case II:** Suppose now that $r \neq 0$. Then

$$\frac{1}{2n} \int_0^n \mathfrak{K}\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\, d\mathfrak{u} \leq \frac{1}{2n} \int_0^n \left[\left(\frac{n-\mathfrak{u}}{2n}\right) \mathfrak{K}^r(\mathbf{j}) + \left(\frac{n+\mathfrak{u}}{2n}\right) \mathfrak{K}^r(\mathbf{i})\right]^{\frac{1}{r}} d\mathfrak{u}. \tag{2.3}$$

Let

$$\left(\frac{n-\mathfrak{u}}{2n}\right) \mathfrak{K}^r(\mathbf{j}) + \left(\frac{n+\mathfrak{u}}{2n}\right) \mathfrak{K}^r(\mathbf{i}) = \mathfrak{y}_1.$$

Thus

$$\begin{aligned}\frac{1}{2n}\int_0^n \mathfrak{K}\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\,d\mathfrak{u} &\leq \frac{1}{\mathfrak{K}^r(\mathbf{i})-\mathfrak{K}^r(\mathbf{j})}\int_{A(\mathfrak{K}^r(\mathbf{j}),\mathfrak{K}^r(\mathbf{i}))}^{\mathfrak{K}^r(\mathbf{i})} \mathfrak{y}_1^{\frac{1}{r}}\,d\mathfrak{y}_1\\ &= E\big(\mathfrak{K}^r(\mathbf{j}),\mathfrak{K}^r(\mathbf{i});r,r+1\big)\\ &\quad+\frac{r(\mathfrak{K}^{r+1}(\mathbf{j})-[A(\mathfrak{K}^r(\mathbf{j}),\mathfrak{K}^r(\mathbf{i}))]^{1+\frac{1}{r}})}{(r+1)(\mathfrak{K}^r(\mathbf{i})-\mathfrak{K}^r(\mathbf{j}))},\end{aligned}$$

where $A(\mathfrak{K}^r(\mathbf{j}),\mathfrak{K}^r(\mathbf{i}))$ is the arithmetic mean of $\mathfrak{K}^r(\mathbf{j})$ and $\mathfrak{K}^r(\mathbf{i})$, and the result is achieved. □

**Lemma 1** *Let* $\mathfrak{K}:\Bbbk\subseteq\mathcal{R}_+=(0,\infty)\to\mathcal{R}$ *be a differentiable function on* $\Bbbk^\circ$, *and let* $\mathbf{j},\mathbf{i}\in\Bbbk^\circ$ *with* $\mathbf{j}<\mathbf{i}$. *If* $\mathfrak{K}'\in L([\mathbf{j},\mathbf{i}])$, *then*

$$\begin{aligned}&\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\\ &\quad=\frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\int_0^n \mathfrak{u}\big[\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{1+\mathfrak{u}}{2}}\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)-\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\big]\,d\mathfrak{u}.\end{aligned}\tag{2.4}$$

*Proof* Let

$$\Bbbk_1=\frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\,d\mathfrak{u}$$

and

$$\Bbbk_2=\frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\,d\mathfrak{u}.$$

By integration by parts we have

$$\begin{aligned}\Bbbk_1&=\frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\,d\mathfrak{u}\\ &=\frac{1}{2n}\int_0^n \mathfrak{u}d\big[\mathfrak{K}\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\big]\\ &=\frac{1}{2}\mathfrak{K}(\mathbf{i})-\frac{1}{2n}\int_0^n \mathfrak{K}\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\,d\mathfrak{u}\\ &=\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\sqrt{\mathbf{j}\mathbf{i}}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1.\end{aligned}\tag{2.5}$$

Analogously, we have

$$\Bbbk_2=\frac{\mathfrak{K}(\mathbf{j})}{2}+\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\sqrt{\mathbf{j}\mathbf{i}}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1.\tag{2.6}$$

From (2.5) and (2.6) we get the required identity. □

**Lemma 2** *For $u, v > 0$, we have*

$$T_0(u,v) = \frac{1}{2n}\int_0^n u^{\frac{n-\mathrm{u}}{2n}} v^{\frac{n+\mathrm{u}}{2n}}\,d\mathrm{u} = \begin{cases} \frac{1}{2}\sqrt{v}[E(u,v;0,\frac{1}{2})]^2, & u \neq v, \\ \frac{1}{4}u, & u = v, \end{cases}$$

$$R_n(u,v) = \frac{1}{2n}\int_0^n \mathrm{u}u^{\frac{n-\mathrm{u}}{2n}} v^{\frac{n+\mathrm{u}}{2n}}\,d\mathrm{u} = \begin{cases} \frac{u-n[E(u,v;0,\frac{1}{2})]^2}{\ln v-\ln u} + E(u,v;0,\frac{1}{2}), & u \neq v, \\ \frac{1}{4}u, & u = v, \end{cases}$$

*and*

$$S_n(u,v) = \frac{1}{2n}\int_0^n \mathrm{u}^2 u^{\frac{n-\mathrm{u}}{2n}} v^{\frac{n+\mathrm{u}}{2n}}\,d\mathrm{u} = \begin{cases} \frac{4n^2[E(u,v;0,\frac{1}{2})]^2-u(\ln v-\ln u+1)}{(\ln v-\ln u)^2} - \frac{(4+\ln v-\ln u)E(u,v,0,1)}{\ln v-\ln u}, & u \neq v, \\ \frac{1}{6}u, & u = v. \end{cases}$$

*Proof* The proof follows from a straightforward computation. □

**Lemma 3** *For $u, v > 0$ and $r \in \mathcal{R}$ with $r \neq 0$, we have*

$$\frac{1}{2n}\int_0^n \left[\left(\frac{n-\mathrm{u}}{2n}\right)u^r + \left(\frac{n+\mathrm{u}}{2n}\right)v^r\right]^{\frac{1}{r}} d\mathrm{u} = \theta(u,v;r),$$

$$\frac{1}{2n}\int_0^n \mathrm{u}\left[\left(\frac{n-\mathrm{u}}{2n}\right)u^r + \left(\frac{n+\mathrm{u}}{2n}\right)v^r\right]^{\frac{1}{r}} d\mathrm{u} = \theta_{n,1}(u,v;r),$$

*and*

$$\frac{1}{2n}\int_0^n \mathrm{u}^2\left[\left(\frac{n-\mathrm{u}}{2n}\right)u^r + \left(\frac{n+\mathrm{u}}{2n}\right)v^r\right]^{\frac{1}{r}} d\mathrm{u} = \theta_{n,2}(u,v;r),$$

*where*

$$\theta(u,v;r) = \begin{cases} E(u,v;r,r+1) + \frac{r[u^{r+1}-[A(u,v)]^{1+\frac{1}{r}}]}{(r+1)(v^r-u^r)}, & r \neq -1, \\ \frac{\ln v-\ln[A(u^{-1},v^{-1})]}{v^{-1}-u^{-1}}, & r = -1, \end{cases}$$

$$\theta_{n,1}(u,v;r) = \begin{cases} \frac{2n[E^r(u,v;r,2r+1)-A(u^r,v^r)E(u,v;r,r+1)]}{v^r-u^r} & u \neq v, \\ \quad + \frac{2nr[(r+1)u^{2r+1}-(2r+1)u^{r+1}A(u^r,v^r)+r[A(u^r,v^r)]^{2+\frac{1}{r}}]}{(r+1)(2r+1)(v^r-u^r)^2}, & r \neq -1, -\frac{1}{2}, \\ \frac{2n[v^{-1}+A(u^{-1},v^{-1})\ln[A(u^{-1},v^{-1})]+A(u^{-1},v^{-1})\ln v-A(u^{-1},v^{-1})]}{(v^{-1}-u^{-1})^2}, & u \neq v, r = -1, \\ \frac{n[2\ln[A(u^{-\frac{1}{2}},v^{-\frac{1}{2}})]+2v^{\frac{1}{2}}A(u^{-\frac{1}{2}},v^{-\frac{1}{2}})-\ln v-2]}{(v^{-\frac{1}{2}}-u^{-\frac{1}{2}})^2}, & u \neq v, r = -\frac{1}{2}, \\ \frac{1}{4}u, & u = v, \end{cases}$$

$$\theta_{n,2}(u,v;r)=\begin{cases} \frac{4n^2r[(r+1)(2r+1)u^{3r+1}-2r^2[A(u^r,v^r)]^{3+\frac{1}{r}}]}{(r+1)(2r+1)(3r+1)(v^r-u^r)^3} \\ \quad +\frac{4n^2r[(2r+1)A(u^r,v^r)-2u(r+1)]u^{r+1}A(u^r,v^r)}{(r+1)(2r+1)(v^r-u^r)^3} \\ \quad +\frac{4n^2[[A(u^r,v^r)]^2E(u,v;r,r+1)+[E(u,v;r,2r+1)]^{r+1}]}{(v^r-u^r)^2}, & u\neq v,\ r\neq -1,-\frac{1}{2},-\frac{1}{3}, \\ \frac{2n^2[1-4vA(u^{-1},v^{-1})+v^2[A(u^{-1},v^{-1})]^2[3-2\ln[A(u^{-1},v^{-1})]]]}{v^2(v^{-1}-u^{-1})^3} \\ \quad -\frac{4n^2[A(u^{-1},v^{-1})]^2\ln v}{(v^{-1}-u^{-1})^3}, & u\neq v, r=-1, \\ \frac{4n^2[A(u^{-\frac{1}{2}},v^{-\frac{1}{2}})\ln v+2A(u^{-\frac{1}{2}},v^{-\frac{1}{2}})\ln[A(u^{-\frac{1}{2}},v^{-\frac{1}{2}})]]}{(v^{-\frac{1}{2}}-u^{-\frac{1}{2}})^3} \\ \quad -\frac{4n^2v^{-\frac{1}{2}}(v[A(u^{-\frac{1}{2}},v^{-\frac{1}{2}})]^2-1)}{(v^{-\frac{1}{2}}-u^{-\frac{1}{2}})^3}, & u\neq v, r=-\frac{1}{2}, \\ \frac{2n^2uv\{6\ln[A(u^{-\frac{1}{3}},v^{-\frac{1}{3}})]+3v^{\frac{2}{3}}[A(u^{-\frac{1}{3}},v^{-\frac{1}{3}})]^2\}}{3(v^{\frac{1}{3}}-u^{\frac{1}{3}})^3} \\ \quad +\frac{2\ln v-12v^{\frac{1}{3}}A(u^{-\frac{1}{3}},v^{-\frac{1}{3}})+9}{3(v^{\frac{1}{3}}-u^{\frac{1}{3}})^3}, & u\neq v, r=-\frac{1}{3}, \\ \frac{1}{6}u, & u=v. \end{cases}$$

*Proof* The proof is obvious when $u=v$ and when $u\neq v$ and $r=-1,-\frac{1}{2},-\frac{1}{3}$.

