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# Developments of some new results that weaken certain conditions of fractional type differential equations

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**Abstract**

We introduce double and triple $F$-expanding mappings. We prove related fixed point theorems. Based on our obtained results, we also prove the existence of a solution for fractional type differential equations by using a weaker condition than the sufficient small Lipschitz constant studied by Mehmood and Ahmad (AIMS Math. 5:385–398, 2019) and Hanadi et al. (Mathematics 8:1168, 2020). As applications, we ensure the existence of a unique solution of a boundary value problem for a second-order differential equation.

**MSC:** 47H10; 47H19; 54H25

## 1 Introduction

In 2012, Wardowski [3] generalized the Banach contraction principle by introducing a new type of contractions, called $F$-contractions, and established a unique related fixed point theorem. This modification of BCP motivated many researchers to study further possibilities of its extensions [4–25]. In 2017, Gornicki [26] presented some new fixed point results for $F$-expanding mappings. We modify this setting by introducing multiple $F$ functions. The usage of multiple $F$ functions permits to find solutions for an extensive range of integral equations.

The nonlinear fractional differential equations have a valuable role in various fields of science, such as engineering, biology, fluid mechanics, physics, chemistry, bio-physics. For more details, see [21, 22, 27–37]. After establishing the fixed point theorems for expanding type mappings, we provide some new sufficient conditions for the existence of solutions of an integral boundary value problem for a scalar nonlinear Caputo fractional differential equation with fractional order in $(1, 2)$. We also compare the obtained result with known ones in the literature. Furthermore, we use our obtained results to find a solution of an engineering problem, in which the transformed mathematical model of a problem representing activation of a spring affected by an external force is a boundary value problem for a second-order differential equation.

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## 2 Preliminaries

In this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{R}_+$ denote the set of natural numbers, $\mathbb{N} \cup \{0\}$, real numbers, and positive real numbers, respectively. Throughout the paper, every set $X$ taken into account is nonempty. Wardowski [3] defined the concept of $F$-contractions as follows.

**Definition 2.1** ([3]) Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be an $F$-contraction if there is a real number $\tau > 0$ such that, for all $x, y \in X$,

$$\left[ \mathcal{D}(Tx, Ty) > 0 \text{ implies } \tau + F\big(\mathcal{D}(Tx, Ty)\big) \leq F\big(\mathcal{D}(x, y)\big) \right],$$

where $F : \mathbb{R}_+ = (0, \infty) \mapsto \mathbb{R}$ is a function satisfying the following conditions:

(F1) $F$ is strictly increasing, that is, for all $x, y \in \mathbb{R}_+$, $x < y, F(x) < F(y)$;

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \to \infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} F(\alpha_n) = -\infty$;

(F3) There is $k \in (0, 1)$ such that $\lim_{\alpha \mapsto 0^+} \alpha^k F(\alpha) = 0$.

Denote by $\mathcal{F}$ the set of all functions satisfying conditions (F1)–(F3).

*Example* 2.1 Let $F_i : \mathbb{R}_+ \mapsto \mathbb{R}$ $(i = 1, 2, 3, 4)$ be defined by

(i) $F_1(t) = \ln t$.

(ii) $F_2(t) = t + \ln t$.

(iii) $F_3(t) = -\frac{1}{\sqrt{t}}$.

(iv) $F_4(t) = \ln(t^2 + t)$.

Then $F_1, F_2, F_3, F_4 \in \mathcal{F}$.

*Remark* 2.1 From the conditions of $F$-contractions, it is easy to conclude that every $F$-contraction mapping is necessarily continuous.

Further, Wardowski [3] stated a modified version of the Banach contraction principle as follows.

**Theorem 2.1** ([3]) *Let $(X, \mathcal{D})$ be a complete metric space and $T : X \mapsto X$ be an $F$-contraction. Then $T$ has a unique fixed point, say $x^* \in X$, and for every $x \in X$, the sequence $\{T^n x\}_{n \in N}$ converges to $x^*$.*

For details on $F$-contraction mappings, see [8, 9, 38, 39]. The concept of $F$-expanding mappings is given as follows.

**Definition 2.2** ([26]) Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be $F$-expanding if there are $F \in \mathcal{F}$ and a real number $\tau > 0$ such that, for all $x, y \in X$,

$$\left[ \mathcal{D}(Tx, Ty) > 0 \text{ implies } F\big(\mathcal{D}(Tx, Ty)\big) \geq F\big(\mathcal{D}(x, y)\big) + \tau \right].$$

## 3 Main results

Firstly, we introduce two types of double $F$-expanding mappings that generalized $F$-expanding mappings.

**Definition 3.1** Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be double $F$-expanding of type I, if there exist a real number $\tau > 0$ and $F_1, F_2 \in \mathcal{F}$ such that, for all $x, y \in X$, we have

$$
\begin{aligned}
\mathcal{D}(T^2 x, T^2 y) > 0 \quad &\text{and} \quad \mathcal{D}(Tx, Ty) > 0 \\
\text{imply } \min\{ &F_2(\mathcal{D}(T^2 x, T^2 y)), F_1(\mathcal{D}(Tx, Ty)) \} \\
&\geq \alpha_2 F_2(\mathcal{D}(x, y)) + \alpha_1 F_1(\mathcal{D}(x, y)) + \tau,
\end{aligned}
\tag{1}
$$

where

$$
\left\{
\begin{aligned}
&\alpha_1 = 0, \alpha_2 = 1, \text{ if } F_2(\mathcal{D}(T^2 x, T^2 y)) \leq F_1(\mathcal{D}(Tx, Ty)) \\
&\alpha_1 = 1, \alpha_2 = 0, \text{ if } F_2(\mathcal{D}(T^2 x, T^2 y)) > F_1(\mathcal{D}(Tx, Ty))
\end{aligned}
\right\} .
$$

*Remark* 3.1 For some $x, y \in X$, the conditions of Definition 3.1 yield either $F_2(\mathcal{D}(T^2 x, T^2 y)) \geq F_2(\mathcal{D}(x, y)) + \tau$ or $F_1(\mathcal{D}(Tx, Ty)) \geq F_1(\mathcal{D}(x, y)) + \tau$.

**Definition 3.2** Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be double $F$-expanding of type II, if there exist $\tau > 0$ and $F_1, F_2 \in \mathcal{F}$ such that, for all $x, y \in X$,

$$
\begin{aligned}
\mathcal{D}(T^2 x, T^2 y) > 0 \quad &\text{and} \quad \mathcal{D}(Tx, Ty) > 0 \\
\text{imply } \min\{ &F_2(\mathcal{D}(T^2 x, T^2 y)), F_1(\mathcal{D}(Tx, Ty)) \} \\
&\geq \alpha_2 F_2(\mathcal{D}(x, y)) + \alpha_1 F_1(\mathcal{D}(x, y)) + \tau,
\end{aligned}
\tag{2}
$$

where either $\alpha_1 = 0$ or $\alpha_2 = 0$ and $\alpha_1 + \alpha_2 = 1$.

*Remark* 3.2 For all $x, y \in X$, the double $F$-expanding mapping of type II will deal with one of the following two cases (R1) and (R2):

$$
\min\{ F_2(\mathcal{D}(T^2 x, T^2 y)), F_1(\mathcal{D}(Tx, Ty)) \} \geq F_2(\mathcal{D}(x, y)) + \tau,
\tag{R1}
$$

$$
\min\{ F_2(\mathcal{D}(T^2 x, T^2 y)), F_1(\mathcal{D}(Tx, Ty)) \} \geq F_1(\mathcal{D}(x, y)) + \tau.
\tag{R2}
$$

For some, $x, y \in X$, (R1) further yields $F_2(\mathcal{D}(T^2 x, T^2 y)) \geq F_2(\mathcal{D}(x, y)) + \tau$, or, $F_1(\mathcal{D}(Tx, Ty)) \geq F_2(\mathcal{D}(x, y)) + \tau$.

Similarly, for some $x, y \in X$, (R2) yields $F_2(\mathcal{D}(T^2 x, T^2 y)) \geq F_1(\mathcal{D}(x, y)) + \tau$, or, $F_1(\mathcal{D}(Tx, Ty)) \geq F_1(\mathcal{D}(x, y)) + \tau$.

Next, we introduce triple $F$-expanding mappings.

**Definition 3.3** Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be a triple $F$-expanding mapping, if there exist $\tau > 0$ and $F, F_1, F_2 \in \mathcal{F}$ such that, for all $x, y \in X$, we have

$$
\begin{aligned}
\mathcal{D}(T^2 x, T^2 y) > 0 \quad &\text{and} \quad \mathcal{D}(Tx, Ty) > 0 \\
\text{imply } \min\{ &F_2(\mathcal{D}(T^2 x, T^2 y)), F_1(\mathcal{D}(Tx, Ty)) \} \geq F(\mathcal{D}(x, y)) + \tau.
\end{aligned}
\tag{3}
$$

*Example* 3.1  Take $F_1(\alpha) = \ln\alpha$ and $F_2(\alpha) = \ln k\alpha, k > 0$. Then $F_1$ and $F_2 \in \mathcal{F}$.

The double $F$-expanding mapping of type I will take the form

$$\min\{\ln k(\mathcal{D}(T^2x, T^2y)), \ln(\mathcal{D}(Tx, Ty))\} \geq \alpha_2 \ln k(\mathcal{D}(x,y)) + \alpha_1 \ln(\mathcal{D}(x,y)) + \tau.$$

Condition (F1) allows us to write

$$\ln\min\{k\mathcal{D}(T^2x, T^2y), \mathcal{D}(Tx, Ty)) \geq \ln k^{\alpha_2}(\mathcal{D}(x,y))^{\alpha_1+\alpha_2} + \tau.$$

By the assumption of the definition, we have $\alpha_1 + \alpha_2 = 1$, so

$$\min\{k\mathcal{D}(T^2x, T^2y), \mathcal{D}(Tx, Ty)) \geq e^\tau k^{\alpha_2}\mathcal{D}(x,y).$$

Note that if we suppose (as a particular case) that, for all $x, y \in X, \mathcal{D}(Tx, Ty) < k\mathcal{D}(T^2x, T^2y)$, then we have $\mathcal{D}(Tx, Ty) \geq e^\tau \mathcal{D}(x,y)$ with $\alpha_2 = 0$. That is, $T$ is an expanding mapping. Further, if for all $x, y \in X, \mathcal{D}(Tx, Ty) > k\mathcal{D}(T^2x, T^2y)$, then we have $\mathcal{D}(T^2x, T^2y) \geq e^\tau \mathcal{D}(x,y)$. Hence, $T$ is neither a contraction nor an expanding mapping.

