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Existence of nonoscillatory solutions tending to zero of fourth-order nonlinear neutral dynamic equations on time scales

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Abstract

In this paper, a class of fourth-order nonlinear neutral dynamic equations on time scales is investigated. We obtain some sufficient conditions for the existence of nonoscillatory solutions tending to zero with some characteristics of the equations by Krasnoselskii's fixed point theorem. Finally, two interesting examples are presented to show the significance of the results.

MSC: 34N05; 34C10; 39A13

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1 Introduction

In this paper, we consider the existence of nonoscillatory solutions tending to zero of a fourth-order nonlinear neutral dynamic equation

$$R_4\big(t,x(t)\big)+f\big(t,x\big(h(t)\big)\big)=0 \tag{1}$$

on a time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$, where

$$R_k\big(t,x(t)\big)=\begin{cases} x(t)+p(t)x(g(t)), & k=0,\\ r_{4-k}(t)R_{k-1}^{\Delta}(t,x(t)), & k=1,2,3,\\ R_3^{\Delta}(t,x(t)), & k=4 \end{cases}$$

and $t\in[t_0,\infty)_{\mathbb{T}}$ with $t_0\in\mathbb{T}$. Moreover, throughout this paper we satisfy the conditions as follows:

(C1) $r_i\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $i=1,2,3$;
(C2) $p\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[0,\infty))$ and $\lim_{t\to\infty}p(t)=p_0\in[0,1)$;
(C3) $g,h\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ and $\lim_{t\to\infty}g(t)=\lim_{t\to\infty}h(t)=\infty$;
(C4) $f\in C([t_0,\infty)_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$ and $xf(t,x)>0$ for $x\neq0$;

(C5) $\eta_i = \lim_{t\to\infty} H_i(g(t))/H_i(t) \in (0,1]$, where

$$H_1(t) = \int_t^\infty \frac{\Delta u_3}{r_3(u_3)}, \qquad H_2(t) = \int_t^\infty \int_{u_3}^\infty \frac{\Delta u_2 \Delta u_3}{r_2(u_2) r_3(u_3)},$$

and

$$H_3(t) = \int_t^\infty \int_{u_3}^\infty \int_{u_2}^\infty \frac{\Delta u_1 \Delta u_2 \Delta u_3}{r_1(u_1) r_2(u_2) r_3(u_3)}$$

if $H_i(t_0) < \infty$, $i = 1, 2, 3$, respectively.

In recent years, the research on nonoscillation of dynamic equations on time scales has made some progress. The scientists have provided some sufficient conditions which guarantee that the equations have nonoscillatory solutions with certain characteristics. We refer the reader to [1–6] for details of the theory of time scale, and [7–22] with the references cited therein for the achievements on the existence of nonoscillatory solutions of nonlinear neutral dynamic equations on time scales.

A solution x of (1) is called eventually positive (or eventually negative) if there exists $T \in [t_0, \infty)_{\mathbb{T}}$ satisfying $x(t) > 0$ (or $x(t) < 0$) for $t \in [T, \infty)_{\mathbb{T}}$. The existence and asymptotic behavior of nonoscillatory solutions of a class of nonlinear neutral dynamic equations on time scales similar to (1) have been studied successively. Without loss of generality, only the eventually positive solutions are considered. For the first-order case, Zhu and Wang [22] investigated

$$\big(x(t) + p(t)x\big(g(t)\big)\big)^\Delta + f\big(t, x\big(h(t)\big)\big) = 0.$$

Gao and Wang [8], Deng and Wang [7] considered the second-order case

$$\big(r(t)\big(x(t) + p(t)x\big(g(t)\big)\big)^\Delta\big)^\Delta + f\big(t, x\big(h(t)\big)\big) = 0$$

under different assumptions $\int_{t_0}^\infty 1/r(t)\Delta t < \infty$ and $\int_{t_0}^\infty 1/r(t)\Delta t = \infty$, respectively. Then, the third-order case

$$\big(r_1(t)\big(r_2(t)\big(x(t) + p(t)x\big(g(t)\big)\big)^\Delta\big)^\Delta\big)^\Delta + f\big(t, x\big(h(t)\big)\big) = 0 \tag{2}$$

was studied in [15, 19, 21], and the higher-order case was considered in [17, 18, 20]. To have a deeper understanding of the asymptotic behavior of nonoscillatory solutions of these equations, Qiu [16] studied (1) with some conditions. In their works, different groups of eventually positive solutions of the equations are summarized. For each case, an appropriate Banach space is introduced and Krasnoselskii's fixed point theorem is employed to present some sufficient conditions (or necessary and sufficient conditions) for the existence of these solutions.

We note that the case tending to zero is an important type for nonoscillatory solutions of the equations. However, the asymptotic behavior of this type is more complicated than those of other cases. It is obvious that the results of the existence for nonoscillatory solutions tending to zero are not satisfactory in [7, 8, 15, 20, 22]. Some special sufficient conditions are provided but not enough to be applied universally. Therefore, new methods should be found to study nonoscillatory solutions tending to zero of the equations. Mojsej and Tartal'ová [23] were concerned with a third-order nonlinear differential equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f\bigl(x(t)\bigr) = 0, \quad t \geq a, \tag{3}$$

where f satisfies the Lipschitz condition. The authors obtained some nice sufficient conditions to ensure that (3) has a solution x with $\lim_{t\to\infty} x(t) = 0$ meeting some characteristics. Inspired by [23], Qiu [24] investigated the nonoscillatory solutions tending to zero of (2) when $g(t) \geq t$ for $t \in [t_0,\infty)_{\mathbb{T}}$, by employing a Banach space

$$\mathrm{BC}[T_0,\infty)_{\mathbb{T}} = \Bigl\{x \in \mathrm{C}\bigl([T_0,\infty)_{\mathbb{T}},\mathbb{R}\bigr) : \sup_{t\in[T_0,\infty)_{\mathbb{T}}} \bigl|x(t)\bigr| < \infty\Bigr\}, \tag{4}$$