Suppose $u\neq v$ and $r\neq -1,-\frac{1}{2},-\frac{1}{3}$. Then we have

$$\begin{aligned} &\frac{1}{2n}\int_0^1 \mathfrak{u}\left[\left(\frac{n-\mathfrak{u}}{2n}\right)u^r+\left(\frac{n+\mathfrak{u}}{2n}\right)v^r\right]^{\frac{1}{r}}\,d\mathfrak{u} \\ &\quad =\frac{2nr^2[A(u^r,v^r)]^{2+\frac{1}{r}}-2nrv^{r+1}(2r+1)A(u^r,v^r)+2nr(r+1)v^{2r+1}}{(r+1)(2r+1)(v^r-u^r)^2} \\ &\quad =\frac{2nr^2[A(u^r,v^r)]^{2+\frac{1}{r}}}{(r+1)(2r+1)(v^r-u^r)^2} \\ &\qquad +\frac{2n[E(u,v;r,2r+1)]^{r+1}-2nA(u^r,v^r)E(u,v;r,r+1)}{v^r-u^r} \\ &\qquad +\frac{2nr(r+1)u^{2r+1}-2nr(2r+1)u^{r+1}A(u^r,v^r)}{(r+1)(2r+1)(v^r-u^r)^2} \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2n}\int_0^1 \mathfrak{u}^2\left[\left(\frac{n-\mathfrak{u}}{2n}\right)u^r+\left(\frac{n+\mathfrak{u}}{2n}\right)v^r\right]^{\frac{1}{r}}\,d\mathfrak{u} \\ &\quad =-\frac{8n^2r^3[A(u^r,v^r)]^{3+\frac{1}{r}}}{(r+1)(2r+1)(3r+1)(v^r-u^r)^3}+\frac{4rn^2v^{r+1}[A(u^r,v^r)]^2}{(r+1)(v^r-u^r)^3} \\ &\qquad -\frac{8n^2rv^{2r+1}A(u^r,v^r)}{(2r+1)(v^r-u^r)^3}+\frac{4n^2rv^{3r+1}}{(3r+1)(v^r-u^r)^3} \\ &\quad =\frac{4n^2r[(r+1)(2r+1)u^{3r+1}-2r^2[A(u^r,v^r)]^{3+\frac{1}{r}}]}{(r+1)(2r+1)(3r+1)(v^r-u^r)^3} \\ &\qquad +\frac{4n^2r[(2r+1)A(u^r,v^r)-2u(r+1)]u^{r+1}A(u^r,v^r)}{(r+1)(2r+1)(v^r-u^r)^3} \\ &\qquad +\frac{4n^2[[A(u^r,v^r)]^2E(u,v;r,r+1)+[E(u,v;r,2r+1)]^{r+1}]}{(v^r-u^r)^2}. \end{aligned}$$

□

We now establish new Hermite–Hadamard-type inequalities for geometrically $r$-convex functions. We believe that our results provide a refinement of the results proved in [25].

**Lemma 4** *For $u, v > 0$,*

$$\int_0^1 u^{\frac{1-\mathfrak{u}}{2}} v^{\frac{1+\mathfrak{u}}{2}}\,d\mathfrak{u} \le \int_0^1 u^{1-\mathfrak{u}} v^{\mathfrak{u}}\,d\mathfrak{u},$$

$$\int_0^1 \mathfrak{u} u^{\frac{1-\mathfrak{u}}{2}} v^{\frac{1+\mathfrak{u}}{2}}\,d\mathfrak{u} \le \int_0^1 \mathfrak{u} u^{1-\mathfrak{u}} v^{\mathfrak{u}}\,d\mathfrak{u},$$

*and*

$$\int_0^1 \mathfrak{u}^2 u^{\frac{1-\mathfrak{u}}{2}} v^{\frac{1+\mathfrak{u}}{2}}\,d\mathfrak{u} \le \int_0^1 \mathfrak{u}^2 u^{1-\mathfrak{u}} v^{\mathfrak{u}}\,d\mathfrak{u}.$$

*Proof* It is obvious. □

**Lemma 5** *For $u, v > 0$ and $r \in \mathcal{R}$ with $r \neq 0$, $\mathfrak{u} \in [0,1]$, we have*

$$\int_0^1 \left[\left(\frac{1-\mathfrak{u}}{2}\right)u^r + \left(\frac{1+\mathfrak{u}}{2}\right)v^r\right]^{\frac{1}{r}} d\mathfrak{u} \le \int_0^1 \left[(1-\mathfrak{u})u^r + \mathfrak{u}v^r\right]^{\frac{1}{r}} d\mathfrak{u},$$

$$\int_0^1 \mathfrak{u}\left[\left(\frac{1-\mathfrak{u}}{2}\right)u^r + \left(\frac{1+\mathfrak{u}}{2}\right)v^r\right]^{\frac{1}{r}} d\mathfrak{u} \le \int_0^1 \mathfrak{u}\left[(1-\mathfrak{u})u^r + \mathfrak{u}v^r\right]^{\frac{1}{r}} d\mathfrak{u},$$

*and*

$$\int_0^1 \mathfrak{u}^2\left[\left(\frac{1-\mathfrak{u}}{2}\right)u^r + \left(\frac{1+\mathfrak{u}}{2}\right)v^r\right]^{\frac{1}{r}} d\mathfrak{u} \le \int_0^1 \mathfrak{u}^2\left[(1-\mathfrak{u})u^r + \mathfrak{u}v^r\right]^{\frac{1}{r}} d\mathfrak{u}.$$

*Proof* It is obvious. □

**Theorem 1** *Let $\mathfrak{K} : \mathbb{k} \subseteq \mathcal{R}_+ = (0,\infty) \to \mathcal{R}$ be a differentiable function on $\mathbb{k}^\circ$, where $\mathbf{j}, \mathbf{i} \in \mathbb{k}^\circ$ with $\mathbf{j} < \mathbf{i}$ and $r \in \mathcal{R}$, $r \neq 0$. Suppose that $\mathfrak{K}' \in L([\mathbf{j},\mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically r-convex on $[\mathbf{j},\mathbf{i}]$ for $q \ge 1$. Then*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i} - \ln\mathbf{j}} \int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \\
&\quad\le \frac{(\ln\mathbf{i} - \ln\mathbf{j})}{4n^2}\Big\{\big[\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q, \big|\mathfrak{K}'(\mathbf{i})\big|^q; r\big)\big]^{1-\frac{1}{q}} \\
&\qquad\times \big[n(\mathbf{j}+\mathbf{i})\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q, \big|\mathfrak{K}'(\mathbf{i})\big|^q; r\big) + (\mathbf{i}-\mathbf{j}) \\
&\qquad\times \theta_{n,2}\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q, \big|\mathfrak{K}'(\mathbf{i})\big|^q; r\big)\big]^{\frac{1}{q}} + \big[\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q, \big|\mathfrak{K}'(\mathbf{j})\big|^q; r\big)\big]^{1-\frac{1}{q}} \\
&\qquad\times \big[n(\mathbf{j}+\mathbf{i})\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q, \big|\mathfrak{K}'(\mathbf{j})\big|^q; r\big) \\
&\qquad+ (\mathbf{j}-\mathbf{i})\theta_{n,2}\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q, \big|\mathfrak{K}'(\mathbf{j})\big|^q; r\big)\big]^{\frac{1}{q}}\Big\}. 
\end{aligned} \tag{2.7}$$

*Proof* From Lemma 1 and the power-mean inequality we have

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i} - \ln\mathbf{j}} \int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \\
&\quad\le \frac{\ln\mathbf{i} - \ln\mathbf{j}}{4n^2}
\end{aligned}$$

$$\times \left\{ \left( \int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}} \right)^{1-\frac{1}{q}} \left( \int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}} \left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q \right)^{\frac{1}{q}} + \left( \int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n-\mathfrak{u}}{2n}} \right)^{1-\frac{1}{q}} \left( \int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n-\mathfrak{u}}{2n}} \left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q \right)^{\frac{1}{q}} \right\}. \tag{2.8}$$

Since $|\mathfrak{K}'|^q$ is geometrically $r$-convex on $[\mathbf{j},\mathbf{i}]$ for $q \geq 1$, using Lemma 3, we have

$$\begin{aligned}
&\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}} \left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q \\
&\quad\leq \int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}} \left[ \left(\frac{n-\mathfrak{u}}{2n}\right) \left|\mathfrak{K}'(\mathbf{j})\right|^{rq} + \left(\frac{1+\mathfrak{u}}{2n}\right) \left|\mathfrak{K}'(\mathbf{i})\right|^{rq} \right]^{\frac{1}{r}} d\mathfrak{u} \\
&\quad\leq \int_0^n \mathfrak{u}\left(\frac{n-\mathfrak{u}}{2n}\mathbf{j} + \frac{n+\mathfrak{u}}{2n}\mathbf{i}\right) \\
&\qquad\times \left[ \left(\frac{n-\mathfrak{u}}{2n}\right) \left|\mathfrak{K}'(\mathbf{j})\right|^{rq} + \left(\frac{n+\mathfrak{u}}{2n}\right) \left|\mathfrak{K}'(\mathbf{i})\right|^{rq} \right]^{\frac{1}{r}} d\mathfrak{u} \\
&\quad= n(\mathbf{j}+\mathbf{i})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right) + (\mathbf{i}-\mathbf{j})\theta_{n,2}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right)
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n-\mathfrak{u}}{2n}} \left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q \\
&\quad\leq \int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n-\mathfrak{u}}{2n}} \left[ \left(\frac{n+\mathfrak{u}}{2n}\right) \left|\mathfrak{K}'(\mathbf{j})\right|^{rq} + \left(\frac{n-\mathfrak{u}}{2n}\right) \left|\mathfrak{K}'(\mathbf{i})\right|^{rq} \right]^{\frac{1}{r}} d\mathfrak{u} \\
&\quad\leq \int_0^n \mathfrak{u}\left(\frac{n+\mathfrak{u}}{2n}\mathbf{j} + \frac{n-\mathfrak{u}}{2n}\mathbf{i}\right) \left[ \left(\frac{n+\mathfrak{u}}{2n}\right) \left|\mathfrak{K}'(\mathbf{j})\right|^{rq} + \left(\frac{n-\mathfrak{u}}{2n}\right) \left|\mathfrak{K}'(\mathbf{i})\right|^{rq} \right]^{\frac{1}{r}} d\mathfrak{u} \\
&\quad= n(\mathbf{j}+\mathbf{i})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right) \\
&\qquad+ (\mathbf{j}-\mathbf{i})\theta_{n,2}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right).
\end{aligned} \tag{2.10}$$

Using (2.9) and (2.10) in (2.8), we get the required result. □

**Corollary 1** *We observe that for $n = 1$, we obtain*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln \mathbf{i} - \ln \mathbf{j}} \int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1} d\mathfrak{x}_1\right| \\
&\quad\leq \frac{(\ln \mathbf{i} - \ln \mathbf{j})}{4} \left\{ \left[\theta_{1,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right)\right]^{1-\frac{1}{q}} \right. \\
&\qquad\times \left[(\mathbf{j}+\mathbf{i})\theta_{1,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right) + (\mathbf{i}-\mathbf{j}) \right. \\
&\qquad\times \left.\theta_{1,2}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right)\right]^{\frac{1}{q}} + \left[\theta_{1,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right)\right]^{1-\frac{1}{q}} \\
&\qquad\times \left[(\mathbf{j}+\mathbf{i})\theta_{1,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right) \right. \\
&\qquad\left.\left. + (\mathbf{j}-\mathbf{i})\theta_{1,2}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right)\right]^{\frac{1}{q}} \right\},
\end{aligned} \tag{2.11}$$

*where $\theta_{1,1}(|\mathfrak{K}'(\mathbf{i})|^q, |\mathfrak{K}'(\mathbf{j})|^q; r)$ and $\theta_{1,2}(|\mathfrak{K}'(\mathbf{i})|^q, |\mathfrak{K}'(\mathbf{j})|^q; r)$ can be evaluated using Lemma 3.*