*Example* 3.2  Take $F_1(\alpha) = \ln\alpha$ and $F_2(\alpha) = \ln k\alpha, k > 1$. Then $F_1, F_2 \in \mathcal{F}$.

Then, by the definition of a double $F$-expanding mapping of type II, for all $x, y \in X$, we have

$$\min\{F_2(\mathcal{D}(T^2x, T^2y)), F_1(\mathcal{D}(Tx, Ty))\} \geq \alpha_2 F_2(\mathcal{D}(x,y)) + \alpha_1 F_1(\mathcal{D}(x,y)) + \tau.$$

Conditions (R1) and (R2) allow us to write

$$\min\{F_2(\mathcal{D}(T^2x, T^2y)), F_1(\mathcal{D}(Tx, Ty))\} \geq F_1(\mathcal{D}(x,y)) + \tau \tag{4}$$

or

$$\min\{F_2(\mathcal{D}(T^2x, T^2y)), F_1(\mathcal{D}(Tx, Ty))\} \geq F_2(\mathcal{D}(x,y)) + \tau. \tag{5}$$

Relation (4) yields that

$$\min\{\ln(k\mathcal{D}(T^2x, T^2y)), \ln(\mathcal{D}(Tx, Ty))\} \geq \ln(\mathcal{D}(x,y)) + \tau.$$

Condition (F1) allows us to write

$$\ln\min\{k\mathcal{D}(T^2x, T^2y), \mathcal{D}(Tx, Ty)\} \geq \ln(\mathcal{D}(x,y)) + \tau$$

so that

$$\min\{k\mathcal{D}(T^2x, T^2y), \mathcal{D}(Tx, Ty)\} \geq e^\tau(\mathcal{D}(x,y)). \tag{6}$$

Relation (5) implies that

$$\min\{\ln(k\mathcal{D}(T^2x, T^2y)), \ln(\mathcal{D}(Tx, Ty))\} \geq \ln(k\mathcal{D}(x,y)) + \tau.$$

With the usage of condition (F1), we can write

$$\min\{k\mathcal{D}(T^2x, T^2y), \mathcal{D}(Tx, Ty)\} \geq e^\tau (k\mathcal{D}(x,y)). \tag{7}$$

One may observe that, for all $x, f(x) = y \in X$, relations (6) and (7) produce the sequences in which the iterates of $T$ may have several combinations of expansions and contractions.

*Example* 3.3 Given $F_2(\alpha) = \ln k_2\alpha$, $F_1(\alpha) = \ln k_1\alpha$, and $F(\alpha) = \ln k\alpha$, where $k, k_1, k_2 > 0$, then $F, F_1, F_2 \in \mathcal{F}$.

Then, by the definition of a triple $F$-expanding mapping, we have for all $x, y \in X$

$$\min\{F_2(\mathcal{D}(T^2x, T^2y)), F_1(\mathcal{D}(Tx, Ty))\} \geq F(\mathcal{D}(x,y)) + \tau$$

or

$$\min\{\ln(k_2\mathcal{D}(T^2x, T^2y)), \ln(k_1\mathcal{D}(Tx, Ty))\} \geq \ln(k\mathcal{D}(x,y)) + \tau.$$

Condition (F1) allows us to write

$$\ln\min\{k_2\mathcal{D}(T^2x, T^2y), k_1\mathcal{D}(Tx, Ty)) \geq \ln k\mathcal{D}(x,y) + \tau$$

or

$$\min\{k_2\mathcal{D}(T^2x, T^2y), k_1\mathcal{D}(Tx, Ty)) \geq e^\tau k\mathcal{D}(x,y) + \tau.$$

Then either $k_2(\mathcal{D}(T^2x, T^2y)) \geq ke^\tau(\mathcal{D}(x,y))$ or $k_1(\mathcal{D}(Tx, Ty)) \geq ke^\tau(\mathcal{D}(x,y))$.

We can define $k_2 = 2\alpha_2, k_1 = 2\alpha_1, k_1 + k_2 = 2$, where $\alpha_1 + \alpha_2 = 1$.

So that we have either

$$\alpha_2(\mathcal{D}(T^2x, T^2y)) \geq \frac{1}{2}ke^\tau(\mathcal{D}(x,y))$$

or

$$\alpha_1(\mathcal{D}(Tx, Ty)) \geq \frac{1}{2}ke^\tau(\mathcal{D}(x,y)).$$

Both of the above inequalities can be written as

$$\alpha_2(\mathcal{D}(T^2x, T^2y)) + \alpha_1(\mathcal{D}(Tx, Ty)) \geq ke^\tau(\mathcal{D}(x,y)).$$

That is a reversal of a mean Lipschitzian mapping, and so the fixed point of (3) will be the fixed point of $T$.

**Theorem 3.1** *Let $(X, \mathcal{D})$ be a complete metric space. Suppose that a surjective continuous mapping $T : X \mapsto X$ is a double $F$-expanding mapping of type I, and for all $t_1, t_2 \in \mathbb{R}_+$, there are $\sigma > 0$ and $\tau > \sigma$ such that*

$$F_2(t_2) < F_1(t_1) \quad implies \quad F_1(t_1) \leq F_2(t_2) + \sigma. \tag{A}$$

*Then $T$ has a unique fixed point in $X$, and for every $x_0 \in X$, the sequence $\{T^m x_0\}_{m=1}^\infty$ converges in $X$.*

*Proof* Consider a sequence $\{x_1, x_2, \ldots\}$ such that, for any $x_0 \in X$, we have $x_{m+1} = Tx_m = T^{m+1}x_0$ for all $m \in \mathbb{N}_0$. If, for some $m \in \mathbb{N}$, $\mathcal{D}(x_m, Tx_m) = 0$, $T$ admits a fixed point.

Let $\mathcal{D}(x_m, Tx_m) = \mathcal{D}(Tx_{m-1}, Tx_m) > 0$ for all $m \in \mathbb{N}$.

We will prove that $\lim_{m \to \infty} \mathcal{D}(x_m, Tx_m) = 0$.

For any $m \in \mathbb{N}$, we can write

$$\min\{F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}$$
$$\geq \alpha_2 F_2(\mathcal{D}(x_{m-1}, x_m)) + \alpha_1 F_1(\mathcal{D}(x_{m-1}, x_m)) + \tau. \tag{8}$$

Now, we will discuss the two possible cases (C) and (D):

$$F_1(\mathcal{D}(Tx_{m-1}, Tx_m)) = \min\{F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}, \tag{C}$$
$$F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)) = \min\{F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}. \tag{D}$$

If (C) holds, then by the conditions of Definition 3.1, inequality (8) will take the form

$$\min\{F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \geq F_1(\mathcal{D}(x_{m-1}, x_m)) + \tau. \tag{9}$$

Relation (9) further yields

$$\min\{F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}$$
$$> \min\{F_2(\mathcal{D}(Tx_{m-1}, Tx_m)), F_1(\mathcal{D}(x_{m-1}, x_m))\} + \tau. \tag{10}$$

Therefore, the possible existence of (C) implies the existence of (10).

Similarly, if (D) holds, inequality (8) will take the form

$$\min\{F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \geq F_2(\mathcal{D}(x_{m-1}, x_m)) + \tau. \tag{11}$$

If $F_2(\mathcal{D}(x_{m-1}, x_m)) \geq F_1(\mathcal{D}(x_{m-1}, x_m))$, then (11) can be written as

$$\min\{F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \geq F_1(\mathcal{D}(x_{m-1}, x_m)) + \tau. \tag{12}$$

If $F_2(\mathcal{D}(x_{m-1}, x_m)) < F_1(\mathcal{D}(x_{m-1}, x_m))$, then condition (A) allows us to write

$$\min\{F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \geq F_1(\mathcal{D}(x_{m-1}, x_m)) - \sigma + \tau. \tag{13}$$

Combining inequalities (12) and (13),

$$\min\{F_2(\mathcal{D}(T^2x_{m-1}, T^2x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \geq F_1(\mathcal{D}(x_{m-1}, x_m)) + \eta_m \sigma + \tau,$$

where

$$\eta_m = \begin{cases} -1 & \text{if } F_2(t) < F_1(t), \\ 0 & \text{if } F_2(t) \geq F_1(t), \end{cases} \quad t \in \mathbb{R}_+.$$

The above inequality can be written as

$$
\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}
$$
$$
\geq \min\{F_2(\mathcal{D}(Tx_{m-1}, Tx_m)), F_1(\mathcal{D}(x_{m-1}, x_m))\} + \eta_m \sigma + \tau. \tag{14}
$$

Therefore, the existence of (D) implies the existence of inequality (14).

Both cases (C) and (D) yield inequalities (10) and (13) that can be written in the combined form as follows:

$$
\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}
$$
$$
\geq \min\{F_2(\mathcal{D}(Tx_{m-1}, Tx_m)), F_1(\mathcal{D}(x_{m-1}, x_m))\} + \eta_m \varsigma_m \sigma + \tau,
$$

where $\varsigma_m$ is either 0 or 1.