where $\mathrm{C}([T_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the set of all continuous functions that map $[T_0,\infty)_{\mathbb{T}}$ into $\mathbb{R}$ and $\|x\| = \sup_{t\in[T_0,\infty)_{\mathbb{T}}} |x(t)|$. According to Krasnoselskii's fixed point theorem, some new results are presented. However, considering the cases such as $g(t) = t - 2$, $g(t) = t/3$, and $g(t) = t + \cos t$ for $t \in [t_0,\infty)_{\mathbb{T}}$, the conclusions in [24] are not applicable when $g(t) \geq t$ is not fulfilled eventually, especially for [7, 8, 15–22]. Afterwards, Qiu et al. [25] studied (2) under $g(t) \leq t$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and partially solved the problem. In this paper, we continue to relax the constraint and unite the cases of the function g. Provided that H_i have been defined for $i = 1,2,3$, note that they are all strictly decreasing on $[t_0,\infty)_{\mathbb{T}}$. For the case that $g(t) \geq t$ is not satisfied eventually, the condition $\eta_i = 1$ should be satisfied for $i = 1,2,3$, respectively.

In the following, Krasnoselskii's fixed point theorem (see [26]) is presented in Lemma 1.1, which will be used in the next section. Then, we show the relation between R_0 and x in Lemma 1.2 (see [24, Lemma 2.5]).

Lemma 1.1 *Suppose that U is a contraction mapping, V is completely continuous, and $Ux + Vy \in \Omega$ holds for all $x,y \in \Omega$, where $U,V : \Omega \to X$ are two operators, X is a Banach space, and Ω is a bounded, convex, and closed subset of X, then $U + V$ has a fixed point in Ω.*

Lemma 1.2 *Suppose that x is an eventually positive solution of* (1). *If there exists a constant $a \geq 0$ satisfying $\lim_{t\to\infty} R_0(t,x(t)) = a$, then we have*

$$\lim_{t\to\infty} x(t) = \frac{a}{1+p_0}.$$

2 Main results

In this section, we present some sufficient conditions for the existence of eventually positive solutions of (1) under different assumptions. Firstly, suppose that the function $f(t,x)$ is nondecreasing with respect to x, then we have Theorems 2.1–2.4.

Theorem 2.1 *Assume that the function $f(t,x)$ is nondecreasing with respect to x, $H_1(t_0) < \infty$, and*

$$\int_{t_0}^{\infty}\int_{t_0}^{u_2}\int_{t_0}^{u_1} \frac{f(u_0,2H_1(h(u_0)))}{r_1(u_1)r_2(u_2)} \Delta u_0 \Delta u_1 \Delta u_2 < \infty, \tag{5}$$

then there exists $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that (1) has two eventually positive solutions x_1 and x_2 tending to zero, which satisfy that $R_0(t,x_i(t)) > 0$, $R_1(t,x_i(t)) < 0$, $i = 1,2$, $R_2(t,x_1(t)) < 0$, $R_3(t,x_1(t)) < 0$, $R_2(t,x_2(t)) > 0$, and $R_3(t,x_2(t)) > 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$.

Proof Take p_1 satisfying $p_0 < p_1 < (1 + 4p_0)/5 < 1$, then there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$\frac{5p_1 - 1}{4} \leq p(t) \leq p_1 < 1, \qquad p(t)\frac{H_1(g(t))}{H_1(t)} \geq \frac{(5p_1 - 1)\eta_1}{4}, \quad t \in [T_0,\infty)_{\mathbb{T}}, \tag{6}$$

and

$$\int_{T_0}^{\infty}\int_{T_0}^{u_2}\int_{T_0}^{u_1}\frac{f(u_0,2H_1(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 \leq \frac{1-p_1\eta_1}{4}.$$

Choose $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Define a Banach space $\mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as (4), $\Omega_1 = \{x \in \mathrm{BC}[T_0,\infty)_{\mathbb{T}} : H_1(t) \leq x(t) \leq 2H_1(t)\}$, and two operators $U_1, V_1 : \Omega_1 \to \mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as follows:

$$(U_1x)(t) = \begin{cases} (U_1x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3p_1\eta_1H_1(t)/2 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$

$$(V_1x)(t) = \begin{cases} (V_1x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3H_1(t)/2 & \\ + \int_t^{\infty}\int_{T_1}^{u_3}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3, & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

The proof that U_1 and V_1 satisfy the conditions in Lemma 1.1 is similar to those of [7, Theorem 2.5], [8, Theorem 2], [15, Theorem 3.1], and [22, Theorem 8], so it is omitted here. Therefore, there exists $x_1 \in \Omega_1$ such that $(U_1 + V_1)x_1 = x_1$, and then, for $t \in [T_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} x_1(t) = {}& \frac{3(1+p_1\eta_1)}{2}H_1(t) - p(t)x_1\big(g(t)\big) \\ &+ \int_t^{\infty}\int_{T_1}^{u_3}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3. \end{aligned} \tag{7}$$

Since

$$\begin{aligned} &\int_t^{\infty}\int_{T_1}^{u_3}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3 \\ &\quad < H_1(t)\int_{T_1}^{\infty}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 \\ &\quad \leq H_1(t)\int_{T_1}^{\infty}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,2H_1(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 \end{aligned}$$

for $t \in [T_1,\infty)_{\mathbb{T}}$ and

$$\lim_{t\to\infty} H_1(t)\int_{T_1}^{\infty}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,2H_1(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 = 0$$

in view of (5), by Lemma 1.2, we derive

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} R_0\bigl(t,x_1(t)\bigr) = 0. \tag{8}$$