**Corollary 2** *Suppose the assumptions of Theorem 1 are satisfied. If $q = 1$, then*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln \mathbf{i} - \ln \mathbf{j}} \int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\, d\mathfrak{x}_1\right| \\
&\quad \le \frac{(\ln \mathbf{i} - \ln \mathbf{j})}{4n^2} \\
&\qquad \times \left\{n(\mathbf{j} + \mathbf{i})\left[\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|, \left|\mathfrak{K}'(\mathbf{i})\right|; r\right) + \theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|, \left|\mathfrak{K}'(\mathbf{j})\right|; r\right)\right]\right. \\
&\qquad \left. + (\mathbf{i} - \mathbf{j})\left[\theta_{n,2}\left(\left|\mathfrak{K}'(\mathbf{j})\right|, \left|\mathfrak{K}'(\mathbf{i})\right|; r\right) - \theta_{n,2}\left(\left|\mathfrak{K}'(\mathbf{i})\right|, \left|\mathfrak{K}'(\mathbf{j})\right|; r\right)\right]\right\}. \qquad (2.12)
\end{aligned}$$

**Corollary 3** *Letting $n = 1$ and $q = 1$ in Theorem 1 gives*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln \mathbf{i} - \ln \mathbf{j}} \int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\, d\mathfrak{x}_1\right| \\
&\quad \le \frac{(\ln \mathbf{i} - \ln \mathbf{j})}{4} \\
&\qquad \times \left\{(\mathbf{j} + \mathbf{i})\left[\theta_{1,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|, \left|\mathfrak{K}'(\mathbf{i})\right|; r\right) + \theta_{1,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|, \left|\mathfrak{K}'(\mathbf{j})\right|; r\right)\right]\right. \\
&\qquad \left. + (\mathbf{i} - \mathbf{j})\left[\theta_{1,2}\left(\left|\mathfrak{K}'(\mathbf{j})\right|, \left|\mathfrak{K}'(\mathbf{i})\right|; r\right) - \theta_{1,2}\left(\left|\mathfrak{K}'(\mathbf{i})\right|, \left|\mathfrak{K}'(\mathbf{j})\right|; r\right)\right]\right\}. \qquad (2.13)
\end{aligned}$$

**Theorem 2** *Let $\mathfrak{K} : \Bbbk \subseteq \mathcal{R}_+ = (0, \infty) \to \mathcal{R}$ be a differentiable function on $\Bbbk^\circ$, where $\mathbf{j}, \mathbf{i} \in \Bbbk^\circ$ with $\mathbf{j} < \mathbf{i}$ and $r \in \mathcal{R}$, $r \neq 0$. Suppose that $\mathfrak{K}' \in L([\mathbf{j}, \mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically $r$-convex on $[\mathbf{j}, \mathbf{i}]$ for $q > 1$. Then*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln \mathbf{i} - \ln \mathbf{j}} \int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\, d\mathfrak{x}_1\right| \\
&\quad \le \frac{(\ln \mathbf{i} - \ln \mathbf{j})}{2n}\left\{\left[R_n\left(\mathbf{j}^{\frac{q}{q-1}}, \mathbf{i}^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}}\left[\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right)\right]^{\frac{1}{q}}\right. \\
&\qquad \left. + \left[R_n\left(\mathbf{i}^{\frac{q}{q-1}}, \mathbf{j}^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}}\left[\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right)\right]^{\frac{1}{q}}\right\}. \qquad (2.14)
\end{aligned}$$

*Proof* From Lemma 1 and Hölder's inequality we have

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln \mathbf{i} - \ln \mathbf{j}} \int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\, d\mathfrak{x}_1\right| \\
&\quad \le \frac{\ln \mathbf{i} - \ln \mathbf{j}}{4n^2}\left\{\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{q(n-\mathfrak{u})}{2n(q-1)}}\mathbf{i}^{\frac{q(n+\mathfrak{u})}{2n(q-1)}}\, d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{q(n+\mathfrak{u})}{2n(q-1)}}\mathbf{i}^{\frac{q(n-\mathfrak{u})}{2n(q-1)}}\, d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}\right\}. \qquad (2.15)
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^n \mathfrak{u}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u} \\
&\quad \le \int_0^n \mathfrak{u}\left[\left(\frac{n-\mathfrak{u}}{2n}\right)\left|\mathfrak{K}'(\mathbf{j})\right|^{rq} + \left(\frac{n+\mathfrak{u}}{2n}\right)\left|\mathfrak{K}'(\mathbf{i})\right|^{rq}\right]^{\frac{1}{r}} d\mathfrak{u} \\
&\quad = 2n\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right), \qquad (2.16)
\end{aligned}$$

$$\begin{aligned}
&\int_0^n \mathfrak{u}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\\
&\quad\le \int_0^n \mathfrak{u}\left[\left(\frac{n+\mathfrak{u}}{2n}\right)\big|\mathfrak{K}'(\mathbf{j})\big|^{rq}+\left(\frac{n-\mathfrak{u}}{2n}\right)\big|\mathfrak{K}'(\mathbf{i})\big|^{rq}\right]^{\frac{1}{r}}d\mathfrak{u}\\
&\quad= 2n\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q,\big|\mathfrak{K}'(\mathbf{j})\big|^q;r\big),
\end{aligned}\tag{2.17}$$

$$\int_0^n \mathfrak{u}\mathbf{j}^{\frac{q(n-\mathfrak{u})}{2n(q-1)}}\mathbf{i}^{\frac{q(n+\mathfrak{u})}{2n(q-1)}}\,d\mathfrak{u}=2nR_n\big(\mathbf{j}^{\frac{q}{q-1}},\mathbf{i}^{\frac{q}{q-1}}\big),\tag{2.18}$$

and

$$\int_0^n \mathfrak{u}\mathbf{j}^{\frac{q(n+\mathfrak{u})}{2n(q-1)}}\mathbf{i}^{\frac{q(n-\mathfrak{u})}{2n(q-1)}}\,d\mathfrak{u}=2nR_n\big(\mathbf{i}^{\frac{q}{q-1}},\mathbf{j}^{\frac{q}{q-1}}\big).\tag{2.19}$$

Inequality (2.14) is proved by applying (2.16)–(2.19) in (2.15). □

**Theorem 3** *Let $\mathfrak{K}:\Bbbk\subseteq\mathcal{R}_+=(0,\infty)\to\mathcal{R}$ be a differentiable function on $\Bbbk^\circ$, where $\mathbf{j},\mathbf{i}\in\Bbbk^\circ$ with $\mathbf{j}<\mathbf{i}$ and $r\in\mathcal{R}$, $r\neq 0$. Suppose that $\mathfrak{K}'\in L([\mathbf{j},\mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically $r$-convex on $[\mathbf{j},\mathbf{i}]$ for $q>1$. Then*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right|\\
&\quad\le\frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\big\{\big[\vartheta_n(\mathbf{j},\mathbf{i})\big]^{1-\frac{1}{q}}\big[n(\mathbf{j}+\mathbf{i})\theta\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q,\big|\mathfrak{K}'(\mathbf{i})\big|^q;r\big)\\
&\qquad+(\mathbf{i}-\mathbf{j})\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q,\big|\mathfrak{K}'(\mathbf{i})\big|^q;r\big)\big]^{\frac{1}{q}}+\big[\vartheta_n(\mathbf{i},\mathbf{j})\big]^{1-\frac{1}{q}}\\
&\qquad\times\big[n(\mathbf{j}+\mathbf{i})\theta\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q,\big|\mathfrak{K}'(\mathbf{j})\big|^q;r\big)\\
&\qquad+(\mathbf{j}-\mathbf{i})\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q,\big|\mathfrak{K}'(\mathbf{j})\big|^q;r\big)\big]^{\frac{1}{q}}\big\}.
\end{aligned}\tag{2.20}$$

*Proof* From Lemma 1 and Hölder's inequality we have

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right|\\
&\quad\le\frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\\
&\qquad\times\left\{\left(\int_0^n\mathfrak{u}^{\frac{q}{q-1}}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\int_0^n\mathfrak{u}^{\frac{q}{q-1}}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}\right\}.
\end{aligned}\tag{2.21}$$

Since $|\mathfrak{K}'|^q$ is geometrically $r$-convex on $[\mathbf{j},\mathbf{i}]$ for $q>1$, we obtain

$$\begin{aligned}
&\int_0^n\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\\
&\quad\le\int_0^n\left(\frac{n-\mathfrak{u}}{2n}\mathbf{j}+\frac{n+\mathfrak{u}}{2n}\mathbf{i}\right)\left[\frac{n-\mathfrak{u}}{2n}\big|\mathfrak{K}'(\mathbf{j})\big|^{rq}+\frac{n+\mathfrak{u}}{2n}\big|\mathfrak{K}'(\mathbf{i})\big|^{rq}\right]^{\frac{1}{r}}d\mathfrak{u}\\
&\quad=n(\mathbf{j}+\mathbf{i})\theta\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q,\big|\mathfrak{K}'(\mathbf{i})\big|^q;r\big)+(\mathbf{i}-\mathbf{j})\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q,\big|\mathfrak{K}'(\mathbf{i})\big|^q;r\big)
\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}
&\int_0^n \mathbf{j}^{\frac{n+\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n-\mathfrak{u}}{2n}} \left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u} \\
&\leq \int_0^n \left(\frac{n+\mathfrak{u}}{2n}\mathbf{j} + \frac{n-\mathfrak{u}}{2n}\mathbf{i}\right)\left[\frac{n+\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{j})\right|^{rq} + \frac{n-\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{i})\right|^{rq}\right]^{\frac{1}{r}} d\mathfrak{u} \\
&= n(\mathbf{j}+\mathbf{i})\theta\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right) + (\mathbf{j}-\mathbf{i})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right). \qquad (2.23)
\end{aligned}$$

We also observe that

$$\begin{aligned}
\int_0^n \mathfrak{u}^{\frac{q}{q-1}} \mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}} d\mathfrak{u} &\leq \int_0^n \mathfrak{u}^{\frac{q}{q-1}} \left[\frac{n-\mathfrak{u}}{2n}\mathbf{j} + \frac{n+\mathfrak{u}}{2n}\mathbf{i}\right] d\mathfrak{u} \\
&= \frac{n^{\frac{2q-1}{q-1}}(q-1)[(q-1)\mathbf{i} + (5q-3)\mathbf{j}]}{2(3q-2)(2q-1)} = \vartheta_n(\mathbf{j},\mathbf{i}), \qquad (2.24)
\end{aligned}$$

and we similarly obtain

$$\int_0^n \mathfrak{u}^{\frac{q}{q-1}} \mathbf{j}^{\frac{n-\mathfrak{u}}{2n}} \mathbf{i}^{\frac{n+\mathfrak{u}}{2n}} d\mathfrak{u} = \frac{n^{\frac{2q-1}{q-1}}(q-1)[(q-1)\mathbf{j} + (5q-3)\mathbf{i}]}{2(3q-2)(2q-1)} = \vartheta_n(\mathbf{i},\mathbf{j}). \qquad (2.25)$$