The above inequality can also be written as

$$
\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}
$$
$$
\geq \min\{F_2(\mathcal{D}(T^2 x_{m-2}, T^2 x_{m-1})), F_1(\mathcal{D}(Tx_{m-2}, Tx_{m-1}))\} + \eta_m \varsigma_m \sigma + \tau.
$$

Repeating this process, we have

$$
\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}
$$
$$
\geq \min\{F_2(\mathcal{D}(T^2 x_{m-3}, T^2 x_{m-2})), F_1(\mathcal{D}(Tx_{m-3}, Tx_{m-2}))\}
$$
$$
\eta_{m-1} \varsigma_{m-1} \sigma + \eta_m \varsigma_m \sigma - 2\tau
$$

$$
\vdots
$$

$$
\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}
$$
$$
\geq \min\{F_2(\mathcal{D}(T^2 x_1, T^2 x_0)), F_1(\mathcal{D}(Tx_1, Tx_0))\} + \sum_{j=1}^{m} \eta_j \varsigma_j \sigma + m\tau
$$

or

$$
\lim_{m\to\infty} \min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} - \sum_{j=1}^{m} \eta_j \varsigma_j \sigma - m\tau
$$
$$
\geq \min\{F_2(\mathcal{D}(T^2 x_1, T^2 x_0)), F_1(\mathcal{D}(Tx_1, Tx_0))\}. \tag{15}
$$

Next, we will show that $T$ is bijective.

Let $Tx = Ty$, so that $\mathcal{D}(Tx, Ty) = 0$.

If $\mathcal{D}(x, y) > 0$, we have

$$
\min\{F_2(\mathcal{D}(T^2 x, T^2 y)), F_1(\mathcal{D}(Tx, Ty))\} \geq \alpha_2 F_2(\mathcal{D}(x, y)) + \alpha_1 F_1(\mathcal{D}(x, y)) + t.
$$

So,

$$
\min\{F_2(0), F_1(0)\} \geq \alpha_2 F_2(\mathcal{D}(x, y)) + \alpha_1 F_1(\mathcal{D}(x, y)) + t,
$$

which implies that

$$F_2(0) \geq F_2\big(\mathcal{D}(x,y)\big) + t, \quad \text{or} \quad F_1(0) \geq F_1\big(\mathcal{D}(x,y)\big) + t.$$

Both of the above inequalities are contradictions, and so we have $\mathcal{D}(x,y) = 0$ if and only if $x = y$. Therefore, $T$ is bijective.

Consider a mapping $\Omega$ such that $T\Omega = \Omega T = I$, where $I$ is the identity mapping.

For a sequence $\{x_1, x_2, \ldots, x_{m+3}, \ldots\} = \{x_1, Tx_1, \ldots T^{m+2}x_1, \ldots\}$, we can choose $x_{m+3} = u_1$, so that $\Omega^{m+2}u_1 = u_{m+3} = x_1$, $\Omega^{m+1}u_1 = u_{m+2} = x_2$, $\Omega^m u_1 = u_{m+1} = x_3$, $\Omega^{m-1}u_1 = u_m = x_4$. Moreover, $x_{m+3} = u_1$. It implies that $x_{m+2} = u_2$, $x_{m+1} = u_3$, $x_m = u_4$. Hence, inequality (20) yields

$$\min\big\{F_2\big(\mathcal{D}(u_3, u_2)\big), F_1\big(\mathcal{D}(u_4, u_3)\big)\big\} - \lim_{m\to\infty} \sum_{j=1}^{m} \eta_j \varsigma_j \sigma - m\tau$$

$$\geq \lim_{m\to\infty} \min\big\{F_2\big(\mathcal{D}(u_{m+2}, u_{m+1})\big), F_1\big(\mathcal{D}(u_{m+3}, u_{m+2})\big)\big\}$$

or

$$\min\big\{F_2\big(\mathcal{D}(\Omega^2 u_1, \Omega^2 u_0)\big), F_1\big(\mathcal{D}(\Omega^3 u_1, \Omega^3 u_0)\big)\big\} - \lim_{m\to\infty} \sum_{j=1}^{m} \eta_j \varsigma_j \sigma - m\tau$$

$$\geq \lim_{m\to\infty} \min\big\{F_2\big(\mathcal{D}(\Omega^2 u_m, \Omega^2 u_{m-1})\big), F_1\big(\mathcal{D}(\Omega^3 u_m, \Omega^3 u_{m-1})\big)\big\}.$$

Since $\sigma < \tau$, we have

$$\lim_{m\to\infty} \min\big\{F_2\big(\mathcal{D}(\Omega^2 u_m, \Omega^2 u_{m-1})\big), F_1\big(\mathcal{D}(\Omega^3 u_m, \Omega^3 u_{m-1})\big)\big\} = -\infty. \tag{16}$$

Now, equation (16) implies that

$$\lim_{m\to\infty} F_2\big(\mathcal{D}(\Omega^2 u_{m-1}, \Omega^2 u_m)\big) = -\infty \tag{E}$$

or

$$\lim_{m\to\infty} F_1\big(\mathcal{D}(\Omega^3 u_{m-1}, \Omega^3 u_m)\big) = -\infty. \tag{F}$$

Condition (F2) among (E) yields that

$$\lim_{m\to\infty} \mathcal{D}\big(\Omega^2 u_{m-1}, \Omega^2 u_m\big) = 0,$$

or equivalently,

$$\lim_{m\to\infty} \mathcal{D}\big(\Omega^2 u_{m-1}, \Omega^2 u_m\big) = \lim_{m\to\infty} \mathcal{D}(u_{m+1}, \Omega u_{m+1}) = \lim_{m\to\infty} \mathcal{D}(u_m, \Omega u_m) = 0.$$

Condition (F2) yields

$$\lim_{m\to\infty} \mathcal{D}\big(\Omega^3 u_{m-1}, \Omega^3 u_m\big) = \lim_{m\to\infty} \mathcal{D}(u_{m+2}, \Omega u_{m+2}) = \lim_{m\to\infty} \mathcal{D}(u_m, \Omega u_m) = 0.$$

Therefore, from (21) we get

$$\lim_{m \to \infty} \mathcal{D}(u_m, \Omega u_m) = 0. \tag{17}$$

Now, we will prove that the sequence $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence. On the contrary, suppose that there exist $\varepsilon > 0$ and sequences $\{\mathfrak{g}(m)\}_{n=1}^{\infty}$ and $\{\mathfrak{h}(m)\}_{m=1}^{\infty}$ of natural numbers such that

$$\mathfrak{g}(m) > \mathfrak{h}(m) > m, \qquad \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)}) \geq \varepsilon,$$

$$\mathcal{D}(u_{\mathfrak{g}(m)-1}, u_{\mathfrak{h}(m)}) < \varepsilon \quad \text{for all } m \in \mathbb{N}. \tag{18}$$

We further suppose that $\mathfrak{h}(m)$ is greater than $\mathfrak{g}(m)$ by $\mathfrak{l}(m)$.

Now, we can write

$$\varepsilon \leq \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)})$$

$$\leq \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{g}(m)-1}) + \mathcal{D}(u_{\mathfrak{g}(m)-1}, u_{\mathfrak{h}(m)})$$

$$< \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{g}(m)-1}) + \varepsilon$$

$$= \mathcal{D}(u_{\mathfrak{g}(m)-1}, Tu_{\mathfrak{g}(m)-1}) + \varepsilon.$$

That is,

$$\varepsilon \leq \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)}) < \mathcal{D}(u_{\mathfrak{g}(m)-1}, Tu_{\mathfrak{g}(m)-1}) + \varepsilon. \tag{19}$$

The above inequality along with (19) yields

$$\lim_{m \to \infty} \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)}) = \varepsilon.$$

Further, from (17) there exists $N \in \mathbb{N}$ such that

$$\mathcal{D}(u_{\mathfrak{g}(m)}, Tu_{\mathfrak{g}(m)}) < \frac{\varepsilon}{4} \quad \text{and} \quad \mathcal{D}(u_{\mathfrak{h}(m)}, Tu_{\mathfrak{h}(m)}) < \frac{\varepsilon}{4} \quad \text{for all } m \geq N. \tag{20}$$

Next, we claim that

$$\mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)}) = \mathcal{D}(u_{\mathfrak{g}(m)+1}, u_{\mathfrak{h}(m)+1}) > 0 \quad \text{for all } m \geq N. \tag{21}$$

On the contrary, suppose that there exists $r \geq N$ such that

$$\mathcal{D}(u_{\mathfrak{g}(r)+1}, u_{\mathfrak{h}(r)+1}) = 0. \tag{22}$$

It follows from (18), (20), and (22) that

$$\varepsilon \leq \mathcal{D}(u_{\mathfrak{g}(r)}, u_{\mathfrak{h}(r)}) \leq \mathcal{D}(u_{\mathfrak{g}(r)}, u_{\mathfrak{g}(r)+1}) + \mathcal{D}(u_{\mathfrak{g}(r)+1}, u_{\mathfrak{h}(r)})$$

$$\leq \mathcal{D}(u_{\mathfrak{g}(r)}, u_{\mathfrak{g}(r)+1}) + \mathcal{D}(u_{\mathfrak{g}(r)+1}, u_{\mathfrak{h}(r)+1}) + \mathcal{D}(u_{\mathfrak{h}(r)+1}, u_{\mathfrak{h}(r)})$$

$$= \mathcal{D}(u_{\mathfrak{g}(r)}, Tu_{\mathfrak{g}(r)}) + \mathcal{D}(u_{\mathfrak{g}(r)+1}, u_{\mathfrak{h}(r)+1}) + \mathcal{D}(u_{\mathfrak{h}(r)}, Tu_{\mathfrak{h}(r)})$$

$$< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

That is a contradiction.

Next, we suppose that, for some $x_{\mathfrak{b}(m)}, x_{\mathfrak{r}(m)} \in X$, we have $u_{\mathfrak{g}(m)} = x_{\mathfrak{b}(m)}$ and $u_{\mathfrak{h}(m)} = x_{\mathfrak{r}(m)}$ such that $u_{\mathfrak{g}(m)-1} = T x_{\mathfrak{b}(m)}, u_{\mathfrak{h}(m)-1} = T x_{\mathfrak{r}(m)}, u_{\mathfrak{g}(m)-2} = T^2 x_{\mathfrak{b}(m)}, u_{\mathfrak{h}(m)-2} = T^2 x_{\mathfrak{r}(m)}$.