Moreover, for $t \in [T_1,\infty)_{\mathbb{T}}$, it follows that

$$\begin{aligned}
R_0\bigl(t,x_1(t)\bigr) &= \frac{3(1+p_1\eta_1)}{2}H_1(t) \\
&\quad + \int_t^\infty \int_{T_1}^{u_3} \int_{T_1}^{u_2} \int_{T_1}^{u_1} \frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3 > 0, \\
R_1\bigl(t,x_1(t)\bigr) &= -\frac{3(1+p_1\eta_1)}{2} \\
&\quad - \int_{T_1}^t \int_{T_1}^{u_2} \int_{T_1}^{u_1} \frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)} \Delta u_0 \Delta u_1 \Delta u_2 < 0, \\
R_2\bigl(t,x_1(t)\bigr) &= -\int_{T_1}^t \int_{T_1}^{u_1} \frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)} \Delta u_0 \Delta u_1 < 0,
\end{aligned}$$

and

$$R_3\bigl(t,x_1(t)\bigr) = -\int_{T_1}^t f\bigl(u_0,x\bigl(h(u_0)\bigr)\bigr)\Delta u_0 < 0.$$

On the other hand, we define another operator $\overline{V}_1 : \Omega_1 \to \mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as follows:

$$(\overline{V}_1 x)(t) = \begin{cases} (\overline{V}_1 x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3H_1(t)/2 & \\ \quad + \int_t^\infty \int_{u_3}^\infty \int_{T_1}^{u_2} \int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3, & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

Similarly, there exists $x_2 \in \Omega_1$ such that $(U_1 + \overline{V}_1)x_2 = x_2$, and then, for $t \in [T_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned}
x_2(t) &= \frac{3(1+p_1\eta_1)}{2}H_1(t) - p(t)x_2\bigl(g(t)\bigr) \\
&\quad + \int_t^\infty \int_{u_3}^\infty \int_{T_1}^{u_2} \int_{T_1}^{u_1} \frac{f(u_0,x_2(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)} \Delta u_0 \Delta u_1 \Delta u_2 \Delta u_3.
\end{aligned}$$

It follows that

$$\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} R_0\bigl(t,x_2(t)\bigr) = 0. \tag{9}$$

For $t \in [T_1,\infty)_{\mathbb{T}}$, we obtain

$$R_0\bigl(t,x_2(t)\bigr) > 0, \qquad R_1\bigl(t,x_2(t)\bigr) < 0, \qquad R_2\bigl(t,x_2(t)\bigr) > 0, \qquad R_3\bigl(t,x_2(t)\bigr) > 0.$$

The proof is complete. □

Theorem 2.2 *Assume that the function $f(t,x)$ is nondecreasing with respect to x, $H_1(t_0) < \infty$, and*

$$\int_{t_0}^{\infty}\int_{u_2}^{\infty}\int_{t_0}^{u_1}\frac{f(u_0,2H_1(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 < \infty, \tag{10}$$

then there exists $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that (1) has two eventually positive solutions x_1 and x_2 tending to zero, which satisfy that $R_0(t,x_i(t)) > 0$, $R_1(t,x_i(t)) < 0$, $i = 1,2$, $R_2(t,x_1(t)) < 0$, $R_3(t,x_1(t)) > 0$, $R_2(t,x_2(t)) > 0$, and $R_3(t,x_2(t)) < 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$.

Proof Take p_1 satisfying $p_0 < p_1 < (1+4p_0)/5 < 1$, then there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that (6) holds and

$$\int_{T_0}^{\infty}\int_{u_2}^{\infty}\int_{T_0}^{u_1}\frac{f(u_0,2H_1(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 \le \frac{1-p_1\eta_1}{4}.$$

Define the same T_1, $\mathrm{BC}[T_0,\infty)_{\mathbb{T}}$, Ω_1, and U_1 as in Theorem 2.1, and an operator $V_1' : \Omega_1 \to \mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as follows:

$$\bigl(V_1'x\bigr)(t) = \begin{cases} (V_1'x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3H_1(t)/2 & \\ \quad + \int_t^{\infty}\int_{T_1}^{u_3}\int_{u_2}^{\infty}\int_{T_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3, & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

Then there exists $x_1 \in \Omega_1$ such that $(U_1 + V_1')x_1 = x_1$. For $t \in [T_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} x_1(t) = {} & \frac{3(1+p_1\eta_1)}{2}H_1(t) - p(t)x_1\bigl(g(t)\bigr) \\ & + \int_t^{\infty}\int_{T_1}^{u_3}\int_{u_2}^{\infty}\int_{T_1}^{u_1}\frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3, \end{aligned}$$

which means that (8) and

$$R_0\bigl(t,x_1(t)\bigr) > 0, \qquad R_1\bigl(t,x_1(t)\bigr) < 0, \qquad R_2\bigl(t,x_1(t)\bigr) < 0, \qquad R_3\bigl(t,x_1(t)\bigr) > 0$$

for $t \in [T_1,\infty)_{\mathbb{T}}$.

Define another operator $\overline{V}_1' : \Omega_1 \to \mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as follows:

$$\bigl(\overline{V}_1'x\bigr)(t) = \begin{cases} (\overline{V}_1'x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3H_1(t)/2 & \\ \quad + \int_t^{\infty}\int_{u_3}^{\infty}\int_{u_2}^{\infty}\int_{T_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3, & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

Then there exists $x_2 \in \Omega_1$ such that $(U_1 + \overline{V}_1')x_2 = x_2$, and then, for $t \in [T_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} x_2(t) = {} & \frac{3(1+p_1\eta_1)}{2}H_1(t) - p(t)x_2\bigl(g(t)\bigr) \\ & + \int_t^{\infty}\int_{u_3}^{\infty}\int_{u_2}^{\infty}\int_{T_1}^{u_1}\frac{f(u_0,x_2(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3. \end{aligned}$$

It follows that (9) holds and

$$R_0(t,x_2(t)) > 0, \qquad R_1(t,x_2(t)) < 0, \qquad R_2(t,x_2(t)) > 0, \qquad R_3(t,x_2(t)) > 0$$

for $t \in [T_1,\infty)_{\mathbb{T}}$. This completes the proof. □

Theorem 2.3 *Assume that the function $f(t,x)$ is nondecreasing with respect to x, $H_2(t_0) < \infty$, and*

$$\int_{t_0}^{\infty}\int_{t_0}^{u_1}\frac{f(u_0,2H_2(h(u_0)))}{r_1(u_1)}\Delta u_0\Delta u_1 < \infty, \tag{11}$$

then there exists $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that (1) has two eventually positive solutions x_1 and x_2 tending to zero, which satisfy that $R_0(t,x_i(t)) > 0$, $R_1(t,x_i(t)) < 0$, $R_2(t,x_i(t)) > 0$, $i = 1,2$, $R_3(t,x_1(t)) > 0$, and $R_3(t,x_2(t)) < 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$.