Applying (2.22)–(2.25) in (2.21), we obtain the required inequality (2.20). □

**Theorem 4** *Let $\mathfrak{K}: \Bbbk \subseteq \mathcal{R}_+ = (0,\infty) \to \mathcal{R}$ be a differentiable function on $\Bbbk^\circ$, where $\mathbf{j}, \mathbf{i} \in \Bbbk^\circ$ with $\mathbf{j} < \mathbf{i}$ and $r \in \mathcal{R}$, $r \neq 0$. Suppose that $\mathfrak{K}' \in L([\mathbf{j},\mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically r-convex on $[\mathbf{j},\mathbf{i}]$ for $q \geq 1$. Then*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i} - \ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1} d\mathfrak{x}_1\right| \\
&\leq \frac{\ln\mathbf{i} - \ln\mathbf{j}}{4n^3} \\
&\quad \times \left\{\left[2n^2 R_0(\mathbf{j},\mathbf{i}) - 2nR_n(\mathbf{j},\mathbf{i})\right]^{1-\frac{1}{q}}\left[n^2(\mathbf{j}+\mathbf{i})\theta\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right)\right.\right. \\
&\quad \left. - 2n\mathbf{j}\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right) + (\mathbf{j}-\mathbf{i})\theta_{n,2}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right)\right]^{\frac{1}{q}} \\
&\quad + \left[2nR_n(\mathbf{j},\mathbf{i})\right]^{1-\frac{1}{q}}\left[n(\mathbf{j}+\mathbf{i})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right)\right. \\
&\quad \left. + (\mathbf{i}-\mathbf{j})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q; r\right)\right]^{\frac{1}{q}} + \left[2nR_n(\mathbf{i},\mathbf{j})\right]^{1-\frac{1}{q}} \\
&\quad \times \left[n(\mathbf{j}+\mathbf{i})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right) + (\mathbf{i}-\mathbf{j})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right)\right]^{\frac{1}{q}} \\
&\quad + \left[2n^2 R_0(\mathbf{i},\mathbf{j}) - 2nR_n(\mathbf{i},\mathbf{j})\right]^{1-\frac{1}{q}}\left[n^2(\mathbf{j}+\mathbf{i})\theta\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right) - 2n\mathbf{i}\right. \\
&\quad \left.\left. \times \theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right) + (\mathbf{j}-\mathbf{i})\theta_{n,2}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q; r\right)\right]^{\frac{1}{q}}\right\}. \qquad (2.26)
\end{aligned}$$

*Proof* From Lemma 1 and the improved power-mean inequality we have

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i} - \ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1} d\mathfrak{x}_1\right| \\
&\leq \frac{\ln\mathbf{i} - \ln\mathbf{j}}{4n^3}
\end{aligned}$$

$$\times\left\{\left(\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}$$
$$+\left(\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}\right\}. \tag{2.27}$$

Since $|\mathfrak{K}'|^q$ is geometrically $r$-convex on $[\mathbf{j},\mathbf{i}]$ for $q>1$, by Lemma 3 we obtain

$$\begin{aligned}
&\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\\
&\quad\le\int_0^n (n-\mathfrak{u})\left(\frac{n-\mathfrak{u}}{2n}\mathbf{j}+\frac{n+\mathfrak{u}}{2n}\mathbf{i}\right)\left[\frac{n-\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{j})\right|^{rq}+\frac{n+\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{i})\right|^{rq}\right]^{\frac{1}{r}}d\mathfrak{u}\\
&\quad= n^2(\mathbf{j}+\mathbf{i})\theta\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q,\left|\mathfrak{K}'(\mathbf{i})\right|^q;r\right)-2n\mathbf{j}\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q,\left|\mathfrak{K}'(\mathbf{i})\right|^q;r\right)\\
&\qquad+(\mathbf{j}-\mathbf{i})\theta_{n,2}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q,\left|\mathfrak{K}'(\mathbf{i})\right|^q;r\right),
\end{aligned}\tag{2.28}$$

$$\begin{aligned}
&\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\\
&\quad\le\int_0^n (n-\mathfrak{u})\left(\frac{n+\mathfrak{u}}{2n}\mathbf{j}+\frac{n-\mathfrak{u}}{2n}\mathbf{i}\right)\left[\frac{n+\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{j})\right|^{rq}+\frac{n-\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{i})\right|^{rq}\right]^{\frac{1}{r}}d\mathfrak{u}\\
&\quad= n^2(\mathbf{j}+\mathbf{i})\theta\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q,\left|\mathfrak{K}'(\mathbf{j})\right|^q;r\right)-2n\mathbf{i}\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q,\left|\mathfrak{K}'(\mathbf{j})\right|^q;r\right)\\
&\qquad+(\mathbf{i}-\mathbf{j})\theta_{n,2}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q,\left|\mathfrak{K}'(\mathbf{j})\right|^q;r\right),
\end{aligned}\tag{2.29}$$

$$\begin{aligned}
&\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\\
&\quad\le\int_0^n \mathfrak{u}\left(\frac{n-\mathfrak{u}}{2n}\mathbf{j}+\frac{n+\mathfrak{u}}{2n}\mathbf{i}\right)\left[\frac{n-\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{j})\right|^{rq}+\frac{n+\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{i})\right|^{rq}\right]^{\frac{1}{r}}d\mathfrak{u}\\
&\quad= n(\mathbf{j}+\mathbf{i})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q,\left|\mathfrak{K}'(\mathbf{i})\right|^q;r\right)\\
&\qquad+(\mathbf{i}-\mathbf{j})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q,\left|\mathfrak{K}'(\mathbf{i})\right|^q;r\right),
\end{aligned}\tag{2.30}$$

and

$$\begin{aligned}
&\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\\
&\quad\le\int_0^n \mathfrak{u}\left(\frac{n+\mathfrak{u}}{2n}\mathbf{j}+\frac{n-\mathfrak{u}}{2n}\mathbf{i}\right)\left[\frac{n+\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{j})\right|^{rq}+\frac{n-\mathfrak{u}}{2n}\left|\mathfrak{K}'(\mathbf{i})\right|^{rq}\right]^{\frac{1}{r}}d\mathfrak{u}\\
&\quad= n(\mathbf{j}+\mathbf{i})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q,\left|\mathfrak{K}'(\mathbf{j})\right|^q;r\right)\\
&\qquad+(\mathbf{j}-\mathbf{i})\theta_{n,1}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q,\left|\mathfrak{K}'(\mathbf{j})\right|^q;r\right).
\end{aligned}\tag{2.31}$$

We also observe from Lemma 2 that

$$\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u} = 2n^2R_0(\mathbf{j},\mathbf{i}) - 2nR_n(\mathbf{j},\mathbf{i}) \tag{2.32}$$

and

$$\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u} = 2n^2R_0(\mathbf{i},\mathbf{j}) - 2nR_n(\mathbf{i},\mathbf{j}). \tag{2.33}$$

Applying (2.28)–(2.33) in (2.27), we obtain the required inequality (2.26). □

**Corollary 4** *Suppose that the assumptions of Theorem 4 are satisfied and $q = 1$. Then*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \\
&\quad\le \frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^3}\big\{2n^2(\mathbf{j}+\mathbf{i})\theta\big(\big|\mathfrak{K}'(\mathbf{j})\big|,\big|\mathfrak{K}'(\mathbf{i})\big|;r\big) \\
&\qquad + (n-1)(\mathbf{j}-\mathbf{i})\big[\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{i})\big|,\big|\mathfrak{K}'(\mathbf{j})\big|;r\big) - \theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{j})\big|,\big|\mathfrak{K}'(\mathbf{i})\big|;r\big)\big] \\
&\qquad + (\mathbf{j}-\mathbf{i})\big[\theta_{n,2}\big(\big|\mathfrak{K}'(\mathbf{j})\big|,\big|\mathfrak{K}'(\mathbf{i})\big|;r\big) - \theta_{n,2}\big(\big|\mathfrak{K}'(\mathbf{i})\big|,\big|\mathfrak{K}'(\mathbf{j})\big|;r\big)\big]\big\}.
\end{aligned} \tag{2.34}$$

**Theorem 5** *Let $\mathfrak{K}:\Bbbk\subseteq\mathcal{R}_+=(0,\infty)\to\mathcal{R}$ be a differentiable function on $\Bbbk^\circ$, where $\mathbf{j},\mathbf{i}\in\Bbbk^\circ$ with $\mathbf{j}<\mathbf{i}$ and $r\in\mathcal{R}$, $r\neq 0$. Suppose that $\mathfrak{K}'\in L([\mathbf{j},\mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically r-convex on $[\mathbf{j},\mathbf{i}]$ for $q>1$. Then*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \\
&\quad\le \frac{\ln\mathbf{i}-\ln\mathbf{j}}{(2n)^{2-\frac{1}{q}}}\big\{\big[\lambda_n(\mathbf{j},\mathbf{i};p)\big]^{\frac{1}{p}}\big[\theta\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q,\big|\mathfrak{K}'(\mathbf{i})\big|^q;r\big) \\
&\qquad + \theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q,\big|\mathfrak{K}'(\mathbf{i})\big|^q;r\big)\big]^{\frac{1}{q}} + \big[\mu_n(\mathbf{j},\mathbf{i};p)\big]^{\frac{1}{p}} \\
&\qquad \times\big[\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q,\big|\mathfrak{K}'(\mathbf{i})\big|^q;r\big)\big]^{\frac{1}{q}} + \big[\lambda_n(\mathbf{i},\mathbf{j};p)\big]^{\frac{1}{p}} \\
&\qquad \times\big[\theta\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q,\big|\mathfrak{K}'(\mathbf{j})\big|^q;r\big) + \theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q,\big|\mathfrak{K}'(\mathbf{j})\big|^q;r\big)\big]^{\frac{1}{q}} \\
&\qquad + \big[\mu_n(\mathbf{i},\mathbf{j};p)\big]^{\frac{1}{p}}\big[\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q,\big|\mathfrak{K}'(\mathbf{j})\big|^q;r\big)\big]^{\frac{1}{q}}\big\},
\end{aligned} \tag{2.35}$$

*where $p^{-1}+q^{-1}=1$.*

*Proof* From Lemma 1 and the Hölder–İşcan inequality we have

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \\
&\quad\le \frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2} \\
&\qquad\times\left\{\left(\int_0^n (1-\mathfrak{u})\big(\mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)^p\,d\mathfrak{u}\right)^{\frac{1}{p}}\left(\int_0^n (1-\mathfrak{u})\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_0^n \mathfrak{u}\big(\mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)^p\,d\mathfrak{u}\right)^{\frac{1}{p}}\left(\int_0^n \mathfrak{u}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}$$
$$+\left(\int_0^n (1-\mathfrak{u})\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)^p\,d\mathfrak{u}\right)^{\frac{1}{p}}\left(\int_0^n (1-\mathfrak{u})\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}$$
$$+\left(\int_0^n \mathfrak{u}\big(\mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)^p\,d\mathfrak{u}\right)^{\frac{1}{p}}\left(\int_0^n \big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}\Bigg\}. \tag{2.36}$$

Since $|\mathfrak{K}'|^q$ is geometrically $r$-convex on $[\mathbf{j},\mathbf{i}]$ for $q>1$, by Lemma 3 we obtain

$$\begin{aligned}
&\int_0^n (1-\mathfrak{u})\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\\
&\quad\le \int_0^n (1-\mathfrak{u})\left[\frac{n-\mathfrak{u}}{2n}\big|\mathfrak{K}'(\mathbf{j})\big|^{rq}+\frac{n+\mathfrak{u}}{2n}\big|\mathfrak{K}'(\mathbf{i})\big|^{rq}\right]^{\frac{1}{r}}d\mathfrak{u}\\
&\quad= 2n\theta\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q,\big|\mathfrak{K}'(\mathbf{i})\big|^q;r\big)+2n\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{j})\big|^q,\big|\mathfrak{K}'(\mathbf{i})\big|^q;r\big)
\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}
&\int_0^n (1-\mathfrak{u})\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\\
&\quad\le \int_0^n (1-\mathfrak{u})\left[\frac{n+\mathfrak{u}}{2n}\big|\mathfrak{K}'(\mathbf{j})\big|^{rq}+\frac{n-\mathfrak{u}}{2n}\big|\mathfrak{K}'(\mathbf{i})\big|^{rq}\right]^{\frac{1}{r}}d\mathfrak{u}\\
&\quad= 2n\theta\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q,\big|\mathfrak{K}'(\mathbf{j})\big|^q;r\big)+2n\theta_{n,1}\big(\big|\mathfrak{K}'(\mathbf{i})\big|^q,\big|\mathfrak{K}'(\mathbf{j})\big|^q;r\big).
\end{aligned} \tag{2.38}$$