Therefore, relation (21) with the assumption of the theorem gives

$$\mathcal{D}(x_{\mathfrak{b}(m)}, x_{\mathfrak{r}(m)}) = \mathcal{D}(x_{\mathfrak{b}(m)+1}, x_{\mathfrak{r}(m)+1}) > 0 \quad \text{implies}$$

$$\min\{F_2(\mathcal{D}(T^2 x_{\mathfrak{b}(m)}, T^2 x_{\mathfrak{r}(m)})), F_1(\mathcal{D}(T x_{\mathfrak{b}(m)}, T x_{\mathfrak{r}(m)}))\}$$

$$\geq \alpha_2 F_2(x_{\mathfrak{b}(m)}, x_{\mathfrak{r}(m)}) + \alpha_1 F_1(\mathcal{D}(x_{\mathfrak{b}(m)}, x_{\mathfrak{r}(m)})) + \tau. \tag{23}$$

Now, we will deal with two possible cases of (23):

$$F_2(\mathcal{D}(T^2 x_{\mathfrak{b}(m)}, T^2 x_{\mathfrak{r}(m)})) \geq F_2(x_{\mathfrak{b}(m)}, x_{\mathfrak{r}(m)}) + \tau$$

or

$$F_1(\mathcal{D}(T x_{\mathfrak{b}(m)}, T x_{\mathfrak{r}(m)})) \geq F_1(x_{\mathfrak{b}(m)}, x_{\mathfrak{r}(m)}) + \tau.$$

Both of the above inequalities will take the form

$$F_2(\mathcal{D}(\Omega^2 u_{\mathfrak{g}(m-4)}, \Omega^2 x_{\mathfrak{h}(m-4)})) \geq F_2(\Omega^4 u_{\mathfrak{g}(m-4)}, \Omega^4 u_{\mathfrak{h}(m-4)}) + \tau$$

or

$$F_1(\mathcal{D}(\Omega^3 q_{\mathfrak{g}(m-4)}, \Omega^3 h_{\mathfrak{r}(m-4)})) \geq F_1(\Omega^4 u_{\mathfrak{g}(m)}, \Omega^4 u_{\mathfrak{h}(m)}) + \tau.$$

So that we have the following contradictions: $F_2(\varepsilon) \geq F_2(\varepsilon) + \tau$ or $F_1(\varepsilon) \geq F_1(\varepsilon) + \tau$.

Therefore, $\{u_m\}_{n=1}^{\infty}$ is a Cauchy sequence. The completeness of $(X, \mathcal{D})$ proves that $\{u_m\}_{n=1}^{\infty}$ converges to some point $u^*$ in $X$. Now, the continuity of $\Omega$ implies that

$$\mathcal{D}(\Omega u, u) = \lim_{m \to \infty} \mathcal{D}(\Omega u_m, u_m) = \lim_{m \to \infty} \mathcal{D}(u_{m+1}, u_m) = \mathcal{D}(u^*, u^*) = 0.$$

Therefore, $\Omega$ has a fixed point $u^*$ in $X$ and $\Omega u^* = u^*$ so that $u^* = T u^*$. Now, for the uniqueness, let us suppose that $T$ has more than one fixed point. That is, there exist two distinct $u, v \in X$ such that $Tu = u \neq v = Tv$.

Therefore, $\mathcal{D}(u, v) = \mathcal{D}(Tu, Tv) = \mathcal{D}(T^2 u, T^2 v) > 0$ with relation (1) implies that either

$$F_1(\mathcal{D}(Tu, Tv)) \geq F_1(\mathcal{D}(u, v)) + \tau > F_1(\mathcal{D}(u, v)) = F_1(\mathcal{D}(Tu, Tv)) \tag{24}$$

or

$$F_2(\mathcal{D}(T^2 u, T^2 v)) \geq F_2(\mathcal{D}(u, v)) + \tau > F_2(\mathcal{D}(u, v)) = F_2(\mathcal{D}(T^2 u, T^2 v)). \tag{25}$$

Both relations (24) and (25) are the contradictions, and so we have a unique fixed point. $\square$

**Theorem 3.2** *Let $(X, \mathcal{D})$ be a complete metric space. Suppose that a surjective continuous mapping $T : X \mapsto X$ is a double F-expanding mapping of type II, and for all $t, t_1, t_2 \in \mathbb{R}_+$,*

*there is $\sigma > 0$ such that $\sigma < \tau$ and*

$$F_2(t) < F_1(t) \quad implies \; F_1(t) \le F_2(t) + \sigma, \tag{B}$$

$$F_1(t) < F_2(t) \quad implies \; F_2(t) \le F_1(t) + \sigma. \tag{B'}$$

*Then $T$ has a unique fixed point in $X$, and for every $x_0 \in X$, the sequence $\{T^m x_0\}_{m=1}^{\infty}$ converges to a definite number.*

*Proof* Consider a sequence $\{x_1, x_2, \ldots\}$ such that $x_{m+1} = Tx_m = T^{m+1} x_0$ for any $x_0 \in X$, for all $m \in \mathbb{N}_0$. If, for some $m \in \mathbb{N}$, $\mathcal{D}(x_m, Tx_m) = 0$, $T$ will admit a fixed point. Let $\mathcal{D}(x_m, Tx_m) = \mathcal{D}(Tx_{m-1}, Tx_m) > 0$ for all $m \in \mathbb{N}$.

If, for all $x, y \in X$, relation (R2) holds, then the analysis of the previous theorem yields

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}$$
$$\ge \min\{F_2(\mathcal{D}(Tx_{m-1}, Tx_m)), F_1(\mathcal{D}(x_{m-1}, x_m))\} + \tau.$$

The above inequality can be written as follows:

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}$$
$$\ge \min\{F_2(\mathcal{D}(T^2 x_{m-2}, T^2 x_{m-1})), F_1(\mathcal{D}(Tx_{m-2}, Tx_{m-1}))\} + \tau.$$

Repeating this process, we have

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}$$
$$\ge \min\{F_2(\mathcal{D}(T^2 x_{m-3}, T^2 x_{m-2})), F_1(\mathcal{D}(Tx_{m-3}, Tx_{m-2}))\} + 2\tau,$$

$$\vdots$$

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}$$
$$\ge \min\{F_2(\mathcal{D}(T^2 x_1, T^2 x_0)), F_1(\mathcal{D}(Tx_1, Tx_0))\} + m\tau.$$

Now, the analysis similar to the previous theorem yields

$$\min\{F_2(\mathcal{D}(u_3, u_2)), F_1(\mathcal{D}(u_4, u_3))\} - m\tau$$
$$\ge \lim_{m \to \infty} \min\{F_2(\mathcal{D}(u_{m+2}, u_{m+1})), F_1(\mathcal{D}(u_{m+3}, u_{m+2}))\}$$

or

$$\min\{F_2(\mathcal{D}(\Omega^2 u_1, \Omega^2 u_0)), F_1(\mathcal{D}(\Omega^3 u_1, \Omega^3 u_0))\} - m\tau$$
$$\ge \lim_{m \to \infty} \min\{F_2(\mathcal{D}(\Omega^2 u_m, \Omega^2 u_{m-1})), F_1(\mathcal{D}(\Omega^3 u_m, \Omega^3 u_{m-1}))\},$$
$$\lim_{m \to \infty} \min\{F_2(\mathcal{D}(\Omega^2 u_m, \Omega^2 u_{m-1})), F_1(\mathcal{D}(\Omega^3 u_m, \Omega^3 u_{m-1}))\} = -\infty. \tag{26}$$

If, for all $x, y \in X$, relation (R1) holds, then we can write

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \ge F_1(\mathcal{D}(x_{m-1}, x_m)) + \eta_m \sigma + \tau,$$

where

$$\eta_m = \begin{cases} -1 & \text{if } F_2(t) < F_1(t), \\ 0 & \text{if } F_2(t) \geq F_1(t), \end{cases} \quad t \in \mathbb{R}_+.$$

The above inequality will take the form

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(T x_{m-1}, T x_m))\}$$
$$\geq \min\{F_2(\mathcal{D}(T x_{m-1}, T x_m)), F_1(\mathcal{D}(x_{m-1}, x_m))\} + \eta_m \sigma + \tau.$$

That can be written as

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(T x_{m-1}, T x_m))\}$$
$$\geq \min\{F_2(\mathcal{D}(T^2 x_{m-2}, T^2 x_{m-1})), F_1(\mathcal{D}(T x_{m-2}, T x_{m-1}))\} + \eta_m \sigma + \tau.$$

Repeating this process, we have

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(T x_{m-1}, T x_m))\}$$
$$\geq \min\{F_2(\mathcal{D}(T^2 x_{m-3}, T^2 x_{m-2})), F_1(\mathcal{D}(T x_{m-3}, T x_{m-2}))\}$$
$$+ \eta_{m-1} \sigma + \eta_m \sigma + 2\tau$$

$$\vdots$$

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(T x_{m-1}, T x_m))\}$$
$$\geq \min\{F_2(\mathcal{D}(T^2 x_1, T^2 x_0)), F_1(\mathcal{D}(T x_1, T x_0))\} + \sum_{j=1}^{m} \eta_j \sigma + m\tau.$$

So that

$$\min\{F_2(\mathcal{D}(\Omega^2 u_1, \Omega^2 u_0)), F_1(\mathcal{D}(\Omega^3 u_1, \Omega^3 u_0))\} - \lim_{m \to \infty} \sum_{j=1}^{m} \eta_j \sigma - m\tau$$
$$\geq \lim_{m \to \infty} \min\{F_2(\mathcal{D}(\Omega^2 u_m, \Omega^2 u_{m-1})), F_1(\mathcal{D}(\Omega^3 u_m, \Omega^3 u_{m-1}))\}.$$

Since $\sigma < \tau$, we have

$$\lim_{m \to \infty} \min\{F_2(\mathcal{D}(\Omega^2 u_m, \Omega^2 u_{m-1})), F_1(\mathcal{D}(\Omega^3 u_m, \Omega^3 u_{m-1}))\} = -\infty. \tag{27}$$

Therefore, relations (26) and (27) imply that

$$\lim_{m \to \infty} F_2(\mathcal{D}(\Omega^2 u_{m-1}, \Omega^2 u_m)) = -\infty.$$

or,

$$\lim_{m \to \infty} F_1(\mathcal{D}(\Omega^3 u_{m-1}, \Omega^3 u_m)) = -\infty.$$

Now, using the analysis of the previous theorem, relations (R1) and (R2) yield that

$$F_2(\varepsilon) \geq F_2(\varepsilon) + \tau, \tag{28}$$

$$F_1(\varepsilon) \geq F_1(\varepsilon) + \tau, \tag{29}$$

$$F_2(\varepsilon) \geq F_1(\varepsilon) + \tau, \tag{30}$$

$$F_1(\varepsilon) \geq F_2(\varepsilon) + \tau. \tag{31}$$

As $\tau > 0$, relations (28) and (29) are contradictions.