Proof Take p_1 as in Theorem 2.1. Then there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$\frac{5p_1-1}{4} \le p(t) \le p_1 < 1, \qquad p(t)\frac{H_2(g(t))}{H_2(t)} \ge \frac{(5p_1-1)\eta_2}{4}, \quad t \in [T_0,\infty)_{\mathbb{T}},$$

and

$$\int_{T_0}^{\infty}\int_{T_0}^{u_1}\frac{f(u_0,2H_2(h(u_0)))}{r_1(u_1)}\Delta u_0\Delta u_1 \le \frac{1-p_1\eta_2}{4}.$$

Choose the same T_1, $\mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as in Theorem 2.1, $\Omega_2 = \{x \in \mathrm{BC}[T_0,\infty)_{\mathbb{T}} : H_2(t) \le x(t) \le 2H_2(t)\}$, and two operators $U_2, V_2 : \Omega_2 \to \mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as follows:

$$(U_2x)(t) = \begin{cases} (U_2x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3p_1\eta_2H_2(t)/2 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$

$$(V_2x)(t) = \begin{cases} (V_2x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3H_2(t)/2 \\ \quad + \int_t^{\infty}\int_{u_3}^{\infty}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3, & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

Similarly, U_2 and V_2 satisfy the conditions in Lemma 1.1. Then there exists $x_1 \in \Omega_2$ such that $(U_2+V_2)x_1 = x_1$. For $t \in [T_1,\infty)_{\mathbb{T}}$, it follows that

$$\begin{aligned} x_1(t) = {}& \frac{3(1+p_1\eta_2)}{2}H_2(t) - p(t)x_1(g(t)) \\ &+ \int_t^{\infty}\int_{u_3}^{\infty}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3. \end{aligned}$$

Since

$$\begin{aligned} &\int_t^{\infty}\int_{u_3}^{\infty}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3 \\ &\quad < H_2(t)\int_{T_1}^{\infty}\int_{T_1}^{u_1}\frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1 \end{aligned}$$

$$\leq H_2(t)\int_{T_1}^{\infty}\int_{T_1}^{u_1}\frac{f(u_0,2H_2(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1$$

for $t\in[T_1,\infty)_{\mathbb{T}}$ and

$$\lim_{t\to\infty}H_2(t)\int_{T_1}^{\infty}\int_{T_1}^{u_1}\frac{f(u_0,2H_2(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1=0$$

by virtue of (11), by Lemma 1.2, we obtain (8) and

$$R_0\bigl(t,x_1(t)\bigr)>0,\qquad R_1\bigl(t,x_1(t)\bigr)<0,\qquad R_2\bigl(t,x_1(t)\bigr)>0,\qquad R_3\bigl(t,x_1(t)\bigr)>0$$

for $t\in[T_1,\infty)_{\mathbb{T}}$.

On the other hand, define $\overline{V}_2:\Omega_2\to \mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as follows:

$$(\overline{V}_2x)(t)=\begin{cases}(\overline{V}_2x)(T_1), & t\in[T_0,T_1)_{\mathbb{T}},\\ 3H_2(t)/2 & \\ \quad+\int_t^{\infty}\int_{u_3}^{\infty}\int_{u_2}^{\infty}\int_{T_1}^{u_1}\frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3, & t\in[T_1,\infty)_{\mathbb{T}}.\end{cases}$$

Similarly, there exists $x_2\in\Omega_2$ such that $(U_2+\overline{V}_2)x_2=x_2$, and then, for $t\in[T_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned}x_2(t)&=\frac{3(1+p_1\eta_2)}{2}H_2(t)-p(t)x_2\bigl(g(t)\bigr)\\&\quad+\int_t^{\infty}\int_{u_3}^{\infty}\int_{u_2}^{\infty}\int_{T_1}^{u_1}\frac{f(u_0,x_2(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3,\end{aligned}$$

which implies that (9) holds. For $t\in[T_1,\infty)_{\mathbb{T}}$, we derive

$$R_0\bigl(t,x_2(t)\bigr)>0,\qquad R_1\bigl(t,x_2(t)\bigr)<0,\qquad R_2\bigl(t,x_2(t)\bigr)>0,\qquad R_3\bigl(t,x_2(t)\bigr)<0.$$

The proof is complete. □

Theorem 2.4 *Assume that the function $f(t,x)$ is nondecreasing with respect to x, $H_3(t_0)<\infty$, and*

$$\int_{t_0}^{\infty}f\bigl(u_0,2H_3\bigl(h(u_0)\bigr)\bigr)\Delta u_0<\infty,\tag{12}$$

then there exists $T_1\in[t_0,\infty)_{\mathbb{T}}$ such that (1) has an eventually positive solution x tending to zero, which satisfies that $R_0(t,x(t))>0$, $R_1(t,x(t))<0$, $R_2(t,x(t))>0$, and $R_3(t,x(t))<0$ for $t\in[T_1,\infty)_{\mathbb{T}}$.