We also observe that

$$\begin{aligned}
\lambda_n(\mathbf{j},\mathbf{i};p) &= \int_0^n (1-\mathfrak{u})\big(\mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)^p\,d\mathfrak{u}\\
&\le \int_0^n \mathfrak{u}^p(1-\mathfrak{u})\left(\frac{n-\mathfrak{u}}{2n}\mathbf{j}^p+\frac{n+\mathfrak{u}}{2n}\mathbf{i}^p\right)d\mathfrak{u}\\
&= \frac{n^{p+1}[(3+p-n(p+1))\mathbf{j}^p+((p+3)(2p+3)-n(p+1)(2p+5))\mathbf{i}^p]}{2(p+1)(p+2)(p+3)},
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
\lambda_n(\mathbf{i},\mathbf{j};p) &= \int_0^n (1-\mathfrak{u})\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)^p\,d\mathfrak{u}\\
&\le \int_0^n \mathfrak{u}^p(1-\mathfrak{u})\left(\frac{n+\mathfrak{u}}{2n}\mathbf{j}^p+\frac{n-\mathfrak{u}}{2n}\mathbf{i}^p\right)d\mathfrak{u}\\
&= \frac{n^{p+1}[(3+p-n(p+1))\mathbf{i}^p+((p+3)(2p+3)-n(p+1)(2p+5))\mathbf{j}^p]}{2(p+1)(p+2)(p+3)},
\end{aligned} \tag{2.40}$$

$$\begin{aligned}
\int_0^n \mathfrak{u}\big(\mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)^p\,d\mathfrak{u} &\le \int_0^n \mathfrak{u}^{p+1}\left(\frac{n-\mathfrak{u}}{2n}\mathbf{j}^p+\frac{n+\mathfrak{u}}{2n}\mathbf{i}^p\right)\\
&= \frac{n^{p+2}[\mathbf{j}^p+\mathbf{i}^p(2p+5)]}{2(p+2)(p+3)} = \mu_n(\mathbf{j},\mathbf{i};p),
\end{aligned} \tag{2.41}$$

and

$$\int_0^n \mathfrak{u}\big(\mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)^p\,d\mathfrak{u} \le \int_0^n \mathfrak{u}^{p+1}\left(\frac{n+\mathfrak{u}}{2n}\mathbf{j}^p + \frac{n-\mathfrak{u}}{2n}\mathbf{i}^p\right)$$
$$= \frac{n^{p+2}[\mathbf{i}^p + \mathbf{j}^p(2p+5)]}{2(p+2)(p+3)} = \mu_n(\mathbf{i},\mathbf{j};p). \tag{2.42}$$

Applying (2.37)–(2.42) in (2.36), we obtain the required inequality (2.35). □

**Theorem 6** *Let $\mathfrak{K}:\Bbbk \subseteq \mathcal{R}_+ = (0,\infty) \to \mathcal{R}$ be a differentiable function on $\Bbbk^\circ$, where $\mathbf{j},\mathbf{i} \in \Bbbk^\circ$ with $\mathbf{j} < \mathbf{i}$. Suppose that $\mathfrak{K}' \in L([\mathbf{j},\mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically-convex on $[\mathbf{j},\mathbf{i}]$ for $q \ge 1$. Then*

$$\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right|$$
$$\le \frac{\ln\mathbf{i}-\ln\mathbf{j}}{2n}\Big\{\big[R_n(\mathbf{j},\mathbf{i})\big]^{1-\frac{1}{q}}\big[R_n\big(\mathbf{j}|\mathfrak{K}'(\mathbf{j})|^q,\mathbf{i}|\mathfrak{K}'(\mathbf{i})|^q\big)\big]^{\frac{1}{q}}$$
$$+\big[R_n(\mathbf{i},\mathbf{j})\big]^{1-\frac{1}{q}}\big[R_n\big(\mathbf{i}|\mathfrak{K}'(\mathbf{i})|^q,\mathbf{j}|\mathfrak{K}'(\mathbf{j})|^q\big)\big]^{\frac{1}{q}}\Big\}. \tag{2.43}$$

*Proof* From Lemma 1 and the power-mean inequality we have

$$\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}} \frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right|$$
$$\le \frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\int_0^n \mathfrak{u}\Big[\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\big| + \mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\big|\Big]\,d\mathfrak{u}$$
$$\le \frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\left\{\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}\right\}. \tag{2.44}$$

Using the geometric convexity of $|\mathfrak{K}'|^q$ on $[\mathbf{j},\mathbf{i}]$ for $q \ge 1$ and Lemma 2, we have

$$\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}$$
$$\le \int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\big(|\mathfrak{K}'(\mathbf{j})|^q\big)^{\frac{n-\mathfrak{u}}{2n}}\big(|\mathfrak{K}'(\mathbf{i})|^q\big)^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u}$$
$$= 2n\left(\frac{1}{2n}\int_0^n \mathfrak{u}\big(\mathbf{j}|\mathfrak{K}'(\mathbf{j})|^q\big)^{\frac{n-\mathfrak{u}}{2n}}\big(\mathbf{i}|\mathfrak{K}'(\mathbf{i})|^q\big)^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)$$
$$= 2nR_n\big(\mathbf{j}|\mathfrak{K}'(\mathbf{j})|^q,\mathbf{i}|\mathfrak{K}'(\mathbf{i})|^q\big). \tag{2.45}$$

Similarly, we have

$$\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big|\mathfrak{K}'\big(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big)\big|^q\,d\mathfrak{u}$$
$$\le \int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\big(|\mathfrak{K}'(\mathbf{j})|^q\big)^{\frac{n+\mathfrak{u}}{2n}}\big(|\mathfrak{K}'(\mathbf{i})|^q\big)^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u}$$

$$= 2n\left(\frac{1}{2n}\int_0^n \mathfrak{u}\left(\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)^{\frac{n+\mathfrak{u}}{2n}}\left(\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q\right)^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)$$
$$= 2nR_n\left(\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q, \mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q\right). \tag{2.46}$$

Moreover, from Lemma 2 we also obtain

$$\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u} = 2nR_n(\mathbf{j},\mathbf{i})$$

and

$$\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u} = 2nR_n(\mathbf{i},\mathbf{j}).$$

Using the last two inequalities, (2.45) and (2.46) in (2.44), we obtain the required inequality (2.43). □

**Corollary 5** *Under the assumptions of Theorem 6, if $q = 1$, then*

$$\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i} - \ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right|$$
$$\leq \frac{\ln\mathbf{i} - \ln\mathbf{j}}{2n}\left\{R_n\left(\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|, \mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|\right) + R_n\left(\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|, \mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|\right)\right\}. \tag{2.47}$$

**Theorem 7** *Let $\mathfrak{K} : \Bbbk \subseteq \mathcal{R}_+ = (0,\infty) \to \mathcal{R}$ be a differentiable function on $\Bbbk^\circ$, where $\mathbf{j},\mathbf{i} \in \Bbbk^\circ$ with $\mathbf{j} < \mathbf{i}$. Suppose that $\mathfrak{K}' \in L([\mathbf{j},\mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically-convex on $[\mathbf{j},\mathbf{i}]$ for $q > 1$. Then*

$$\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i} - \ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right|$$
$$\leq \frac{\ln\mathbf{i} - \ln\mathbf{j}}{2n}\left\{\left[R_n\left(\mathbf{j}^{\frac{q}{q-1}}, \mathbf{i}^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}}\left[R_n\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q\right)\right]^{\frac{1}{q}}\right.$$
$$\left. + \left[R_n\left(\mathbf{i}^{\frac{q}{q-1}}, \mathbf{j}^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}}\left[R_n\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q\right)\right]^{\frac{1}{q}}\right\}. \tag{2.48}$$

*Proof* From Lemma 1 and the Hölder inequality we have

$$\left|\frac{\mathfrak{K}(\mathbf{i}) + \mathfrak{K}(\mathbf{j})}{2} - \frac{1}{\ln\mathbf{i} - \ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right|$$
$$\leq \frac{\ln\mathbf{i} - \ln\mathbf{j}}{4n^2}\int_0^n \mathfrak{u}\left[\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right| + \mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|\right]d\mathfrak{u}$$
$$\leq \frac{\ln\mathbf{i} - \ln\mathbf{j}}{4n^2}\left\{\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{q(n-\mathfrak{u})}{2n(q-1)}}\mathbf{i}^{\frac{q(n+\mathfrak{u})}{2n(q-1)}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{q(n+\mathfrak{u})}{2n(q-1)}}\mathbf{i}^{\frac{q(n-\mathfrak{u})}{2n(q-1)}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}\right\}. \tag{2.49}$$

Using the geometric convexity of $|\mathfrak{K}'|^q$ on $[\mathbf{j},\mathbf{i}]$ for $q>1$, we have

$$\begin{aligned}&\int_0^n \mathfrak{u}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\\&\quad\le\int_0^n \mathfrak{u}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)^{\frac{n-\mathfrak{u}}{2n}}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q\right)^{\frac{n+\mathfrak{u}}{2n}} d\mathfrak{u}=2nR_n\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q,\left|\mathfrak{K}'(\mathbf{i})\right|^q\right).\end{aligned}\tag{2.50}$$

Similarly, we have

$$\begin{aligned}\int_0^n \mathfrak{u}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}&\le\int_0^n \mathfrak{u}\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)^{\frac{n+\mathfrak{u}}{2n}}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q\right)^{\frac{n-\mathfrak{u}}{2n}} d\mathfrak{u}\\&=2nR_n\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q,\left|\mathfrak{K}'(\mathbf{j})\right|^q\right).\end{aligned}\tag{2.51}$$

Also, we observe that

$$\int_0^n \mathfrak{u}\mathbf{j}^{\frac{q(n-\mathfrak{u})}{2n(q-1)}}\mathbf{i}^{\frac{q(n+\mathfrak{u})}{2n(q-1)}} d\mathfrak{u}=2nR_n\left(\mathbf{j}^{\frac{q}{q-1}},\mathbf{i}^{\frac{q}{q-1}}\right)\tag{2.52}$$

and

$$\int_0^n \mathfrak{u}\mathbf{j}^{\frac{q(n+\mathfrak{u})}{2n(q-1)}}\mathbf{i}^{\frac{q(n-\mathfrak{u})}{2n(q-1)}} d\mathfrak{u}=2nR_n\left(\mathbf{i}^{\frac{q}{q-1}},\mathbf{j}^{\frac{q}{q-1}}\right).\tag{2.53}$$

Using (2.50)–(2.53) in (2.49), we obtain the required inequality (2.48). □

**Theorem 8** *Let $\mathfrak{K}:\mathbb{k}\subseteq\mathcal{R}_+=(0,\infty)\to\mathcal{R}$ be a differentiable function on $\mathbb{k}^\circ$, where $\mathbf{j},\mathbf{i}\in\mathbb{k}^\circ$ with $\mathbf{j}<\mathbf{i}$. Suppose that $\mathfrak{K}'\in L([\mathbf{j},\mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically-convex on $[\mathbf{j},\mathbf{i}]$ for $q>1$. Then*

$$\begin{aligned}&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}d\mathfrak{x}_1\right|\\&\quad\le\frac{(\ln\mathbf{i}-\ln\mathbf{j})}{2^{2-\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{\left[T_0\left(\mathbf{j}^q\left|\mathfrak{K}'(\mathbf{j})\right|^q,\mathbf{i}^q\left|\mathfrak{K}'(\mathbf{i})\right|^q\right)\right]^{\frac{1}{q}}\right.\\&\quad\quad\left.+\left[T_0\left(\mathbf{i}^q\left|\mathfrak{K}'(\mathbf{i})\right|^q,\mathbf{j}^q\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{2.54}$$