Now, we consider relation (30)

$$F_2(\varepsilon) \geq F_1(\varepsilon) + \tau.$$

Using condition (B'), we have

$$F_1(\varepsilon) + \sigma \geq F_1(\varepsilon) + \tau.$$

It is a contradiction, as $\tau > \sigma$.

Similarly, $F_1(\varepsilon) \geq F_2(\varepsilon) + \tau$, which is a contradiction: $\sigma \leq \tau$.

The contradictions of relations (28)–(31) prove that $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence. The completeness of $(X, \mathcal{D})$ proves that $\{u_m\}_{m=1}^{\infty}$ converges to some point $u^*$ in $X$.

Now, the continuity of $\Omega$ implies

$$\mathcal{D}(\Omega u, u) = \lim_{m \to \infty} \mathcal{D}(\Omega u_m, u_m) = \lim_{m \to \infty} \mathcal{D}(u_{m+1}, u_m) = \mathcal{D}(u^*, u^*) = 0.$$

Therefore, $\Omega$ has a fixed point $u^*$ in $X$ and $\Omega u^* = u^*$. So that $u^* = Tu^*$. Now, for uniqueness, let us suppose that $T$ has more than one fixed point. That is, there exist two distinct $u, v \in X$ such that $Tu = u \neq v = Tv$. Therefore, $\mathcal{D}(u, v) = \mathcal{D}(Tu, Tv) = \mathcal{D}(T^2u, T^2v) > 0$, and the assumption of the theorem leads to the following four possibilities:

$$F_1\big(\mathcal{D}(u,v)\big) = F_1\big(\mathcal{D}(Tu, Tv)\big) \geq F_1\big(\mathcal{D}(u,v)\big) + \tau, \tag{32}$$

$$F_2\big(\mathcal{D}(u,v)\big) = F_2\big(\mathcal{D}(T^2u, T^2v)\big) \geq F_2\big(\mathcal{D}(u,v)\big) + \tau, \tag{33}$$

$$F_2\big(\mathcal{D}(u,v)\big) = F_2\big(\mathcal{D}(T^2u, T^2v)\big) \geq F_1\big(\mathcal{D}(u,v)\big) + \tau \tag{34}$$

$$\text{implies } \sigma + F_1\big(\mathcal{D}(u,v)\big) \geq F_1\big(\mathcal{D}(u,v)\big) + \tau,$$

$$F_1\big(\mathcal{D}(u,v)\big) = F_1\big(\mathcal{D}(Tu, Tv)\big) \geq F_2\big(\mathcal{D}(u,v)\big) + \tau \tag{35}$$

$$\text{implies } \sigma + F_1\big(\mathcal{D}(u,v)\big) \geq F_1\big(\mathcal{D}(u,v)\big) + \tau.$$

All the four relations (32)–(35) are contradictions, and so we have a unique fixed point. □

**Theorem 3.3** *Let $(X, \mathcal{D})$ be a complete metric space. Suppose that a continuous mapping $T : X \mapsto X$ is a triple F-contraction, and for all $t, t_1, t_2 \in \mathbb{R}_+$, there exist $\sigma > 0$ and $\tau > \sigma$ such that $F(t_1) < F_i(t_2)$ implies $F_i(t_2) \leq F(t_1) + \sigma, i = 1, 2$.*

*Then $T$ has a unique fixed point in $X$, and for every $x_0 \in X$, the sequence $\{T^m x_0\}_{m=1}^{\infty}$ converges to a definite number.*

*Proof* Consider a sequence $\{x_1, x_2, \ldots\}$ such that $x_{m+1} = Tx_m = T^{m+1}x_0$ for some $x_0 \in X$ and for all $m \in \mathbb{W}$. If, for some $m \in \mathbb{N}$, $\mathcal{D}(x_m, Tx_m) = 0$, $T$ has a fixed point.

Let $\mathcal{D}(x_m, Tx_m) = \mathcal{D}(Tx_{m-1}, Tx_m) > 0$ for all $m \in \mathbb{N}$.

We will prove that $\lim_{m \to \infty} \mathcal{D}(x_m, Tx_m) = 0$.

For any $m \in \mathbb{N}$, we have

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \geq F(\mathcal{D}(x_{m-1}, x_m)) + \tau. \tag{36}$$

If $F(\mathcal{D}(x_{m-1}, x_m)) \geq F_1(\mathcal{D}(x_{m-1}, x_m))$, inequality (36) can be written as follows:

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \geq F_1(\mathcal{D}(x_{m-1}, x_m)) + \tau. \tag{37}$$

If $F(\mathcal{D}(x_{m-1}, x_m)) < F_1(\mathcal{D}(x_{m-1}, x_m))$, condition (A) allows us to write

$$F_1(\mathcal{D}(x_{m-1}, x_m)) \leq F(\mathcal{D}(x_{m-1}, x_m)) + \sigma.$$

Then inequality (36) can be written as

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \geq F_1(\mathcal{D}(x_{m-1}, x_m)) - \sigma + \tau. \tag{38}$$

Combining inequalities (37) and (38), we have

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} \geq F_1(\mathcal{D}(x_{m-1}, x_m)) + \varsigma_i \sigma + \tau, \tag{39}$$

where $\varsigma_i$ is either 0 or $-1$.

Next, we will consider the following two possible cases:

$$F_1(\mathcal{D}(x_{m-1}, x_m)) = \min\{F_2(\mathcal{D}(Tx_{m-1}, Tx_m)), F_1(\mathcal{D}(x_{m-1}, x_m))\} \tag{40}$$

or

$$F_2(\mathcal{D}(Tx_{m-1}, Tx_m)) = \min\{F_2(\mathcal{D}(Tx_{m-1}, Tx_m)), F_1(\mathcal{D}(x_{m-1}, x_m))\}. \tag{41}$$

If (40) is true, inequality (39) will take the form

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}$$
$$\geq \min\{F_2(\mathcal{D}(Tx_{m-1}, Tx_m)), F_1(\mathcal{D}(x_{m-1}, x_m))\} + \varsigma_i \sigma + \tau. \tag{42}$$

If (41) is true, we have $F_2(\mathcal{D}(Tx_{m-1}, Tx_m)) < F_1(\mathcal{D}(x_{m-1}, x_m))$.

So that relation (41) can be written as

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\} > F_2(\mathcal{D}(Tx_{m-1}, Tx_m)) + \varsigma_i \sigma + \tau.$$

Moreover, condition (41) will change the above inequality in the following form:

$$\min\{F_2(\mathcal{D}(T^2 x_{m-1}, T^2 x_m)), F_1(\mathcal{D}(Tx_{m-1}, Tx_m))\}$$

$$> \min\big\{F_2\big(\mathcal{D}(Tx_{m-1}, Tx_m)\big), F_1\big(\mathcal{D}(x_{m-1}, x_m)\big)\big\} + \varsigma_i\sigma + \tau. \tag{43}$$

Combining (42) and (43), we have

$$\min\big\{F_2\big(\mathcal{D}\big(T^2x_{m-1}, T^2x_m\big)\big), F_1\big(\mathcal{D}(Tx_{m-1}, Tx_m)\big)\big\}$$
$$\geq \min\big\{F_2\big(\mathcal{D}(Tx_{m-1}, Tx_m)\big), F_1\big(\mathcal{D}(x_{m-1}, x_m)\big)\big\} + \varsigma_i\sigma + \tau.$$

That is equivalent to

$$\min\big\{F_2\big(\mathcal{D}\big(T^2x_{m-1}, T^2x_m\big)\big), F_1\big(\mathcal{D}(Tx_{m-1}, Tx_m)\big)\big\}$$
$$\geq \min\big\{F_2\big(\mathcal{D}\big(T^2x_{m-2}, T^2x_{m-1}\big)\big), F_1\big(\mathcal{D}(Tx_{m-2}, Tx_{m-1})\big)\big\} + \varsigma_i\sigma + \tau.$$

Repeating this process, we have

$$\min\big\{F_2\big(\mathcal{D}\big(T^2x_{m-1}, T^2x_m\big)\big), F_1\big(\mathcal{D}(Tx_{m-1}, Tx_m)\big)\big\}$$
$$\geq \min\big\{F_2\big(\mathcal{D}\big(T^2x_{m-3}, T^2x_{m-2}\big)\big), F_1\big(\mathcal{D}(Tx_{m-3}, Tx_{m-2})\big)\big\} + \varsigma_{i-1}\sigma + \varsigma_i\sigma + 2\tau$$

$$\vdots$$

$$\min\big\{F_2\big(\mathcal{D}\big(T^2x_{m-1}, T^2x_m\big)\big), F_1\big(\mathcal{D}(Tx_{m-1}, Tx_m)\big)\big\}$$
$$\geq \min\big\{F_2\big(\mathcal{D}\big(T^2x_1, T^2x_0\big)\big), F_1\big(\mathcal{D}(Tx_1, Tx_0)\big)\big\} + \sigma \sum_{j=1}^{m} \varsigma_j + m\tau.$$

Now, the analysis similar to the previous theorem yields that

$$\min\big\{F_2\big(\mathcal{D}(u_3, u_2)\big), F_1\big(\mathcal{D}(u_4, u_3)\big)\big\} - \lim_{m\to\infty} \sum_{j=1}^{m} \varsigma_j\sigma - m\tau$$
$$\geq \lim_{m\to\infty} \min\big\{F_2\big(\mathcal{D}(u_{m+2}, u_{m+1})\big), F_1\big(\mathcal{D}(u_{m+3}, u_{m+2})\big)\big\}$$

or

$$\min\big\{F_2\big(\mathcal{D}\big(\Omega^2u_1, \Omega^2u_0\big)\big), F_1\big(\mathcal{D}\big(\Omega^3u_1, \Omega^3u_0\big)\big)\big\} - \lim_{m\to\infty} \sum_{j=1}^{m} \delta_j\sigma - m\tau$$
$$\geq \lim_{m\to\infty} \min\big\{F_2\big(\mathcal{D}\big(\Omega^2u_m, \Omega^2u_{m-1}\big)\big), F_1\big(\mathcal{D}\big(\Omega^3u_m, \Omega^3u_{m-1}\big)\big)\big\}.$$