Proof Take p_1 as in Theorem 2.1. Then there exists $T_0\in[t_0,\infty)_{\mathbb{T}}$ such that

$$\frac{5p_1-1}{4}\leq p(t)\leq p_1<1,\qquad p(t)\frac{H_3(g(t))}{H_3(t)}\geq\frac{(5p_1-1)\eta_3}{4},\quad t\in[T_0,\infty)_{\mathbb{T}},$$

and

$$\int_{T_0}^{\infty}f\bigl(u_0,2H_3\bigl(h(u_0)\bigr)\bigr)\Delta u_0\leq\frac{1-p_1\eta_3}{4}.$$

Choose the same T_1, $\mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as in Theorem 2.1, $\Omega_3 = \{x \in \mathrm{BC}[T_0,\infty)_{\mathbb{T}} : H_3(t) \le x(t) \le 2H_3(t)\}$, and two operators $U_3, V_3 : \Omega_3 \to \mathrm{BC}[T_0,\infty)_{\mathbb{T}}$ as follows:

$$(U_3x)(t) = \begin{cases} (U_3x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3p_1\eta_3H_3(t)/2 - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$

$$(V_3x)(t) = \begin{cases} (V_3x)(T_1), & t \in [T_0,T_1)_{\mathbb{T}}, \\ 3H_3(t)/2 \\ \quad + \int_t^\infty \int_{u_3}^\infty \int_{u_2}^\infty \int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3, & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

Similarly, U_3 and V_3 satisfy the conditions in Lemma 1.1. Then there exists $x \in \Omega_3$ such that $(U_3 + V_3)x = x$. For $t \in [T_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} x(t) &= \frac{3(1+p_1\eta_3)}{2}H_3(t) - p(t)x\bigl(g(t)\bigr) \\ &\quad + \int_t^\infty \int_{u_3}^\infty \int_{u_2}^\infty \int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3. \end{aligned}$$

Since

$$\begin{aligned} &\int_t^\infty \int_{u_3}^\infty \int_{u_2}^\infty \int_{T_1}^{u_1} \frac{f(u_0,x(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3 \\ &\quad < H_3(t)\int_{T_1}^\infty f\bigl(u_0,x\bigl(h(u_0)\bigr)\bigr)\Delta u_0 \le H_3(t)\int_{T_1}^\infty f\bigl(u_0,2H_3\bigl(h(u_0)\bigr)\bigr)\Delta u_0 \end{aligned}$$

for $t \in [T_1,\infty)_{\mathbb{T}}$ and

$$\lim_{t\to\infty} H_3(t)\int_{T_1}^\infty f\bigl(u_0,2H_3\bigl(h(u_0)\bigr)\bigr)\Delta u_0 = 0$$

by virtue of (12), similarly, we can conclude (8) and

$$R_0\bigl(t,x(t)\bigr) > 0, \qquad R_1\bigl(t,x(t)\bigr) < 0, \qquad R_2\bigl(t,x(t)\bigr) > 0, \qquad R_3\bigl(t,x(t)\bigr) < 0$$

for $t \in [T_1,\infty)_{\mathbb{T}}$. This completes the proof. □

Secondly, we obtain Theorems 2.5–2.8 based on the assumption that the function $f(t,x)$ satisfies the Lipschitz condition on an interval.

Theorem 2.5 *Assume that $H_1(t_0) < \infty$. If there exist a constant $L > 0$ and two functions $q \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and $f_0 \in \mathrm{C}([0,2H_1(t_0)],\mathbb{R})$ such that*

$$xf(t,x) \le xq(t)f_0(x), \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{13}$$

$$\bigl|f(t,x_1) - f(t,x_2)\bigr| \le L\cdot q(t)|x_1 - x_2|, \quad x_1,x_2 \in \bigl[0,2H_1(t_0)\bigr], \tag{14}$$

and

$$\int_{t_0}^\infty \int_{t_0}^{u_2} \int_{t_0}^{u_1} \frac{q(u_0)}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 < \infty, \tag{15}$$

then there exists $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that (1) *has two eventually positive solutions x_1 and x_2 tending to zero, which satisfy that $R_0(t,x_i(t)) > 0$, $R_1(t,x_i(t)) < 0$, $i = 1,2$, $R_2(t,x_1(t)) < 0$, $R_3(t,x_1(t)) < 0$, $R_2(t,x_2(t)) > 0$, and $R_3(t,x_2(t)) > 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$.*

Proof Take p_1 satisfying $p_0 < p_1 < (1+4p_0)/5 < 1$. There also exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that (6) holds and

$$\int_{T_0}^{\infty}\int_{T_0}^{u_2}\int_{T_0}^{u_1}\frac{q(u_0)}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 \le \min\left\{\frac{1-p_1\eta_1}{4K},1\right\},$$

where $K = \max\{|f_0(x)| : x \in [0,2H_1(t_0)]\} > 0$. Then define the same T_1, $\mathrm{BC}[T_0,\infty)_{\mathbb{T}}$, Ω_1, U_1, and V_1 as in Theorem 2.1. Proceeding as in the proof of Theorem 2.1, there exists $x_1 \in \Omega_1$ such that $(U_1 + V_1)x_1 = x_1$, and we arrive at (7). Since

$$\begin{aligned}
&\int_t^{\infty}\int_{T_1}^{u_3}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{f(u_0,x_1(h(u_0)))}{r_1(u_1)r_2(u_2)r_3(u_3)}\Delta u_0\Delta u_1\Delta u_2\Delta u_3\\
&\quad< H_1(t)\int_{T_1}^{\infty}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{q(u_0)f_0(x_1(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2\\
&\quad\le K\cdot H_1(t)\int_{T_1}^{\infty}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{q(u_0)}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2
\end{aligned}$$

for $t \in [T_1,\infty)_{\mathbb{T}}$ and

$$\lim_{t\to\infty} K\cdot H_1(t)\int_{T_1}^{\infty}\int_{T_1}^{u_2}\int_{T_1}^{u_1}\frac{q(u_0)}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 = 0$$

in view of (15), by Lemma 1.2, we obtain (8) and

$$R_0\bigl(t,x_1(t)\bigr) > 0,\qquad R_1\bigl(t,x_1(t)\bigr) < 0,\qquad R_2\bigl(t,x_1(t)\bigr) < 0,\qquad R_3\bigl(t,x_1(t)\bigr) < 0$$

for $t \in [T_1,\infty)_{\mathbb{T}}$. Similarly, we deduce the remaining conclusions as in Theorem 2.1. This completes the proof. □

In views of Theorems 2.2–2.5, we can also obtain Theorems 2.6–2.8 respectively when $f(t,x)$ satisfies the Lipschitz condition on an interval. The proofs are similar to those of Theorems 2.2–2.4 and thus are omitted.