*Proof* From Lemma 1 and Hölder's inequality we have

$$\begin{aligned}&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}d\mathfrak{x}_1\right|\\&\quad\le\frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\int_0^n \mathfrak{u}\left[\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)+\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right]d\mathfrak{u}\\&\quad\le\frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2}\left(\int_0^n \mathfrak{u}^{\frac{q}{q-1}}d\mathfrak{u}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^n\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)^q\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}\right.\\&\quad\quad\left.+\left(\int_0^n\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)^q\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{2.55}$$

Since $|\mathfrak{K}'|^q$ is geometrically convex on $[\mathbf{j},\mathbf{i}]$ for $q>1$, we obtain

$$\begin{aligned}
&\int_0^n \bigl(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigr)^q \bigl|\mathfrak{K}'\bigl(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigr)\bigr|^q\,d\mathfrak{u} \\
&\quad\le \int_0^n \bigl(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigr)^q \bigl(\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)^{\frac{n-\mathfrak{u}}{2n}} \bigl(\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u} \\
&\quad= \int_0^n \bigl(\mathbf{j}^q\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)^{\frac{n-\mathfrak{u}}{2n}} \bigl(\mathbf{i}^q\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u} \\
&\quad= 2nT_0\bigl(\mathbf{j}^q\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q,\mathbf{i}^q\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)
\end{aligned}\tag{2.56}$$

and

$$\begin{aligned}
&\int_0^n \bigl(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\bigr)^q \bigl|\mathfrak{K}'\bigl(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\bigr)\bigr|^q\,d\mathfrak{u} \\
&\quad\le \int_0^n \bigl(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\bigr)^q \bigl(\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)^{\frac{n+\mathfrak{u}}{2n}} \bigl(\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u} \\
&\quad= \int_0^n \bigl(\mathbf{j}^q\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)^{\frac{n+\mathfrak{u}}{2n}} \bigl(\mathbf{i}^q\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u} \\
&\quad= 2nT_0\bigl(\mathbf{i}^q\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q,\mathbf{j}^q\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr).
\end{aligned}\tag{2.57}$$

Applying (2.56) and (2.57) in (2.55), we obtain the required inequality (2.54). □

**Theorem 9** *Let $\mathfrak{K}:\Bbbk\subseteq\mathcal{R}_+=(0,\infty)\to\mathcal{R}$ be a differentiable function on $\Bbbk^\circ$, where $\mathbf{j},\mathbf{i}\in\Bbbk^\circ$ with $\mathbf{j}<\mathbf{i}$. Suppose that $\mathfrak{K}'\in L([\mathbf{j},\mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically convex on $[\mathbf{j},\mathbf{i}]$ for $q\ge 1$. Then*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \\
&\quad\le \frac{\ln\mathbf{i}-\ln\mathbf{j}}{2n^2}\bigl\{\bigl[nR_0(\mathbf{j},\mathbf{i})-R_n(\mathbf{j},\mathbf{i})\bigr]^{1-\frac{1}{q}} \\
&\qquad\times\bigl[nR_0\bigl(\mathbf{j}\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q,\mathbf{i}\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)-R_n\bigl(\mathbf{j}\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q,\mathbf{i}\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)\bigr]^{\frac{1}{q}} \\
&\qquad+\bigl[R_n(\mathbf{j},\mathbf{i})\bigr]^{1-\frac{1}{q}}\bigl[R_n\bigl(\mathbf{j}\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q,\mathbf{i}\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)\bigr]^{\frac{1}{q}}+\bigl[R_n(\mathbf{j},\mathbf{i})\bigr]^{1-\frac{1}{q}} \\
&\qquad\times\bigl[R_n\bigl(\mathbf{i}\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q,\mathbf{j}\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)\bigr]^{\frac{1}{q}}+\bigl[nR_0(\mathbf{i},\mathbf{j})-R_n(\mathbf{i},\mathbf{j})\bigr]^{1-\frac{1}{q}} \\
&\qquad\times\bigl[nR_0\bigl(\mathbf{i}\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q,\mathbf{j}\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)-R_n\bigl(\mathbf{i}\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q,\mathbf{j}\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)\bigr]^{\frac{1}{q}}\bigr\}.
\end{aligned}\tag{2.58}$$

*Proof* From Lemma 1 and the improved power-mean inequality we have

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \\
&\quad\le \frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^3} \\
&\qquad\times\left\{\left(\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigl|\mathfrak{K}'\bigl(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigr)\bigr|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}$$
$$+\left(\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}$$
$$+\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\right)^{\frac{1}{q}}\Bigg\}. \tag{2.59}$$

Since $|\mathfrak{K}'|^q$ is geometrically convex on $[\mathbf{j},\mathbf{i}]$ for $q \geq 1$, using Lemma 3, we obtain

$$\begin{aligned}
&\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\\
&\quad\leq \int_0^n (n-\mathfrak{u})\left(\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)^{\frac{n-\mathfrak{u}}{2n}}\left(\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q\right)^{\frac{n+\mathfrak{u}}{2n}} d\mathfrak{u}\\
&\quad= 2n^2R_0\left(\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q,\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q\right)-2nR_n\left(\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q,\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q\right), \qquad (2.60)\\
&\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\\
&\quad\leq \int_0^n (n-\mathfrak{u})\left(\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)^{\frac{n+\mathfrak{u}}{2n}}\left(\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q\right)^{\frac{n-\mathfrak{u}}{2n}} d\mathfrak{u}\\
&\quad= 2n^2R_0\left(\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q,\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)-2nR_n\left(\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q,\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q\right), \qquad (2.61)\\
&\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\\
&\quad\leq \int_0^n \mathfrak{u}\left(\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)^{\frac{n+\mathfrak{u}}{2n}}\left(\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^{rq}\right)^{\frac{n-\mathfrak{u}}{2n}} d\mathfrak{u}\\
&\quad= 2nR_n\left(\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q,\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q\right), \qquad (2.62)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^n \mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u}\\
&\quad\leq \int_0^n \mathfrak{u}\left(\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)^{\frac{n-\mathfrak{u}}{2n}}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q\right)^{\frac{n+\mathfrak{u}}{2n}} d\mathfrak{u}\\
&\quad= 2nR_n\left(\mathbf{i}\left|\mathfrak{K}'(\mathbf{i})\right|^q,\mathbf{j}\left|\mathfrak{K}'(\mathbf{j})\right|^q\right). \qquad (2.63)
\end{aligned}$$

We also observe from Lemma 2 that

$$\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u} = 2n^2R_0(\mathbf{j},\mathbf{i})-2nR_n(\mathbf{j},\mathbf{i}) \tag{2.64}$$

and

$$\int_0^n (n-\mathfrak{u})\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\,d\mathfrak{u} = 2n^2R_0(\mathbf{i},\mathbf{j})-2nR_n(\mathbf{i},\mathbf{j}). \tag{2.65}$$

Applying (2.60)–(2.65) in (2.59), we obtain the required inequality (2.58). □

**Corollary 6** *Under the assumptions of Theorem 9 and $q = 1$, we have the following inequality*:

$$\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \leq \frac{\ln\mathbf{i}-\ln\mathbf{j}}{2n}\bigl\{R_0\bigl(\mathbf{j}\bigl|\mathfrak{K}'(\mathbf{j})\bigr|,\mathbf{i}\bigl|\mathfrak{K}'(\mathbf{i})\bigr|\bigr)+R_0\bigl(\mathbf{i}\bigl|\mathfrak{K}'(\mathbf{i})\bigr|,\mathbf{j}\bigl|\mathfrak{K}'(\mathbf{j})\bigr|\bigr)\bigr\}. \tag{2.66}$$

**Theorem 10** *Let $\mathfrak{K}:\Bbbk\subseteq\mathcal{R}_+=(0,\infty)\to\mathcal{R}$ be a differentiable function on $\Bbbk^\circ$, where $\mathbf{j}$, $\mathbf{i}\in\Bbbk^\circ$ with $\mathbf{j}<\mathbf{i}$. Suppose that $\mathfrak{K}'\in L([\mathbf{j},\mathbf{i}])$ and $|\mathfrak{K}'|^q$ is geometrically convex on $[\mathbf{j},\mathbf{i}]$ for $q>1$. Then*

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \\
&\quad\leq \frac{\ln\mathbf{i}-\ln\mathbf{j}}{(2n)^{2-\frac{1}{q}}}\bigl\{\bigl[\lambda_n(\mathbf{j},\mathbf{i};p)\bigr]^{\frac{1}{p}}\bigl[T_0\bigl(\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q,\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr) \\
&\qquad+R_n\bigl(\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q,\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)\bigr]^{\frac{1}{q}}+\bigl[\mu_n(\mathbf{j},\mathbf{i};p)\bigr]^{\frac{1}{p}} \\
&\qquad\times\bigl[R_n\bigl(\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q,\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)\bigr]^{\frac{1}{q}}+\bigl[\lambda_n(\mathbf{i},\mathbf{j};p)\bigr]^{\frac{1}{p}} \\
&\qquad\times\bigl[T_0\bigl(\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q,\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)+R_n\bigl(\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q,\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)\bigr]^{\frac{1}{q}} \\
&\qquad+\bigl[\mu_n(\mathbf{i},\mathbf{j};p)\bigr]^{\frac{1}{p}}\bigl[R_n\bigl(\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q,\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)\bigr]^{\frac{1}{q}}\bigr\},
\end{aligned} \tag{2.67}$$

*where $p^{-1}+q^{-1}=1$.*

*Proof* From Lemma 1 and the Hölder–İşcan inequality we have

$$\begin{aligned}
&\left|\frac{\mathfrak{K}(\mathbf{i})+\mathfrak{K}(\mathbf{j})}{2}-\frac{1}{\ln\mathbf{i}-\ln\mathbf{j}}\int_{\mathbf{j}}^{\mathbf{i}}\frac{\mathfrak{K}(\mathfrak{x}_1)}{\mathfrak{x}_1}\,d\mathfrak{x}_1\right| \\
&\quad\leq\frac{\ln\mathbf{i}-\ln\mathbf{j}}{4n^2} \\
&\qquad\times\left\{\left(\int_0^n(1-\mathfrak{u})\bigl(\mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigr)^p\,d\mathfrak{u}\right)^{\frac{1}{p}}\left(\int_0^n(1-\mathfrak{u})\bigl|\mathfrak{K}'\bigl(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigr)\bigr|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}\right. \\
&\qquad+\left(\int_0^n\mathfrak{u}\bigl(\mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigr)^p\,d\mathfrak{u}\right)^{\frac{1}{p}}\left(\int_0^n\mathfrak{u}\bigl|\mathfrak{K}'\bigl(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigr)\bigr|^q\,d\mathfrak{u}\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_0^n(1-\mathfrak{u})\bigl(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\bigr)^p\,d\mathfrak{u}\right)^{\frac{1}{p}}\left(\int_0^n(1-\mathfrak{u})\bigl|\mathfrak{K}'\bigl(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\bigr)\bigr|^q\,d\mathfrak{u}\right)^{\frac{1}{q}} \\
&\qquad\left.+\left(\int_0^n\mathfrak{u}\bigl(\mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\bigr)^p\,d\mathfrak{u}\right)^{\frac{1}{p}}\left(\int_0^n\bigl|\mathfrak{K}'\bigl(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\bigr)\bigr|^q\,d\mathfrak{u}\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{2.68}$$