As $\tau > \sigma$, we have

$$\lim_{m\to\infty} \min\big\{F_2\big(\mathcal{D}\big(\Omega^2u_m, \Omega^2u_{m-1}\big)\big), F_1\big(\mathcal{D}\big(\Omega^3u_m, \Omega^3u_{m-1}\big)\big)\big\} = -\infty. \tag{44}$$

Therefore, relation (44) implies that

$$\lim_{m\to\infty} F_2\big(\mathcal{D}\big(\Omega^2u_{m-1}, \Omega^2u_m\big)\big) = -\infty$$

or

$$\lim_{m \to \infty} F_1\big(\mathcal{D}\big(\Omega^3 u_{m-1}, \Omega^3 u_m\big)\big) = -\infty.$$

Using the analysis of the previous theorem, we will have

$$\min\{F_2(\varepsilon), F_1(\varepsilon)\} \geq F(\varepsilon) + \tau.$$

That yields either $F_2(\varepsilon) \geq F(\varepsilon) + \tau$ or $F_1(\varepsilon) \geq F(\varepsilon) + \tau$. Now, with the usage of condition (B), we can write $F_2(\varepsilon) \geq F(\varepsilon) + \tau$ or $F_2(\varepsilon) > F(\varepsilon) + \sigma$, which implies, $F_2(\varepsilon) > F_2(\varepsilon)$. It is a contradiction. Similarly, $F_1(\varepsilon) \geq F(\varepsilon) + \tau$. Also, it yields a contradiction.

Thus, $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence. The completeness of $(X, \mathcal{D})$ proves that $\{u_m\}_{m=1}^{\infty}$ converges to some point $u^*$ in $X$. Now, the continuity of $\Omega$ implies

$$\mathcal{D}(\Omega u, u) = \lim_{m \to \infty} \mathcal{D}(\Omega u_m, u_m) = \lim_{m \to \infty} \mathcal{D}(\Omega u_{m+1}, u_m) = \mathcal{D}\big(u^*, u^*\big) = 0.$$

Therefore, $\Omega$ has a fixed point $u^*$ in $X$, that is, $\Omega u^* = u^*$. So that $u^* = Tu^*$. Now, for uniqueness, let us suppose that $T$ has more than one fixed point. That is, there exist two distinct $u, v \in X$ such that $Tu = u \neq v = Tv$. Therefore, $\mathcal{D}(u, v) = \mathcal{D}(Tu, Tv) = \mathcal{D}(T^2 u, T^2 v) > 0$ with $\min\{F_2(\mathcal{D}(T^2 u, T^2 v)), F_1(\mathcal{D}(Tu, Tv))\} \geq F(\mathcal{D}(u, v)) + \tau$ implies that either

$$F_1\big(\mathcal{D}(u, v)\big) = F_1\big(\mathcal{D}(Tu, Tv)\big) \geq F\big(\mathcal{D}(u, v)\big) + \tau > F\big(\mathcal{D}(u, v)\big) + \sigma > F_1\big(\mathcal{D}(u, v)\big) \qquad (45)$$

or

$$F_2\big(\mathcal{D}(u, v)\big) = F_2\big(\mathcal{D}(T^2 u, T^2 v)\big) \geq F\big(\mathcal{D}(u, v)\big) + \tau > F\big(\mathcal{D}(u, v)\big) + \sigma > F_2\big(\mathcal{D}(u, v)\big). \quad (46)$$

Since both relations (45) and (46) are contradictions, the mapping $T$ has a unique fixed point. □

## 4 Applications to Caputo fractional differential equations

As applications of our work, we will study the existence of solutions of Caputo fractional differential equations of the fractional order in (1, 2) with an integral boundary condition. The main condition in the problems studied in [1] (see Theorem 3.2 therein) and [2] (see Theorem 12 therein) is associated with sufficient small Lipschitz constant. We will use a less restrictive condition than the Lipschitz condition by applying our obtained fixed point theorem.

For $1 < \mathfrak{r} < 2$ and a Caputo fractional derivative ${}_{\alpha}^{C}D_t^{\mathfrak{r}} x(t) = \frac{1}{\Gamma(2-\mathfrak{r})} \int_{\alpha}^{t} (t-s)^{1-\mathfrak{r}} x''(s)\,ds$, consider a nonlinear Caputo fractional differential equation

$$_{\alpha}^{C}D_{\mathfrak{t}}^{\mathfrak{r}}\big(x(\mathfrak{t})\big) = f\big(\mathfrak{t}, x(\mathfrak{t})\big) \quad \text{for } \mathfrak{t} \in (\alpha, \beta) \tag{47}$$

with an integral boundary condition

$$x(\alpha) = 0, x(\beta) = \int_{\alpha}^{\lambda} x(s)\,ds \quad (\alpha < \lambda < \beta), \tag{48}$$

where, for some $\lambda \in (\alpha, \beta)$, $x(\lambda) \in \mathbb{R}$ and $\alpha, \beta$ are given real numbers such that $0 \leq \alpha < \beta$.

Let $\Omega = C([\alpha, \beta], \mathbb{R})$ be endowed with a norm $\|x\|_{[\alpha, \beta]} = \sup_{s \in [\alpha, \beta]} |x(s)|$.

For any $x, y \in \Omega$, we define $\mathcal{W}(x, y) = \|x - y\|_{[\alpha, \beta]}$.

In order to assure the existence of solution of nonlinear Caputo fractional differential equation (47), we consider the following fractional differential equation:

$$
{}^{C}_{\alpha}D^{\mathfrak{r}}_{t}(x(t)) = g(t) \quad \text{for } t \in (\alpha, \beta).
\tag{49}
$$

Kilbas [40] proved that the following function

$$
\begin{aligned}
x(t) ={}& \frac{1}{\Gamma(\mathfrak{r})} \int_{\alpha}^{t} (t-s)^{\mathfrak{r}-1} g(s)\, ds \\
&+ \frac{2(t-\alpha)}{((\lambda-\alpha)^2 - 2(\beta-\alpha))\Gamma(\mathfrak{r})} \int_{\alpha}^{\beta} (\beta-s)^{\mathfrak{r}-1} g(s)\, ds \\
&- \frac{2(t-\alpha)}{((\lambda-\alpha)^2 - 2(\beta-\alpha))\Gamma(\mathfrak{r})} \int_{\alpha}^{\lambda} \int_{\alpha}^{s} (s-\xi)^{\mathfrak{r}-1} g(\xi)\, d\xi\, ds
\end{aligned}
\tag{50}
$$

represents the solution of boundary value problem (48) and (49) for $g \in \Omega$, based on the following presentation of the solution:

$$
x(t) = \frac{1}{\Gamma(\mathfrak{r})} \int_{\alpha}^{t} (t-s)^{\mathfrak{r}-1} g(s)\, ds - d_1 - d_2(t-\alpha).
$$

Next, we will define a mild solution of (47) and (48).

**Definition 4.1** The function $x \in \Omega$ is a mild solution of boundary value problem (47) and (48) if it satisfies

$$
\begin{aligned}
x(t) ={}& \frac{1}{\Gamma(\mathfrak{r})} \int_{\alpha}^{t} (t-s)^{\mathfrak{r}-1} \mathcal{T}\big(s, x(s)\big)\, ds \\
&+ \frac{2(t-\alpha)}{((\lambda-\alpha)^2 - 2(\beta-\alpha))\Gamma(\mathfrak{r})} \int_{\alpha}^{\beta} (\beta-s)^{\mathfrak{r}-1} \mathcal{T}\big(s, x(s)\big)\, ds \\
&- \frac{2(t-\alpha)}{((\lambda-\alpha)^2 - 2(\beta-\alpha)\Gamma(\mathfrak{r})} \int_{\alpha}^{\lambda} \int_{\alpha}^{s} (s-\xi)^{\mathfrak{r}-1} \mathcal{T}\big(\xi, x(\xi)\big)\, d\xi\, ds, \quad t \in [\alpha, \beta].
\end{aligned}
\tag{51}
$$

For any function $u \in \Omega$, we define a surjective mapping $\Upsilon : \Omega \to \Omega$ by

$$
\begin{aligned}
\Upsilon(u)(t) ={}& \frac{1}{\Gamma(\mathfrak{r})} \int_{\alpha}^{t} (t-s)^{\mathfrak{r}-1} \mathcal{T}\big(s, u(s)\big)\, ds \\
&+ \frac{2(t-\alpha)}{((\lambda-\alpha)^2 - 2(\beta-\alpha)\Gamma(\mathfrak{r})} \int_{\alpha}^{\beta} (\beta-s)^{\mathfrak{r}-1} \mathcal{T}\big(s, u(s)\big)\, ds \\
&- \frac{2(t-\alpha)}{((\lambda-\alpha)^2 - 2(\beta-\alpha))\Gamma(\mathfrak{r})} \int_{\alpha}^{\lambda} \int_{\alpha}^{s} (s-\xi)^{\mathfrak{r}-1} \mathcal{T}\big(\xi, u(\xi)\big)\, d\xi\, ds
\end{aligned}
\tag{52}
$$

for $t \in [\alpha, \beta]$.

Now, we establish the existence result as follows.