Theorem 2.6 *Assume that $H_1(t_0) < \infty$. If there exist a constant $L > 0$ and two functions $q \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and $f_0 \in \mathrm{C}([0,2H_1(t_0)],\mathbb{R})$ satisfying* (13), (14), *and*

$$\int_{t_0}^{\infty}\int_{u_2}^{\infty}\int_{t_0}^{u_1}\frac{q(u_0)}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2 < \infty,$$

then there exists $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that (1) *has two eventually positive solutions x_1 and x_2 tending to zero, which satisfy that $R_0(t,x_i(t)) > 0$, $R_1(t,x_i(t)) < 0$, $i = 1,2$, $R_2(t,x_1(t)) < 0$, $R_3(t,x_1(t)) > 0$, $R_2(t,x_2(t)) > 0$, and $R_3(t,x_2(t)) < 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$.*

Theorem 2.7 *Assume that $H_2(t_0) < \infty$. If there exist a constant $L > 0$ and two functions $q \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and $f_0 \in \mathrm{C}([0,2H_2(t_0)],\mathbb{R})$ satisfying (13),*

$$\bigl|f(t,x_1)-f(t,x_2)\bigr| \le L\cdot q(t)|x_1-x_2|, \quad x_1,x_2 \in \bigl[0,2H_2(t_0)\bigr],$$

and

$$\int_{t_0}^{\infty}\int_{t_0}^{u_1}\frac{q(u_0)}{r_1(u_1)}\Delta u_0\Delta u_1 < \infty,$$

then there exists $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that (1) has two eventually positive solutions x_1 and x_2 tending to zero, which satisfy that $R_0(t,x_i(t)) > 0$, $R_1(t,x_i(t)) < 0$, $R_2(t,x_i(t)) > 0$, $i = 1,2$, $R_3(t,x_1(t)) > 0$, and $R_3(t,x_2(t)) < 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$.

Theorem 2.8 *Assume that $H_3(t_0) < \infty$. If there exist a constant $L > 0$ and two functions $q \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and $f_0 \in \mathrm{C}([0,2H_3(t_0)],\mathbb{R})$ satisfying (13),*

$$\bigl|f(t,x_1)-f(t,x_2)\bigr| \le L\cdot q(t)|x_1-x_2|, \quad x_1,x_2 \in \bigl[0,2H_3(t_0)\bigr],$$

and

$$\int_{t_0}^{\infty} q(u_0)\Delta u_0 < \infty,$$

then there exists $T_1 \in [t_0,\infty)_{\mathbb{T}}$ such that (1) has an eventually positive solution x tending to zero, which satisfies that $R_0(t,x(t)) > 0$, $R_1(t,x(t)) < 0$, $R_2(t,x(t)) > 0$, and $R_3(t,x(t)) < 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$.

In addition, we also have the following conclusion.

Theorem 2.9 *Assume that one of the following conditions*

$$\int_{t_0}^{\infty}\frac{\Delta u_3}{r_3(u_3)} = \infty, \tag{16}$$

$$\int_{t_0}^{\infty}\int_{u_3}^{\infty}\frac{\Delta u_2\Delta u_3}{r_2(u_2)r_3(u_3)} = \infty, \tag{17}$$

and

$$\int_{t_0}^{\infty}\int_{u_3}^{\infty}\int_{u_2}^{\infty}\frac{\Delta u_1\Delta u_2\Delta u_3}{r_1(u_1)r_2(u_2)r_3(u_3)} = \infty \tag{18}$$

holds, then (1) has no eventually positive solution x, for which R_1, R_2, and R_3 are all eventually negative.

Proof Suppose that x is an eventually positive solution of (1) and there exists $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that, for $t \in [T_0,\infty)_{\mathbb{T}}$, we have

$$x(t) > 0, \quad R_1\bigl(t,x(t)\bigr) < 0, \qquad R_2\bigl(t,x(t)\bigr) < 0, \qquad R_3\bigl(t,x(t)\bigr) < 0.$$

There also exists $T_1 \in (T_0,\infty)_{\mathbb{T}}$ such that $g(t) \geq T_0$ and $h(t) \geq T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Substituting u_0 for t in (1) and integrating (1) with respect to u_0 from T_1 to u_1, where $u_1 \in [\sigma(T_1),\infty)_{\mathbb{T}}$, we have

$$R_3\bigl(u_1,x(u_1)\bigr) - R_3\bigl(T_1,x(T_1)\bigr) = -\int_{T_1}^{u_1} f\bigl(u_0,x\bigl(h(u_0)\bigr)\bigr)\Delta u_0 < 0,$$

which implies that

$$R_2^{\Delta}\bigl(u_1,x(u_1)\bigr) < \frac{R_3(T_1,x(T_1))}{r_1(u_1)}. \tag{19}$$

Integrating (19) with respect to u_1 from T_1 to u_2, where $u_2 \in [\sigma(T_1),\infty)_{\mathbb{T}}$, we have

$$R_2\bigl(u_2,x(u_2)\bigr) < R_2\bigl(T_1,x(T_1)\bigr) + R_3\bigl(T_1,x(T_1)\bigr)\int_{T_1}^{u_2}\frac{\Delta u_1}{r_1(u_1)}.$$