Since $|\mathfrak{K}'|^q$ is geometrically convex on $[\mathbf{j},\mathbf{i}]$ for $q>1$, using Lemma 2, we obtain

$$\begin{aligned}
&\int_0^n(1-\mathfrak{u})\bigl|\mathfrak{K}'\bigl(\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\bigr)\bigr|^q\,d\mathfrak{u} \\
&\quad\leq\int_0^n(1-\mathfrak{u})\bigl(\bigl|\mathfrak{K}'(\mathbf{j})\bigr|^q\bigr)^{\frac{n-\mathfrak{u}}{2n}}\bigl(\bigl|\mathfrak{K}'(\mathbf{i})\bigr|^q\bigr)^{\frac{n+\mathfrak{u}}{2n}}\,d\mathfrak{u}
\end{aligned}$$

$$= 2nT_0\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q\right) + 2nR_n\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q, \left|\mathfrak{K}'(\mathbf{i})\right|^q\right) \tag{2.69}$$

and

$$\begin{aligned}
&\int_0^n (1-\mathfrak{u})\left|\mathfrak{K}'\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)\right|^q d\mathfrak{u} \\
&\quad\le \int_0^n (1-\mathfrak{u})\left(\left|\mathfrak{K}'(\mathbf{j})\right|^q\right)^{\frac{n+\mathfrak{u}}{2n}}\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q\right)^{\frac{n-\mathfrak{u}}{2n}} d\mathfrak{u} \\
&\quad= 2nT_0\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q\right) + 2nR_n\left(\left|\mathfrak{K}'(\mathbf{i})\right|^q, \left|\mathfrak{K}'(\mathbf{j})\right|^q\right).
\end{aligned} \tag{2.70}$$

We also observe that

$$\begin{aligned}
\lambda_n(\mathbf{j},\mathbf{i};p) &= \int_0^n (1-\mathfrak{u})\left(\mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)^p d\mathfrak{u} \\
&\le \int_0^n \mathfrak{u}^p(1-\mathfrak{u})\left(\frac{n-\mathfrak{u}}{2n}\mathbf{j}^p + \frac{n+\mathfrak{u}}{2n}\mathbf{i}^p\right) d\mathfrak{u} \\
&= \frac{n^{p+1}[(3+p-n(p+1))\mathbf{j}^p + ((p+3)(2p+3)-n(p+1)(2p+5))\mathbf{i}^p]}{2(p+1)(p+2)(p+3)},
\end{aligned} \tag{2.71}$$

$$\begin{aligned}
\lambda_n(\mathbf{i},\mathbf{j};p) &= \int_0^n (1-\mathfrak{u})\left(\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)^p d\mathfrak{u} \\
&\le \int_0^n \mathfrak{u}^p(1-\mathfrak{u})\left(\frac{n+\mathfrak{u}}{2n}\mathbf{j}^p + \frac{n-\mathfrak{u}}{2n}\mathbf{i}^p\right) d\mathfrak{u} \\
&= \frac{n^{p+1}[(3+p-n(p+1))\mathbf{i}^p + ((p+3)(2p+3)-n(p+1)(2p+5))\mathbf{j}^p]}{2(p+1)(p+2)(p+3)},
\end{aligned} \tag{2.72}$$

$$\begin{aligned}
\int_0^n \mathfrak{u}\left(\mathfrak{u}\mathbf{j}^{\frac{n-\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n+\mathfrak{u}}{2n}}\right)^p d\mathfrak{u} &\le \int_0^n \mathfrak{u}^{p+1}\left(\frac{n-\mathfrak{u}}{2n}\mathbf{j}^p + \frac{n+\mathfrak{u}}{2n}\mathbf{i}^p\right) \\
&= \frac{n^{p+2}[\mathbf{j}^p + \mathbf{i}^p(2p+5)]}{2(p+2)(p+3)} = \mu_n(\mathbf{j},\mathbf{i};p),
\end{aligned} \tag{2.73}$$

and

$$\begin{aligned}
\int_0^n \mathfrak{u}\left(\mathfrak{u}\mathbf{j}^{\frac{n+\mathfrak{u}}{2n}}\mathbf{i}^{\frac{n-\mathfrak{u}}{2n}}\right)^p d\mathfrak{u} &\le \int_0^n \mathfrak{u}^{p+1}\left(\frac{n+\mathfrak{u}}{2n}\mathbf{j}^p + \frac{n-\mathfrak{u}}{2n}\mathbf{i}^p\right) \\
&= \frac{n^{p+2}[\mathbf{i}^p + \mathbf{j}^p(2p+5)]}{2(p+2)(p+3)} = \mu_n(\mathbf{i},\mathbf{j};p).
\end{aligned} \tag{2.74}$$

Applying (2.69)–(2.74) in (2.68), we obtain the required inequality (2.67). □

*Remark* 4 From Lemmas 4 and 5 it obviously follows that for $n = 1$, the results presented in this paper provide improvements of the results established in [25].

## 3 Applications

In this section, we apply some of the established inequalities of Hermite–Hadamard type to construct inequalities for special definite integrals that cannot be evaluated analytically.

**Theorem 11** [7] *Let $\phi$ be a twice continuously differentiable real quasiconvex function on an open convex set $S \subseteq \mathcal{R}^n$. If there exists a real number $r^*$ such that*

$$r^* = \sup_{\mathfrak{x}_1 \in S, \|z\|=1} -\frac{z^T \nabla^2 \phi(\mathfrak{x}_1) z}{[z^T \nabla \phi(\mathfrak{x}_1) z]^2} \tag{3.1}$$

*whenever $z^T \nabla \phi(\mathfrak{x}_1) z \neq 0$, then $\phi$ is r-convex for every $r \geq r^*$. The function $\phi$ is r-concave if the supremum in* (3.1) *is replaced by infimum.*

*Remark* 5 If $\phi$ is $r$-convex and increasing on an open convex set $S \subseteq \mathcal{R}^n$, then $\phi$ is geometrically $r$-convex on $S$.

**Theorem 12** *Let $0 < \mathbf{j} < \mathbf{i} < \frac{\pi}{2}$, $r \in \mathcal{R}$, and let n be a positive integer. Then*

$$\begin{aligned}
&\frac{(\ln \mathbf{i} - \ln \mathbf{j}) \ln(\sec \mathbf{i} \sec \mathbf{j})}{2} - \frac{r(\ln \mathbf{i} - \ln \mathbf{j})^2}{4n^2} \\
&\quad \times \left\{ n(\mathbf{j} + \mathbf{i}) \left[ \theta_{n,1}\left( \frac{\tan \mathbf{j}}{r}, \frac{\tan \mathbf{i}}{r}; -r \right) + \theta_{n,1}\left( \frac{\tan \mathbf{i}}{r}, \frac{\tan \mathbf{j}}{r}; -r \right) \right] \right. \\
&\quad \left. + (\mathbf{i} - \mathbf{j}) \left[ \theta_{n,2}\left( \frac{\tan \mathbf{j}}{r}, \frac{\tan \mathbf{i}}{r}; -r \right) - \theta_{n,2}\left( \frac{\tan \mathbf{i}}{r}, \frac{\tan \mathbf{j}}{r}; -r \right) \right] \right\} \\
&\leq \int_{\mathbf{j}}^{\mathbf{i}} \frac{\ln(\sec \mathfrak{x}_1)}{\mathfrak{x}_1} \, d\mathfrak{x}_1 \leq \frac{(\ln \mathbf{i} - \ln \mathbf{j}) \ln(\sec \mathbf{i} \sec \mathbf{j})}{2} + \frac{r(\ln \mathbf{i} - \ln \mathbf{j})^2}{4n^2} \\
&\quad \times \left\{ n(\mathbf{j} + \mathbf{i}) \left[ \theta_{n,1}\left( \frac{\tan \mathbf{j}}{r}, \frac{\tan \mathbf{i}}{r}; -r \right) + \theta_{n,1}\left( \frac{\tan \mathbf{i}}{r}, \frac{\tan \mathbf{j}}{r}; -r \right) \right] \right. \\
&\quad \left. + (\mathbf{i} - \mathbf{j}) \left[ \theta_{n,2}\left( \frac{\tan \mathbf{j}}{r}, \frac{\tan \mathbf{i}}{r}; -r \right) - \theta_{n,2}\left( \frac{\tan \mathbf{i}}{r}, \frac{\tan \mathbf{j}}{r}; -r \right) \right] \right\},
\end{aligned} \tag{3.2}$$

*where $\theta_{n,1}$ and $\theta_{n,2}$ are defined as in Lemma* 3.

*Proof* Let $\mathfrak{K}(\mathfrak{x}_1) = \frac{\ln(\sec \mathfrak{x}_1)}{r}$ for $\mathfrak{x}_1 \in (0, \frac{\pi}{2})$ and $r \in \mathcal{R}$ with $r \neq 0$. Then

$$\mathfrak{K}'(\mathfrak{x}_1) = \frac{\tan \mathfrak{x}_1}{r}.$$

Thus

$$\left| \mathfrak{K}'(\mathfrak{x}_1) \right| = \frac{\tan \mathfrak{x}_1}{r}.$$

By using Theorem 11 we get that $r^* = -r$ is a $(-r)$-convex function increasing on $(0, \frac{\pi}{2})$ and hence on $[\mathbf{j}, \mathbf{i}] \subset (0, \frac{\pi}{2})$. We get inequality (3.2) from the inequality of Corollary 2. □

**Theorem 13** *Let $0 < \mathbf{j} < \mathbf{i} < 1$, $r \in \mathcal{R}$, $r \neq 0$, and let n be a positive integer. Then*

$$\begin{aligned}
&\frac{(\ln \mathbf{i} - \ln \mathbf{j})(e^{\mathbf{i}} + e^{\mathbf{j}})}{2} - \frac{(\ln \mathbf{i} - \ln \mathbf{j})^2}{4rn^3} \\
&\quad \times \left\{ 2n^2(\mathbf{j} + \mathbf{i}) \theta\left( re^{\mathbf{j}}, re^{\mathbf{i}}; -\frac{1}{r} \right) + (n-1)(\mathbf{j} - \mathbf{i}) \right.
\end{aligned}$$

$$\times\left[\theta_{n,1}\left(re^{\mathbf{i}},re^{\mathbf{j}};-\frac{1}{r}\right)-\theta_{n,1}\left(re^{\mathbf{j}},re^{\mathbf{i}};-\frac{1}{r}\right)\right]$$
$$+(\mathbf{j}-\mathbf{i})\left[\theta_{n,2}\left(re^{\mathbf{j}},re^{\mathbf{i}};-\frac{1}{r}\right)-\theta_{n,2}\left(re^{\mathbf{i}},re^{\mathbf{j}};-\frac{1}{r}\right)\right]\Bigg\}$$
$$\leq\int_{\mathbf{j}}^{\mathbf{i}}\frac{e^{\mathfrak{x}_1}}{\mathfrak{x}_1}\,d\mathfrak{x}_1\leq\frac{(\ln\mathbf{i}-\ln\mathbf{j})(e^{\mathbf{i}}+e^{\mathbf{j}})}{2}$$
$$+\frac{(\ln\mathbf{i}-\ln\mathbf{j})^2}{4rn^3}\Bigg\{2n^2(\mathbf{j}+\mathbf{i})\theta\left(re^{\mathbf{j}},re^{\mathbf{i}};-\frac{1}{r}\right)$$
$$+(n-1)(\mathbf{j}-\mathbf{i})\left[\theta_{n,1}\left(re^{\mathbf{i}},re^{\mathbf{j}};-\frac{1}{r}\right)-\theta_{n,1}\left(re^{\mathbf{j}},re^{\mathbf{i}};-\frac{1}{r}\right)\right]$$
$$+(\mathbf{j}-\mathbf{i})\left[\theta_{n,2}\left(re^{\mathbf{j}},re^{\mathbf{i}};-\frac{1}{r}\right)-\theta_{n,2}\left(re^{\mathbf{i}},re^{\mathbf{j}};-\frac{1}{r}\right)\right]\Bigg\}. \tag{3.3}$$