**Theorem 4.1** *Suppose that*

(*i*) *There exist a constant $K > 0$ with*

$$\frac{K(\beta - \alpha)\mathfrak{r}}{\Gamma(1 + \mathfrak{r})}\left(1 + \frac{2K(\beta - \alpha)}{(2(\beta - \alpha) - (\lambda - \alpha)^2)}\left(1 + \frac{\lambda - \alpha}{1 + \mathfrak{r}}\right)\right) \in (0, \infty) \tag{53}$$

*and a function $\mathcal{T} \in C([\alpha, \beta] \times \mathbb{R}, \mathbb{R})$ such that*

$$\left|\mathcal{T}(t, x) - \mathcal{T}(t, y)\right| \le K|x - y|^r, \quad x, y \in \mathbb{R}, t \in [\alpha, \beta],$$

*where $r \in (0, 1]$;*

(*ii*) *There exists a function $x_0 \in \Omega$ such that $\mathcal{W}(x_0, \Upsilon(x_0)) > 0$, where the operator $\Upsilon$ is defined by* (52);

(*iii*) *For any two functions $x, y \in \Omega$ such that $\mathcal{W}(x, y) > 0$, the inequality $\mathcal{W}(\Upsilon(x), \Upsilon(y)) > 0$ holds*;

*Then boundary value problem* (47) *and* (48) *has a mild solution.*

*Proof* Note that any fixed point of the mapping $\Upsilon$ is a mild solution of boundary value problem (47) and (48). Now, let $x, y \in \Omega$ be such that $\mathcal{W}(x, y) > 0$. By condition (i), we obtain

$$\begin{aligned}
&\left|\Upsilon(x)(t) - \Upsilon(y)(t)\right| \\
&\quad \le \frac{1}{\Gamma(\mathfrak{r})} \int_\alpha^t (t - s)^{\mathfrak{r}-1}\left|\mathcal{T}(s, x(s)) - \mathcal{T}(s, y(s))\right|\mathcal{W}s \\
&\qquad + \frac{2(t - \alpha)}{(2(\beta - \alpha) - (\lambda - \alpha)^2)\Gamma(\mathfrak{r})} \int_\alpha^\beta (1 - s)^{\mathfrak{r}-1}\left|\mathcal{T}(s, x(s)) - \mathcal{T}(s, y(s))\right|\mathcal{W}s \\
&\qquad + \frac{2(t - \alpha)}{(2(\beta - \alpha) - (\lambda - \alpha)^2)\Gamma(\mathfrak{r})} \int_\alpha^\lambda \left(\int_\alpha^s (s - t)^{\mathfrak{r}-1}\left|\mathcal{T}(t, x(t)) - \mathcal{T}(t, y(t))\right|\mathcal{W}t\right)\mathcal{W}s \\
&\quad \le \frac{K}{\Gamma(\mathfrak{r})} \int_\alpha^t (t - s)^{\mathfrak{r}-1}\left|x(s) - y(s)\right|^r ds \\
&\qquad + \frac{2K(t - \alpha)}{(2(\beta - \alpha) - (\lambda - \alpha)^2)\Gamma(\mathfrak{r})} \int_\alpha^\beta (\beta - s)^{\mathfrak{r}-1}\left|x(s) - y(s)\right|^r ds \\
&\qquad + \frac{2K(t - \alpha)}{(2(\beta - \alpha) - (\lambda - \alpha)^2)\Gamma(\mathfrak{r})} \int_\alpha^\lambda \left(\int_\alpha^s (s - \xi)^{\mathfrak{r}-1}\left|x(\xi) - y(\xi)\right|^r d\xi\right) ds \\
&\quad \le \left(\frac{K(t - \alpha)^{\mathfrak{r}}}{\mathfrak{r}\Gamma(\mathfrak{r})} + \frac{2K(t - \alpha)}{(2(\beta - \alpha) - (\lambda - \alpha)^2)\Gamma(\mathfrak{r})}\left(\frac{(\beta - \alpha)^{\mathfrak{r}}}{\mathfrak{r}} + \frac{(\lambda - \alpha)^{1+\mathfrak{r}}}{\mathfrak{r}(1 + \mathfrak{r})}\right)\right)\|x - y\|_\infty^r \\
&\quad \le \frac{K(\beta - \alpha)^{\mathfrak{r}}}{\Gamma(1 + \mathfrak{r})}\left(1 + \frac{2K(\beta - \alpha)}{(2(\beta - \alpha) - (\lambda - \alpha)^2)}\left(1 + \frac{\lambda - \alpha}{1 + \mathfrak{r}}\right)\right)\|x - y\|_\infty^r \\
&\quad = \Lambda \|x - y\|_\infty^r, \quad t \in [\alpha, \beta],
\end{aligned}$$

where

$$\Lambda = \frac{K(\beta - \alpha)^{\mathfrak{r}}}{\Gamma(1 + \mathfrak{r})}\left(1 + \frac{2K(\beta - \alpha)}{(2(\beta - \alpha) - (\lambda - \alpha)^2)}\left(1 + \frac{\lambda - \alpha}{1 + \mathfrak{r}}\right)\right) \in (0, \infty).$$

Therefore,

$$\left\| \Upsilon(x) - \Upsilon(y) \right\|_\infty \leq \Lambda \left\| x - y \right\|_\infty^r. \tag{54}$$

The conditions of the theorem imply that $\Upsilon(x) \neq \Upsilon(y)$ if and only if $x \neq y$. Therefore, $\Upsilon : \Omega \to \Omega$ is a bijective mapping. There exists $\chi : \Omega \to \Omega$ such that $\Upsilon \chi = I = \chi \Upsilon$ ($I$ the identity mapping).

Next, we suppose that $f = \Upsilon(x)$ and $h = \Upsilon(y)$ such that $\chi(f) = x$ and $\chi(h) = y$.

So that

$$\| f - h \|_\infty \leq \Lambda \left\| \chi(f) - \chi(h) \right\|_\infty^r.$$

That implies that

$$\left\| \chi(f) - \chi(h) \right\|_\infty \geq \left( \frac{1}{\Lambda} \right)^{1/r} \| f - h \|_\infty^{1/r}. \tag{55}$$

Take $\kappa > \Lambda$, so one writes

$$(\kappa)^{1/r} \left\| \chi(f) - \chi(h) \right\|_\infty \geq \left( \frac{\kappa}{\Lambda} \right)^{1/r} \| f - h \|_\infty^{1/r}. \tag{56}$$

Moreover, (55) further implies that

$$\left\| \chi^2(f) - \chi^2(h) \right\|_\infty \geq \left( \frac{1}{\Lambda} \right)^{1/r} \left\| \chi(f) - \chi(h) \right\|_\infty^{1/r}$$

$$\geq \left( \frac{1}{\Lambda} \right)^{(\frac{1}{r} + \frac{1}{r^2})} \| f - h \|_\infty^{1/r^2}.$$

So that

$$\kappa^{(\frac{1}{r} + \frac{1}{r^2})} \left\| \chi^2(f) - \chi^2(h) \right\|_\infty \geq \left( \frac{\kappa}{\Lambda} \right)^{(\frac{1}{r} + \frac{1}{r^2})} \| f - h \|_\infty^{1/r^2}. \tag{57}$$

Relation (56) yields that

$$\ln (\kappa)^{1/r} \left\| \chi(f) - \chi(h) \right\|_\infty \geq \frac{1}{r} \ln \left( \frac{\kappa}{\Lambda} \right) + \ln \| f - h \|_\infty^{1/r}. \tag{58}$$

Likewise, relation (56) can be written as

$$\ln \kappa^{(1 + \frac{1}{r})} \left\| \chi^2(f) - \chi^2(h) \right\|_\infty^r \geq \ln \left( \frac{\kappa}{\Lambda} \right) + \frac{1}{r} \ln \left( \frac{\kappa}{\Lambda} \right) + \ln \| f - h \|_\infty^{1/r}$$

$$> \frac{1}{r} \ln \left( \frac{\kappa}{\Lambda} \right) + \ln \| f - h \|_\infty^{1/r}. \tag{59}$$

Let $F_2(z) = \ln \kappa^{(1 + \frac{1}{r})}(z)^r, F_1(z) = \ln (\kappa)^{1/r}(z)$, and $F(z) = \ln (z)^{1/r}$. It is easy to verify that $F, F_1, F_2 \in \mathcal{F}$.

Combining relations (57) and (59), we have

$$\min\{F_2\big(\mathcal{W}\big(\chi^2(f),\chi^2(h)\big)\big),F_1\big(\mathcal{W}\big(\chi(f),\chi(h)\big)\big)\} \geq \frac{1}{r}\ln\bigg(\frac{\kappa}{\Lambda}\bigg) + F\big(\mathcal{W}(f,h)\big).$$

Therefore, $\chi : \Omega \to \Omega$ is a triple $F$-expanding mapping, and the operator $\chi$ has a fixed point in $\Omega$. That is, there exists a function $x^* \in C([\alpha,\beta],\mathbb{R})$ such that $x^* = \chi(x^*)$. This implies that $\Omega(x^*) = x^*$. The function $x^*$ is a mild solution of boundary value problem (47) and (48). □

*Example* 4.1 Consider the nonlinear Caputo fractional differential equation

$$_2^C D_t^{1.75}\big(x(t)\big) = \frac{1}{\sqrt{t+14}}\arctan\big(\sqrt{|x(t)|} + e^t\cos t\big) + \sin t \quad \text{for } t \in (2,3) \tag{60}$$

with the integral boundary conditions

$$x(2) = 0, \qquad x(3) = \int_0^{2.5} x(s)\,ds. \tag{61}$$

Here, $\mathfrak{r} = 1.75$. Also,

$$\mathcal{T}(t,x) = \frac{1}{\sqrt{t+14}}\arctan\big(\sqrt{|x|} + e^t\cos t\big) + \sin t$$

and

$$\big|\mathcal{T}(t,x) - \mathcal{T}(t,y)\big| \leq \bigg(\frac{\pi}{4} + 1\bigg)\sqrt{|x-y|},$$

where

$$\begin{aligned}
\Lambda &= \frac{K(\beta - \alpha)^{\mathfrak{r}}}{\Gamma(1 + \mathfrak{r})}\bigg(1 + \frac{2K(\beta - \alpha)}{(2(\beta - \alpha) - (\lambda - \alpha)^2)}\bigg(1 + \frac{\lambda - \alpha}{1 + \mathfrak{r}}\bigg)\bigg) \\
&= \frac{(\frac{\pi}{4} + 1)}{\Gamma(2.75)}\bigg(1 + \frac{2(\frac{\pi}{4} + 1)}{1.75}\frac{3.25}{2.75}\bigg) \in (0,\infty).
\end{aligned}$$

Note that $\Lambda > 1$. Therefore, Theorem 4.1 guarantees the existence of a solution of boundary value problem (60) and (61).