By analogy, we obtain

$$\begin{aligned} R_0\bigl(t,x(t)\bigr) < {}& R_0\bigl(T_1,x(T_1)\bigr) + R_1\bigl(T_1,x(T_1)\bigr)\int_{T_1}^{t}\frac{\Delta u_3}{r_3(u_3)} \\ &+ R_2\bigl(T_1,x(T_1)\bigr)\int_{T_1}^{t}\int_{T_1}^{u_3}\frac{\Delta u_2\Delta u_3}{r_2(u_2)r_3(u_3)} \\ &+ R_3\bigl(T_1,x(T_1)\bigr)\int_{T_1}^{t}\int_{T_1}^{u_3}\int_{T_1}^{u_2}\frac{\Delta u_1\Delta u_2\Delta u_3}{r_1(u_1)r_2(u_2)r_3(u_3)}. \end{aligned}$$

If one of (16)–(18) holds, then we derive $R_0(t,x(t)) \to -\infty$ as $t \to \infty$. However, we have $R_0(t,x(t)) = x(t) + p(t)x(g(t)) > 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. It causes a contradiction. This completes the proof. □

3 Examples

In this section, two interesting examples are provided to illustrate the conclusions.

Example 3.1 Let $\mathbb{T} = \bigcup_{n=1}^{\infty}[2n-1,2n]$. For $t \in [3,\infty)_{\mathbb{T}}$, consider

$$\biggl(t^{\alpha}\biggl(t^{\beta}\biggl(t^2\biggl(x(t) + p(t)x\biggl(t - \frac{\cos\pi t}{\pi}\biggr)\biggr)^{\Delta}\biggr)^{\Delta}\biggr)^{\Delta}\biggr)^{\Delta} + t\cdot x(t-2) = 0, \tag{20}$$

where p satisfies (C2). Here, we have $r_1(t) = t^{\alpha}$, $r_2(t) = t^{\beta}$, $r_3(t) = t^2$, $g(t) = t - \cos(\pi t)/\pi$, $h(t) = t-2$, $f(t,x) = t\cdot x$, and $t_0 = 3$. Moreover, we obtain

$$\int_{t_0}^{\infty}\frac{\Delta u_3}{r_3(u_3)} = \int_3^{\infty}\frac{\Delta u_3}{u_3^2} < \infty, \qquad H_1(t) = \int_t^{\infty}\frac{\Delta u_3}{u_3^2} < \frac{1}{2}, \quad t \in [3,\infty)_{\mathbb{T}},$$

and $\eta_1 = \lim_{t\to\infty} H_1(g(t))/H_1(t) = 1$. Hence, it fulfills conditions (C1)–(C5). Since $f(t,x)$ is nondecreasing with respect to x, when $\alpha > 3$ and $\beta > 1$, or $\alpha \leq 3$ and $\beta > 4-\alpha$, we have

$$\int_{t_0}^{\infty}\int_{t_0}^{u_2}\int_{t_0}^{u_1}\frac{f(u_0,2H_1(h(u_0)))}{r_1(u_1)r_2(u_2)}\Delta u_0\Delta u_1\Delta u_2$$

$$\le \int_3^\infty \int_3^{u_2} \int_3^{u_1} \frac{u_0}{u_1^\alpha u_2^\beta} \Delta u_0 \Delta u_1 \Delta u_2 < \frac{1}{2} \int_3^\infty \int_3^{u_2} \frac{\Delta u_1 \Delta u_2}{u_1^{\alpha-2} u_2^\beta} < \infty,$$

which means that (20) has two eventually positive solutions x_1 and x_2 tending to zero in terms of Theorem 2.1. Moreover, there exists $T_1 \in [t_0,\infty)_\mathbb{T}$ such that $R_0(t,x_i(t)) > 0$, $R_1(t,x_i(t)) < 0$, $i = 1,2$, $R_2(t,x_1(t)) < 0$, $R_3(t,x_1(t)) < 0$, $R_2(t,x_2(t)) > 0$, and $R_3(t,x_2(t)) > 0$ for $t \in [T_1,\infty)_\mathbb{T}$.

When $\alpha > 3$ and $\beta > 4-\alpha$, it follows that

$$\int_{t_0}^\infty \int_{u_2}^\infty \int_{t_0}^{u_1} \frac{f(u_0, 2H_1(h(u_0)))}{r_1(u_1)r_2(u_2)} \Delta u_0 \Delta u_1 \Delta u_2 < \frac{1}{2} \int_3^\infty \int_{u_2}^\infty \frac{\Delta u_1 \Delta u_2}{u_1^{\alpha-2} u_2^\beta} < \infty.$$

Hence, we deduce that (20) has two eventually positive solutions satisfying the conclusions of Theorem 2.2.

When $\alpha > 3$ and $\beta > 1$, we obtain $H_2(3) < \infty$. Then there exists a constant $M > 0$ such that

$$H_2(t) = \int_t^\infty \int_{u_3}^\infty \frac{\Delta u_2 \Delta u_3}{u_2^\beta u_3^2} \le M,$$

from which it follows that

$$\begin{aligned} &\int_{t_0}^\infty \int_{t_0}^{u_1} \frac{f(u_0, 2H_2(h(u_0)))}{r_1(u_1)} \Delta u_0 \Delta u_1 \\ &\quad \le 2M \int_3^\infty \int_3^{u_1} \frac{u_0}{u_1^\alpha} \Delta u_0 \Delta u_1 < 4M \int_3^\infty \frac{\Delta u_1}{u_1^{\alpha-2}} < \infty. \end{aligned}$$

Note that $\eta_2 = 1$. Therefore, (20) has two eventually positive solutions fulfilling the results of Theorem 2.3.