*Proof* Let $\mathfrak{K}(\mathfrak{x}_1)=re^{\mathfrak{x}_1}$ for $\mathfrak{x}_1\in(0,1)$, $r\in\mathcal{R}$ with $r\neq 0$. Then

$$\left|\mathfrak{K}'(\mathfrak{x}_1)\right|=re^{\mathfrak{x}_1}.$$

By using Theorem 11 we get that $r^*=-\frac{1}{r}$. Thus

$$\left|\mathfrak{K}'(\mathfrak{x}_1)\right|=re^{\mathfrak{x}_1}$$

is a $(-\frac{1}{r})$-convex function increasing on $(0,1)$ and hence on $[\mathbf{j},\mathbf{i}]\subset(0,1)$. We get inequality (3.2) from the inequality of Corollary 2. □

**Theorem 14** *Let* $0<\mathbf{j}<\mathbf{i}<\infty$, $r\in\mathcal{R}$, $r\in[-1,0)\cup(0,1]$, $q\geq 1$, *and let* $n$ *be a positive integer. Then*

$$\left|A\left(\mathbf{j}^r,\mathbf{i}^r\right)-L\left(\mathbf{j}^r,\mathbf{i}^r\right)\right|\leq\frac{\ln\mathbf{i}-\ln\mathbf{j}}{2n}$$
$$\times\left\{\left[R_n(\mathbf{j},\mathbf{i})\right]^{1-\frac{1}{q}}\left[R_n\left(|r|^q\mathbf{j}^{q(r-1)+1},|r|^q\mathbf{i}^{q(r-1)+1}\right)\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[R_n(\mathbf{i},\mathbf{j})\right]^{1-\frac{1}{q}}\left[R_n\left(|r|^q\mathbf{i}^{q(r-1)+1},|r|^q\mathbf{j}^{q(r-1)+1}\right)\right]^{\frac{1}{q}}\right\}. \tag{3.4}$$

*Proof* Let $\mathfrak{K}(\mathfrak{x}_1)=\mathfrak{x}_1^r$ for $\mathfrak{x}_1>0$, $r\in[-1,0)\cup(0,1]$. Then

$$\left|\mathfrak{K}'\left(\mathfrak{x}_1^{\lambda}\mathfrak{y}_1^{1-\lambda}\right)\right|^q=|r|^q\left[\mathfrak{x}_1^{q(r-1)}\right]^{\lambda}\left[\mathfrak{y}_1^{q(r-1)}\right]^{1-\lambda}\leq\left[|r|^q\mathfrak{x}_1^{q(r-1)}\right]^{\lambda}\left[|r|^q\mathfrak{y}_1^{q(r-1)}\right]^{1-\lambda}$$

for $\lambda\in[0,1]$, $\mathfrak{x}_1,\mathfrak{y}_1>0$, and $q\geq 1$, that is, $|\mathfrak{K}'(\mathfrak{x}_1)|^q=|r|^q\mathfrak{x}_1^{q(r-1)}$ is geometrically convex on $[\mathbf{j},\mathbf{i}]$ for $q\geq 1$ and $r\in[-1,0)\cup(0,1]$, where $\mathbf{j}$, $\mathbf{i}>0$. Hence inequality (3.4) follows from Theorem 6. □

**Corollary 7** *Suppose that the conditions of Theorem 14 are fulfilled and* $q=1$. *Then*

$$\left|A\left(\mathbf{j}^r,\mathbf{i}^r\right)-L\left(\mathbf{j}^r,\mathbf{i}^r\right)\right|\leq\frac{\ln\mathbf{i}-\ln\mathbf{j}}{2n}\left\{R_n\left(|r|\mathbf{j}^r,|r|\mathbf{i}^r\right)+R_n\left(|r|\mathbf{i}^r,|r|\mathbf{j}^r\right)\right\}. \tag{3.5}$$

**Theorem 15** *Let* $0 < \mathbf{j} < \mathbf{i} < \infty$, $q > 1$, *and let n be a positive integer. Then*

$$\begin{aligned} &\left| A\left(e^{\mathbf{i}}, e^{\mathbf{j}}\right) - \frac{1}{\ln \mathbf{i} - \ln \mathbf{j}} \int_{\mathbf{j}}^{\mathbf{i}} \frac{e^{\mathfrak{x}_1}}{\mathfrak{x}_1} \, d\mathfrak{x}_1 \right| \\ &\quad \leq \frac{\ln \mathbf{i} - \ln \mathbf{j}}{2n} \left\{ \left[R_n\left(\mathbf{j}^{\frac{q}{q-1}}, \mathbf{i}^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}} \left[R_n\left(e^{q\mathbf{j}}, e^{q\mathbf{i}}\right)\right]^{\frac{1}{q}} \right. \\ &\qquad \left. + \left[R_n\left(\mathbf{i}^{\frac{q}{q-1}}, \mathbf{j}^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}} \left[R_n\left(e^{q\mathbf{i}}, e^{q\mathbf{j}}\right)\right]^{\frac{1}{q}} \right\}. \end{aligned} \tag{3.6}$$

*Proof* Let $\mathfrak{K}(\mathfrak{x}_1) = e^{\mathfrak{x}_1}$ for $\mathfrak{x}_1 > 0$. Then

$$\left|\mathfrak{K}'\left(\mathfrak{x}_1^{\lambda} \mathfrak{y}_1^{1-\lambda}\right)\right|^q = \left[e^{\mathfrak{x}_1^{\lambda} \mathfrak{y}_1^{1-\lambda}}\right]^q \leq \left[e^{\lambda \mathfrak{x}_1 + (1-\lambda)\mathfrak{y}_1}\right]^q = \left(\left|\mathfrak{K}'(\mathfrak{x}_1)\right|^q\right)^{\lambda} \left(\left|\mathfrak{K}'(\mathfrak{y}_1)\right|^q\right)^{1-\lambda}$$

for $\lambda \in [0,1]$, $\mathfrak{x}_1, \mathfrak{y}_1 > 0$, and $q > 1$, that is, $|\mathfrak{K}'(\mathfrak{x}_1)|^q = e^{q\mathfrak{x}_1}$ is geometrically convex on $[\mathbf{j}, \mathbf{i}]$ for $q > 1$, where $\mathbf{j}, \mathbf{i} > 0$. Hence inequality (3.6) follows from Theorem 7. □

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**Authors' contributions**
MAL carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. All authors read and approved the final manuscript.

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**References**

1. Bai, R.-F., Qi, F., Xi, B.-Y.: Hermite–Hadamard type inequalities for the *m*- and $(\alpha, m)$-logarithmically convex functions. Filomat **27**(1), 1–7 (2013) https://doi.org/10.2298/FIL1301001B
2. Chun, L., Qi, F.: Integral inequalities of Hermite–Hadamard type for functions whose third derivatives are convex. J. Inequal. Appl. **2013**, 451 (2013) https://doi.org/10.1186/1029-242X-2013-451
3. Dragomir, S.S.: Hermite–Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces. Linear Algebra Appl. **436**(5), 1503–1515 (2012)
4. Dragomir, S.S.: Refinements of the Hermite–Hadamard integral inequality for log-convex functions. Austral. Math. Soc. Gaz. **28**(3), 129–134 (2001)
5. Gill, P.M., Pearce, C.E.M., Pečarić, J.: Hadamard's inequality for *r*-convex functions. J. Math. Anal. Appl. **215**(2), 461–470 (1997) https://doi.org/10.1006/jmaa.1997.5645
6. Hwang, D.-Y., Dragomir, S.S.: Inequalities for the weighted mean of *r*-preinvex functions on an invex set. Journal of Mathematical Inequalities **12**(4), 1097–1106 (2018) https://doi.org/10.7153/jmi-2018-12-84
7. Huang, C.-H., Huang, H.-L., Chen, J.-S.: Examples of *r*-convex functions and characterizations of *r*-convex functions associated with second-order cone. Linear and Nonlinear Analysis **3**(3), 367–384 (2017)
8. Huy, V.N., Chung, N.T.: Some generalizations of the Fejér and Hermite–Hadamard inequalities in Hölder spaces. J. Appl. Math. Inform. **29**(3–4), 859–868 (2011)
9. Hua, J., Xi, B.-Y., Qi, F.: Hemite–Hadamard type inequalities for geometrically-arithmetically *s*-convex functions. Commun. Korean Math. Soc. **29**(1), 51–63 (2014)
10. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1934)
11. Ion, D.A.: Some estimates on the Hermite–Hadamard inequality through quasi-convex functions. An. Univ. Craiova Ser. Mat. Inform. **34**, 83–88 (2007)
12. İşcan, İ.: Hemite-Hadamard type inequalities for GA-*s*-convex functions. Le Matematiche, **LXIX** (2014)–Fasc. II, 129–146
13. Ji, A.P., Zhang, T.Y., Qi, F.: Integral Inequalities of Hermite–Hadamard Type for $(\alpha, m)$-GA-Convex Functions. Journal of Function Spaces and Applications **2013**, Article ID 823856 (2013)

14. Klaričić Bakula, M., Pečarić, J.: Note on some Hadamard-type inequalities. JIPAM. J. Inequal. Pure Appl. Math. **5**(3), Article ID 74 (2004)
15. Klaričić Bakula, M., Özdemir, M.E., Pečarić, J.: Hadamard type inequalities for $m$-convex and $(\alpha, m)$-convex functions. JIPAM. J. Inequal. Pure Appl. Math. **9**(4), Article ID 96 (2008)
16. Latif, M.A.: New Hermite–Hadamard type integral inequalities for GA-convex functions with applications. Analysis **34**(4), 379–389 (2014)
17. Park, J.: Hermite–Hadamard-type inequalities for real $\alpha$-star $s$-convex mappings. J. Appl. Math. Inform. **28**(5–6), 1507–1518 (2010)
18. Pearce, C.E.M., Pečarić, J., Šimić, V.: Stolarsky means and Hadamard's inequality. J. Math. Anal. Appl. **220**(1), 99–109 (1998) https://doi.org/10.1006/jmaa.1997.5822
19. Qi, F., Xi, B.-Y.: Some integral inequalities of Simpson type for GA-$\varepsilon$-convex functions. Georgian Math. J. **20**(4), 775–788 (2013) https://doi.org/10.1515/gmj-2013-0043
20. Sarikaya, M.Z.: On new Hermite Hadamard Fejér type integral inequalities. Stud. Univ. Babeş-Bolyai Math. **57**(3), 377–386 (2012)
21. Sarikaya, M.Z., Saglam, A., Yildirim, H.: On some Hadamard-type inequalities for $h$-convex functions. J. Math. Inequal. **2**(3), 335–341 (2008)
22. Shuang, Y., Yin, H.P., Qi, F.: Hermite–Hadamard type integral inequalities for geometric-arithmetically $s$-convex functions. Analysis **33**, 1001–1010 (2013)
23. Sun, M.-B., Yang, X.-P.: Inequalities for the weighted mean of $r$-convex functions. Proc. Amer. Math. Soc. **133**(6), 1639–1646 (2005) https://doi.org/10.1090/S0002-9939-05-07835-4
24. Unyong, B., Govindan, V., Bowmiya, S., Rajchakit, G., Gunasekaran, N., Vadivel, R., Lim, C.P., Agarwal, P.: Generalized linear differential equation using Hyers–Ulam stability approach. AIMS Mathematics **6**(2), 1607–1623 (2021). https://doi.org/10.3934/math.2021096
25. Xi, B.-Y., Qi, F.: Hemite–Hadamard type inequalities for geometrically $r$-convex functions. Studia Scientiarum Mathematicarum Hungarica **51**(4), 530–546 (2014)
26. Zhang, T.Y., Ji, A.P., Qi, F.: Some inequalities of Hermite–Hadamard type for GA-Convex functions with applications to means. Le Matematiche **48**(1), 229–239 (2013)