*Remark* 4.1 Note that boundary value problem (60) and (61) is also studied in [1] (see Example 5 therein) and [2] (see Example 3.3 therein). Based on the obtained fixed point theorems, we used the weaker conditions for the right-hand side part of the equation and found the existence of a fixed point for $K > 1$ and $\Lambda > 1$.

## 5 An application to integral equations
As another application of our work, we will consider an engineering problem in which the transformed mathematical model of a problem representing activation of spring affected

by an external force is a boundary value problem for a second-order differential equation. That is,

$$\ddot{u} + \frac{c}{m}\dot{u} = H\big(\mathfrak{w}, u(\mathfrak{w})\big); \quad u(0) = 0, \dot{u}(0) = a,$$

where $H : [0, I] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $I > 0$.

Conversion of the given problem into the following integral equation is well known:

$$u(\mathfrak{r}) = \int_0^{\mathfrak{r}} H\big(\mathfrak{w}, u(\mathfrak{w})\big) G(\mathfrak{r}, \mathfrak{w}), \quad \mathfrak{r} \in [0, I]. \tag{62}$$

Define the Green function $G(\mathfrak{r}, \mathfrak{w})$ as

$$G(\mathfrak{r}, \mathfrak{w}) = \begin{cases} (\mathfrak{r} + \mathfrak{w})e^{\tau(\mathfrak{r}-\mathfrak{w})} & \text{if } 0 \le \mathfrak{w} \le \mathfrak{r} \le I, \\ 0 & \text{if } 0 \le \mathfrak{r} \le \mathfrak{w} \le I, \end{cases}$$

with a constant $\tau = \tau(c, m) > 0$.

Let $X$ be the set of all continuous functions from $[0, I]$ into $\mathbb{R}^+$. For any arbitrary $U \in X$, define

$$\|U\|_\tau = \sup_{\mathfrak{r} \in [0,I]} \big\{ |U(\mathfrak{r})| e^{-2\tau\mathfrak{r}} \big\}.$$

Define $\mathcal{D} : X \times X \to [0, \infty)$ by

$$\mathcal{D}(x, y) = \max\big\{ \|x\|_\tau, \|y\|_\tau \big\}$$

for all $x, y \in X$.

Now, in order to find the existence of a solution to the integral equation, we consider a function $g : X \to X$ defined by

$$g\big(u(\mathfrak{r})\big) = \int_0^{\mathfrak{r}} H\big(\mathfrak{w}, u(\mathfrak{w})\big) G(\mathfrak{r}, \mathfrak{w}) \tag{63}$$

for all $u \in X$ and $\mathfrak{r} \in [0, I]$.

We will prove that there exists some $v \in X$ such that $g(v(\mathfrak{r})) = v(\mathfrak{r})$. That is, the fixed point of the $F$-expanding mapping is a solution of integral equation (62).

**Theorem 5.1** *The nonlinear integral equation* (58) *has a solution if the following conditions hold*:

(a) $H(\mathfrak{w}, u(\mathfrak{w}))$ *is an increasing function in its second variable*;

(b) $H(\mathfrak{w}, u) \ge k\tau^2 e^\tau u, k > 0$ *such that* $\mathfrak{r}\tau \ge 1 + \frac{1}{2e^2}$, *where* $\tau > 0$, $\mathfrak{r}, \mathfrak{w} \in [0, I]$, *and* $u \in \mathbb{R}^+$;

(c) $g : X \to X$ *is a surjective continuous mapping*.

*Proof* For all $v \in X$, we have

$$\big|g\big(v(\mathfrak{w})\big)\big| = g\big(v(\mathfrak{w})\big) = \int_0^{\mathfrak{r}} H\big(\mathfrak{w}, v(\mathfrak{w})\big) G(\mathfrak{r}, \mathfrak{w}) \, d\mathfrak{w}.$$

Now, using conditions (a) and (b), we can write

$$\left| g\big(v(\mathfrak{w})\big) \right| \geq k \int_0^{\mathfrak{r}} \tau^2 e^\tau \left| v(\mathfrak{w}) \right| G(\mathfrak{r}, \mathfrak{w}) \, d\mathfrak{w}$$

$$= k \int_0^{\mathfrak{r}} \tau^2 e^\tau \left| v(\mathfrak{w}) \right| (\mathfrak{r} + \mathfrak{w}) e^{\tau(\mathfrak{r} - \mathfrak{w})} \, d\mathfrak{w}$$

$$= k \int_0^{\mathfrak{r}} \tau^2 e^\tau e^{2\tau \mathfrak{w}} e^{-2\tau \mathfrak{w}} \left| v(\mathfrak{w}) \right| (\mathfrak{r} + \mathfrak{w}) e^{\tau(\mathfrak{r} - \mathfrak{w})} \, d\mathfrak{w}$$

$$= k \int_0^{\mathfrak{r}} \tau^2 e^\tau e^{2\tau \mathfrak{w}} \|v\|_\tau (\mathfrak{r} + \mathfrak{w}) e^{\tau(\mathfrak{r} - \mathfrak{w})} \, d\mathfrak{w}$$

$$= k \tau^2 e^\tau e^{\tau \mathfrak{r}} \|v\|_\tau \int_0^{\mathfrak{r}} e^{2\tau \mathfrak{w}} (\mathfrak{r} + \mathfrak{w}) e^{\tau \mathfrak{w}} \, d\mathfrak{w}$$

$$= k \tau^2 e^{\tau \mathfrak{r} + \tau} \|v\|_\tau \left( 2\frac{\mathfrak{r} e^{\tau \mathfrak{r}}}{\tau} - \frac{\mathfrak{r}}{\tau} - \frac{e^{\tau \mathfrak{r}}}{\tau^2} + \frac{1}{\tau^2} \right)$$

$$= k e^{\tau \mathfrak{r} + \tau} \|v\|_\tau \big( 2\mathfrak{r} \tau e^{\tau \mathfrak{r}} - \mathfrak{r} \tau - e^{\tau \mathfrak{r}} + 1 \big)$$

$$= k e^{2\tau \mathfrak{r}} e^\tau \|v\|_\tau \big( 2\mathfrak{r} \tau - \mathfrak{r} \tau e^{-\tau \mathfrak{r}} - 1 + e^{-\tau \mathfrak{r}} \big).$$

Therefore,

$$e^{-2\tau \mathfrak{r}} \left| g\big(v(\mathfrak{w})\big) \right| \geq k e^\tau \|v\|_\tau \big( 2t\tau - 1 + (1 - \mathfrak{r} \tau) e^{-\tau \mathfrak{r}} \big).$$

Using condition (b), one gets

$$2\mathfrak{r} \tau - 1 + (1 - \mathfrak{r} \tau) e^{-\tau \mathfrak{r}} \geq 1.$$

That is,

$$\left\| g\big(v(\mathfrak{w})\big) \right\|_\tau \geq k e^\tau \|v\|_\tau.$$

Likewise, for $w \in X$, we can find that

$$\left\| g(w) \right\|_\tau \geq k e^\tau \|w\|_\tau.$$

Since

$$\max \big\{ \|gv\|_\tau, \|gw\|_\tau \big\} \geq \max \big\{ k e^\tau \|v\|_\tau, k e^\tau \|w\|_\tau \big\},$$

we have

$$\mathcal{D}(gv, gw) \geq k e^\tau \max \big\{ \|v\|_\tau, \|w\|_\tau \big\}.$$

Equivalently,

$$\mathcal{D}(gv, gw) \geq k e^\tau \mathcal{D}(v, w). \tag{64}$$

Therefore,

$$\ln\left(\frac{1}{k}\mathcal{D}(gv,gw)\right) \geq \ln\big(\mathcal{D}(v,w)\big) + \tau. \tag{65}$$

Now, relation (64) also implies that

$$\mathcal{D}\big(g^2v,g^2w\big) \geq k^2 e^{2\tau}\mathcal{D}(v,w).$$

That is,

$$\frac{1}{k^2}\mathcal{D}\big(g^2v,g^2w\big) \geq e^{2\tau}\mathcal{D}(v,w).$$

Thus,

$$\ln\left(\frac{1}{k^2}\mathcal{D}\big(g^2v,g^2w\big)\right) \geq \ln\big(\mathcal{D}(v,w)\big) + 2\tau.$$

We deduce

$$\ln\left(\frac{1}{k^2}\mathcal{D}\big(g^2v,g^2w\big)\right) > \ln\big(\mathcal{D}(v,w)\big) + \tau. \tag{66}$$

Define $F_1(x) = \ln\frac{x}{k}, F_2(x) = \ln\frac{x}{k^2}$, and $F(x) = \ln x$. So that inequalities (65) and (66) take the form

$$F_1\big(\mathcal{D}(Tv,Tw)\big) \geq F\big(\mathcal{D}(v,w)\big) + \tau \quad \text{and} \quad F_2\big(\mathcal{D}\big(T^2v,T^2w\big)\big) > F\big(\mathcal{D}(v,w)\big) + \tau.$$

Both of the above inequalities can be written as

$$\min\big\{F_1\big(\mathcal{D}(Tv,Tw)\big), F_2\big(\mathcal{D}\big(T^2v,T^2w\big)\big)\big\} \geq F\big(\mathcal{D}(v,w)\big) + \tau.$$

Therefore, the fixed point of the above triple $F$-contraction is the solution of problem (62). □

## 6 Conclusion

In this article, we generalized $F$-expanding mappings by using multiple $F$ functions and certain conditions on the mapping. These conditions allow us to deal with the class of mappings whose iterates expand in general, but some of their iterates may contract as well. Moreover, with the usage of multiple $F$ functions, we presented an idea that allows to use weaker conditions for several fractional type differential equations. The new generalizations of $F$-expanding mappings and the corresponding results will break open new grounds for the researchers working in the field as they will be able to find the existence of solutions of an extensive range of differential equations.

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