When $\alpha > 1$ and $\beta > 3-\alpha$, we obtain $H_3(3) < \infty$,

$$H_3(t) = \int_t^\infty \int_{u_3}^\infty \int_{u_2}^\infty \frac{\Delta u_1 \Delta u_2 \Delta u_3}{u_1^\alpha u_2^\beta u_3^2} = O\bigl(t^{1-\alpha-\beta}\bigr), \qquad \eta_3 = 1,$$

and

$$\int_{t_0}^\infty f\bigl(u_0, 2H_3\bigl(h(u_0)\bigr)\bigr) \Delta u_0 = \int_3^\infty O\biggl(\frac{1}{u_0^{\alpha+\beta-2}}\biggr) \Delta u_0 < \infty,$$

where the conclusions of Theorem 2.4 are satisfied.

On the other hand, consider conditions (16)–(18). Obviously, (16) does not hold here. Then (17) is satisfied when $\alpha \in \mathbb{R}$ and $\beta \le 1$, and (18) holds when $\alpha \le 1$ and $\beta \in \mathbb{R}$, or $\alpha > 1$ and $\beta \le 2-\alpha$. By virtue of Theorem 2.9, if these conditions of α and β are satisfied, then we can conclude that (20) has no eventually positive solution x, for which R_1, R_2, and R_3 are all eventually negative.

Example 3.2 Let $\mathbb{T} = [1,\infty)_\mathbb{R}$. For $t \in \mathbb{T}$, consider

$$\bigl(t^\alpha \bigl(t^\beta \bigl(t^3 \bigl(x(t) + p(t)x(t+1)\bigr)'\bigr)'\bigr)'\bigr)' + \frac{x^3(t)}{t} = 0, \tag{21}$$

where p satisfies (C2). Here, we have $r_1(t) = t^\alpha$, $r_2(t) = t^\beta$, $r_3(t) = t^3$, $g(t) = t + 1$, $h(t) = t$, $f(t,x) = x^3/t$, and $t_0 = 1$. Then, we take $q(t) = 1/t$ and $f_0(x) = x^3$.

Firstly, for $\alpha \ge 1$ and $\beta > 1$, we have

$$H_1(t_0) = \int_{t_0}^{\infty} \frac{du_3}{r_3(u_3)} = \int_1^{\infty} \frac{du_3}{u_3^3} = \frac{1}{2}, \qquad \eta_1 = 1,$$

$$\begin{aligned}\bigl|f(t,x_1) - f(t,x_2)\bigr| &= \frac{1}{t} \cdot |x_1 - x_2| \cdot \bigl|x_1^2 + x_1x_2 + x_2^2\bigr| \\ &\le 3 \cdot \frac{1}{t} \cdot |x_1 - x_2|, \quad x_1, x_2 \in [0,1],\end{aligned}$$

and

$$\int_{t_0}^{\infty} \int_{t_0}^{u_2} \int_{t_0}^{u_1} \frac{q(u_0)}{r_1(u_1)r_2(u_2)} \,du_0\,du_1\,du_2 = \int_1^{\infty} \int_1^{u_2} \int_1^{u_1} \frac{du_0\,du_1\,du_2}{u_0 u_1^\alpha u_2^\beta} < \infty,$$

from which we get the conclusion of Theorem 2.5. On the other hand, for $\alpha > 1$ and $\beta \ge 1$, we derive

$$\int_{t_0}^{\infty} \int_{u_2}^{\infty} \int_{t_0}^{u_1} \frac{q(u_0)}{r_1(u_1)r_2(u_2)} \,du_0\,du_1\,du_2 = \int_1^{\infty} \int_{u_2}^{\infty} \int_1^{u_1} \frac{du_0\,du_1\,du_2}{u_0 u_1^\alpha u_2^\beta} < \infty.$$

Hence, the result of Theorem 2.6 is obtained.

Secondly, for $\alpha > 1$ and $\beta > 1$, there exists a constant $M > 0$ such that

$$H_2(t_0) = \int_{t_0}^{\infty} \int_{u_3}^{\infty} \frac{du_2\,du_3}{r_2(u_2)r_3(u_3)} = \int_1^{\infty} \int_{u_3}^{\infty} \frac{du_2\,du_3}{u_2^\beta u_3^3} \le M, \qquad \eta_2 = 1.$$

Moreover, it follows that

$$\bigl|f(t,x_1) - f(t,x_2)\bigr| \le 12M^2 \cdot \frac{1}{t} \cdot |x_1 - x_2|, \quad x_1, x_2 \in [0, 2M]$$

and

$$\int_{t_0}^{\infty} \int_{t_0}^{u_1} \frac{q(u_0)}{r_1(u_1)} \,du_0\,du_1 = \int_1^{\infty} \int_1^{u_1} \frac{du_0\,du_1}{u_0 u_1^\alpha} < \infty.$$

Then we obtain the conclusion of Theorem 2.7.

Finally, we find that

$$\int_{t_0}^{\infty} q(u_0)\,du_0 = \int_1^{\infty} \frac{du_0}{u_0} = \infty,$$

so the result of Theorem 2.8 seems not to be deduced. However, for $\alpha > 1$ and $\beta > 1$, in view of Theorem 2.4, we have $H_3(1) < \infty$,

$$H_3(t) = \int_t^{\infty} \int_{u_3}^{\infty} \int_{u_2}^{\infty} \frac{du_1\,du_2\,du_3}{u_1^\alpha u_2^\beta u_3^3} = O\bigl(t^{-\alpha-\beta}\bigr), \qquad \eta_3 = 1,$$

and

$$\int_{t_0}^{\infty} f\big(u_0, 2H_3\big(h(u_0)\big)\big)\,du_0 = \int_3^{\infty} \frac{du_0}{O(u_0^{3\alpha+3\beta+1})} < \infty.$$

Therefore, we still derive the result of Theorem 2.4 (or Theorem 2.8).

4 Conclusion

In this paper, we successfully obtain some new results for the existence of nonoscillatory solutions tending to zero of a class of fourth-order nonlinear neutral dynamic equations on time scales. Moreover, compared with the existing references, the assumptions of functions f and g are more relaxed. According to this technique, we can continue to study the existence of nonoscillatory solutions tending to zero of similar forms of higher-order nonlinear neutral dynamic equations on time scales.

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